Séminaires & Congrès
4, 2000, p. 277–305

ZITTERBEWEGUNG

by

George Sparling

Abstract. — We discuss conjectured relations between twistor theory and superstring theory, built around the idea that time asymmetry is crucial. In the context of a simple example, a number of techniques are described, which should shed light on these conjectures.

***Résumé* (Dynamique du cône de lumière).** — Nous étudions des relations conjecturées entre la théorie des twisteurs et la théorie des supercordes, construites autour de l'idée que l'asymétrie du temps est cruciale. Dans le cadre d'un exemple simple, un certain nombre de techniques sont décrites qui devraient éclaircir ces conjectures.

1. Introduction

The twistor theory associated to flat spacetime may be summarized as follows [**1**–**5**]. First the geometry. We start with a complex vector space $\boldsymbol{T}$, called twistor space, of four complex dimensions, equipped with a pseudo-hermitian sesquilinear form K of signature $(2,2)$. For $1 \leq n \leq 4$, denote by $\boldsymbol{G}_n$ the Grassmannian of all subspaces of $\boldsymbol{T}$ of dimension n. Then we have a decomposition $\boldsymbol{G}_n = \bigcup_{p+q+r=n} \boldsymbol{G}_{(p,q,r)}$, where for each $V \in \boldsymbol{G}_n$, p, q and r are non-negative integers such that $p+q+r=n$ and $p \leq 2$ is the maximal dimension of a subspace of V on which K is positive definite, whereas $q \leq 2$ is the maximal dimension of a subspace of V on which K is negative definite. Each $\boldsymbol{G}_{(p,q,r)}$ is an orbit of the natural action of the pseudo-unitary group $\boldsymbol{U}(K)$, associated to K, acting on $\boldsymbol{G}_n$ ($\boldsymbol{U}(K)$ is isomorphic to $U(2,2)$). When $n=1$, we put $\boldsymbol{PT} = \boldsymbol{G}_1$, $\boldsymbol{PT}^+ = \boldsymbol{G}_{(1,0,0)}$, $\boldsymbol{PT}^- = \boldsymbol{G}_{(0,1,0)}$ and $\boldsymbol{PN} = \boldsymbol{G}_{(0,0,1)}$, so $\boldsymbol{PT} = \boldsymbol{PT}^+ \cup \boldsymbol{PT}^- \cup \boldsymbol{PN}$. $\boldsymbol{PT}$ is a complex projective three-space and $\boldsymbol{PT}^\pm$ are open submanifolds of $\boldsymbol{PT}$, separated by the closed submanifold $\boldsymbol{PN}$, which has real dimension five. In the language of $\boldsymbol{CR}$ geometry, $\boldsymbol{PN}$ is the hyperquadric in

2000 ***Mathematics Subject Classification.*** — 32G05, 32J81, 32V05.
Key words and phrases. — Time asymmetry, $\boldsymbol{CR}$ geometry, hypersurface twistors, pseudo-Kähler structures.

complex projective three-space, with Levi form of signature $(1,1)$. When $n=2$, the decomposition of the four complex dimensional space $\boldsymbol{CM}=\boldsymbol{G}_2$ has six pieces. Three are open submanifolds: $\boldsymbol{M}^{++}=\boldsymbol{G}_{(2,0,0)}$, $\boldsymbol{M}^{--}=\boldsymbol{G}_{(0,2,0)}$ and $\boldsymbol{M}^{+-}=\boldsymbol{G}_{(1,1,0)}$. Another is a closed subset, of real dimension four: $\boldsymbol{M}=\boldsymbol{G}_{(0,0,2)}$. The other two, $\boldsymbol{M}^{+}=\boldsymbol{G}_{(1,0,0)}$ and $\boldsymbol{M}^{-}=\boldsymbol{G}_{(0,1,0)}$ each have real dimension seven and have $\boldsymbol{M}$ as their boundary. The boundary of $\boldsymbol{M}^{++}$ is $\boldsymbol{M}^{+}\cup\boldsymbol{M}$, of $\boldsymbol{M}^{--}$ is $\boldsymbol{M}^{-}\cup\boldsymbol{M}$ and of $\boldsymbol{M}^{+-}$ is $\boldsymbol{M}^{+}\cup\boldsymbol{M}^{-}\cup\boldsymbol{M}$. Each point of $\boldsymbol{CM}$ is a projective line in $\boldsymbol{PT}$. Then $\boldsymbol{M}^{++}$, $\boldsymbol{M}^{--}$, $\boldsymbol{M}^{+-}$, $\boldsymbol{M}^{\pm}$ and $\boldsymbol{M}$ are, respectively, the spaces of projective lines that lie entirely in $\boldsymbol{PT}^{+}$, lie entirely in $\boldsymbol{PT}^{-}$, cross from $\boldsymbol{PT}^{+}$ to $\boldsymbol{PT}^{-}$, touch $\boldsymbol{PN}$ at one point and otherwise lie in $\boldsymbol{PT}^{\pm}$, lie entirely inside $\boldsymbol{PN}$. $\boldsymbol{G}_3$ is isomorphic to dual projective twistor space, the projective dual of $\boldsymbol{PT}$ and has three pieces, $\boldsymbol{G}_{(2,1,0)}$, $\boldsymbol{G}_{(1,2,0)}$ and $\boldsymbol{G}_{(1,1,1)}$.

The Klein correspondence embeds $\boldsymbol{CM}$ as a quadric hypersurface in the projective space of $\Omega^2\boldsymbol{T}$, the exterior product of $\boldsymbol{T}$ with itself. As such it inherits a natural conformally flat complex holomorphic conformal structure. Two points of $\boldsymbol{CM}$ are null related if and only if their corresponding lines in $\boldsymbol{PT}$ intersect. Then $\boldsymbol{CM}$ is the complexification of $\boldsymbol{M}$ and the conformal structure of $\boldsymbol{M}$ is real and Lorentzian. $\boldsymbol{M}$ is a conformal compactification of real Minkowski spacetime. If a specific point $\boldsymbol{I}$ of $\boldsymbol{M}$, is singled out, then on the complement $\boldsymbol{M_I}$ of the null cone of $\boldsymbol{I}$, $\boldsymbol{M}$ has a canonical flat Lorentzian metric, and $\boldsymbol{M_I}$ (of topology $\boldsymbol{R}^4$) may be regarded as Minkowski spacetime.

With respect to the real Minkowski space $\boldsymbol{M_I}$, the imaginary part y, of the position vector of a finite point of $\boldsymbol{CM}$, is canonical. Then $\boldsymbol{M}^{++}$, $\boldsymbol{M}^{--}$, $\boldsymbol{M}^{+-}$, $\boldsymbol{M}^{+}$ and $\boldsymbol{M}^{-}$ are the sets of all points of $\boldsymbol{CM}$, for which y is respectively, past pointing and timelike, future pointing and timelike, spacelike, past pointing and null, future pointing and null.

Each point of $\boldsymbol{PT}$ (called a projective twistor) may be represented as a completely null two-surface in $\boldsymbol{CM}$. This surface intersects $\boldsymbol{M}$, if and only if the projective twistor lies in $\boldsymbol{PN}$ and then the intersection is a null geodesic. The induced mapping from $\boldsymbol{PN}$ to the space of null geodesics in $\boldsymbol{M}$ turns out to be a natural isomorphism, yielding the key fact that the space of null geodesics in $\boldsymbol{M}$ is naturally a $\boldsymbol{CR}$ manifold, such that there is a one-to-one correspondence between points of $\boldsymbol{M}$ and Riemann spheres embeddded in the $\boldsymbol{CR}$ manifold. The null cone of $\boldsymbol{I}$, called scri, is an asymptotic null hypersurface for the Minkowski spacetime. There are now three different kinds of null cones: the null cone of a finite point (a point of $\boldsymbol{M_I}$), scri itself, which has no finite points and the null cone of a point of scri, distinct from $\boldsymbol{I}$. This latter kind of cone intersects scri in a null geodesic and intersects $\boldsymbol{M_I}$ in a null hyperplane.

Analytically, we find that the information in solutions of certain relativistic field equations on $\boldsymbol{M}$ or on $\boldsymbol{CM}$ is encoded in global structure in $\boldsymbol{PT}$: for example, the first sheaf cohomology group of suitable domains in $\boldsymbol{PT}$ with coefficients in the sheaf of germs of holomorphic functions on $\boldsymbol{PT}$ corresponds to the space of solutions of the anti-self-dual Maxwell equations on the corresponding domain in $\boldsymbol{CM}$. In particular for the domains $\boldsymbol{PT}^+$ and $\boldsymbol{PT}^-$, the solutions are global on $\boldsymbol{M}^{++}$ and $\boldsymbol{M}^{--}$ respectively. For solutions in $\boldsymbol{M}$ only we use instead $\boldsymbol{CR}$ cohomology on subsets of $\boldsymbol{PN}$. This has the key advantage that non-analytic solutions are encompassed. If we pass to suitable vector bundles over $\boldsymbol{PT}$, or over $\boldsymbol{PN}$, then we encode the information of solutions of the anti-self-dual Yang-Mills equations. Also each holomorphic surface in $\boldsymbol{PT}$ intersects $\boldsymbol{PN}$ in a three-space. This space gives rise to a shear-free null congruence in Minkowski spacetime and all analytic shear free congruences are obtained this way. Non-analytic shear-free null congruences can be constructed. In general, they appear to be represented by holomorphic surfaces in either $\boldsymbol{PT}^+$, or $\boldsymbol{PT}^-$, that extend to the boundary $\boldsymbol{PN}$, but no further: such surfaces are said to be one-sided embeddable.

Given this elegant theory for flat spacetime, it is natural to ask to extend the theory to curved spacetime. Here a fundamental obstacle immediately arises, even for real analytic spacetimes. The twistors in flat spacetime are interpreted as completely null two-surfaces and it is easy to prove that such surfaces can exist, in the required generality, if and only if the spacetime is conformally flat. In the language of the Frobenius theorem, the twistor surfaces are described by a system of one-forms and the integrability of the system forces conformal flatness. Penrose realized that if the dimension was reduced by one, then the integrability problem would be overcome and a twistor theory could then be constructed **[3]**. Specifically, the curved analogue of the twistor distribution is integrable when restricted to the spin bundle over a hypersurface in spacetime, so each hypersurface in spacetime has an associated twistor theory.

If the spacetime is asympotically flat, then there are attached to the spacetime, two asymptotic null cones, one in the future and one in the past, called scri plus and scri minus, respectively. Newman and Penrose were able to completely analyze the twistor structures of these spaces, called $\boldsymbol{H}$-spaces **[6, 7, 8, 10, 12]**. Each projective twistor is represented in the surface by an appropriate complex null geodesic curve (if p^a is tangent to the curve and if n^a is the normal to the surface, then necessarily the outer product $p^{[a}n^{b]}$ is either self-dual or anti-self-dual; for twistors this outer product must be anti-self-dual; the self-dual alternative gives the "dual" or "conjugate" twistor space; the information in each space is the same). Then the space of such curves is three complex dimensional, as in the flat case. The space is fibered over a complex projective one-space (a Riemann sphere) and in favorable circumstances, there is a four complex parameter set of sections of the fibering (so each section is a Riemann sphere embedded in the projective twistor space) **[21]**. This gives a curved analogue of the space $\boldsymbol{CM}$ of flat twistor space. Just as for flat space, a complex conformal structure is determined by the incidence condition for the holomorphic sections and a preferred holomorphic metric may be defined in this conformal class. This metric is then shown to be vacuum and to have anti-self-dual Weyl curvature. Finally there is a non-projective twistor space obtained by propagating a spinor along the projective twistor curve and this non-projective space has a pseudo-Kähler structure, K, whose associated metric is Ricci flat. We then have curved analogues of some of the various spaces $\boldsymbol{G}_{(p,q,r)}$ discussed above. In particular, the vanishing of K determines a $\boldsymbol{CR}$ hypersurface in the twistor space, which, in turn, may be interpeted as the bundle of null directions over the asymptotic null hypersurface of the spacetime.

