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INVARIANT OPERATORS OF THE FIRST ORDER ON MANIFOLDS WITH A GIVEN PARABOLIC STRUCTURE

by

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Abstract. — The goal of this paper is to describe explicitly all invariant first order operators on manifolds equipped with parabolic geometries. Both the results and the methods present an essential generalization of Fegan's description of the first order invariant operators on conformal Riemannian manifolds. On the way to the results, we present a short survey on basic structures and properties of parabolic geometries, together with links to further literature.

Résumé **(Opérateurs invariants d'ordre 1 sur des variétés paraboliques).** — Le but de l'article est de décrire explicitement tous les opérateurs différentiels invariants d'ordre un sur les variétés munies d'une structure de géométrie parabolique (les espaces généralisés d'Élie Cartan). Les résultats, ainsi que les méthodes, généralisent un résultat de Fegan sur la classification des opérateurs différentiels d'ordre un sur une variété munie d'une structure conforme. Au passage, nous donnons un bref resumé des propriétés fondamentales des espaces généralisés d'É. Cartan et du calcul différentiel sur ces espaces.

1. Setting of the problem

Invariant operators appear in many areas of global analysis, geometry, mathematical physics, etc. Their analytical properties depend very much on the symmetry groups, which in turn determine the type of the background geometries of the underlying manifolds. The most appealing example is the so called conformal invariance of many distinguished operators like Dirac, twistor, and Yamabe operators in Riemannian geometry which lead to the study of all these operators in the framework of the natural bundles for conformal Riemannian geometries. Of course, mathematicians suggested a few schemes to classify all such operators and to discuss their properties from a universal point of view, usually consisting of a combination of geometric and

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algebraic tools. See e.g. [**41, 42, 43, 6, 7, 8, 33, 9, 28, 10**]. All of them combine, in different ways, ideas of representation theory of Lie algebras with differential geometry and global analysis.

In the context of problems in twistor theory and its various generalizations, the more general framework of representation theory of parabolic subgroups in semisimple Lie groups was suggested and links to the infinite dimensional representation theory were exploited, see e.g. the pioneering works [**4, 5**]. The close relation to the Tanaka's theory (cf. [**39, 40, 17, 44, 32, 13**]) was established and we may witness a fruitful interaction of all these ideas and the classical representation theory nowadays, see e.g. [**2, 3, 12, 14, 15, 16, 18, 22, 23, 24, 25**].

1.1. Parabolic geometries. — The name *parabolic geometry* was introduced in [**26**], following Fefferman's concept of *parabolic invariant theory*, cf. [**19, 20**], and it seems to be commonly adopted now. The general background for these geometries goes back to Klein's definition of geometry as the study of homogeneous spaces, which play the role of the flat models for geometries in the Cartan's point of view. Thus, following Cartan, the (curved) geometry in question on a manifold M is given by a first order object on a suitable bundle of frames, an absolute parallelism $\omega : T\mathcal{G} \to \mathfrak{g}$ for a suitable Lie algebra $\mathfrak{g}$ defined on a principal fiber bundle $\mathcal{G} \to M$ with structure group P whose Lie algebra is contained in $\mathfrak{g}$. On the Klein's homogeneous spaces themselves, there is the canonical choice — the left–invariant Maurer–Cartan form ω while on general $\mathcal{G}$, ω has to be equivariant with respect to the adjoint action and to recover the fundamental vector fields. These objects are called *Cartan connections* and they play the role of the Levi–Civita connections in Riemannian geometry in certain extent. A readable introduction to this background in a modern setting is to be found in [**35**]. The parabolic geometries, real or complex, are just those corresponding to the choices of parabolic subgroups in real or complex Lie groups, respectively.

Each linear representation $\mathbb{E}$ of the (parabolic) structure group P gives rise to the homogeneous vector bundle $E(G/P)$ over the corresponding homogeneous space G/P, and similarly there are the natural vector bundles $\mathcal{G}\times_P\mathbb{E}$ associated to each parabolic geometry on a manifold M. Analogously, more general natural bundles $\mathcal{G}\times_P\mathbb{S}$ are obtained from actions of P on manifolds $\mathbb{S}$.

Morphisms $\varphi : (\mathcal{G},\omega) \to (\mathcal{G}',\omega')$ are principal fiber bundle morphisms with the property $\varphi^*\omega' = \omega$. Obviously, the construction of the natural bundles is functorial and so we obtain the well defined action of morphisms of parabolic geometries on the sheaves of local sections of natural bundles. In particular, the *invariant operators* on manifolds with parabolic geometries are then defined as those operators on such sections commuting with the above actions.

1.2. First order linear operators. — In this paper, first order linear differential operators between natural vector bundles $E(M)$, $E'(M)$ are just those differential operators which are given by linear morphisms $J^1E(M) \to E'(M)$. For example, for conformal Riemannian geometries this means that the (conformal) metrics may enter in any differential order in their definition.

The mere existence of the absolute parallelism ω among the defining data for a parabolic geometry on M yields an identification of all first jet prolongations J^1EM of natural bundles with natural bundles $\mathcal{G} \times_P J^1\mathbb{E}$ for suitable representations $J^1\mathbb{E}$ of P, see 2.4 below. Moreover, there is the well known general relation between invariant differential operators on homogeneous vector bundles and the intertwining morphisms between the corresponding jet modules. Thus, we see immediately that each first order invariant operator between homogeneous vector bundles over G/P extends canonically to the whole category of parabolic geometries of type (G, P). We may say that they are given explicitly by their symbols (which are visible on the flat model G/P) and by the defining Cartan connections ω.

On the other hand, the invariants of the geometries may enter into the expressions of the invariant operators, i.e. we should consider also all possible contributions from the curvature of the Cartan connection ω. This leads either to operators which are not visible at all on the (locally) flat models, or to those which share the symbols with the above ones and again the difference cannot be seen on the flat models.

In this paper we shall not deal with such curvature contributions. In fact, we classify all invariant first order operators between the homogeneous bundles over the flat models, which is a purely algebraic question. In the above mentioned sense, they all extend canonically to all curved geometries.

At the same time, there are strict analogies to the Weyl connections from conformal Riemannian geometries available for all parabolic geometries and so we shall also be able to provide explicit universal formulae for all such operators from the classification list in terms of these linear connections on the underlying manifolds.

This was exactly the output of Fegan's approach in the special case of $G = \mathrm{SO}(m+1, 1)$, P the Poincaré conformal group, which corresponds to the conformal Riemannian geometries, [**21**]. Since the conformal Riemannian geometries are uniformly one–flat (i.e. the canonical torsion vanishes), this also implies that all first order operators on (curved) conformal manifolds, which depend on the conformal metrics up to the first order, are uniquely given by their restrictions to the flat conformal spheres. We recover and vastly extend his approach. In particular, we prove the complete algebraic classification for all parabolic subgroups in semisimple Lie groups G. Moreover, rephrasing the first order dependence on the structure itself by the assumption on the homogeneity of the operator, we obtain the unique extension of our operators for all parabolic geometries with vanishing part of torsion of homogeneity one.

We also show that compared to the complexity of the so called standard operators of all orders in the Bernstein–Gelfand–Gelfand sequences, constructed first in [**16**] and developed much further in [**11**], the original Fegan's approach to first order operators is surprisingly powerful in the most general context.

Although the algebraic classification of the invariant operators does not rely on the next section devoted to a survey on general parabolic geometries, we prefer to include a complete line of arguments leading to full understanding of the curved extensions of the operators and their explicit formulae in terms of the underlying Weyl connections.

2. Parabolic geometries, Weyl connections, and jet modules

2.1. Regular infinitesimal flag structures. — The homogeneous models for parabolic geometries are the (real or complex) generalized flag manifolds G/P with G semisimple, P parabolic. It is well known that on the level of the Lie algebras, the choice of such a pair $(\mathfrak{g},\mathfrak{p})$ is equivalent to a choice of the so called $|k|$–grading of a semisimple $\mathfrak{g}$

$$\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_k$$
$$\mathfrak{p} = \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_k$$
$$\mathfrak{g}_- = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1} \simeq \mathfrak{g}/\mathfrak{p}.$$

Then the Cartan–Killing form provides the identification $\mathfrak{g}_i^* = \mathfrak{g}_{-i}$ and there is the Hodge theory on the cohomology $H^*(\mathfrak{g}_-, \mathbb{W})$ for any $\mathfrak{g}$–module $\mathbb{W}$, cf. [**40, 44, 13, 16**].

Now, the Maurer–Cartan form ω distributes these gradings to all frames $u \in G$ and all P–equivariant data are projected down to the flag manifolds G/P. This construction goes through for each Cartan connection of type (G, P) and so there is the filtration

$$TM = T^{-k}M \supset T^{-k+1}M \supset \cdots \supset T^{-1}M \tag{1}$$

on the tangent bundle TM of each manifold M underlying the principal fiber bundle $\mathcal{G} \to M$ with Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$, induced by the inverse images of the P–invariant filtration of $\mathfrak{g}$. Moreover, the same absolute parallelism ω induces the reduction of the structure group of the associated graded tangent bundle

$$\operatorname{Gr} TM = (T^{-k}M/T^{-k+1}) \oplus \cdots \oplus (T^{-2}M/T^{-1}M) \oplus T^{-1}M$$

to the reductive part G_0 of P. In particular, this reduction introduces an algebraic bracket on $\operatorname{Gr} TM$ which is the transfer of the G_0–equivariant Lie bracket in $\mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1}$.

Next, let M be any manifold, $\dim M = \dim \mathfrak{g}_-$. An *infinitesimal flag structure of type* (G, P) on M is given by a filtration (1) on TM together with the reduction of the associated graded tangent bundle to the structure group G_0 of the form $\operatorname{Gr} T_xM \simeq \operatorname{Gr} \mathfrak{g}_-$, with the freedom in G_0, at each $x \in M$.

Let us write $\{\ ,\ \}_{\mathfrak{g}_0}$ for the induced algebraic bracket on $\operatorname{Gr} TM$. The infinitesimal flag structure is called *regular* if $[T^iM, T^jM] \subset T^{i+j}M$ for all $i, j < 0$ and the algebraic bracket $\{\ ,\ \}_{\mathrm{Lie}}$ on $\mathrm{Gr}\, TM$ induced by the Lie brackets of vector fields on M coincides with $\{\ ,\ \}_{\mathfrak{g}_0}$. It is not difficult to observe that the infinitesimal structures underlying Cartan connections ω are regular if and only if there are only positive homogeneous components of the curvature κ of ω, cf. [**34, 14**].

The remarkable conclusion resulting from the general theory claims that for each regular infinitesimal flag structure of type (G,P) on M, under suitable normalization of the curvature κ (its co–closedness), there is a unique Cartan bundle $\mathcal{G} \to M$ and a unique Cartan connection ω on $\mathcal{G}$ of type (G,P) which induces the given infinitesimal flag structure, up to isomorphisms of parabolic geometries and with a few exceptions, see [**40, 32, 13**] or [**14**], sections 2.7–2.11., for more details.

2.2. Examples. — The simplest and best known situation occurs for $|1|$–graded algebras, i.e. $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$. Then the filtration is trivial, $TM = T^{-1}M$, and the regular infinitesimal flag structures coincide with standard G_0–structures, i.e. reductions of the structure group of TM to G_0. The examples include the conformal, almost Grassmannian, and almost quaternionic structures. The projective structures correspond to $\mathfrak{g} = \mathfrak{sl}(m+1,\mathbb{R})$, $\mathfrak{g}_0 = \mathfrak{gl}(m,\mathbb{R})$, and this is one of the exceptions where some more structure has to be chosen in order to construct the canonical Cartan connection ω. The series of papers [**15**] is devoted to all these geometries.

Next, the $|2|$–graded examples include the so called parabolic contact geometries and, in particular, the hypersurface type non–degenerate CR-structures. See e.g. [**44, 14**] for more detailed discussions. Further examples of geometries are given by the Borel subalgebras in semisimple Lie algebras, and they are modeled on the full flag manifolds G/P.

2.3. The invariant differential. — The Cartan connection ω defines the *constant vector fields* $\omega^{-1}(X)$ on $\mathcal{G}$, $X \in \mathfrak{g}$. They are defined by $\omega(\omega^{-1}(X)(u)) = X$, for all $u \in \mathcal{G}$. In particular, $\omega^{-1}(Z)$ is the fundamental vector field if $Z \in \mathfrak{p}$. The constant fields $\omega^{-1}(X)$ with $X \in \mathfrak{g}_-$ are called *horizontal*.

Now, let us consider any natural vector bundle $EM = \mathcal{G} \times_P \mathbb{E}$. Its sections may be viewed as P–equivariant functions $s : \mathcal{G} \to \mathbb{E}$ and the Lie derivative of functions with respect to the constant horizontal vector fields defines the *invariant derivative* (with respect to ω)

$$\nabla^\omega : C^\infty(\mathcal{G},\mathbb{E}) \to C^\infty(\mathcal{G},\mathfrak{g}_-^* \otimes \mathbb{E})$$
$$\nabla^\omega s(u)(X) = \mathcal{L}_{\omega^{-1}(X)} s(u).$$

We also write $\nabla^\omega_X s$ for values with the fixed argument $X \in \mathfrak{g}_-$.

The invariant differentiation is a helpful substitute for the Levi–Civita connections in Riemannian geometry, but it has an unpleasant drawback: it does not produce P–equivariant functions even if restricted to equivariant $s \in C^\infty(\mathcal{G},\mathbb{E})^P$. One possibility how to deal with that is to extend the derivative to all constant fields, i.e. to consider

$\nabla : C^\infty(\mathcal{G},\mathbb{E}) \to C^\infty(\mathcal{G},\mathfrak{g}^*\otimes\mathbb{E})$ which preserves the equivariance. This is a helpful approach in the so called twistor and tractor calculus, see e.g. [**12, 11**]. In this paper, however, we shall stick to horizontal arguments only.

An easy computation reveals the (generalized) Ricci and Bianchi identities and a quite simple calculus is available, cf. [**16, 14, 11**].

2.4. Jet modules. — Let us consider a fixed P–module $\mathbb{E}$ and write λ for the action of $\mathfrak{p}$ on $\mathbb{E}$. The action of $g \in G$ on the sections of $E(G/P)$ is given by $s \mapsto s \circ \ell_{g^{-1}}$, where ℓ is the left multiplication on G, and this defines also the action of P on the one–jets $j^1_o s$ at the origin. A simple check reveals the formula for the induced action of the Lie algebra $\mathfrak{p}$ on the vector space $J^1\mathbb{E} = \mathbb{E} \oplus (\mathfrak{g}_-^* \otimes \mathbb{E})$ of all such jets:

$$Z\cdot(v,\varphi) = \big(\lambda(Z)(v), \lambda(Z)\circ\varphi - \varphi\circ\operatorname{ad}_-(Z) + \lambda(\operatorname{ad}_{\mathfrak{p}}(Z)(\_))(v)\big) \tag{2}$$

where the subscripts at the adjoint operator indicate the splitting of the values according to the components of $\mathfrak{g}$. In particular, the action of the reductive part G_0 of P is given by the obvious tensor product, while the nilpotent part mixes the values with the derivatives. We call the resulting P–module $J^1\mathbb{E}$ the *first jet prolongation* of the module $\mathbb{E}$. Moreover, each P–module homomorphism $\alpha : \mathbb{E} \to \mathbb{F}$ extends to a P–module homomorphism $J^1\alpha : J^1\mathbb{E} \to J^1\mathbb{F}$ by composition on values.

Another simple computation shows that the invariant differentiation ∇^ω defines the mapping $\iota : C^\infty(\mathcal{G},\mathbb{E}_\lambda)^P \to C^\infty(\mathcal{G}, J^1\mathbb{E}_\lambda)^P$

$$\iota(s)(u) = (s(u), (X \longmapsto \nabla^\omega s(u)(X)))$$

which yields diffeomorphisms $J^1EM \simeq \mathcal{G}\times_P J^1\mathbb{E}$, for all parabolic geometries $(\mathcal{G},\omega)$. Moreover, for each fiber bundle morphism $f : EM \to FM$ given by a P–module homomorphism $\alpha : \mathbb{E} \to \mathbb{F}$, the first jet prolongation $J^1 f$ corresponds to the P–module homomorphism $J^1\alpha$. See e.g. [**16, 37**] for more detailed exposition.

Iteration of the above consideration leads to the crucial identification of semi–holonomic prolongations $\bar{J}^k EM$ of natural vector bundles with natural vector bundles associated to semi–holonomic jet modules $\bar{J}^k\mathbb{E}$. Thus, P–module homomorphisms $\Psi : \bar{J}^k\mathbb{E} \to \mathbb{F}$ always provide invariant operators by composition with the iterated invariant derivative ∇^ω. Such operators are called *strongly invariant*, cf. [**16**]. This is at the core of the general construction of the invariant operators of all orders in [**15, 16**]. In this paper, however, only first order operators are treated and so we skip more explicit description of the higher order jet modules.

2.5. Weyl connections. — Let $(\mathcal{G},\omega)$ be a parabolic geometry on a smooth manifold M, P the structure group of $\mathcal{G}$ and G_0 its reductive part. Let us write P_+ for the exponential image of $\mathfrak{p}_+ = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$ and consider the quotient bundle $\mathcal{G}_0 = \mathcal{G}/P_+$. Thus we have the tower of principal fiber bundles

$$\mathcal{G} \xrightarrow{\ \pi\ } \mathcal{G}_0 \xrightarrow{\ p_0\ } M$$

with structure groups P_+ and G_0 and, of course, there is the action of G_0 on the total space of $\mathcal{G}$.

For each smooth parabolic geometry, there exist global G_0–equivariant sections σ of π and the space of all of them is an affine space modeled on $\Omega^1(M)$, the one forms on the underlying manifold, see [**14**]. Each such section σ is called a *Weyl structure* for the parabolic geometry on M.

Each Weyl structure σ provides the reduction of the structure group P to its reductive part G_0 and the pullback of the Cartan connection, which splits according to the values:

$$\sigma^*\omega = \sigma^*(\omega_-) + \sigma^*(\omega_0) + \sigma^*(\omega_+).$$

The negative part $\sigma^*\omega_-$ yields the identification of TM and $\operatorname{Gr} TM$ and may be also viewed as the soldering form of $\mathcal{G}_0$. The $\mathfrak{g}_0$ component is a linear connection on M and we call it the *Weyl connection*. Let us also notice that the non–positive parts provide a Cartan connection of the type $(G/P_+, P/P_+)$. In particular, the usual Weyl connections are recovered for the conformal Riemannian geometries.

Now, consider a P–module $\mathbb{E}$ and the natural bundle EM. Chosen a Weyl structure σ, we obtain $EM = \mathcal{G}_0 \times_{G_0} \mathbb{E}$ and we have introduced two differentials on sections: the invariant differential

$$(\nabla^\omega s)\circ\sigma : (u, X) \longmapsto \mathcal{L}_{\omega^{-1}(X)} s(\sigma(u))$$

and the covariant differential of the Weyl connection

$$\nabla^\sigma(s\circ\sigma) : (u, X) \longmapsto \mathcal{L}_{(\sigma^*(\omega_-+\omega_0))^{-1}(X)}(s\circ\sigma)(u).$$

If the action of the nilpotent part P_+ on $\mathbb{E}$ is trivial (in particular if $\mathbb{E}$ is irreducible), then the restriction of the invariant differential to the image of σ clearly coincides with the covariant differential with respect to the Weyl connection.