The success of the asymptotic twistor theory of Newman and Penrose raises the question of extending the theory to the finite realm. Here one notes that the asymptotic twistor theory is still rather special in that first, scri is a null hypersurface and secondly, that it is shearfree. For a null hypersurface the hypersurface twistor curves are complex null geodesics in the surface, if and only if the surface is shearfree. Geometrically, shearfreeness amounts to the fact that the complexification of scri is foliated by a one complex parameter set of completely null two-surfaces, which cannot exist away from infinity except for certain hypersurfaces in algebraically special spacetimes. Nevertheless one might anticipate that some sort of deformation of the Newman-Penrose theory is required. Indeed, for twistor spaces associated to spacelike hypersurfaces, this is the case, if analyticity is assumed [**9**].

In recent seminars, I have suggested that the Newman-Penrose picture breaks down, at least, for the properly constructed twistor spaces of finite null cones [**17–20**], the mechanism for the breakdown being provided by the Sachs equations [**32**]. These ideas are detailed in the appendix here. Instead I suggest that the twistor spaces of these null cones will be complex manifolds more like those that appear in string theory and that these twistor spaces will then provide a link between the string theory and spacetime theory. Specifically in string theory, complex manifolds with isolated compact Riemann spheres (or surfaces of higher genus) play an important role. Essentially, I am saying that the spheres of string theory are to be identified conceptually and theoretically with isolated spheres in the null hypersurface twistor spaces. String theorists assert that their theory incorporates gravity. To the limited extent that I understand their theory, I would respond that they may well have gravitational degrees of freedom in the theory, in the sense for example that they consistently construct models of gravitating particles, but they do not yet incorporate all the subtleties of the Einstein theory and that it may be that a more complete theory will require a unification of string-theoretic, twistor-theoretic and other ideas. *In the new theory, time asymmetry would be natural.* Also even "local" physics would depend via the structure of null cone hypersurface twistor spaces on the global past of the locality. This would apparently mean that there would be very subtle deviations from $\boldsymbol{PCT}$ invariance in local physics, the main point here being that the global structure of past null cones differs from that of future null cones.

In trying to analyze whether or not these conjectures are in any way sensible, we should be careful to frame the discussion properly. Also we should realize that we are in a no-lose situation. Any progress in this analysis, whether positive or negative, relative to these conjectures, will result in substantial gains in knowledge. Certainly global questions come into play; for example $\boldsymbol{C}^3$ and complex projective three-space differ only at "infinity", but the former has no embedded Riemann spheres, whilst the latter has a four-parameter set. Also, as in flat space, there are many kinds of null cone hypersurfaces; the theory of each kind will have its own flavor. The list includes the past and future null cones of a finite point; null cones avoiding singularities, null cones of points in or on horizons; cosmological null cones; "virtual" null cones: scri plus, scri minus, horizons, null cones of singular points, of points of scri, of points beyond scri. Unfortunately, when trying to construct examples, one is practically forced to use analytic spacetimes, whereas the key to the Einstein theory is its hyperbolic nature, which truly can be exposed only in a non-analytic framework. So one must instead adopt the following philosphical schema: when working with analytic spacetimes, avoid any construction that has no hope of a non-analytic analogue; also avoid bringing in any information which in a non-analytic situation would violate causality. In particular, this entails that we should emphasize the role of the $\boldsymbol{CR}$ twistor manifolds at every opportunity.

The present work gives the first example of the twistor theory of null hypersurfaces, for the case of a shearing null hypersurface. Even in the very simple case, discussed here, the computations are somewhat non-trivial and at various steps were aided by the Maple algebraic computing system. The title of this work refers to the idea prevalent in quantum field theory that dynamics proceeds along the null cone, progress in a timelike direction being made as a zigzag along various null cones, alternately future and past pointing. If my conjectures have any sense, the analogous idea in string theory is chains or ensembles of manifolds of Calabi-Yau type, connected by webs of mirror symmetries. Here I confine myself to working out some of the relevant formulas of the twistor theory. In particular an example of twistor scattering is constructed, I believe for the first time in the literature. The scattering in question depends essentially on the spacetime not being conformally flat. Two null cones intersect in a two-surface. A twistor curve of one cone meets the two-surface at one point. The attached spinor to the curve then naturally gives rise to a new twistor curve on the second null cone. This gives rise to a local diffeomorphism between the two twistor spaces, this diffeomorphism being the Zitterbewegung.

It seems possible, although I do not yet have a proof, that this scattering will be feasible even in the non-analytic case, at least for suitable spacetimes and thus be consistent with my overall philosophy. This would entail that the twistor $\boldsymbol{CR}$ structures of these null cones, even in the non-analytic case, would be at least one-sided embeddable: the null twistors of cones intersecting a given cone, coming from the domain of dependence of the given cone, would provide the non-null twistors for the one-sided embedding of the $\boldsymbol{CR}$ structure associated to the given hypersurface.

The metric studied here is the metric $2(dudv - (dx)^2 - u^{-1}(dy)^2)$. This is non-flat of type $\boldsymbol{N}$ and although it is not vacuum, it is conformal to vacuum, which is all that the twistor theory really needs. It is perhaps the fact that it is only conformal to vacuum that explains why I and other twistor theorists have not examined this metric in detail before. Rather strangely, the conformal factor to take the metric to vacuum is transcendental in u, involving the factor $u^{\sqrt{10}/2}$. In section two below, the connection and curvature of the metric are obtained. In sections three and four, the geodesic equations are solved and the null cones are constructed. In section five, spinors are introduced and the spin connection and curvature are obtained. In section six, it is shown how to rescale the metric to obtain a vacuum metric. In section seven, the Cartan conformal connection is obtained and in that language, it is again shown how the metric is conformal to vacuum.

In section eight, the spin connection is lifted to the spin bundle and the Fefferman conformal structure of the hypersurface twistor structures is found. It is a key fact that the structure of each surface is controlled by the tensor of equation (8.5), restricted to the hypersurface. In section nine, the restriction of the tensor to the spin bundle above any null cones is given and it is shown how the tensor blows up as the spinor points up the null cone. In section ten, the vector field defining the twistor structure of each null cone is written down and the vector field is shown to be explicitly integrable. However at this point a snag arises, in that the final integrals (for the quantities X and Y of equation (10.4)) are elliptic. To avoid dealing with these elliptic integrals at this stage, we restrict our investigations to asymptotic null cones: these are the limits of ordinary null cones as the u-co-ordinate of the vertex goes to zero. They form a space of co-dimension one in the space of all null cones, so are similar in nature to the null cones of points of scri in Minkowski space.

The remaining sections deal only with these limiting null cones. This has the drawback that any two of these cones have the same time orientation, so that their intersection is never compact. We have yet to find a calculable example where two shearing null cones intersect in a compact region.

In section eleven, the twistor space is studied in detail and it is shown that a six-fold covering of the twistor space may be realized as the compact algebraic hypersurface $TW^6 + G^7 - Z^2H^5 = 0$, in the complex four-dimensional projective space, with co-ordinates given by the ratios of the quantities (T, W, Z, G, H). This is the first main result of this work. So at the complex level, we are now able to examine every aspect of the usual twistor constructions in this space: sheaf cohomology, coherent sheaves, etc. However our main concern is with the $\boldsymbol{CR}$ aspects of this space. So, in section twelve, we calculate the pseudo-Kähler scalar of the twistor space, following the prescription of Penrose [**3**]. The corresponding Kähler metric is explicitly given.

The Ricci curvature of the metric is calculated and found to be non-zero. This is perhaps unexpected in that, in all previously known cases, the metric was found to be Ricci flat. However it is in line with an, as yet unpublished, difficult calculation of the author and Lionel Mason: we showed that the Fefferman-Graham obstruction of the Fefferman conformal structures of general twistor hypersurface structures is non-zero and is the magnetic part of the Weyl curvature evaluated on the hypersurface [**35**]. It remains to understand the meaning of the Ricci curvature. In the formula for the Kähler scalar, transcendental powers with exponent $\sqrt{10}$ again arise, indicating that the scalar stores the information that the spacetime metric is conformal to vacuum. This is not too surprising, since the formula uses the Cartan conformal connection, which is sensitive to the Ricci tensor.

Finally, in section thirteen, the Zitterbewegung is calculated explicitly. It is shown that given the twistor and its scattered twistor, their twistor curves intersect at a unique point and, for this example, the scattering equations are completely algebraic. Summarizing, we have constructed explicitly a complex manifold that should contain a hypersurface of dimension nine in the projective "twistor-string space", or equivalently a hypersurface of dimension eleven of the non-projective "twistor-string space", or from the real point of view, a hypersurface of dimension eight of the nine-dimensional space of projective null cone hypersurface twistors, or of dimension ten of the eleven-dimensional space of non-projective null cone hypersurface twistors. Ultimately by studying the elliptic integrals of section ten we should be able to extend these constructions off the hypersurface.

2. The metric and its curvature

We consider a spacetime $(\boldsymbol{M}, g)$, where the manifold $\boldsymbol{M}$ is topologically $\boldsymbol{R}^4$, with co-ordinates $(u, v, x, y) \in \boldsymbol{R}^4$ (where $u > 0$) and with metric, g:

$$\begin{aligned} & g = 2(dudv - (dx)^2 - u^{-1}(dy)^2) = 2(ln - \xi^2 - \eta^2), \\ & l = du, \;\; n = dv, \;\; \xi = dx, \;\; \eta = u^{-1/2}dy. \end{aligned} \tag{2.1}$$

For the exterior derivatives of the tetrad forms, (l, n, ξ, η), we have:

$$dl = dn = d\xi = 0, \;\; d\eta = -(2u)^{-1} l\eta. \tag{2.2}$$

The tetrad vector fields are given as follows:

$$l^* = \partial_v, \;\; n^* = \partial_u, \;\; \xi^* = -2^{-1}\partial_x, \;\; \eta^* = -2^{-1}u^{1/2}\partial_y. \tag{2.3}$$

The Levi-Civita connection, d, associated to the metric is given as follows:

$$dl_a = 0, \;\; dn_a = \frac{1}{u}\eta\eta_a, \;\; d\xi_a = 0, \;\; d\eta_a = \frac{1}{2u}\eta l_a. \tag{2.4}$$

Here and in the following, we use abstract tensor (or spinor) indices. Also d is the covariant exterior derivative. This connection is metric preserving:

$$dg_{ab} = 2d(l_{(a}n_{b)} - \xi_a\xi_b - \eta_a\eta_b) = 0$$

and is manifestly torsion-free, so it is the Levi-Civita connection. Applying d to equation (2.4), we get:

$$d^2 l_a = 0, \;\; d^2 n_a = -\frac{3}{2u^2} l\eta\eta_a, \;\; d^2\xi_a = 0, \;\; d^2\eta_a = -\frac{3}{4u^2} l\eta l_a. \tag{2.5}$$

In general, we have $d^2 v_a = -R_{ab}v^b$, where the curvature two-form R_{ab} is given as follows:

$$R_{ab} = -\frac{3}{u^2}(l\eta l_{[a}\eta_{b]}). \tag{2.6}$$

Introduce the canonical one-form:

$$\theta^a = ln^a + nl^a - 2\xi\xi^a - 2\eta\eta^a. \tag{2.7}$$

Then the torsion-free condition is expressed by the formula $d\theta^a = 0$. Dually, introduce the derivation of forms δ_a, such that $\delta_a\theta^b = \delta_a^b$, where δ_a^b is the Kronecker delta tensor. Then we have the Ricci form $R_b = \delta^a R_{ab}$ and the Ricci scalar, $R = \delta^b R_b$ given as follows:

$$R_b = \delta^a R_{ab} = \frac{3}{4u^2} l l_b, \;\; R = 0. \tag{2.8}$$

The Weyl two-form is $C_{ab} = R_{ab} - \theta_{[a}R_{b]} + \frac{R}{6}\theta_a\theta_b$. It obeys the trace-free condition $\delta^a C_{ab} = 0$. Here C_{ab} is given as follows:

$$C_{ab} = \frac{3}{2u^2}(l\xi l_{[a}\xi_{b]} - l\eta l_{[a}\eta_{b]}). \tag{2.9}$$

3. The geodesic spray

The canonical one-form α on the cotangent bundle of the spacetime may be written: $\alpha = ql + rn + s\xi + t\eta$, where $(q, r, s, t) \in \boldsymbol{R}^4$ are fibre co-ordinates for the cotangent bundle. Then the symplectic form $\omega = d\alpha = -ldq - ndr - \xi ds - \eta dt - \dfrac{t}{2u} l\eta$. The Poisson form, P, which inverts ω is given as follows:

$$P = n^*\partial_q + l^*\partial_r - 2\xi^*\partial_s - 2\eta^*\partial_t - \frac{t}{2u}\partial_q\partial_t. \tag{3.1}$$