Obviously, each first order differential operator $C^\infty(EM) \to C^\infty(FM)$ may be written down by means of the invariant differential. If it is invariant, then it comes from a P–module homomorphism $J^1\mathbb{E} \to \mathbb{F}$, but then it must be given by the same formula in terms of all Weyl connections. On the other hand, a change of the Weyl structure σ implies also the change of the Weyl connection. The general formula for the difference in terms of the one–forms modeling the space of Weyl structures is given in [**14**], Proposition 3.9. We shall need a very special case only which will be easily deduced below. In particular, we shall see that if a formula for first order operator in terms of the Weyl connections does not depend on the choice, then it is given by a homomorphism. This shows that the usual definition of the invariance in conformal Riemannian geometry coincides with our general categorical definition in the first order case. There are strong indications that this observation is valid even for non–linear operators of all orders, cf. [**36**].

3. Algebraic characterization of first order operators

3.1. Restricted jets. — The distinguished subspaces $T^{-1}M$ in the tangent spaces of manifolds with parabolic geometries suggest to deal with partially defined derivatives — those in directions in $T^{-1}M$ only.

In computations below, we shall often use actions of $\mathfrak{p}$ on various modules. To avoid an awkward notation, the action will be denoted by the symbol $\cdot$, it is easy to see from the context what are the modules considered. We shall also write $\mathbb{E}_\lambda$ for the $\mathfrak{p}$–module corresponding to the representation $\lambda : \mathfrak{p} \to GL(\mathbb{E}_\lambda)$, and $E_\lambda M \to M$ will be the corresponding natural vector bundle over M. (In some context, λ may also be the highest weight determining an irreducible module.)

First we rewrite slightly the $\mathfrak{p}$–action (2) on $J^1\mathbb{E}_\lambda = \mathbb{E}_\lambda \oplus (\mathfrak{g}_-^* \otimes \mathbb{E}_\lambda)$. Recall that the Killing form provides the dual pairing $\mathfrak{g}_-^* \simeq \mathfrak{p}_+$ and so we have for all $Y \otimes v \in p_+ \otimes \mathbb{E}_\lambda$, $X \in \mathfrak{g}_-$, $Z \in \mathfrak{p}$

$$\begin{aligned}(Y \otimes v)(\mathrm{ad}_-(Z)(X)) = \langle \mathrm{ad}_-(Z)(X), Y\rangle v = \\ = \langle [Z, X], Y\rangle v = -\langle X, [Z, Y]\rangle v = -([Z, Y] \otimes v)(X).\end{aligned}$$

For a fixed dual linear basis $\xi_\alpha \in \mathfrak{g}_-$, $\eta^\alpha \in \mathfrak{p}_+$ we can also rewrite the term

$$\lambda(\mathrm{ad}_{\mathfrak{p}}(Z)(X))(v) = \sum_\alpha \eta^\alpha \otimes [Z, \xi_\alpha]_{\mathfrak{p}} \cdot v.$$

Thus the 1–jet action of $Z \in \mathfrak{p}$ on $J^1\mathbb{E}_\lambda = \mathbb{E}_\lambda \oplus (\mathfrak{p}_+ \otimes \mathbb{E}_\lambda)$ is

$$J^1\lambda(Z)(v_0, Y_1 \otimes v_1) = \big(Z \cdot v_0, Y_1 \otimes Z \cdot v_1 + [Z, Y_1] \otimes v_1 + \textstyle\sum_\alpha \eta^\alpha \otimes [Z, \xi_\alpha]_{\mathfrak{p}} \cdot v_0\big).$$

Let $\mathfrak{p}_+^2$ denote the subspace $[\mathfrak{p}_+, \mathfrak{p}_+]$ in $\mathfrak{p}$. There is the $\mathfrak{p}$–invariant vector subspace $\{0\} \oplus (\mathfrak{p}_+^2 \otimes \mathbb{E}_\lambda) \subset J^1\mathbb{E}_\lambda$ and we define the $\mathfrak{p}$-module

$$J^1_{\mathcal{R}}\mathbb{E}_\lambda = J^1\mathbb{E}_\lambda/(\{0\} \oplus (\mathfrak{p}_+^2 \otimes \mathbb{E}_\lambda)) \simeq \mathbb{E}_\lambda \oplus ((\mathfrak{p}_+/\mathfrak{p}_+^2) \otimes \mathbb{E}_\lambda) \simeq \mathbb{E}_\lambda \oplus (\mathfrak{g}_{-1}^* \otimes \mathbb{E}_\lambda).$$

The induced action of $Z \in \mathfrak{p}$ on $J^1_{\mathcal{R}}\mathbb{E}$ is

$$J^1_{\mathcal{R}}\lambda(Z)(v_0, Y_1 \otimes v_1) = \big(Z.v_0, Y_1 \otimes Z.v_1 + [Z, Y_1]_{\mathfrak{g}_1} \otimes v_1 + \textstyle\sum_{\alpha'} \eta^{\alpha'} \otimes [Z, \xi_{\alpha'}]_{\mathfrak{p}} \cdot v_0\big)$$

where $\eta^{\alpha'}$ and $\xi_{\alpha'}$ are dual bases of $\mathfrak{g}_{\pm 1}$ and $Y \in \mathfrak{g}_1; v_0, v_1 \in \mathbb{E}_\lambda$. The latter formula gets much simpler if λ is a G_0-representation extended trivially to the whole P. Then for each $W \in \mathfrak{g}_0$, $Z \in \mathfrak{g}_1$

$$\begin{aligned}J^1_{\mathcal{R}}\lambda(W)(v_0, Y_1 \otimes v_1) &= (W \cdot v_0, Y_1 \otimes W \cdot v_1 + [W, Y_1] \otimes v_1) \\ J^1_{\mathcal{R}}\lambda(Z)(v_0, Y_1 \otimes v_1) &= \big(0, \textstyle\sum_{\alpha'} \eta^{\alpha'} \otimes [Z, \xi_{\alpha'}] \cdot v_0\big)\end{aligned}$$

while the action of $[\mathfrak{p}_+, \mathfrak{p}_+]$ is trivial. Exactly as with the functor J^1, the action of $J^1_{\mathcal{R}}$ on $(G_0, \mathfrak{p})$–module homomorphisms is given by the composition.

The associated fiber bundle $J^1_{\mathcal{R}}EM : \mathcal{G} \times_P J^1_{\mathcal{R}}\mathbb{E}_\lambda$ is called the *restricted first jet prolongation* of the natural bundle EM. The invariant differential provides a natural mapping $J^1EM \to J^1_{\mathcal{R}}EM$.

The inductive construction of the semi–holonomic jet prolongations of $(G_0, \mathfrak{p})$–modules can be now repeated with the functor $J^1_{\mathcal{R}}$. The resulting $\mathfrak{p}$–modules are the equalizers of the two natural projections $J^1_{\mathcal{R}}(\bar{J}^k_{\mathcal{R}}\mathbb{E}_\lambda) \to \bar{J}^k_{\mathcal{R}}\mathbb{E}_\lambda$ and, as $\mathfrak{g}_0$-modules, they are equal to

$$\bar{J}^k_{\mathcal{R}}\mathbb{E}_\lambda = \bigoplus_{i=0}^{k} (\otimes^i \mathfrak{g}_1 \otimes \mathbb{E}_\lambda).$$

This construction leads to *restricted semi-holonomic prolongations* of $E_\lambda M$ and $\mathbb{E}_\lambda$ but we shall need only the first order case here.

3.2. Lemma. — *Let $\mathbb{E}$ and $\mathbb{F}$ be irreducible P–modules. Then a G_0 module homomorphism $\Psi : J^1\mathbb{E} \to \mathbb{F}$ is a P–module homomorphism if and only if Ψ factors through $J^1_{\mathcal{R}}\mathbb{E}$ and for all $Z \in \mathfrak{g}_1$*

$$\Psi\left(\sum_{\alpha'} \eta^{\alpha'} \otimes [Z, \xi_{\alpha'}] \cdot v_0\right) = 0, \tag{3}$$

where $\eta^{\alpha'}$, $\xi_{\alpha'}$ is a dual basis of $\mathfrak{g}_{\pm 1}$.

Proof. — Since both $\mathbb{E}$ and $\mathbb{F}$ are irreducible, the action of $\mathfrak{p}_+$ on both is trivial. Thus, each P–homomorphism Ψ must vanish on the image of the P–action on $J^1\mathbb{E}$. Moreover, either $\mathbb{E}$ is isomorphic to $\mathbb{F}$ (and then Ψ is given by the projection to values composed with the identity), or Ψ is supported in the G_0–submodule $\mathfrak{p}_+ \otimes \mathbb{E}$. Further, recall there is the grading element E in the center of $\mathfrak{g}_0$ which acts by j on each $\mathfrak{g}_j \subset \mathfrak{g}$. The intertwining with the grading element implies that Ψ is in fact supported in $\mathfrak{g}_j \otimes \mathbb{E}$ for suitable $j > 0$.

Now, let us fix dual basis η^α, ξ_α of $\mathfrak{p}_+$ and $\mathfrak{g}_-$. For all $Z \in \mathfrak{g}_i$, $i > 0$, and $(v_0, Y \otimes v_1) \in J^1\mathbb{E}_\lambda$, the formula (2) yields the condition

$$0 = \Psi\left([Z, Y] \otimes v_1 + \sum_\alpha \eta^\alpha \otimes [Z, \xi_\alpha]_{\mathfrak{g}_0} \cdot v_0\right).$$

In particular, let us insert $v_0 = 0$ and recall that the whole $\mathfrak{p}_+$ is spanned by $\mathfrak{g}_1$. Thus we obtain $\Psi(\mathfrak{g}_j \otimes \mathbb{E}) = 0$ for all $j > 1$ and this means that Ψ factors through the restricted jets, as required.

Now, looking again at the jet–action (2), we derive the condition (3). On the other hand, each G_0–homomorphism which factors through the derivative part of the restricted jets and satisfies (3) clearly is a P–module homomorphism. □

In the Lemma above, we have considered an endomorphism of Φ from $\mathfrak{g}_1 \otimes \mathbb{E}_\lambda$ defined by

$$\Phi(Z \otimes v) := \sum_{\alpha'} \eta^{\alpha'} \otimes [Z, \xi_{\alpha'}] \cdot v. \tag{4}$$

The Lemma is saying that the G_0-homomorphism Ψ is a P-module homomorphism if and only if it annihilates the image of Φ. By the Schur lemma, the map Φ is a multiple of identity on any irreducible piece in the tensor product. In the next section, we shall compute the corresponding values of Φ on irreducible components using known formulae for Casimir operators.

3.3. The explicit formulae. — The above explicit description of the P–module homomorphisms Ψ represent at the same time explicit formulae for the invariant operators in terms of the Weyl connections. Indeed, we have simply to write down the composition $\Psi \circ \nabla$ using the frame form of the covariant derivative with respect to any of the Weyl connections. By the general theory discussed in Section 2, such formula does not depend on the choice of the Weyl connection ∇ and all invariant first order operators have this form, up to possible curvature contributions.

4. Casimir computations

In Lemma 3.2, we derived an algebraic condition for first order invariant operators on sections of natural bundles for a given parabolic geometry. Here we want to translate this algebraic condition into an explicit formula for highest weights of considered modules using Casimir computations.

4.1. Representations of reductive groups. — Irreducible representations of a (complex) semisimple Lie algebra $\mathfrak{g}$ are classified by their highest weights $\lambda \in \mathfrak{h}^*$, where $\mathfrak{h}$ is a chosen Cartan subalgebra of $\mathfrak{g}$.

A reductive algebra $\mathfrak{g}_0 = \mathfrak{a} \oplus \mathfrak{g}_0^s$ is a direct sum of a commutative algebra $\mathfrak{a}$ and a semisimple algebra $\mathfrak{g}_0^s$ (which can be trivial). Irreducible representations of $\mathfrak{g}_0$ are tensor products of irreducible representations of both summands, irreducible representations of $\mathfrak{a}$ are characterized by an element of $\mathfrak{a}^*$.

In the paper, we shall consider the situation where $\mathfrak{g}$ is a $|k|$-graded (complex) semisimple Lie algebra and $\mathfrak{g}_0$ is its reductive part. The grading element E has eigenvalues j on $\mathfrak{g}_j$ and a Cartan algebra $\mathfrak{h}$ and the set Σ of simple roots can be chosen in such a way that $E \in \mathfrak{h} \subset \mathfrak{g}_0$ and all positive root spaces of $\mathfrak{g}$ are contained in the parabolic subalgebra $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{p}_+$. In this situation, irreducible representations of $\mathfrak{g}_0$ are characterized by an element $\lambda \in \mathfrak{h}^*$ with the property that λ restricted to $\mathfrak{h} \cap \mathfrak{g}_0^s$ is a dominant integral weight for $\mathfrak{g}_0^s$. Such a highest weight λ will be called dominant weight for $\mathfrak{p}$. Moreover, we have at our disposal invariant (nondegenerate) forms $(\cdot, \cdot)$ for $\mathfrak{g}$, their restrictions to $\mathfrak{h}$ are nondegenerate as well. It will be convenient (see e.g. **[9, 15]**) to normalize the choice of the invariant form by the requirement $(E, E) = 1$ (so that it is the Killing form scaled by the factor $(2 \dim \mathfrak{g}_+)^{-1}$). The restriction of this form to $\mathfrak{g}_0$ is nondegenerate and the spaces $\mathfrak{g}_j$ are dual to $\mathfrak{g}_{-j}, j > 0$.

4.2. A formula for the Casimir operator. — Let us suppose that a parabolic subalgebra $\mathfrak{p}$ in a (complex) semisimple Lie algebra $\mathfrak{g}$ is given. We need below a formula for the value of the quadratic Casimir element c on an irreducible representation of the reductive part $\mathfrak{g}_0$ of $\mathfrak{p}$ characterized by a weight $\lambda \in \mathfrak{h}^*$. Such a formula is well known for the case of semisimple Lie algebra and can be easily adapted for our case.

Lemma. — *Let $\mathfrak{g}_0$ be the reductive part of a (complex) graded semisimple Lie algebra $\mathfrak{g}$. Let Π_0 be the set of all positive roots $\alpha \in \mathfrak{h}^*$ for $\mathfrak{g}$ for which $\mathfrak{g}_\alpha \subset \mathfrak{g}_0$ and let us define ρ_0 by $\rho_0 = \frac{1}{2}\sum_{\alpha\in\Pi_0}\alpha$ (for the Borel case $\rho_0 = 0$).*

Let c be the quadratic Casimir element in the universal enveloping algebra of $\mathfrak{g}_0$ (with respect to the chosen invariant form $(\cdot,\cdot)$ on $\mathfrak{g}$) and let $\mathbb{E}_\lambda, \lambda \in \mathfrak{h}^*$ be an irreducible representation of $\mathfrak{g}_0$. Then the value of c on $\mathbb{E}_\lambda$ is given by

$$c = (\lambda, \lambda + 2\rho_0).$$

Proof. — Due to the fact that $\mathfrak{g}_0$ is the reductive part of $\mathfrak{g}$ and that we use the invariant form $(\cdot,\cdot)$ for the whole algebra $\mathfrak{g}$, the proof follows the same lines of argument as in the semisimple case (see [**27**], p.118]).

Let $\{h_a\}$, resp. $\{\tilde{h}_a\}$ will be dual bases for $\mathfrak{h}$ and let for any positive root with $\mathfrak{g}_\alpha \subset \mathfrak{g}_0$, elements x_α, resp. z_α be generators of $\mathfrak{g}_\alpha$, resp. $\mathfrak{g}_{-\alpha}$ dual with respect to $(\cdot,\cdot)$. Then the Casimir element c for $\mathfrak{g}_0$ is given by

$$c = \sum_a \tilde{h}_a h_a + \sum_{\alpha\in\Pi_0}(x_\alpha z_\alpha + z_\alpha x_\alpha).$$

Let v_λ be a highest weight vector in $\mathbb{E}_\lambda$. The action of the first summand $\sum_a \tilde{h}_a h_a$ on v_λ is multiplication by the element (λ,λ) and the action of $x_\alpha z_\alpha + z_\alpha x_\alpha$ is given by multiplication by (λ,α). The action of c on the whole space is the same as on v_λ by the Schur lemma. □

4.3. Casimir computations. — In the algebraic condition for invariant first order operators (see Section 3), the operator Φ defined by the formula

$$\Phi(Z\otimes v)(X) = [Z,X]\cdot v = \left(\sum_{\alpha'}\eta^{\alpha'}\otimes[Z,\xi_{\alpha'}]v\right)(X),\ Z\in\mathfrak{g}_1,\ X\in\mathfrak{g}_{-1},\ v\in\mathbb{E}_\lambda$$

was used. We shall now give an explicit description of the action of the operator Φ.

Lemma. — *Let $\mathbb{E}_\lambda$ be an irreducible representation of $\mathfrak{g}_0$ characterized by $\lambda \in \mathfrak{h}^*$ and let $\mathfrak{g}_1 = \sum_j \mathfrak{g}_1^j$ be a decomposition of $\mathfrak{g}_1$ into irreducible $\mathfrak{g}_0$-submodules. Highest weights of individual components $\mathfrak{g}_1^j$ will be denoted by α_j. Suppose that $\mathfrak{g}_1\otimes\mathbb{E}_\lambda = \sum_j\sum_{\mu_j}\mathbb{E}^j_{\mu_j}$ be a decomposition of the product into irreducible $\mathfrak{g}_0$-modules and π_{λ,μ_j} be the corresponding projections. Let ρ_0 be the half sum of positive roots for $\mathfrak{g}_0^s$ as defined in the previous lemma.*

Then for all $v \in \mathbb{E}_\lambda$,

$$\Phi(Z\otimes v)(X) = [Z,X]\cdot v = \sum_j\sum_{\mu_j} c_{\lambda\mu_j}\pi_{\lambda\mu_j}(Z\otimes v)(X),$$

where

$$c_{\lambda\mu_j} = \frac{1}{2}[(\mu_j, \mu_j + 2\rho_0) - (\lambda, \lambda + 2\rho_0) - (\alpha_j, \alpha_j + 2\rho_0)].$$

Proof. — It is sufficient to prove the claim for each individual component $\mathfrak{g}_1^j$ separately, hence we shall consider one of these components and we shall drop the index j everywhere. Let $\{\xi_\alpha\}$, resp. $\{\eta_\alpha\}$ be dual bases of $\mathfrak{g}_{-1}$, resp. $\mathfrak{g}_1$. Similarly, let $\{Y_a\}$, resp. $\{\tilde{Y}_a\}$ be dual bases of $\mathfrak{g}_0$. The invariance of the scalar product implies

$$[Z, \xi_\alpha] = \sum_a (\tilde{Y}_a, [Z, \xi_\alpha]) Y_a = \sum_a ([\tilde{Y}_a, Z], \xi_\alpha) Y_a,$$

and

$$\Phi(Z \otimes v) = \sum_i \eta_\alpha \otimes [Z, \xi_\alpha] \cdot v = \sum_i \eta_\alpha \otimes \left(\sum_a ([\tilde{Y}_a, Z], \xi_\alpha) Y_a \right) \cdot v = \sum_a [\tilde{Y}_a, Z] \otimes Y_a \cdot v.$$

The same formula holds also in the case when the role of bases $\{Y_a\}$ and $\{\tilde{Y}_a\}$ is exchanged.