The Hamiltonian for the geodesic spray is the function $H = qr - \dfrac{s^2}{4} - \dfrac{t^2}{4}$. The Hamiltonian vector field giving the geodesic spray is the vector field $H^* = P(dH)$ and is given as follows:

$$H^* = ql^* + rn^* + s\xi^* + t\eta^* + \frac{t}{4u}(2r\partial_t + t\partial_q). \tag{3.2}$$

The geodesic equations are then as follows:

$$\begin{aligned}\dot{q} &= \frac{t^2}{4u}, \quad \dot{r} = 0, \quad \dot{s} = 0, \quad \dot{t} = \frac{rt}{2u},\\ \dot{u} &= r, \quad \dot{v} = q, \quad \dot{x} = -\frac{s}{2}, \quad \dot{y} = -\frac{tu^{1/2}}{2}.\end{aligned} \tag{3.3}$$

Generically, we may take $r \neq 0$ and u as parameter. The solutions of the geodesic equations then follow:

$$\begin{aligned}x &= x_0 - \frac{s}{2r}(u-u_0), \quad y = y_0 + A_0(u^2 - u_0^2),\\ v &= v_0 + 2A_0^2(u^2-u_0^2) + (u-u_0)(\frac{H}{r^2} + \frac{s^2}{4r^2}),\\ q &= 4A_0^2 ru + \frac{H}{r} + \frac{s^2}{4r}, \quad t = -4A_0 ru^{1/2}.\end{aligned} \tag{3.4}$$

Here u_0, v_0, x_0, y_0, A_0, r, s and H are constants. For timelike geodesics, $H > 0$ and for null geodesics, $H = 0$. The special case when $r = 0$ is also easily solved, with the following result: u, r, s and t are necessarily constant. If $t \neq 0$, then $H < 0$ so the geodesic is necessarily spacelike; the variable q may be used as a parameter along the geodesic and we have $v = v_0 + (8u)^{-1}t^2(q^2 - q_0^2)$, $y = y_0 - 2t^{-1}u^{3/2}(q - q_0)$ and $x = -2ust^{-2}(q-q_0) + x_0$, where q_0, v_0, x_0 and y_0 are constants. If $t = 0$, but $s \neq 0$, again the geodesic is spacelike; also q and y are constant and x may be used as a parameter. Then $v = -2qs^{-1}(x - x_0) + v_0$, with v_0 and x_0 constant. If $s = t = 0$, but $q \neq 0$, then v is arbitrary and we have q, x and y constant; the geodesic is null. Finally if $q = s = t = 0$, the geodesic reduces to a point.

4. The null cones

Consider the collection of all null geodesics passing through the point (u_0, v_0, x_0, y_0), where $u_0 > 0$. These are given by equation (3.4), with $H = 0$. Eliminating the quantities A_0 and s/r from the equations for x, y and v, we obtain the following equation for the null cone:

$$0 = (u-u_0)(v-v_0) - (x-x_0)^2 - 2\frac{(y-y_0)^2}{u+u_0}. \tag{4.1}$$

Note that from the discussion following equation (3.4), the only null geodesic through (u_0, v_0, x_0, y_0) that is not described by equation (3.4) is the geodesic with $u = u_0$, $x = x_0$, $y = y_0$ and v arbitrary. Clearly this geodesic lies on the hypersurface given by equation (4.1), so equation (4.1) does describe the complete null cone. Note that the formal limit as $u_0 \to 0$ makes sense in equation (4.1), even though the the "null cone" has no vertex in this case (since the metric is not defined when $u = 0$). For later work, it is more convenient to divide this equation by $u - u_0$ and write the equation in the form: $N = 0$, where the function N is given for any $0 < u \neq u_0$ by the formula:

$$N = v - v_0 - \frac{(x-x_0)^2}{u-u_0} - 2\frac{(y-y_0)^2}{u^2-u_0^2}. \tag{4.2}$$

The differential of this equation gives the equation of the normal dN of the null cone:

$$\begin{aligned} dN &= n + l(B^2+C^2) + 2B\xi + 2C\eta, \\ B &= -\frac{x-x_0}{u-u_0}, \quad C = -2\frac{u^{1/2}(y-y_0)}{u^2-u_0^2}. \end{aligned} \tag{4.3}$$

Again we may take the limit formally as $u_0 \to 0$, showing that the hypersurface given by the following equation is everywhere regular and null:

$$0 = v - v_0 - u^{-1}(x-x_0)^2 - 2u^{-2}(y-y_0)^2. \tag{4.4}$$

This hypersurface is topologically $\boldsymbol{R}^3$: it resembles the space obtained by deleting a generator from the top half of a null cone. The hypersurface twistor spaces of the hypersurfaces of (4.4) will be constructed below.

5. The spin connection

We pass to spinors by introducing a spin basis o_A and ι_A and a conjugate spin basis $o_{A'}$ and $\iota_{A'}$, related to the tetrad $(l_a, n_a, \xi_a, \eta_a)$ as follows:

$$o_A o_{A'} = l_a, \quad \iota_A \iota_{A'} = n_a, \quad o_A \iota_{A'} = \xi_a + i\eta_a, \quad \iota_A o_{A'} = \xi_a - i\eta_a. \tag{5.1}$$

We have $o_A\iota_B - o_B\iota_A = \varepsilon_{AB} = -\varepsilon_{BA}$ and $o_{A'}\iota_{B'} - o_{B'}\iota_{A'} = \varepsilon_{A'B'} = -\varepsilon_{B'A'}$, where the spinor symplectic forms ε_{AB} and $\varepsilon_{A'B'}$ are related to the metric by the formula: $g_{ab} = \varepsilon_{AB}\varepsilon_{A'B'}$. We raise or lower spinor indices according to the scheme: $v^A\varepsilon_{AB} = v_B$, $v_B\varepsilon^{AB} = v^A$, $v^{A'}\varepsilon_{A'B'} = v_{B'}$ and $v_{B'}\varepsilon^{A'B'} = v^{A'}$, for any spinors v^A and $v^{A'}$. Note that $o_A\iota^A = 1$ and $o_{A'}\iota^{A'} = 1$. The spin connection is now given as follows:

$$\begin{aligned} do_A &= 0, \quad d\iota_A = -\frac{i}{2u}\eta o_A, \quad do_{A'} = 0, \quad d\iota_{A'} = \frac{i}{2u}\eta o_{A'}, \\ d^2 o_A &= 0, \quad d^2\iota_A = \frac{3i}{4u^2} l\eta o_A, \quad d^2 o_{A'} = 0, \quad d^2\iota_{A'} = -\frac{3i}{4u^2} l\eta o_{A'}. \end{aligned} \tag{5.2}$$

The spinor curvature two-forms, R_{AB} and $R_{A'B'}$ are determined by the equations $d^2 v_A = R_{AB}v^B$ and $d^2 v_{A'} = R_{A'B'}v^{B'}$, for any spinors v^A and $v^{A'}$. From equation (5.2), they are given as follows:

$$R_{AB} = \frac{3i}{4u^2} l\eta o_A o_B, \quad R_{A'B'} = -\frac{3i}{4u^2} l\eta o_{A'} o_{B'}. \tag{5.3}$$

The curvature two-form is given in terms of the spinor curvature forms by the formula: $R_{ab} = \varepsilon_{AB}R_{A'B'} + \varepsilon_{A'B'}R_{AB}$. The Ricci spinor form is given by the formula

$$-\delta_{BA'}R_A^B = -\frac{3i}{4u^2} o_A o^B \delta_{BA'}(l\eta) = \frac{3i}{4u^2} l o_A o^B \eta_{BA'} = \frac{3}{8u^2} l l_a.$$

The Weyl spinor form C_{AB} is given by the decomposition $C_A^B = R_A^B + \theta^{BB'}P_{AB'}$, where the (indexed) one-form P_a is chosen such that $\delta_b C_A^B = 0$. Then C_{AB} and P_a are as follows:

$$C_{AB} = \frac{3}{8u^2} o_A o_B (l\xi + il\eta), \quad P_a = \frac{3}{8u^2} l l_a. \tag{5.4}$$

The spinor decompositions of the one-form θ^a and of the two-form $\theta^a\theta^b$ are as follows:

$$\begin{aligned} \theta^a &= l\iota^A\iota^{A'} + no^A o^{A'} - (\xi - i\eta)o^A\iota^{A'} - (\xi + i\eta)\iota^A o^{A'}, \\ \theta^a\theta^b &= \varepsilon^{AB}\Sigma^{A'B'} + \varepsilon^{A'B'}\Sigma^{AB}, \quad \varepsilon_{A'B'}\theta^a\theta^b = 2\Sigma^{AB} = 2\Sigma^{BA}, \\ \Sigma^{AB} &= (\xi n - i\eta n)o^A o^B - (ln + 2i\xi\eta)o^{(A}\iota^{B)} + (l\xi + il\eta)\iota^A\iota^B. \end{aligned} \tag{5.5}$$

We have $C_{AB} = C_{ABCD}\Sigma^{CD}$, where the Weyl spinor C_{ABCD} is given by the formula:

$$C_{ABCD} = \frac{3}{8u^2} o_A o_B o_C o_D.$$

In particular this shows that the metric is everywhere of type N. Note that $2l_{[a}\xi_{b]} + 2il_{[a}\eta_{b]} = o_A o_B \varepsilon_{A'B'}$ and that $C_{ab} = C_{AB}\varepsilon_{A'B'} + C_{A'B'}\varepsilon_{AB}$, where $C_{A'B'}$ is the complex conjugate of C_{AB}.

6. The conformal field equations

Consider a conformally related spin connection, D, of the following form:

$$Dv_A = dv_A + \gamma o_A o_{B'}\theta^b v_B, \quad Dv_{A'} = dv_{A'} + \gamma o_{A'} o_B \theta^b v_{B'}. \tag{6.1}$$

Here the real-valued function γ depends only on the variable u. Then $D\theta^a = 0$, so D is torsion-free and if ρ is a (non-zero) function of u only, we have

$$\rho^2 D(\rho^{-2} g_{ab}) = 2(\gamma - \frac{\rho'}{\rho}) l g_{ab},$$

so D is the Levi-Civita connection of the conformally rescaled metric $\rho^{-2}g_{ab}$, where $\rho' = \gamma\rho$. We have $D^2 v_A = d^2 v_A + o_A(\gamma' + \gamma^2) l o_{B'}\theta^b v_B$, so the new curvature spinor S_{AB} is given by the formula

$$\begin{aligned} S_{AB} &= R_{AB} + o_A(\gamma' + \gamma^2) l o^{B'}\theta_{B'B} \\ &= R_{AB} - 2o_A(\gamma' + \gamma^2) l o^{B'}(\xi\xi_{BB'} + \eta\eta_{BB'}) \\ &= R_{AB} + o_A o_B(u^{-1}\gamma' + \gamma^2) l(\xi - i\eta). \end{aligned}$$

The new Ricci spinor is then given by the formula $ll_a\big(\frac{3}{8u^2} - \gamma' - \gamma^2\big)$. In particular, for a suitable choice of γ, the conformally rescaled metric is Ricci flat. In fact, putting $\gamma = \rho'/\rho$, we need $\rho'' - \frac{3}{8u^2}\rho = 0$. This linear equation has the general solution $\rho = au^{(2+\sqrt{10})/4} + bu^{(2-\sqrt{10})/4}$, where a and b are constants. In particular we have established that the given metric g is conformally related to a Ricci flat metric, albeit with a strange conformal factor. We can confirm this result explicitly as follows.

For real $a \neq 1/2$, consider the conformally related metric $h = \frac{1}{2}u^{-2a}g$. Make the co-ordinate transformation: $u \mapsto u^{1/(1-2a)}$, $v \mapsto 2(1-2a)v + au^{-1}x^2 + (a+\frac{1}{2})u^{-1}y^2$, $x \mapsto xu^{a/(1-2a)}$ and $y \mapsto yu^{(2a+1)/2(1-2a)}$. Then we find:

$$h = 2dudv - (dx)^2 - (dy)^2 + \Big(a(a-1)x^2 + \big(a+\frac{1}{2}\big)\big(a-\frac{3}{2}\big)y^2\Big)\Big(\frac{du}{(1-2a)u}\Big)^2. \tag{6.2}$$

In this form, the conformally rescaled metric is recognizable as a standard null solution of the vacuum Einstein equations, provided the last term is harmonic in the variables x and y, so provided the last term is a multiple of $x^2 - y^2$. This gives the condition on a: $0 = a(a-1) + (a+\frac{1}{2})(a-\frac{3}{2}) = 2((a-\frac{1}{2})^2 - \frac{5}{8})$. So if $a = \frac{2\pm\sqrt{10}}{4}$, the rescaled metric is vacuum, in agreement with the above discusion. Note that the new metric is nowhere flat, since the Weyl curvature never vanishes.

7. The Cartan conformal connection

The Cartan conformal connection may be formulated conveniently in terms of local twistor transport. In a fixed conformal frame, a local twistor z^α is represented by a pair of spinors, $z^\alpha = (z^A,\ z_{A'})$. Denote by D the local twistor connection and by d the trivial extension of the spinor connection to the local twistor bundle, so that $dz^\alpha = (dz^A,\ dz_{A'})$. Then we have:

$$\begin{aligned} Dz^\alpha &= dz^\alpha - \Gamma^\alpha_\beta z^\beta = (dz^A - i\theta^{AB'}z_{B'},\ \ dz_{A'} + iP_{BA'}z^B), \\ \Gamma^A_B &= 0,\ \ \Gamma^{B'A} = i\theta^{B'A},\ \ \Gamma^{B'}_{A'} = 0,\ \ \Gamma_{BA'} = -iP_{BA'}. \end{aligned} \tag{7.1}$$

Note that D preserves twistor conjugation, which sends the twistor z^α to its conjugate $\overline{z}_\alpha = (\overline{z}_A,\ \overline{z}^{A'})$. This conjugation is pseudo-hermitian of type $(2,2)$. The group of the connection is a subgroup of $\mathrm{SU}(2,2)$, which is the spin group for $\mathrm{SO}(2,4)$, which in turn is the group relevant for the traditional construction of the Cartan conformal connection in relativity. The curvature twistor T^α_β is given by the formula: $D^2z^\alpha = T^\alpha_\beta z^\beta$. If we write $d^2z^\alpha = S^\alpha_\beta z^\beta$, then we have $T^\alpha_\beta = S^\alpha_\beta - d\Gamma^\alpha_\beta - \Gamma^\gamma_\beta\Gamma^\alpha_\gamma$. Here $S^A_B = R^A_B$, $S^{B'A} = 0$, $S_{BA'} = 0$ and $S^{B'}_{A'} = -R^{B'}_{A'}$. Then $T^A_B = R^A_B + \theta^{AB'}P_{BB'} = C^A_B$, so T^α_β is given as follows:

$$T^A_B = C^A_B,\ \ T^{B'A} = 0,\ \ T_{BA'} = idP_{BA'},\ \ T^{B'}_{A'} = -C^{B'}_{A'}. \tag{7.2}$$

For the present metric, using equation (5.4), we find $dP_a = 0$, so we have:

$$\begin{aligned} Dz^\alpha &= (dz^a - i\theta^{AB'}z_{B'},\ \ dz_{A'} + \frac{3i}{8u^2}o_{A'}lo_Bz^B), \\ D^2z^\alpha &= \frac{3}{8u^2}((l\xi + il\eta)o^Ao_Bz^B,\ \ (l\xi - il\eta)o_{A'}o^{B'}z_{B'}) = T^\alpha_\beta z^\beta, \\ T^A_B &= \frac{3}{8u^2}(l\xi + il\eta)o_Bo^A = -\overline{T^{A'}_{B'}},\ \ T^{B'A} = 0,\ \ T_{BA'} = 0. \end{aligned} \tag{7.3}$$

Note that the curvature T^α_β is pseudo-hermitian and tracefree. Using the local twistor connection, the conformal field equations boil down to the existence, or otherwise, of a suitable skew twistor $I^{\alpha\beta} = -I^{\beta\alpha}$, such that $DI^{\alpha\beta} = 0$. Here we may put $I^{AB} = p\varepsilon^{AB}$, $I_{A'}^{B} = iqo_{A'}o^{B}$ and $I_{A'B'} = 0$, where p and q are real functions of the variable u only. Note that $I^{\alpha\beta}\overline{I}_{\alpha\beta} = 0$. Then the condition $DI^{\alpha\beta} = 0$ gives the relations: $p' = q$ and $q' = \frac{3}{8}u^{-2}p$, with the general solution $p = au^{(2+\sqrt{10})/4} + bu^{(2-\sqrt{10})/4}$ and $q = p'$, where a and b are arbitrary constants, showing, in particular that the metric is conformal to vacuum, in agreement with the results of the previous section.

8. The spin bundle

We pass to the spin bundle, where a point of the (primed) spin bundle is labelled by its co-ordinates s and t relative to the spin basis $(o_{A'}, \iota_{A'})$. The canonical section $\pi_{A'}$ and its differential $d\pi_{A'}$ are then given as follows:

$$\pi_{A'} = so_{A'} + t\iota_{A'}, \;\; d\pi_{A'} = \big(ds + \frac{i}{2u}t\eta\big)o_{A'} + (dt)\iota_{A'}. \tag{8.1}$$

Dually we have the horizontal vector field ∂_a on the spin bundle, which are required to annihilate $d\pi_{A'}$ and $d\pi_A$ (where π_A represents the complex conjugate of the spinor $\pi_{A'}$). In terms of the co-ordinates s and t, the vector field ∂_a is given explicitly by the formula:

$$\partial_a = l_a n^* + n_a l^* - 2\xi_a\xi^* - 2\eta_a\eta^* - \frac{i}{2u}\eta_a(t\partial_s - \overline{t}\partial_{\overline{s}}). \tag{8.2}$$

Here the tetrad vector fields $(l^*, n^*, \xi^*, \eta^*)$ are given in equation (2.3). There are also canonical vertical vector fields, $\partial^{A'}$ and its conjugate ∂^{A}, such that $\partial^{A'}\pi_{B'} = \delta^{A'}_{B'}$ and $\partial^{A'}\pi_A = 0$. Explicitly, we have the formulas:

$$\partial^{A'} = -o^{A'}\partial_t + \iota^{A'}\partial_s, \;\; \partial^{A} = -o^{A}\partial_{\overline{t}} + \iota^{A}\partial_{\overline{s}}. \tag{8.3}$$

The contact form of the cotangent bundle pulled back to the spin bundle is the one-form:

$$\theta^a\pi_{A'}\pi_A = s\overline{s}l + t\overline{t}n + s\overline{t}(\xi - i\eta) + t\overline{s}(\xi + i\eta). \tag{8.4}$$

The contact form and its exterior derivative are killed by the vector field $\pi^{A'}\pi^{A}\partial_a$, which is the spinor version of the null geodesic spray and by the vector field $i(\pi_{A'}\partial^{A'} - \pi_A\partial^A)$, which generates spinor phase transformations, leaving the vector $\pi_{A'}\pi_A$ invariant. Finally, we introduce the Fefferman tensor, F, which is a symmetric covariant tensor of rank two on the spin bundle, which, when restricted to the spin bundle over each hypersurface in spacetime, determines the twistor CR structure of that hypersurface. We have:

$$F = -i\theta^a(\pi_A d\pi_{A'} - \pi_{A'}d\pi_A). \tag{8.5}$$

The tensor F is annihilated by the vector fields $\pi^{A}\pi^{A'}\partial_a$ and $\pi_{A'}\partial^{A'} + \pi_A\partial^A$.

9. The null cone spinor geometry

Using spinors, the null cone differential of equation (4.3) is written as follows:

$$dN_a = \alpha_A\alpha_{A'},\ \ \alpha_A = \iota_A + (B - iC)o_A,\ \ \alpha_{A'} = \iota_{A'} + (B + iC)o_{A'}. \tag{9.1}$$

Here the functions B and C are as given in equation (4.3). Note that $o_A\alpha^A = 1$. Restricted to the null cone, we have $\theta^a\alpha_A\alpha_{A'} = 0$, whence $\theta^a\alpha_A = \theta\alpha^{A'}$ and $\theta^a\alpha_{A'} = \overline{\theta}\alpha^A$, where θ is a one-form. Explicitly we have on the null cone:

$$\theta = \theta^a\alpha_A o_{A'} = l(B - iC) + \xi - i\eta. \tag{9.2}$$

Computing the derivative of $\alpha_{A'}$ we have:

$$d\alpha_{A'} = o_{A'}\gamma,\ \ \gamma = \frac{1}{4u(u^2 - u_0^2)}(\theta(u - u_0)^2 - \overline{\theta}(u_0^2 + 2u_0u + 5u^2)). \tag{9.3}$$

Using equations (5.3), (9.2) and (9.3), we find the following exterior derivatives:

$$\begin{aligned} d\gamma &= R_{A'B'}\alpha^{A'}\alpha^{B'} = -\frac{3i}{4u^2}l\eta,\\ d\theta &= -l\overline{\gamma} = \frac{l\xi}{u - u_0} - i\frac{l\eta(u_0^2 + 3u^2)}{2u(u^2 - u_0^2)}. \end{aligned} \tag{9.4}$$

Using the form θ, we may write out the canonical one-form θ^a on the null cone as follows:

$$\theta^a = l\alpha^A\alpha^{A'} - \theta o^A\alpha^{A'} - \overline{\theta}\alpha^A o^{A'}. \tag{9.5}$$

The Fefferman conformal structure for the spin bundle over the null cone is given by the formula:

$$F = 2\Im(\theta^a\pi_A d\pi_{A'}) = 2\Im(l\overline{p}(dp - q\gamma) - \theta\overline{q}(dp - q\gamma) - \overline{\theta}\overline{p}dq)).$$

Here we have put $p = \pi_{A'}\alpha^{A'} \neq 0$ and $q = \pi_{A'}o^{A'}$. Now restrict to $|p|^2 = 1$ and put $q = tp$ and $dp = ipdz$, with z real and t complex. Then we have

$$F = 2(l - \overline{t}\theta - t\overline{\theta})dz + 2\Im(-lt\gamma + \theta t\overline{t}\gamma - \overline{\theta}dt).$$

To make this formula more explicit, we first introduce new co-ordinates X and Y, defined, for $u \neq u_0$:

$$X = \frac{x - x_0}{u - u_0},\ \ Y = \frac{y - y_0}{u^2 - u_0^2}. \tag{9.6}$$

Then θ and γ are expressed in these co-ordinates simply as follows:

$$\theta = (u - u_0)dX - \frac{i}{u^{1/2}}(u^2 - u_0^2)dY,\ \ \gamma = -\,dX - \frac{i}{2u^{3/2}}(u_0^2 + 3u^2)dY. \tag{9.7}$$

Next write $t = r + is$, with r and s real numbers. Then F may be written out explicitly:

$$\begin{aligned} F &= 2dudz - 4r(u - u_0)dXdz + 4su^{-1/2}(u^2 - u_0^2)dYdz\\ &\quad + 2sdudX + ru^{-3/2}(u_0^2 + 3u^2)dudY - (r^2 + s^2)u^{-3/2}(u - u_0)^3dXdY\\ &\quad - 2u^{-1/2}(u^2 - u_0^2)dYdr - 2(u - u_0)dXds. \end{aligned} \tag{9.8}$$

The limiting metric, when $u_0 \to 0$, will be denoted F_0. We have:

$$\begin{aligned} F_0 &= 2dudz - 4rudXdz + 4su^{3/2}dYdz + 2sdudX + 3ru^{1/2}dudY \\ &\quad - (r^2+s^2)u^{3/2}dXdY - 2u^{3/2}dYdr - 2udXds. \end{aligned} \tag{9.9}$$