Using the definition of the Casimir operator c and the previous Lemma, it is sufficient to note that

$$\begin{aligned}\sum_a \tilde{Y}_a Y_a \cdot (Z \otimes s) = \sum_a (\tilde{Y}_a Y_a \cdot Z) \otimes s + \sum_a Z \otimes (\tilde{Y}_a Y_a \cdot s) \\ + \sum_a (\tilde{Y}_a \cdot Z) \otimes (Y_a \cdot s) + (Y_a \cdot Z) \otimes (\tilde{Y}_a \cdot s)\end{aligned}$$

(as before, the symbol $\cdot$ here means the action on different modules used in the formula, for example $Y_a \cdot Z \equiv [Y_a, Z]$). □

4.4. A characterization of invariant first order operators. — Now it is possible to give the promised characterization of the first order operators (up to curvature terms in the sense explained in Section 1).

Theorem. — *Let $\mathfrak{g}$ be a (real) graded Lie algebra and $\mathfrak{g}^{\mathbb{C}}$ its graded complexification. Then $\mathfrak{g}_j = \mathfrak{g} \cap \mathfrak{g}_j^{\mathbb{C}}$.*

Let $\mathbb{E}_\lambda$ be a (complex) irreducible representation of $\mathfrak{g}_0$ with highest weight λ and let $\mathfrak{g}_1^{\mathbb{C}} \sum_j \mathfrak{g}_1^j$ be a decomposition of $\mathfrak{g}_1^{\mathbb{C}}$ into irreducible $\mathfrak{g}_0$-submodules and let α_j be highest weights of $\mathfrak{g}_1^j$. Suppose that

$$\mathfrak{g}_1 \otimes_{\mathbb{R}} \mathbb{E}_\lambda = \mathfrak{g}_1^{\mathbb{C}} \otimes_{\mathbb{C}} \mathbb{E}_\lambda = \sum_j \sum_{\mu_j} \mathbb{E}_{\mu_j}^j$$

be a decomposition of the product into irreducible $\mathfrak{g}_0$-modules and let π_{λ,μ_j} be the corresponding projections. Let us denote (as in Lemma 4.2) the half sum of positive roots for $\mathfrak{g}_0$ by ρ_0 and let us define constants c_{λ,μ_j} by

$$c_{\lambda\mu_j} = \frac{1}{2}[(\mu_j, \mu_j + 2\rho_0) - (\lambda, \lambda + 2\rho_0) - (\alpha_j, \alpha_j + 2\rho_0)].$$

Then the operator $D_{j,\mu_j} : \pi_{\lambda,\mu_j} \circ \nabla^\omega$ is an invariant first order differential operator if and only if $c_{\lambda,\mu_j} = 0$. Moreover, all first order invariant operator acting on sections of E_λ are obtained (modulo a scalar multiple and curvature terms) in such way.

Proof. — The first part of the claim follows from the previous Lemmas and results of Section 3. If D is any first order invariant differential operator, then its restriction to the homogeneous model is given by a P–homomorphism from the space of restricted jets of order one to a P–module. This homomorphism then defines a strongly invariant first order operator $\tilde{D}$ on any manifold with a given parabolic structure. The operators D and $\tilde{D}$ can differ only by a scale or possible curvature terms. □

4.5. The Borel case. — There are two extreme cases of the parabolic subalgebras — maximal ones and the Borel subalgebra. We shall first discuss one of these extremal cases. In this subsection, symbol $\mathfrak{g}$ will denote the complex graded Lie algebra which is the complexification of the real graded Lie algebra in question.

Corollary. — *Let Π denote the set of simple roots for $\mathfrak{g}$. Let λ be the highest weight of an irreducible $\mathfrak{g}_0$-module. An invariant first order operator between sections of E_λ and E_μ exists if and only if the following two conditions are satisfied:*

1) There exists a simple root $\alpha \in \Pi$ such that $\mu = \lambda + \alpha$.

2) $(\lambda, \alpha) = 0$.

Proof. — Note first that the set of all roots α with $\mathfrak{g}_\alpha \subset \mathfrak{g}_1$ is exactly the set of all simple roots. Hence $\mathfrak{g}_1$ in the Borel case is a direct sum of irreducible one dimensional subspaces $\mathfrak{g}_\alpha$ with $\alpha \in \Pi$. The tensor product of $\mathbb{E}_\lambda$ with $\mathfrak{g}_\alpha$ is irreducible and isomorphic to $\mathbb{E}_{\lambda+\alpha}$ (because $\mathfrak{g}_\alpha$ is one dimensional), hence no projections are involved.

In the Borel case, the corresponding element ρ_0 is trivial. Hence the condition in Theorem 4.4 reduces to the condition

$$0 = (\lambda + \alpha, \lambda + \alpha) - (\lambda, \lambda) - (\alpha, \alpha) = 2(\lambda, \alpha).$$

□

4.6. The case of a maximal algebra. — Let us now consider an opposite extreme case, where the parabolic subalgebra of $\mathfrak{g}$ is maximal, i.e. it corresponds to a one-point subset of the set of simple roots for $\mathfrak{g}$ (there is just one node crossed in the usual Dynkin notation for parabolic subalgebras). Then $\mathfrak{g}_0 = \mathfrak{a} \oplus \mathfrak{g}_0^s$, $\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{h}^s$ with $\mathfrak{h}^s = \mathfrak{h} \cap \mathfrak{g}_0^s$ and the commutative subalgebra $\mathfrak{a}$ is generated by the grading element E. Moreover, it is easy to see that the decomposition above is orthogonal. Indeed, the space $\mathfrak{h}^s$ is generated by commutators $[x_\alpha, z_\alpha]$, where x_α, resp. z_α are generators of the root space $\mathfrak{g}_\alpha \subset \mathfrak{g}_0$, resp. $\mathfrak{g}_{-\alpha} \subset \mathfrak{g}_0$ and we have $(E, [x_\alpha, z_\alpha]) = ([E, x_\alpha], z_\alpha) = 0$.

Let λ_E be the element of $\mathfrak{h}^*$ representing the grading element E under the duality given by the invariant bilinear form. Note that λ_E belongs (inside the original real graded Lie algebra) to the noncompact part of $\mathfrak{g}$, hence representations of $\mathfrak{g}_0$ with the highest weight $w.\lambda_E$ integrate to representations of P for any $w \in \mathbb{R}$.

The orthogonal decomposition $\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{h}^s$ induces the dual orthogonal decomposition $\mathfrak{h}^* = \mathfrak{a}^* \oplus (\mathfrak{h}^s)^*$, where the embedding of both summands is defined by requirement that $\mathfrak{a}^*$, resp. $(\mathfrak{h}^s)^*$ annihilates $\mathfrak{h}^s$, resp. $\mathfrak{a}$. The one dimensional space $\mathfrak{a}^*$ is generated by λ_E. Any weight $\lambda \in \mathfrak{h}^*$ can be then written as $\lambda = w\lambda_E + \lambda'$ with $w \in \mathbb{C}, \lambda' \in (\mathfrak{h}^s)^*$.

In this case, we shall consider (complex) irreducible representations of $\mathfrak{g}_0$, which are tensor products of one dimensional representation with highest $w.\lambda_E$, $w \in \mathbb{R}$ (w is a generalized conformal weight) with an irreducible representation $V_{\lambda'}$, where λ' is a dominant integral weight for $\mathfrak{g}_0^s$. Any such representation integrates to a representation of P (nilpotent part acting trivially) and we shall denote such representation by $\mathbb{E}_{\lambda'}(w)$.

In [**15**], the case of almost Hermitean symmetric structure was considered. This is just a special case of maximal parabolic subalgebras, which are moreover $|1|$-graded Lie algebras (but note that there is a lot of cases of $|k|$-graded Lie algebras with $k > 1$ which are maximal). In the $|1|$-graded case (see [**15**], Part III; see also [**21**] for the conformal case), it was proved that for any projection to an irreducible piece of the $\mathfrak{g}_0^s$-module $\mathbb{E}_\lambda \otimes \mathfrak{g}_1$, there is a unique conformal weight w such that the resulting first order operator is invariant. The value of w was computed using suitable Casimir expressions. We are going to show that computations and formulae proved there can be extended without any substantial change to the general case of $|k|$-graded Lie algebra.

4.7. The general case. — In the general case, it is possible again to consider the orthogonal decomposition $\mathfrak{g}_0 = \langle E \rangle_{\mathbb{C}} \oplus \mathfrak{g}_0'$, and $\mathfrak{h}^* = \langle \lambda_E \rangle_{\mathbb{C}} \oplus (\mathfrak{h}')^*$, where elements of $(\mathfrak{h}')^*$ annihilate E. Hence again any weight $\lambda \in \mathfrak{h}^*$ can be decomposed as $\lambda = w\lambda_E + \lambda'$ with $w \in \mathbb{C}, \lambda' \in (\mathfrak{h}')^*$ (note that $\mathfrak{g}_0'$ is again reductive but not necessarily semisimple). We are now able to prove a generalization of facts proved first by Fegan in conformal case and then extended to $|1|$-graded case in [**15**].

Corollary. — *Let $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{g}$. Let $\mathbb{E}_\lambda$ be an irreducible representation of $\mathfrak{g}_0$ characterized by $\lambda \in \mathfrak{h}^*$ and let $\mathfrak{g}_1 = \sum_j \mathfrak{g}_1^j$ be a decomposition of $\mathfrak{g}_1$ into irreducible $\mathfrak{g}_0$-submodules. Highest weights of individual components $\mathfrak{g}_1^j$ will be denoted by α_j. Suppose that $\mathfrak{g}_1 \otimes \mathbb{E}_\lambda = \sum_j \sum_{\mu_j} \mathbb{E}_{\mu_j}^j$ be a decomposition of the product into irreducible $\mathfrak{g}_0$-modules and π_{λ,μ_j} be the corresponding projections. Let ρ_0 be the half sum of positive roots for $\mathfrak{g}_0^s$ as defined in Lemma 4.3.*

Suppose that weights λ, α_j and μ_j are split as

$$\lambda = w\lambda_E + \lambda', \alpha_j = \lambda_E + \alpha_j', \mu_j = (w+1)\lambda_E + \mu_j'.$$

Then for all $v \in \mathbb{E}_\lambda(w), Z \in \mathfrak{g}_1$

$$\Phi(Z \otimes v)(X) = [Z, X] \cdot v = \sum_{\mu'} (w - c_{\lambda'\mu'}) \pi_{\lambda'\mu'}(Z \otimes v)(X),$$

where

$$c_{\lambda'\mu'} = -\frac{1}{2}[(\mu', \mu' + 2\rho_0) - (\lambda', \lambda' + 2\rho_0) - (\alpha', \alpha' + 2\rho_0)].$$

Hence the operator $D_{\lambda\mu} = \pi_{\lambda\mu} \circ \nabla^\omega$ *is invariant first order operator if and only if* $w = c_{\lambda'\mu'}$.