Note that for each of these metrics, the metric coefficients depend only on the variables u, r and s, so the vector fields ∂_z, ∂_X and ∂_Y are Killing vectors. Also, by inspection, the vector field $X\partial_X + Y\partial_Y - r\partial_r - s\partial_s$ is a fourth Killing vector. The metric F_0 always has signature $(3,3)$. The metric F also has signature $(3,3)$, unless $u = u_0$, when the signature drops to $(1,1)$. The restriction of the metric F to the subspace spanned by the Killing vectors has signature $(2,2)$ unless $u = u_0$ or $r = s = 0$. The parameter t should be allowed to go to infinity, since the limit as $t \to \infty$ corresponds to the vanishing of the co-ordinate p. We can see the behaviour of the metric as $t \to \infty$ by making co-ordinate replacements $r \mapsto r/(r^2+s^2)$ and $s \mapsto -s/(r^2+s^2)$. Then

$$\begin{aligned} F \longmapsto \frac{1}{r^2+s^2}\Big(& 2dudz - 4r(u-u_0)dXdz - 4su^{-1/2}(u^2-u_0^2)dYdz \\ & - 2sdudX + ru^{-3/2}(u_0^2+3u^2)dudY - u^{-3/2}(u-u_0)^3dXdY \\ & - 2u^{-1/2}(u^2-u_0^2)dYdr + 2(u-u_0)dXds \\ & + 4u^{-1/2}(u^2-u_0^2)dY\frac{r^2dr+rsds}{r^2+s^2} - 4(u-u_0)dX\frac{rsdr+s^2dr}{r^2+s^2}\Big). \end{aligned}$$

Even as a conformal structure the metric coefficients blow up as $(r,s) \to (0,0)$; but if we make a co-ordinate change to polar co-ordinates $r = m\cos\phi$ and $s = m\sin\phi$, with $m \geq 0$, then we have

$$\begin{aligned} m^2F = {} & 2dudz - u^{-3/2}(u-u_0)^3dXdY \\ & + 2dm\big(u^{-1/2}\cos\phi(u^2-u_0^2)dY - 2\sin\phi(u-u_0)dX\big) \\ & + m\cos\phi\big(-4(u-u_0)dXdz + u^{-3/2}(u_0^2+3u^2)dudY + 2(u-u_0)dXd\phi\big) \\ & + m\sin\phi\big(-4u^{-1/2}(u^2-u_0^2)dYdz - 2dudX + 2u^{-1/2}(u^2-u_0^2)dYd\phi\big). \end{aligned}$$

Putting $m = 0$ in this latter expression we arrive at the metric

$$2dudz - u^{-3/2}(u-u_0)^3dXdY + 2dm(u^{-1/2}\cos\phi(u^2-u_0^2)dY - 2\sin\phi(u-u_0)dX).$$

So now the conformal structure is smooth at $m = 0$, but degenerate in the ϕ direction. These pathologies reflect the degeneration of the twistor CR structure as the co-ordinate p goes to zero, so as $\pi_{A'}$ becomes proportional to the spinor tangent to the null cone.

10. The twistor structure

The vector field defining the twistor structure of each null cone is the complex vector field $T = \alpha^A\pi^{A'}\partial_a$, where α^A (given by equation (9.1)) is the tangent spinor to the null cone, ∂_a (given by equation (8.2)) is the horizontal vector field representing the connection on the spin bundle and $\pi_{A'}$ (given by equation (8.1)) is the tautological indexed function on the spin bundle. Combining these various quantities gives the following formula for T:

$$\begin{aligned} T &= (\iota^A + (B - iC)o^A)(so^{A'} + t\iota^{A'})\partial_a = s(B - iC)l^* + tn^* + s(\xi^* - i\eta^*) \\ &\quad + \frac{1}{4u}(s - Bt + iCt)(t\partial_s - \bar{t}\partial_{\bar{s}}) + t(B - iC)(\xi^* + i\eta^*) \\ &= t\partial_u - s\partial_x + s(B - iC)\partial_v + \frac{1}{4u}(s - tB + itC)(2u\partial_x + 2iu^{3/2}\partial_y + t\partial_s - \bar{t}\partial_{\bar{s}}). \end{aligned} \tag{10.1}$$

We check that T is tangent to the null cone: then using co-ordinates (u, x, y) for the null cone, we may drop the terms involving ∂_v from T. Then the integral curves are the solutions of the following differential system:

$$\begin{aligned} \dot{s} &= \frac{t}{4u}(s - tB + itC), \quad \dot{t} = 0, \quad \dot{u} = t, \\ \dot{x} &= \frac{1}{2}(-s - tB + itC), \quad \dot{y} = \frac{i}{2}u^{1/2}(s - tB + itC). \end{aligned} \tag{10.2}$$

To solve this system, define new variables X, Y and S, as follows:

$$X = \frac{x - x_0}{u - u_0}, \quad Y = \frac{y - y_0}{u^2 - u_0^2}, \quad S = s - tB - itC = s + tX + 2itu^{1/2}Y. \tag{10.3}$$

The rationale for introducing X, Y and S is that they are constant along the generators of the null cone (lifted to the spin bundle in the case of S). Also $S = \pi_{A'}\alpha^{A'}$, so S vanishes when the spinor $\pi_{A'}$ points up the null cone. Then the system of equation (10.2) simplifies to the following system:

$$\begin{aligned} \dot{t} &= 0, \quad \dot{S} = tS\Big(\frac{1}{4u} - \frac{1}{2(u - u_0)} - \frac{u}{(u^2 - u_0^2)}\Big), \\ \dot{u} &= t, \quad \dot{X} = -\frac{S}{2(u - u_0)}, \quad \dot{Y} = \frac{iu^{1/2}S}{2(u^2 - u_0^2)}. \end{aligned} \tag{10.4}$$

Note that two independent obvious solutions of this system are $(X, Y, S) = (1, 0, 0)$ and $(X, Y, S) = (0, 1, 0)$. However the third independent solution has

$$S = S_0 u^{1/4}(u + u_0)^{-1/2}(u - u_0)^{-1},$$

with S_0 a non-zero constant. Inserting this expression into the other equations leaves us with elliptic integrals for the quantities X and Y. These equations will not be analyzed further here.

11. A limiting twistor space

We consider the simplified space obtained by putting $u_0 = 0$ in equation (10.4). Thus we analyze the following differential system, giving the twistor curves for the hypersurface of equation (4.4):

$$\dot{t}=0,\quad \dot{u}=t,\quad \dot{S}=-\frac{5tS}{4u},\quad \dot{X}=-\frac{S}{2u},\quad \dot{Y}=\frac{iS}{2u^{3/2}}. \tag{11.1}$$

Here the co-ordinates are complex and we have to decide how to handle the square root of u. One approach is to restrict to regions such that u is always close to being real and positive. An alternative, which we adopt, is to introduce a co-ordinate change, writing $p=u^{-1/4}$. In the physical regime where u and p are real and positive these co-ordinates are equivalent. Finally, we rescale the parameter along the curve by a factor of 4, for convenience. Thus we study the system:

$$\dot{t}=0,\quad \dot{p}=-tp^5,\quad \dot{S}=-5tp^4S,\quad \dot{X}=-2p^4S,\quad \dot{Y}=2ip^6S. \tag{11.2}$$

First we assume that $t\neq 0$. Then we may take $p\neq 0$ as a parameter for the curve and we have the following constants of the motion:

$$T=t,\quad Z=\frac{S}{35p^5},\quad G=\frac{S}{35}-t\frac{X}{14},\quad H=\frac{Sp^2}{35}+t\frac{Y}{10i}. \tag{11.3}$$

Here $(G,H,Z,T)\in \boldsymbol{C}^4$ are complex parameters specifying the twistor curve and the attached spinor $\pi_{A'}=So_{A'}+t\alpha_{A'}$ (more accurately: a local twistor $(0,\pi_{A'})$, parallelly propagated along the twistor curve, by the Cartan connection). The differentials of the functions (G,H,Z,T) are independent if and only if $T\neq 0$, as is easily seen. So provided $T\neq 0$,these parameters also serve as local co-ordinates for the twistor space. They are redundant for the description of the twistor curve only, which gives a point of the projective twistor space and requires only the co-ordinate ratios $(g,h,z)=(G/T,H/T,Z/T)$: scaling T by a non-zero complex number gives the same curve. The degeneracy at $T=0$ is also seen by putting $T=0$ in equation (11.3). We then have the relation $Z^2H^5-G^7=0$, so the surface $T=0$ is not a hypersurface in the parameter space. To construct co-ordinates which are valid around $T=0$, we take $S\neq 0$ and either X or Y as a parameter and we study twistor curves defined near $X=X_1$ for some fixed X_1, or near $Y=Y_2$, for some fixed Y_2.

Introduce the auxiliary functions $q(X,S,t)$ and $r(Y,S,p,t)$ defined by the formulas:

$$q=\big(1-\frac{5t}{2S}(X-X_1)\big)^{1/5},\quad r=\big(1-\frac{7it}{2Sp^2}(Y-Y_2)\big)^{1/7}. \tag{11.4}$$

Here q is holomorphic in (X,S,t) (with $S\neq 0$) and is fixed by the requirement that q takes the value 1 whenever $X=X_1$, whereas r is holomorphic in (Y,S,p,t) (with $S\neq 0$) and is fixed by the requirement that r takes the value 1 whenever $Y=Y_2$. The domain of q is the set U_{X_1} of all triples $(X,S,t)\in\boldsymbol{C}^3$, such that $S\neq 0$ and $1-\dfrac{5t}{2S}(X-X_1)$ is not a non-positive real number. The domain of r is the set V_{Y_2} of all triples $(Y,S,p,t)\in\boldsymbol{C}^4$, such that $Sp^2\neq 0$ and $1-\dfrac{7it}{2Sp^2}(Y-Y_2)$ is not a non-positive real number. On U_{X_1}, we have $|\arg(q)|<\pi/5$ and on V_{Y_2}, we have

$|\arg(r)| < \pi/7$. When X is the parameter, we have the following constants of the motion:

$$\begin{aligned} s_1 &= \frac{S}{35}q^5, \quad p_1 = pq, \quad c_1 = 35\frac{t}{S}q^{-5}, \\ y_1 &= \frac{Y}{10i} + p^2(X - X_1)\frac{(q^6+q^5+q^4+q^3+q^2+q+1)}{14(q^4+q^3+q^2+q+1)}. \end{aligned} \tag{11.5}$$

Here the twistor parameters are (s_1, p_1, c_1, y_1), with $s_1 \neq 0$ and $p_1 \neq 0$. Also (p_1, c_1, y_1) are the projective co-ordinates. The numbers s_1, p_1 and y_1 are respectively just the values of $S/35$, p and $Y/10i$ at $X = X_1$. Alternatively, with y as the parameter, the constants are as follows:

$$\begin{aligned} s_2 &= \frac{Sp^2}{35}r^7, \quad p_2 = pr, \quad d_2 = \frac{35t}{Sp^2}r^{-7}, \\ x_2 &= -\frac{X}{14} + ip^{-2}(Y - Y_2)\frac{(r^4+r^3+r^2+r+1)}{10(r^6+r^5+r^4+r^3+r^2+r+1)}. \end{aligned} \tag{11.6}$$

Here the twistor parameters are (s_2, p_2, d_2, x_2), with $s_2 \neq 0$ and $p_2 \neq 0$. Also (p_2, d_2, x_2) are the projective co-ordinates. The numbers s_2, p_2 and x_2 are respectively just the values of $Sp^2/35$, p and $-X/14$ at $Y = Y_2$. We may compare these co-ordinates with the co-ordinates given by equation (11.3) above. We have:

$$\begin{aligned} T &= c_1 s_1 = c_2 s_2, \quad Z = s_1 p_1^{-5} = s_2 p_2^{-7}, \\ G &= s_1\big(1 - \frac{c_1 X_1}{14}\big) = s_2(p_2^{-2} + c_2 x_2), \\ H &= s_1(p_1^2 + c_1 y_1) = s_2\big(1 - i\frac{c_2 Y_2}{10}\big). \end{aligned} \tag{11.7}$$

Now if $T = 0$, then $c_1 = 0$; still we have the relation $Z^2H^5 - G^7 = 0$, but in the (s_1, p_1, y_1, c_1) co-ordinates, $c_1 = 0$ is an ordinary hypersurface: the vanishing of c_1 imposes no constraints on the other three co-ordinates. Given (T, Z, G, H), we may solve equation (11.7) for s_1, c_1, p_1 and y_1 with the following result:

$$\begin{aligned} s_1 &= G + \frac{TX_1}{14}, \quad c_1 = \frac{T}{G + \frac{TX_1}{14}}, \\ p_1 &= \Big(\frac{G + \frac{TX_1}{14}}{Z}\Big)^{1/5}, \quad y_1 = \frac{H - \big(G + \frac{TX_1}{14}\big)p_1^2}{T}. \end{aligned} \tag{11.8}$$

This solution is valid provided $T \neq 0$, $G \neq -TX_1/14$ and $Z \neq 0$. Note that there are then five solutions, such that if (s_1, c_1, p_1, y_1) is one solution, then all solutions are $(s_1, c_1, \rho p_1, y_1 + \frac{p_1^2}{c_1}(1 - \rho^2))$, where $\rho^5 = 1$ and $\rho \neq 1$. This corresponds to the fact, as seen from equation (11.3) that a twistor curve generically attains the same x-value at five different points along the curve. We may also see this Z_5 structure directly in equation (11.4) above. Using the formulas of (11.4) but with q everywhere replaced by $\rho^k q$ also gives constants of the motion, for any integer k (but only for k a multiple of five are the constants finite when $q = 1$). A similar story holds for the (s_2, p_2, x_2, d_2) co-ordinates. To understand the global nature of the twistor space, we return to our original conserved quantities (T, Z, G, H) and add a new conserved quantity, U, given as follows:

$$U = \frac{S^6}{35^6}\Big(-\frac{iY}{2p^2}+\frac{X}{2}\Big)+\frac{S^5t}{35^5}\Big(-\frac{Y^2}{10p^4}-\frac{3X^2}{28}\Big)+\frac{5S^4t^2}{35^4}\Big(\frac{2iY^3}{10^3p^6}+\frac{7X^3}{14^3}\Big)$$
$$+\frac{5S^3t^3}{35^3}\Big(\frac{Y^4}{10^4p^8}-\frac{7x^4}{14^4}\Big)-\frac{S^2t^4}{35^2}\Big(\frac{iY^5}{10^5p^{10}}-\frac{21X^5}{14^5}\Big)-\frac{St^5X^6}{5(14^6)}+\frac{t^6X^7}{14^7}. \tag{11.9}$$

Note that U is a homogeneous polynomial in the variables (s, t) of degree six. That U is conserved follows immediately from equation (11.3) and the algebraic identity, an immediate consequence of equation (11.9):

$$Ut = \Big(\frac{S}{35p^5}\Big)^2\Big(\frac{Sp^2}{35}-i\frac{tY}{10}\Big)^5-\Big(\frac{S}{35}-\frac{tX}{14}\Big)^7. \tag{11.10}$$

Now given a twistor curve, we may assign to the curve the *five* co-ordinates

$$(U, T, Z, G, H),$$

giving a map from the twistor space to the space $\boldsymbol{C}^5$. But the five co-ordinates are related by the algebraic relation implied by equations (11.3) and (11.9):

$$0 = TU + G^7 - Z^2H^5. \tag{11.11}$$

When $T \neq 0$, these co-ordinates clearly suffice to parametrize the twistor curve. But if $T = 0$, we have

$$Z = \frac{S}{35p^5},\quad G = \frac{S}{35},\quad H = \frac{Sp^2}{35},\quad U = \frac{S^6}{35^6p^2}\Big(-i\frac{Y}{2}+\frac{Xp^2}{2}\Big)$$

and as long as Z is not also zero, these co-ordinates again suffice to parametrize the curve uniquely. Thus away from the set $Z = T = 0$, the twistor space is given by the algebraic variety in $\boldsymbol{C}^5$ of equation (11.11). When both Z and T vanish, the attached spinor $\pi_{A'}$ of the twistor curve vanishes, which entails a separate treatment for the twistor structure. This will not be analyzed further here. Finally we note that the co-ordinates (U, T, Z, G, H) are homogeneous in the variables (s, t) (and therefore in the spinor $\pi_{A'}$ of degrees $(6, 1, 1, 1, 1)$ and with those weights, the surface of (11.11) is homogeneous of total degree seven. Therefore the projective twistor space is the hypersurface with equation (11.11), in the weighted four-dimensional complex projective space of weights $(6, 1, 1, 1, 1)$: the quotient of $\boldsymbol{C}^5$ with the origin deleted by the action $(U, T, Z, G, H) \mapsto (\lambda^6U, \lambda T, \lambda Z, \lambda G, \lambda H)$ for all non-zero complex numbers λ. Note that we may also picture the twistor space in an ordinary projective space: namely as the algebraic hypersurface in $\boldsymbol{CP}^4$, with homogeneous co-ordinates (W, T, Z, G, H) and equation $TW^6 + G^7 - Z^2H^5 = 0$, except that one should quotient by the obvious action of the group of integers modulo six on the variable W.

The null twistor space is the set of curves that contain a real point: i.e for which X, Y and u are real. Put $g = G/T$, $h = H/T$ and $z = Z/T$, whenever $T \neq 0$. We then have

$$p^5 = \frac{g-\overline{g}}{z-\overline{z}} \quad \text{and} \quad p^7 = \frac{h+\overline{h}}{z+\overline{z}},$$

so the null (projective) twistor space is the real hypersurface in $\boldsymbol{C}^3$ given by the equation

$$\Big(\frac{g-\overline{g}}{z-\overline{z}}\Big)^7 = \Big(\frac{h+\overline{h}}{z+\overline{z}}\Big)^5.$$

Clearing denominators, we find the following real polynomial equation of total degree twelve for the hypersurface:

$$M = i(g-\overline{g})^7(z+\overline{z})^5 - i(z-\overline{z})^7(h+\overline{h})^5 = 0. \tag{11.12}$$

Using the co-ordinates (U, T, Z, G, H) and equation (11.10) and clearing denominators, the surface is given by a polynomial of degree twenty-two in the weighted projective space. This should be contrasted with the standard hypersurface of flat twistor space which is the hyperquadric and is described by a quadratic polynomial in the twistor variables. The complexity of the present hypersurface is remarkable, considering the simplicity of the original spacetime. Note that this surface has a high degree of degeneracy when $z = 0$ is approached. This corresponds to the breakdown of the twistor CR structure, when $\pi_{A'}$ points along the null cone.

12. The Kähler scalar and its curvature

In the complexified spacetime, we consider the equation of parallel propagation of a local twistor up the generators of the null cone. From section six above, the required propagation equation is the restriction to the relevant generator of the equations $dz^A = i\theta^{AB'}\pi_{B'}$, $dz_{A'} = -\frac{3i}{8u^2} l o_{A'} o_B z^B$. But the restriction of θ^a to a generator is proportional to $\alpha^A\alpha^{A'}$, $\theta^a = \rho\alpha^A\alpha^{A'}$, for some ρ. By contracting this equation with o_A and $o_{A'}$, we deduce that $\rho = du$. Also along a generator, the pairs of spinors (o_A, α_A) and $(o_{A'}, \alpha_{A'})$ are both parallely propagated and form normalized spin bases: $o_A\alpha^A = o_{A'}\alpha^{A'} = 1$. Put $z^A = Fo^A + G\alpha^A$ and $z_{A'} = Ho_{A'} + J\alpha_{A'}$. Then we have

$$dz^A = (dF)o^A + (dG)\alpha^A = i\theta^{AB'}z_{B'} = i\alpha^A\alpha^{B'}\pi_{B'}du = iH\alpha^A du$$

and

$$(dH)o_{A'} + (dJ)\alpha^{A'} = dz_{A'} = -\frac{3i}{8u^2} l o_{A'} o_B z^B = -\frac{3i}{8u^2} o_{A'} G du.$$

The vector field $L = \alpha^A\alpha^{A'}\partial_a$ obeys $L(X) = L(Y) = L(u) - 1 = 0$, so in the co-ordinate system (u, X, Y), we have $L = \partial_u$. So in the co-ordinate system (u, X, Y), the quantities X and Y are fixed on a generator and all quantities only depend on the variable u. Then the transport equations are as follows:

$$dF = 0, \quad dG = iHdu, \quad dH = \frac{-3i}{8u^2}Gdu, \quad dJ = 0. \tag{12.1}$$

So F and J are constant, $H = -iG'$ and G obeys the equation $G'' = \frac{3}{8u^2}G$, where a prime denotes differentiation with respect to u. Solving this equation gives G as a linear combination of u^α and u^β, where $\alpha = \frac{2+\sqrt{10}}{4}$ and $\beta = \frac{2-\sqrt{10}}{4}$. We choose initial conditions so that at $u = u_1$, we have $(z^A, z_{A'}) = (0, \pi_{A'})$, where for the twistor $Z_1 = (G, H, Z, T) = t(g, h, z, 1)$ (with $T = t$), we have $\pi_{A'} = So_{A'} + t\alpha_{A'}$ and S is given in equation (11.3). The local twistor $(0, \pi_{A'})$ is parallelly propagated along the twistor curve with respect to the Cartan connection. Then the complete solution at any point (u, X, Y) of the generator is as follows:

$$\begin{aligned} z^A &= \frac{i}{\alpha-\beta}\pi_{B'}\alpha^{B'}\alpha^A(u^\alpha u_1^\beta - u^\beta u_1^\alpha), \\ z_{A'} &= \pi_{A'} + \frac{1}{\alpha-\beta}\pi_{B'}\alpha^{B'}o_{A'}(\alpha(u^{-\beta}u_1^\beta - 1) - \beta(u^{-\alpha}u_1^\alpha - 1)). \end{aligned} \tag{12.2}$$

In equation (12.2) it is understood that all spinors are parallelly propagated with respect to the Levi-Civita connection along the generator to the relevant point (i.e. their components are constant in the frames (o_A, α_A) and $(o_{A'}, \alpha_{A'})$).

A dual twistor curve, W_2, with parameters $(G', H', Z', T') = t'(g', h', z', 1)$, is given by formally conjugating equation (11.3), so is as follows:

$$\begin{aligned} S' = 35z't'u^{-5/4}, \quad X = \frac{x-x_0}{u} = 14(-g' + z'u^{-5/4}), \\ Y = \frac{y-y_0}{u^2} = -10i(h' - z'u^{-7/4}). \end{aligned} \tag{12.3}$$

The local twistor associated to the dual twistor along its curve is the (dual) twistor $(\pi_A, 0)$, where $\pi_A = S'o_A + t'\alpha^A$. To find the Kähler scalar $K(Z_1, W_2)$ for the twistor Z_1 and the dual twistor W_2, we find a common generator for the curves of Z_1 and W_2, parallelly propagate the local twistor of Z_1, $(0, \pi_{A'})$, from the curve for Z_1, along that common generator, to the curve for W_2, using the local twistor connection, and then take the dual pairing of the local twistor that results with the twistor $(\pi_A, 0)$. Here, if Z_1 and W_2 share the generator labelled by (X, Y), with Z_1 at $u = u_1$ and W_2 at $u = u_2$, then at $u = u_2$, we pair the dual local twistor $(\pi_A, 0)$ with the twistor of equation (12.2) evaluated at $u = u_2$. This gives the following formula for $K(Z_1, W_2)$:

$$K(Z_1, W_2) = \pi_A z^A = 245i\sqrt{10}zz'tt'(u_1u_2)^{-5/4}(u_2^\alpha u_1^\beta - u_2^\beta u_1^\alpha). \tag{12.4}$$

Then $p = u_1^{-1/4}$ and $p' = u_2^{-1/4}$ are related by the following equations, derived from equations (11.3) and (12.3):

$$\begin{aligned} -\frac{X}{14} &= g - zp^5 = g' - z'p'^5, \\ -i\frac{Y}{10} &= h - zu_1^{-7/4} = -h' + z'u_2^{-7/4}, \\ p' &= (z')^{-1}(zp^5 - g + g')^3(zp^7 - h - h')^{-2}, \\ 0 &= (zp^5 - g + g')^7 + (z')^2(zp^7 - h - h')^5. \end{aligned} \tag{12.5}$$

In particular p obeys an equation of degree thirty-five in the projective twistor and dual projective twistor variables!