Proof. — For simplicity of notation, we shall drop subscripts j everywhere. We have $(\lambda' + w\lambda_E, \lambda' + w\lambda_E + 2\rho_0) = (\lambda', \lambda' + 2\rho_0) + 2w(\lambda_E, \lambda') + w^2$; similar formulae hold for terms with μ (with weight $w+1$) and for α (with weight 1). Using $(w+1)^2 - w^2 - 1 = 2w$, we get

$$(\mu, \mu + 2\rho_0) - (\lambda, \lambda + 2\rho_0) - (\alpha, \alpha + 2\rho_0) = 2w + (\mu', \mu' + 2\rho_0) - (\lambda', \lambda' + 2\rho_0) - (\alpha', \alpha' + 2\rho_0)$$

and the claim follows. □

In general case, the reductive algebra $\mathfrak{g}_0$ is reductive and may be split into its commutative and semisimple part. Suppose that $\mathfrak{g}_0 = \mathfrak{a} \oplus \mathfrak{g}_0'$ is such an orthogonal splitting. It induces the splitting $\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{h}'$ of the Cartan subalgebra. Every weight $\lambda \in \mathfrak{h}^*$ can be hence again split into a sum $\lambda = \lambda_0 + \lambda'$ with $\lambda_0 \in (\mathfrak{a})^*, \lambda' \in (\mathfrak{h}')^*$. The Corollary above is saying that we can, for a given λ and μ to shift λ, resp. μ by a multiple of λ_E to $\tilde{\lambda}$, resp. $\tilde{\mu}$ in such a way that there is an invariant first order operator from $\mathbb{E}_{\tilde{\lambda}}$ to $\mathbb{E}_{\tilde{\mu}}$.

It is possible to consider more general changes of λ, resp. μ by adding to them an arbitrary element $\nu \in (\mathfrak{a})^*$ and to ask whether we can have an invariant operator between spaces with shifted values of highest weights. It is an easy calculation to see that the relation $c_{\lambda\mu_j} = 0$ in Theorem 4.4 yields one linear relation for ν (the quadratic terms cancel each other). Hence we have a linear subspace of codimension 1 in $\mathfrak{a}^*$ of such elements ν.

5. Multiplicity one result

A tensor product of two irreducible representations of the reductive group $\mathfrak{g}_0$ decomposes into irreducible components and the projections to these components are key tools in the construction of invariant first order operators. Important information concerning such decompositions is multiplicity of individual components in their isotopic components. The best situation is when all multiplicities are one, then all irreducible components (as well as the corresponding projections) are defined uniquely, without any ambiguity. In this section, we are going to prove such multiplicity one result for the tensor product used in the definition of invariant operators and we are going to give full information on highest weights of individual components in such decompositions for any classical graded Lie algebra.

5.1. Simple factors of $\mathfrak{g}_0$. — Our starting point for a choice of structure in question is a real graded Lie algebra $\mathfrak{g}$. For the discussion of (complex) finite dimensional representations, we can simplify the situation and to work with the complexification $\mathfrak{g}^{\mathbb{C}}$. There are two main cases to be considered. Either $\mathfrak{g}$ is a real form of $\mathfrak{g}^{\mathbb{C}}$, or it is a complex graded Lie algebra considered as a real one. In the latter case, there is no need to go through complexification in subsequent discussions. So we shall concentrate in this section to the former case.

So let us suppose that $\mathfrak{g}$ is a real form of a complex graded Lie algebra of classical type and that $(\mathfrak{g}_0)^{\mathbb{C}}$ is just $(\mathfrak{g}^{\mathbb{C}})_0$. Hence any (complex) irreducible $\mathfrak{g}_0$–module is at the same time $(\mathfrak{g}^{\mathbb{C}})_0$–module and vice versa. Consequently, the discussion of decomposition of the tensor products of irreducible $\mathfrak{g}_0$–modules with irreducible components of $(\mathfrak{g}^{\mathbb{C}})_1 \simeq (\mathfrak{g}_1)^{\mathbb{C}}$ can be done completely in the setting of complex graded Lie algebras. Hence we shall change the notation and we shall denote in this section by $\mathfrak{g}$ a complex simple graded Lie algebra given by its Dynkin diagram with corresponding crosses. There is a simple and very intuitive way how to find simple components of the semisimple part of $\mathfrak{g}_0$ from the corresponding Dynkin diagrams. Delete all crossed nodes and lines emanating from them. The rest will consist of several connected components which will be again Dynkin diagrams for simple Lie algebras. Then the corresponding semisimple part of $\mathfrak{g}_0^{\mathbb{C}}$ is isomorphic to the product of these factors. We shall give more details (including explanation why this is true) in the discussion of individual cases below.

We are going to study in more details the tensor products $\mathfrak{g}_1 \otimes \mathbb{E}_\lambda$ of $\mathfrak{g}_0$–modules and their decompositions into irreducible components. In general, only the semisimple part of $\mathfrak{g}_0$ is playing a role in the decomposition. Having a better information on the number and types of simple factors of $\mathfrak{g}_0$, we shall describe then the number and the highest weights of irreducible pieces of the $\mathfrak{g}_0$–module $\mathfrak{g}_1$. Even if there is a lot of common features, full details differ substantially in individual cases and we have to discuss all four of them separately.

Most of the simple factors of $\mathfrak{g}_0$ will be of type A_j, exceptionally also B_j, C_j and D_j appear. A general irreducible representation of a product of certain number of simple Lie algebras is a tensor product of irreducible representations of the individual factors in $\mathfrak{g}_0$. Hence to describe a $\mathfrak{g}_0$–module, it is sufficient to give a list of highest weights of the individual factors. For components of $\mathfrak{g}_1$, we shall need only very small number of quite simple representations. We shall now give the list of them and we introduce a notation for their highest weights.

For A_n, we shall need:

– the defining representation $\mathbb{C}_{n+1}$ with the highest weight denoted by α_1;
– its symmetric power $\odot^2(\mathbb{C}_{n+1})$ with the highest weight $2\alpha_1$;

– its exterior power $\Lambda^2(\mathbb{C}_{n+1})$ with the highest weight denoted by α_{11}.

For B_n, C_n and D_n, we shall need only their defining representations, their highest weights will be denoted by β_1, γ_1 and δ_1.

It will also help a lot to use the symbol A_0 for the trivial Lie algebra $\{0\}$ of dimension 0. All its irreducible representations are trivial. Its presence in the product will be just a notational convenience, (these factors can be dropped out, they have no significance in the structure of the algebra but they will be substantial for a description of irreducible pieces of the module $\mathfrak{g}_1$).

A general method used below to clarify these questions is a very nice and explicit description of gradings in terms of block matrices, which can be found in the paper by Yamaguchi ([**44**]), we refer to this paper for further details. It makes also possible to give explicitly the form of all irreducible pieces in the module $\mathfrak{g}_1$.

5.2. *A*-series. — Suppose that $\mathfrak{g} = A_n = \mathfrak{sl}(n+1, \mathbb{C})$. This is the simplest case which is particularly intuitive when described using block matrices. First, it is necessary to understand block forms of maximal graded Lie algebras. In our case, they are specified by their Dynkin diagram $\bullet\!-\cdots-\!\times\!-\cdots-\!\bullet$ with the cross at the j-the node. The corresponding grading is indicated by the following diagram (where numbers $-1, 0, 1$ indicate the grading of the algebra).

$$\begin{array}{r|c|c|}
\cline{2-3}
j & 0 & 1 \\
\cline{2-3}
n+1-j & -1 & 0 \\
\cline{2-3}
\end{array}$$

The general case with several crosses is then given by a simple superposition of the diagrams. There is an example with three crosses:

$$\begin{array}{|c|c|}\hline 0 & 1 \\ \hline -1 & 0 \\ \hline\end{array}\qquad
\begin{array}{|c|c|}\hline 0 & 1 \\ \hline -1 & 0 \\ \hline\end{array}\qquad
\begin{array}{|c|c|}\hline 0 & 1 \\ \hline -1 & 0 \\ \hline\end{array}\qquad \Longrightarrow \qquad
\begin{array}{|c|c|c|c|}\hline 0 & 1 & 2 & 3 \\ \hline -1 & 0 & 1 & 2 \\ \hline -2 & -1 & 0 & 1 \\ \hline -3 & -2 & -1 & 0 \\ \hline\end{array}$$

Let the set $I = \{i_1, \ldots, i_j\}$, $1 \leq i_1 < \cdots < i_j \leq n$ denote the set of crossed nodes in the Dynkin diagram of type A_n. Then the corresponding semisimple part $\mathfrak{g}_0^s$ is equal to the product

$$A_{i_1-1} \times A_{i_2-i_1-1} \times \cdots \times A_{i_j-i_{j-1}-1} \times A_{n-i_j}.$$

There are $j+1$ factors in the product (some of them possibly equal to A_0, these can be dropped as far as the structure of $\mathfrak{g}_0$ is concerned).

Using additional notation $i_0 = 0$, $i_{j+1} = n+1$, we have $\mathfrak{g}_0^s = \Pi_{k=1}^{n+1} A_{i_k - i_{k-1} - 1}$.