Finally we specialize to the case that W_2 is the pseudo-hermitian conjugate of Z_1. Then we obtain the real Kähler scalar $K(Z_1)$ of twistor space, summarized as follows, where we have put $\lambda = T/\overline{T}$, $A = p^{\sqrt{10}}$ and $B = \overline{A}$:

$$\begin{aligned} K(Z_1) &= 245i\sqrt{10}Z\overline{Z}(p\overline{p})^3\frac{A^2-B^2}{AB}, \\ 0 &= (Zp^5 - G + \lambda\overline{G})^7 + (\lambda\overline{Z})^2(Zp^7 - H - \lambda\overline{H})^5, \\ \overline{p} &= (\lambda\overline{Z})^{-1}(Zp^5 - G + \lambda\overline{G})^3(Zp^7 - H - \lambda\overline{H})^{-2}. \end{aligned} \tag{12.6}$$

Using the Maple computing system, the pseudo-Kähler metric M, corresponding to the scalar K, may be calculated. It is given explicitly as follows:

$$\begin{aligned} M &= \frac{4i(A^2+B^2)\Im(M_1+M_2+M_3)}{ABz\overline{z}p^7\overline{p}^7(p^2+\overline{p}^2)^3} \\ &\quad +\frac{2(A^2-B^2)\Re(N_1+N_2+N_3+N_4)}{\sqrt{10}ABz\overline{z}p^7\overline{p}^7(p^2+\overline{p}^2)^3}, \\ M_1 &= 35z^2\overline{z}p^{10}\overline{p}^5(p^2+\overline{p}^2)^2(5\overline{p}^7\overline{t}dtd\overline{z} + 7p^2\overline{p}^5\overline{t}dtd\overline{z} \\ &\quad -2p^7tdzd\overline{t} + 7p^2tdgd\overline{t} - 7p^2\overline{t}dtd\overline{g} - 5tdhd\overline{t} - 5\overline{t}dtd\overline{h}), \\ M_2 &= z^2t\overline{t}p^{10}(p-\overline{p})(p+\overline{p})(-7p^2dg + 5dh + 2p^7dz) \\ &\quad (5\overline{p}^7d\overline{z} + 7p^2\overline{p}^5d\overline{z} - 7p^2d\overline{g} - 5d\overline{h}), \\ M_3 &= -z\overline{z}t\overline{t}\overline{p}^5p^7(175p^3\overline{p}^4dzd\overline{h} + 147p^5\overline{p}^4dzd\overline{g} \\ &\quad +440p^5\overline{p}^2dzd\overline{h} - 78p^7\overline{p}^7dzd\overline{z} + 225p^7dzd\overline{h} \\ &\quad -70p^9\overline{p}^5dzd\overline{z} + 245p^9dzd\overline{g} + 448p^7\overline{p}^2dzd\overline{g} \\ &\quad -140\overline{p}^2dgd\overline{h} - 50dhd\overline{h} - 98p^2\overline{p}^2dgd\overline{g}), \\ N_1 &= 1225z^2\overline{z}^2p^{10}\overline{p}^{10}(p^2+\overline{p}^2)^3dtd\overline{t} \\ N_2 &= 70z^2\overline{z}\overline{p}^5p^{10}(p^2+\overline{p}^2)^2(20\overline{p}^7\overline{t}dtd\overline{z} + 14p^2\overline{p}^5\overline{t}dtd\overline{z} \\ &\quad +6tp^7dzd\overline{t} + 21p^2\overline{t}dtd\overline{g} - 21p^2tdgd\overline{t} + 15tdhd\overline{t} + 15\overline{t}dtd\overline{h}) \\ N_3 &= 4z^2t\overline{t}p^{10}(2p^2-\overline{p}^2)(5dh + 2p^7dz - 7p^2dg) \\ &\quad (5d\overline{h} - \overline{p}^5d\overline{z}(5\overline{p}^2 + 7p^2) + 7p^2d\overline{g}) \\ N_4 &= -2z\overline{z}t\overline{t}\overline{p}^5p^5(\overline{p}^2+p^2)(2p^7\overline{p}^7dzd\overline{z} + 70p^9\overline{p}^5dzd\overline{z} \\ &\quad -98p^2\overline{p}^2dgd\overline{g} - 560p^5\overline{p}^2dzd\overline{h} - 504p^7\overline{p}^2dzd\overline{g} - 770p^9dzd\overline{g} \\ &\quad -690p^7dzd\overline{h} + 100dhd\overline{h} + 35p^2dhd\overline{g} - 35p^2dgd\overline{h}). \end{aligned} \tag{12.7}$$

Although this formula is somewhat fearsome, it is not difficult to check that the restriction of this metric to the null twistor space (when K vanishes and p is real) exactly agrees with the Fefferman metric F_0 of equation (9.9). Lastly, the Ricci curvature is given by the quantity $\partial\overline{\partial}\ln(\det(M))$, where M is regarded as a 4×4

pseudo-hermitian matrix. We find that the Ricci curvature is non-zero, with $\det(M)$ given as follows:

$$\det(M) = 7^6 10^5 z\overline{z}(t\overline{t})^3(p\overline{p})^2 \frac{(A^4 + 38A^2B^2 + B^4)}{A^2B^2(p^2 + \overline{p}^2)^2}. \tag{12.8}$$

Here $B = \overline{A}$ and $A = p^{\sqrt{10}}$, as before.

13. Twistor scattering

We consider the twistor spaces T_0 and T_1 associated to the limiting cones C_0 and C_1, with (u, v, x, y) co-ordinates for the vertices $(0, v_0, x_0, y_0)$ and $(0, v_1, x_1, y_1)$, respectively. Let these cones (complexified) intersect in the region C_{01}. Each element of the primed spin bundle over C_{01} gives the initial spinor for a twistor curve of each space, and for each space, the ensemble of such curves is an open subset of the whole space. Thus the intersection region gives rise to a local holomorphic diffeomorphism from one space to the other. We shall endeavour to calculate the scattering formula, which expresses this local diffeomorphism in terms of co-ordinates.

The cones C_0 and C_1 have the following equations (with u replaced by p^{-4}, as in section 10 above) :

$$\begin{aligned} C_0 : \quad 0 &= v - v_0 - p^4(x - x_0)^2 - 2p^8(y - y_0)^2, \\ C_1 : \quad 0 &= v - v_1 - p^4(x - x_1)^2 - 2p^8(y - y_1)^2. \end{aligned} \tag{13.1}$$

Subtraction gives the following formula for the intersection region C_{01}:

$$0 = V - Xp^4(2x - x_0 - x_1) - 2Yp^8(2y - y_0 - y_1). \tag{13.2}$$

Here and in the following we write V for $v_1 - v_0$, X for $x_1 - x_0$ and Y for $y_1 - y_0$. The twistor curves of C_0 and C_1, with parameters (T_0, Z_0, G_0, H_0) and (T_1, Z_1, G_1, H_1), respectively, are given as follows:

$$\begin{aligned} T_0 &= t, \\ 35Z_0 &= sp^{-5} + tp^{-1}(x - x_0) + 2itp(y - y_0), \\ 70G_0 &= 2s - 3tp^4(x - x_0) + 4itp^6(y - y_0), \\ 70H_0 &= 2sp^2 + 2tp^6(x - x_0) - 3itp^8(y - y_0), \\ T_1 &= t, \\ 35Z_1 &= sp^{-5} + tp^{-1}(x - x_1) + 2itp(y - y_1), \\ 70G_1 &= 2s - 3tp^4(x - x_1) + 4itp^6(y - y_1), \\ 70H_1 &= 2sp^2 + 2tp^6(x - x_1) - 3itp^8(y - y_1). \end{aligned} \tag{13.3}$$

Henceforth take $t \neq 0$, $x_1 \neq x_0$ and $y_1 \neq y_0$ (so $X \neq 0$ and $Y \neq 0$) and introduce the following nomenclature:

$$
\begin{aligned}
a &= X = x_1 - x_0, \quad b = Z = \frac{35}{t}(Z_1 - Z_0), \quad c = 2iY = 2i(y_1 - y_0),\\
q &= p^{-1}, \quad G = \frac{70}{t}(G_1 - G_0), \quad H = \frac{70}{t}(H_1 - H_0),\\
r &= b^2 = Z^2 = \frac{1225}{t^2}(Z_1 - Z_0)^2,\\
w &= -ac = -2iXY = -2i(x_1 - x_0)(y_1 - y_0),\\
\xi &= p^4(2x - x_0 - x_1), \quad \psi = ip^6(2y - y_0 - y_1), \quad \zeta = \frac{35}{t}(Z_0 + Z_1),\\
\gamma &= \frac{70}{t}(G_0 + G_1), \quad \eta = \frac{70}{t}(H_0 + H_1), \quad \sigma = 2\frac{s}{t}.
\end{aligned}
\tag{13.4}
$$

Then by subtracting and adding the corresponding equations of equation (13.3), we must solve the following equations (where we have included also equation (13.2)):

$$
\begin{aligned}
P_1 &: \ 0 = aq^2 + bq + c,\\
P_2 &: \ 0 = Gq^6 + 3bq + 5c,\\
P_3 &: \ 0 = 2Hq^8 - 4bq - 7c,\\
Q_1 &: \ 0 = \zeta q^{-5} - \sigma - \xi - 2\psi,\\
Q_2 &: \ 0 = \gamma - 2\sigma + 3\xi - 4\psi,\\
Q_3 &: \ 0 = \eta q^2 - 2\sigma - 2\xi + 3\psi,\\
R_1 &: \ 0 = V - X\xi + 2iY\psi q^{-2}.
\end{aligned}
\tag{13.5}
$$

We first solve simultaneously the polynomial equations P_1, P_2 and P_3. Introduce the variables T and U defined as follows:

$$
\begin{aligned}
T &= Gc^5 + r^3 + \frac{15}{2}r^2w + 15rw^2 + 5w^3,\\
U &= Hc^7 - \frac{1}{4}(3r^4 + 28r^3w + 84r^2w^2 + 84rw^3 + 14w^4).
\end{aligned}
\tag{13.6}
$$

Then by direct calculation using the Maple algebraic computing program on a Macintosh computer, we find that the resultants P_{12} of P_1 and P_2 and P_{13} of P_1 and P_3 are given as follows:

$$
\begin{aligned}
c^4P_{12} &= T^2 - \frac{r}{4}(r + 4w)^3(2r + 3w)^2,\\
c^6P_{13} &= U^2 - \frac{r}{16}(r + 4w)^3(3r^2 + 10rw + 6w^2)^2.
\end{aligned}
\tag{13.7}
$$

When the resultants P_{12} and P_{13} vanish, we find by polynomial division that the common root q_1 of P_1 and P_2 and the common root q_2 of P_1 and P_3 are given, generically, as follows:

$$
\begin{aligned}
q_1 &= -\frac{b}{2a} - \frac{T}{ab(r+4w)(2r+3w)} \\
q_2 &= -\frac{b}{2a} + \frac{2U}{ab(r+4w)(3r^2+10rw+6w^2)}.
\end{aligned} \tag{13.8}
$$

Finally by polynomial division, we find that generically all three polynomial equations P_1, P_2 and P_3 have a common root if and only if q_1 and q_2 are equal, which gives a linear relationship between T and U:

$$T(3r^2+10rw+6w^2) = -U(4r+6w). \tag{13.9}$$

The non-generic cases are the cases when $b = 0$, $r+4w = 0$, $2r+3w = 0$, or $3r^2+10rw+6w^2 = 0$. These may be solved to give a collection of linear equations for Z in terms of (suitable roots of) XY. They will not be analyzed further here. However we should note that the equation obtained by squaring both sides of equation (13.9) is always a valid equation, since it follows from the required vanishing of the resultants P_{12} and P_{13} (see equation (13.7)). The three equations (13.9) and $P_{12} = P_{13} = 0$ give all but two of the scattering equations. They effectively determine G and H in terms of Z, X and Y. We also have the trivial relations $T_1 = T_0 = t$, leaving us needing one further equation for the quantity Z. This we do by solving equations Q_1, Q_2 and Q_3 and inserting the result into equation R_1. The system of equations Q_1, Q_2 and Q_3 is linear with the following solution:

$$(\sigma,\xi,\psi) = \frac{1}{35}(\zeta q^{-5}+7\gamma+10\eta q^2, 14\zeta q^{-5}-7\gamma, 10\zeta q^{-5}-5\eta q^2). \tag{13.10}$$

Here it is understood that $q = q_1 = q_2$. Using equation (13.10), equation R_1 becomes the following equation:

$$0 = q_1^7(35V+7\gamma X-10i\eta Y)+\zeta(14Zq_1+48iY). \tag{13.11}$$

Generically (i.e. provided $14Zq_1+48iY$ is invertible, so provided $r \neq 0$ and $r+4w \neq 0$), this equation gives a formula for ζ, and hence the last piece of the scattering information, thus completing the required local diffeomorphism of the twistor spaces.