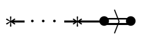
Irreducible representations of $\mathfrak{g}_0$ are tensor products of irreducible modules of individual factors, they are given by their highest weights. There is j irreducible components of the $\mathfrak{g}_0$–module $\mathfrak{g}_1$, as is immediately seen from the corresponding block diagram. The $(j+1)$–tuples of their highest weights are clearly:

$$(\alpha_1, \alpha_1, 0, \dots, 0); (0, \alpha_1, \alpha_1, 0, \dots, 0); \dots; (0, \dots, 0, \alpha_1, \alpha_1).$$

So each component is just the tensor product of two defining representations of neighboring factors in $\mathfrak{g}_0$.

5.3. *B*-series. — As in the previous case, the key information is contained in the block diagrams for maximal graded Lie algebra, described in [**44**]. We shall not reproduce them but we shall only describe the form of the simple factors and highest weights of irreducible parts of $\mathfrak{g}_1$. The method used to get these facts is the same as in the A_n case.

Let again the set $I = \{i_1, \dots, i_j\}$, $1 \le i_1 < \dots < i_j \le n$ denote the set of crossed nodes in the Dynkin diagram of type B_n. We shall consider three different subcases.

1)  $i_j \le n-2$ (here stars indicate nodes with either bullets or crosses).

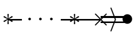
Then the corresponding semisimple part $\mathfrak{g}_0^s$ is equal to the product ($j+1$ factors)

$$\mathfrak{g}_0^s \simeq A_{i_1-1} \times A_{i_2-i_1-1} \times \dots \times A_{i_j-i_{j-1}-1} \times B_{n-i_j}$$

and there are j irreducible pieces in $\mathfrak{g}_1$ with highest weights

$$(\alpha_1, \alpha_1, 0, \dots, 0); (0, \alpha_1, \alpha_1, 0, \dots, 0); \dots; (0, \dots, 0, \alpha_1, \beta_1).$$

(recall that α_1 denotes the highest weight of the defining representation of A_k and β_1 denotes the highest weight of the defining representation of B_k).

2)  $i_j = n-1$

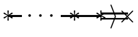
Then $\mathfrak{g}_0^s$ has $j+1$ factors

$$\mathfrak{g}_0^s \simeq A_{i_1-1} \times A_{i_2-i_1-1} \times \dots \times A_{i_j-i_{j-1}-1} \times A_1.$$

The last factor A_1 is isomorphic with B_1. We shall need the defining representation of B_1, which is just the second symmetric power of the defining representation of A_1. Hence as a representation of A_1, it has the highest weight $2\alpha_1$.

There are j irreducible parts in $\mathfrak{g}_1$. The list of their highest weights is:

$$(\alpha_1, \alpha_1, 0, \dots, 0); (0, \alpha_1, \alpha_1, 0, \dots, 0); \dots; (0, \dots, 0, \alpha_1, \alpha_1, 0); (0, \dots, 0, \alpha_1, 2\alpha_1).$$

3)  $i_j = n$

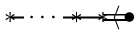
Then

$$\mathfrak{g}_0^s \simeq A_{i_1-1} \times A_{i_2-i_1-1} \times \dots \times A_{i_j-i_{j-1}-1} \times A_0.$$

There are j irreducible pieces in $\mathfrak{g}_1$ with highest weights

$$(\alpha_1, \alpha_1, 0, \dots, 0); (0, \alpha_1, \alpha_1, 0, \dots, 0); \dots; (0, \dots, 0, \alpha_1, \alpha_1).$$

5.4. *C*-series. — Let indices $1 \leq i_1 < \cdots < i_j \leq n$ again indicate the set of crossed nodes in the Dynkin diagram of type C_n in the standard ordering of nodes.

1)  $i_j \leq n-1$

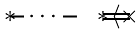
Then $\mathfrak{g}_0^s$ has $j+1$ factors

$$\mathfrak{g}_0^s \simeq A_{i_1-1} \times A_{i_2-i_1-1} \times \cdots \times A_{i_j-i_{j-1}-1} \times C_{n-i_j}.$$

(but note that $C_1 \equiv A_1$).

There are j irreducible parts in $\mathfrak{g}_1$. The list of their highest weights is:

$$(\alpha_1, \alpha_1, 0, \ldots, 0); (0, \alpha_1, \alpha_1, 0, \ldots, 0); \ldots; (0, \ldots, 0, \alpha_1, \alpha_1, 0); (0, \ldots, 0, \alpha_1, \gamma_1).$$

2)  $i_j = n$

This case brings a new feature, let us illustrate it in the case of maximal parabolic subalgebra with the last node crossed. This is a $|1|$–graded case with the block grading as follows:

$$\begin{array}{|c|c|} \hline 0 & 1 \\ \hline -1 & 0 \\ \hline \end{array} \qquad \begin{array}{|c|c|} \hline A & B \\ \hline C & -A' \\ \hline \end{array}$$

where the symbol A' indicates the matrix transposed with respect to the antidiagonal and the matrices B and C satisfy $B = B'$, $C = C'$. Hence $\mathfrak{g}_1$ is the symmetric power $\odot^2(\mathbb{C}_{n+1})$ of the defining representation and its highest weight is $2\alpha_1$.

In the general case, we get in the same way that

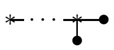
$$\mathfrak{g}_0^s \simeq A_{i_1-1} \times A_{i_2-i_1-1} \times \cdots \times A_{i_{j-1}-i_{j-2}-1} \times A_{n-i_{j-1}}$$

(note that there are only j factors here).

There are j irreducible pieces in $\mathfrak{g}_1$ with highest weights

$$(\alpha_1, \alpha_1, 0, \ldots, 0); (0, \alpha_1, \alpha_1, 0, \ldots, 0); \ldots; (0, \ldots, 0, \alpha_1, \alpha_1)(0, \ldots, 0, 2\alpha_1).$$

5.5. *D*-series. — Let $1 \leq i_1 < \cdots < i_j \leq n$ indicate the set of crossed nodes in the Dynkin diagram of type D_n in the standard ordering of nodes.

1)  $i_j \leq n-2$

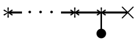
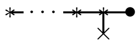
Then

$$\mathfrak{g}_0^s \simeq A_{i_1-1} \times A_{i_2-i_1-1} \times \cdots \times A_{i_j-i_{j-1}-1} \times D_{n-i_j}.$$

Note that if $i_{j-1} = n-2$, then the last factor is $D_2 \equiv A_1 \times A_1$. The list of all j irreducible pieces of $\mathfrak{g}_1$ is again the standard list:

$$(\alpha_1, \alpha_1, 0, \ldots, 0); (0, \alpha_1, \alpha_1, 0, \ldots, 0); \ldots; (0, \ldots, 0, \alpha_1, \delta_1).$$

The proof of the claim is again visible directly from the block forms of maximal parabolic subalgebras given in [**44**].

2)  or  $i_{j-1} \le n-2, \quad i_j = n-1 \quad \text{or} \quad i_j = n$

This is similar to the second case in the C_n series. Let us illustrate it again in the simplest $|1|$-graded case. The graded algebra $\mathfrak{g}$ has the following block form:

$$\begin{array}{|c|c|} \hline 0 & 1 \\ \hline -1 & 0 \\ \hline \end{array} \qquad \begin{array}{|c|c|} \hline A & B \\ \hline C & -A' \\ \hline \end{array}$$

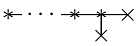
where the symbol A' indicates again the matrix transposed with respect to the antidiagonal and the matrices B and C satisfy $B = -B'$, $C = -C'$. Hence $\mathfrak{g}_1$ is the outer power $\Lambda^2(\mathbb{C}_{n+1})$ of the defining representation of A_n and its highest weight was denoted by α_{11}.

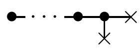
In general case

$$\mathfrak{g}_0^s \simeq A_{i_1-1} \times A_{i_2-i_1-1} \times \cdots \times A_{i_{j-1}-i_{j-2}-1} \times A_{n-i_{j-1}-1}$$

(there are j factors only). The list of all j irreducible pieces of $\mathfrak{g}_1$ is:

$$(\alpha_1, \alpha_1, 0, \ldots, 0); (0, \alpha_1, \alpha_1, 0, \ldots, 0); \ldots; (0, \ldots, 0, \alpha_1, \alpha_1); (0, \ldots, 0, \alpha_{11}).$$

3)  $i_{j-1} = n-1, \quad i_j = n$

This is the most unusual case. Let us illustrate it in the case  , i.e. $I = (n-1, n)$.

The corresponding matrix looks as follows:

$$\begin{array}{r|c|c|c|} \cline{2-4} n-1 & 0 & 1 & 2 \\ \cline{2-4} 2 & -1 & 0 & 1 \\ \cline{2-4} n-1 & -2 & -1 & 0 \\ \cline{2-4} \end{array}$$

In the middle, there is the 2×2 matrix, which is antisymmetric with respect to the antidiagonal (D_1!). The module $\mathfrak{g}_1$ is a $((n-1) \times 2)$–matrix, which is the tensor product of the defining representation for A_{n-2} and D_1. (Note that there are two blocks in the 1 part of the block matrix above but they are inverse transpose of each other.) But D_1 is commutative and its corresponding factor in $\mathfrak{g}_0$ is the trivial algebra A_0. Or even better for our purposes, we can identify D_1 with the product $A_0 \times A_0$. The algebra $\mathfrak{g}_0$ is hence the product $A_{n-2} \times A_0 \times A_0$.

Also in this case, the module $\mathfrak{g}_1$ has $j=2$ irreducible pieces, i.e. both columns of (the left upper part) of $\mathfrak{g}_1$. Their highest weights are equal to $(\alpha_1,\alpha_1,0)$ and $(\alpha_1,0,\alpha_1)$.

In general, for arbitrary I,

$$\mathfrak{g}_0^s \simeq A_{i_1-1}\times A_{i_2-i_1-1}\times\cdots\times A_{i_{j-1}-i_{j-2}-1}\times A_0\times A_0.$$

The list of all j irreducible pieces of $\mathfrak{g}_1$ is as follows:

$$(\alpha_1,\alpha_1,0,\ldots,0);\ldots;(0,\ldots,0,\alpha_1,\alpha_1,0,0);(0,\ldots,0,\alpha_1,\alpha_1,0);(0,\ldots,0,\alpha_1,0,\alpha_1).$$

5.6. Decomposition of tensor products. — We have found above the form of all irreducible pieces of $\mathfrak{g}_1$. They are quite simple modules, hence there is a chance to get a better information on their product with arbitrary other modules. Such a discussion was needed for the study of first order operators in the $|1|$–graded case (see [**15**], part III). We shall summarize now the facts proved there.

Basic tool for understanding tensor products of irreducible modules of a simple Lie algebra $\mathfrak{g}$ is the Klimyk algorithm (see [**27**], Sec.24, Ex.9).

Lemma. — *Let $\mathfrak{h}$ be a Cartan subalgebra of a simple Lie algebra $\mathfrak{g}$. For any weight $\xi\in\mathfrak{h}^*$, let $\{\xi\}$ denote the dominant weight lying on the orbit of ξ under the Weyl group. Let ρ be the half sum of positive roots. If $\{\xi\}$ belongs to the interior of the dominant Weyl chamber, there is the unique $w\in W$ such that $\{\xi\}=w\xi$. Let $t(\xi)$ be equal to the sign of w in this case and zero otherwise.*

Suppose moreover that we know the list $\Pi(\mu)$ of all weights of the irreducible representation $\mathbb{V}_\mu$ with the highest weight μ, including their multiplicities $m_\mu(\nu)$, for $\nu\in\Pi(\mu)$. Let $\mathbb{E}_\lambda$ denote the irreducible representation of $\mathfrak{g}$ with the highest weight λ. Then the formal sum

$$\sum_{\nu\in\Pi(\mu)} m_\mu(\nu)t(\lambda+\rho+\nu)\mathbb{V}_{\{\lambda+\rho+\nu\}-\rho}$$

gives the decomposition of the tensor product $\mathbb{E}_\lambda\otimes\mathbb{E}_\mu$ into isotopic components. The resulting coefficients are always non–negative and give the multiplicity of the corresponding representation in the decomposition. Note that some cancellations happen often.

5.7. A_n decompositions. — For representations of A_n, we need to decompose products $\mathbb{E}_\lambda\otimes\mathbb{E}_{\alpha_1}$, $\mathbb{E}_\lambda\otimes\mathbb{E}_{\alpha_{11}}$ and $\mathbb{E}_\lambda\otimes\mathbb{E}_{2\alpha_1}$.

For the two first cases, we have the following information.

Lemma. — *Let α_1 be the highest weight of the defining representation of A_n and let α_{11} be the highest weight of its outer product. Let $\mu=\alpha_1$, or $\mu=\alpha_{11}$. Let $\mathbb{E}_\lambda$ be an irreducible A_n–module with the highest weight λ.*

Then the decomposition of the product $\mathbb{E}_\lambda\otimes\mathbb{E}_\mu$ is multiplicity free.

Moreover, $\mathbb{V}_\nu$ appears with multiplicity one if and only if ν is dominant integral and there exists a weight β of $\mathbb{E}_\mu$ such that $\nu=\lambda+\beta$.

Proof. — A direct check shows (see e.g. Appendix 2 in [**15**], part III), that all weights of $\mathbb{E}_\lambda$ appear with multiplicity one and that for any weight β of $\mathbb{E}_\mu$, $\beta+\rho$ belongs to the dominant Weyl chamber. Then the same is true for $\lambda+\beta+\rho$ and no action of Weyl group is needed in the Klimyk formula. Moreover, $\lambda+\beta+\rho$ belongs to the interior of the dominant Weyl chamber if and only if $\lambda+\beta$ belongs to the dominant Weyl chamber. □

Remark. — Consider a general tensor product of two modules $\mathbb{E}_\lambda \otimes \mathbb{E}_\mu$. There is a general fact that $\mathbb{E}_\nu$ appears in the decomposition only if ν is of the form $\lambda+\beta$, where β is a weight of $\mathbb{E}_\mu$.

In our case, we know more. The set A of all weights which appear in the decomposition is exactly given by

$$A = \{\nu = \lambda+\beta \mid \beta \text{ is a weight of } \mathbb{E}_\mu,\ \lambda+\beta \text{ is dominant}\}. \tag{5}$$

It is more difficult to decompose the tensor product in the third case which we need.

Lemma. — *Let $\mathbb{E}_\mu$ be the second symmetric power of the defining representation of A_n, i.e $\mu = 2\alpha_1$. The list of all its weights is $\beta = e_i + e_j, 1 \le i \le j \le n$, where e_i, $i = 1, \dots, n$, denotes elements of the canonical basis of $\mathbb{R}_n$. Let $\mathbb{E}_\lambda$ be an irreducible representation of A_n with the highest weight λ.*

Then

$$\mathbb{E}_\lambda \otimes \mathbb{E}_\mu = \sum_{\nu \in A\setminus A'} \mathbb{E}_\nu,$$

where A is defined in (5) and

$$A' = \{\nu = \lambda + e_i + e_{i+1} \mid \lambda_i = \lambda_{i+1} \textit{ and either } \lambda_{i-1} > \lambda_i \textit{ or } i = 1\}.$$

Proof. — Suppose that one of the following cases is true:

1) $\beta = e_i + e_j$ with $i < j$,

2) $\beta = 2e_1$,

3) $\beta = 2e_{i+1}, i = 1, \dots, n-1$ and $\lambda_i > \lambda_{i+1}$.

Then we have again the property that $\beta+\delta$ belongs to the dominant Weyl chamber. Hence again (as in the proof of the previous lemma) we know that $\lambda+\beta$ appears in the decomposition if and only if it is dominant (if and only if it belongs to A).

4) If $\beta = 2e_{i+1}, i = 2, \dots, n-1$ and $\lambda_{i-1} = \lambda_i = \lambda_{i+1}$, then $\lambda+\beta+\delta$ belongs to the boundary of the Weyl chamber and the summand will not appear in the Klimyk formula.

5) If however either $\beta = 2e_{i+1}, i = 2, \dots, n-1$ and $\lambda_{i-1} > \lambda_i = \lambda_{i+1}$, or $\beta = 2e_2$ and $\lambda_1 = \lambda_2$, then $\lambda+\beta+\delta$ should be moved to the interior of the dominant chamber by one reflection with respect to a simple root (permutation of neighboring components) and $\{\lambda+\rho+\nu\} - \rho = \lambda + e_i + e_{i+1}$. This shows that these elements should be removed from the set A. □

5.8. B_n **decompositions.** — For representations of B_n, we need to decompose products $\mathbb{E}_\lambda \otimes \mathbb{E}_{\beta_1}$. It is well known (see e.g. [**21**]) that the following is true.

Lemma. — *Let $\mu = \beta_1$ be the highest weight of the defining representation of B_n. Let $\mathbb{E}_\lambda$ be an irreducible B_n–module with the highest weight λ.*

Then the decomposition of the product $\mathbb{E}_\lambda \otimes \mathbb{E}_\mu$ is multiplicity free and

$$\mathbb{E}_\lambda \otimes \mathbb{E}_\mu = \sum_{\mu \in A \setminus A'} \mathbb{E}_\nu,$$

where A is defined in (5) and $A' = \{\nu = \lambda | \lambda_n = 0\}$.

5.9. C_n **and** D_n **decompositions.** — For representations of C_n, resp. D_n, we need to decompose products $\mathbb{E}_\lambda \otimes \mathbb{E}_\mu$, where $\mu = \gamma_1$, resp. $\mu = \delta_1$.

Lemma. — *Let $\mu = \beta_1$ be the highest weight of the defining representation of B_n. Let $\mathbb{E}_\lambda$ be an irreducible B_n–module with the highest weight λ.*

Then the decomposition of the product $\mathbb{E}_\lambda \otimes \mathbb{E}_\mu$ is multiplicity free and

$$\mathbb{E}_\lambda \otimes \mathbb{E}_\mu = \sum_{\mu \in A} \mathbb{E}_\nu,$$

where A is defined in (5).

Proof. — It is easy to check directly that again $\beta+\delta$ is in the dominant Weyl chamber of all weights of $\mathbb{E}_\mu$. Hence the same proof as above applies. □

5.10. Theorem. — *Let $\mathfrak{g}$ be a classical semisimple graded Lie algebra (i.e. $\mathfrak{g}$ belongs to one of series A – D) and let $\mathfrak{g}_0$ be its reductive part. Suppose further that $\mathbb{E}_\lambda$ be an irreducible $\mathfrak{g}_0$–module with highest weight λ.*

Then all components in the tensor product $\mathfrak{g}_1 \otimes \mathbb{E}_\lambda$ have multiplicity one.

Proof. — Let us consider first an irreducible piece $\mathbb{E}_1$ of $\mathfrak{g}_1$. The detailed discussion of the form of irreducible components of the $\mathfrak{g}_0$–module $\mathfrak{g}_1$ presented above together with the explicit information presented above) shows that for every factor in the product describing the semisimple part of the algebra $\mathfrak{g}_0$, the corresponding tensor product has a decomposition containing only pieces with multiplicity one. The same is hence true for their product.

If $\mathbb{E}_1$, resp. $\mathbb{E}_1'$ are different irreducible pieces of $\mathfrak{g}_1$, we know from their explicit description above, that they are tensor products of irreducible modules of different couples of factors in the decomposition of $\mathfrak{g}_0$ into simple parts. Hence the pieces in the decomposition of $\mathbb{E}_1 \otimes \mathbb{E}_\lambda$, resp. $\mathbb{E}_1' \otimes \mathbb{E}_\lambda$, have different highest weights and cannot be isomorphic. □

Explicit description of individual components of $\mathfrak{g}_1$ and of their tensor products described above gives hence the complete information on irreducible components and their highest weights.

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