This approach to the scattering formulas has the virtue that it shows that given the pair of the twistor and its scattered counterpart, then the two twistors meet at exactly one point. A more straightforward procedure is to solve equations P_1, P_2, P_3, Q_1, Q_2 and Q_3 regarding q as given, with the result that all the scattering is given in terms of rational functions in the variable q. Then the final equation R_1 gives a polynomial relation for q, or for $p = q^{-1}$. This polynomial relation is then as follows:

$$0 = 2Y^2tp^8+40iYZ_0p^7-28XZ_0p^5+p^4tX^2+Vt+28XG_0-40iYH_0. \tag{13.12}$$

The scattering point is as follows:

$$
\begin{aligned}
s &= p^{-2}(Z_0p^7 + 14G_0p^2 + 20H_0),\\
v &= v_0 - \frac{4}{t^2p^8}(Z_0^2p^{14} + 98Z_0G_0p^9 - 100H_0Z_0p^7 - 49G_0^2p^4 + 50H_0^2),\\
x &= x_0 + \frac{14}{tp^4}(Z_0p^5 - G_0),\\
y &= y_0 - \frac{10i}{tp^8}(Z_0p^7 - H_0).
\end{aligned}
\tag{13.13}
$$

Then the twistor scattering is as follows:

$$
\begin{aligned}
T_1 &= T_0 = t, \quad G_1 = G_0 + \frac{t}{70}(3Xp^4 - 4iYp^6),\\
H_1 &= H_0 - \frac{t}{70}(2Xp^6 - 3iYp^8),\\
Z_1 &= Z_0 - \frac{t}{35p}(X + 2iYp^2).
\end{aligned}
\tag{13.14}
$$

For the validity of this solution, we need only that $p \neq 0$ and $t \neq 0$.

14. Appendix: On Time Asymmetry

14.1. Introduction. — A common experience is that time flows in only one direction. It is suggested in this note that there is a deep lying chiral asymmetry in the universe, which may be responsible for the flow of time: specifically the future null and past null cones of spacetime events are to be understood to have the *opposite* chiralities. Concretely this asymmetry is expressed in the language of twistor theory [**1**–**10**]. Twistors come in two mutually dual types, each inherently chiral, of opposite chirality [**1**]. If twistors are used to describe future null cones, then dual twistors will be used to describe past null cones.

14.2. Ghosts. — Twistors typically form complex analytic spaces of either three or four complex dimensions, the former usually being a projective version of the latter. For the purposes of this note it will suffice to consider only the three-dimensional case.

A ghost is by definition a complex analytic variety of three complex dimensions, containing exactly two disjoint holomorphic compact Riemann spheres. It is suggested that in a non-flat vacuum asymptotically flat space-time, the null cone hypersurface twistor spaces of Penrose, for either a future, or past null cone, are ghosts [**5, 14, 15**]. One of the holomorphic spheres of the ghost represents the vertex of the cone. The other represents the vertex of the null cone at infinity. The fact that it is even conceivably possible to have such ghosts requires overcoming the Kodaira theorems that in perturbations of conformally flat spacetimes tend to provide an overabundance of holomorphic curves [**21**]. The key physics here lies in the famous null geodesic deviation equations of Sachs [**32**], which, in particular, show that, in the presence of Weyl curvature, there is decoherence of pencils of light rays along a null cone, vis á vis the situation in (real) conformally flat or (complex) conformally self-dual spacetimes. In terms of the Cauchy-Riemann structure of null hypersurface twistor space [**5, 14**], this decoherence is associated with the degeneration of the structure along the light rays of the cone. These features, which might be regarded as pathological from the point of view of flat or self-dual space-time, allow the twistor spaces for null hypersurfaces of real spacetimes to be depleted of their usual supply of holomorphic curves.

When the ghost space of a future null cone meets that of a past null cone, one finds on the overlap that there is a natural correspondence between the twistor curves of one hypersurface and the *dual* twistor curves of the other surface. This correspondence yields the chirality and the time asymmetry: twistor spaces are used for each future cone and dual twistor spaces for each past cone; one can then consistently term the future-pointing spaces ghosts and the past-pointing anti-ghosts.

The mathematical source for ghost and anti-ghost spaces is the genre of (open subvarieties of) Calabi-Yau manifolds [**22, 23**]. When ghosts and anti-ghosts meet, we have apparently arrived at the situation envisaged in the theory of mirror manifolds and the associated conjectures of Yau [**33, 34**]. Then the act of passing to a mirror corresponds to interchanging past and future. Slight discrepancies in the structures of these spaces relative to their mirrors account for the difference between past and future.

The ideas sketched here are a natural consequence of the author's proposed unification of a triad of powerful theories: twistor theory, superstring theory and the theory of "dessins d'enfants", based on their common themes of quasi-conformal analysis and sheaf theory [**3, 24, 30, 31**]. The realization of such a unification has been a long-standing objective of the author [**16**]. The overall philosophy has been sketched in four recent talks [**17–20**]. The feasibility of the aspect of unification discussed here results from an astounding numerical coincidence: that ten is the sum of six and four. Ten is the usual dimension of the arena of superstring theory, six is the real dimension of projective twistor space and four is the dimension of space-time.

It is to be hoped that a thorough development of the ideas contained here will lead to an understanding of the fundamental role of irreversibility and thermodynamics in physics. For this one will have to link with the groundbreaking work of Bekenstein and Hawking [**25, 26**]. A step towards such a link will be a deep analysis, in the present language, of the Schwarzschild and Kerr solutions of general relativity, which, in their global structure, already encode the essence of temperature, as was shown convincingly by Hawking and his colleagues [**26, 27, 28, 29**]. For a fuller theory, one will suitably extend the concept of ghost to include, for example, cosmic anti-ghosts and singular ghosts, associated to cones terminating at the big bang and at a singularity, respectively. Then Penrose's proposed Weyl Curvature Hypothesis, which may be equivalent to the Second Law of Thermodynamics, would boil down to comprehending the difference between these various kinds of ghosts **[13]**.

Acknowledgments. — The author thanks the *Mathematical Institute, Oxford*, the *E. Schrödinger Institute, Vienna*, the *Collegium, Budapest* and the *Centre International de Rencontres Mathematiques, Luminy*, where some of this work was written and presented, for their hospitality. The author is deeply indebted to his mentors, Roger Penrose and Ted Newman and to the entire collective of twistor theorists. He would particularly like to thank Lionel Mason, Claude Le Brun, Gary Gibbons, Jörg Frauendiener, Pawel Nurowski, Maciej Dunaski, Zoltan Perjes and the members of the Laboratory, especially Tom Metzger, Devendra Kapadia, Jonathan Holland and Phillip Tillman.

References

[1] Penrose, R., "Twistor Algebra", Journal of Mathematical Physics, **8-2** 345-366, 1967.

[2] Penrose, R. and MacCallum, M.A.H., "Twistor Theory: an approach to the quantization of fields and space-time," Physics Reports, Physics Letters C **6** (1970) 241-315.

[3] Penrose, R., "Twistor theory: its aims and achievements," in *Quantum gravity: An Oxford Symposium*, editors C.J. Isham, R. Penrose and D.W. Sciama, Oxford: Clarendon Press, 1975.

[4] Penrose, R. and Rindler, W., *Spinors and Space-Time. Volume 1: Two-spinor Calculus and Relativistic Fields*, Cambridge: Cambridge University Press, 1984.

[5] Penrose, R. and Rindler, W., *Spinors and Space-Time. Volume 2: Spinor and Twistor Methods in Space-Time Geometry*, Cambridge: Cambridge University Press, 1986.

[6] Penrose, R., "Non-linear gravitons and curved twistor theory," General Relativity and Gravitation **7** (1976) 31-52.

[7] Newman, E.T., "Heaven and its properties," General Relativity and Gravitation **7** (1976) 107-127.

[8] Ko, M., Newman, E.T. and Penrose, R., "The Kähler structure of asymptotic twistor space", Journal of Mathematical Physics **18** (1977) 58-64.

[9] Le Brun, C., "Twistor CR Manifolds and Three-Dimensional Conformal Geometry", Transactions of the American Mathematical Society **284** (1984) 601–616.

[10] Le Brun, C., "H-Space with a Cosmological Constant", Proceedings of the Royal Society of London, **A 380**, (1982), 171-185.

[11] Mason, L.J. and Woodhouse, N.M.J., *Integrability, Self-Duality, and Twistor Theory*, London Mathematical Society Monographs, New Series 15, Oxford: Clarendon Press, 1996.

[12] Ward, R.S., "Self-dual space-times with cosmological constant", Communications in Mathematical Physics **78** (1980) 1-17.

[13] Penrose, R., "Gravity and Quantum Mechanics", in *GRG 1992, Proceedings of the 13th International Conference on General Relativity and Gravitation, Cordoba Argentina*, editors R.J. Gleiser, C.N. Kozameh, O.N. Moreschi, Institute of Physics Publishing, Bristol and Philadelphia, 1993.

[14] Sparling, G.A.J., "Abstract/Virtual/Reality/Complexity", in *The Geometric Universe: Science, Geometry and the work of Roger Penrose*, Eds. L. J. Mason, K.P.Tod, S.T.Tsou, N.M.J.Woodhouse, Oxford University Press, 1997.

[15] Sparling, G.A.J., "The twistor theory of hypersurfaces in space-time," to appear in *Further Advances in Twistor Theory, Volume III*, editors L.P. Hughston, L.J. Mason, London: Pitman Press.

[16] Sparling, G.A.J., "Towards the Geometrization of Physics", Nature **321**, 417, 1986.

[17] Sparling, G.A.J., "Dreams of a theory, ... finally!", Oxford Seminar May 1999.

[18] Sparling, G.A.J., "Conformal techniques in relativity" CIRM Conference, Luminy, June 1999.

[19] Sparling, G.A.J., "D-Bar", ESI Conference, Vienna, June 1999.

[20] Sparling, G.A.J., "Kiválthány", Collegium Budapest Conference, July 1999.

[21] Kodaira, K., "A theorem of completeness of characteristic systems for analytic families of compact submanifolds of complex manifolds", Annals of Mathematics **75** (1962) 146-162.

[22] Yau, S.T., "On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampere equation I". Communications in Pure and Applied mathematics **31** 229-411 1978.

[23] Candelas, P., Horowitz, G., Strominger, A., Witten, E., "Vacuum configurations for superstrings", Nuclear Physics B **285**, 46 (1985)

[24] Green, M.B., Schwartz, J.H., Witten, E., *Superstring theory, Volume 1, Introduction* Cambridge Monographs on Mathematical Physics, Cambridge University Press, Cambridge, 1987.

[25] Bekenstein, J.D, "Black holes and Entropy", Physical Review **D7** 2333-2346 1973.

[26] Hawking, S.W., "Black Hole Explosions", Nature **248** 30-31 1974.

[27] Schwarzschild, K., "Über das Gravitationsfeld eines Masspunktes nach der Einsteinschen Theorie", Berliner Sitzungberichte 189 1916.

[28] Kerr, R.P, "Gravitational Field of a spinning mass as an example of algebraically special metrics", Physical Review Letters, **11** 237-238 1963.

[29] Gibbons, G.W., Perry, M.J., "Black holes and thermal Green's functions", Proceedings of the Royal Society of London, **A358** 467-494 1978.

[30] Grothendieck, A., "Esquisse d'un Programme", in *Geometric Galois Actions 1. Around Grothendieck's Esquisse d'un Programme*, eds, Schneps, L., Lochak, P., London Mathematical Society Lecture Note Series **242**, Cambridge University Press, Cambridge, 1997.

[31] Belyi, G, "On Galois extensions of a maximal cyclotomic field", Mathematics of the USSR, Izvestia, **14-2** 247-256 1980.

[32] Sachs, R.K., "Gravitational Waves in General Relativity VI. The Outgoing radiation condition", Proceedings of the Royal Society of London **A264** 309-338 1961.

[33] McLean, R., Communications in Analysis and Geometry, **6** 705-747 1998.

[34] Yau, S.T., Strominger, A., Zaslow, M., Nuclear Physics B **479** 243-259 1996.

[35] Fefferman, C., Graham, R., "Conformal Invariants", in *Élie Cartan et les Mathématiques d'aujourd'hui*, Astérisque, Hors série, 95-116 1985.

G. Sparling, Department of Mathematics, University of Pittsburgh, Pittsburgh, Pennsylvania, USA • *E-mail :* sparling@euler.math.pitt.edu