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# TWISTOR AND KILLING SPINORS IN LORENTZIAN GEOMETRY

*by*

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***Abstract.*** — This paper is a survey of recent results concerning twistor and Killing spinors on Lorentzian manifolds based on lectures given at CIRM, Luminy, in June 1999, and at ESI, Wien, in October 1999. After some basic facts about twistor spinors we explain a relation between Lorentzian twistor spinors with lightlike Dirac current and the Fefferman spaces of strictly pseudoconvex spin manifolds which appear in CR-geometry. Secondly, we discuss the relation between twistor spinors with timelike Dirac current and Lorentzian Einstein Sasaki structures. Then, we indicate the local structure of all Lorentzian manifolds carrying real Killing spinors. In particular, we show a global Splitting Theorem for complete Lorentzian manifolds in the presence of Killing spinors. Finally, we review some facts about parallel spinors in Lorentzian geometry.

***Résumé* (Twisteurs et spineurs de Killing en géométrie lorentzienne).** — Le présent papier est un article de synthèse basé sur les exposés donnés au CIRM, Luminy, en juin 1999, et à l'ESI, Vienne, en octobre 1999, concernant des nouveaux résultats sur les spineurs twisteurs et les spineurs de Killing lorentziens. Après quelques préliminaires sur les spineurs twisteurs, on met en évidence des relations entre les spineurs twisteurs lorentziens admettant un courant de Dirac isotrope et les espaces de Fefferman des variétés spinorielles strictement pseudoconvexes qui apparaissent dans la géométrie CR. De plus, on décrit la relation entre les spineurs twisteurs admettant un courant de Dirac de type temps et les structures de Sasaki-Einstein lorentziennes. On indique aussi la structure locale des variétés lorentziennes admettant des spineurs de Killing réels. En particulier, on obtient un théorème de « splitting » global pour les variétés lorentziennes complètes qui admettent des spineurs de Killing. Enfin, on fait le point sur la théorie des spineurs parallèles en géométrie lorentzienne.

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## 1. Introduction

Twistor spinors were introduced by R.Penrose and his collaborators in General Relativity as solutions of a conformally invariant spinorial field equation (twistor equation) (see [**Pen67**], [**PR86**], [**NW84**]). Twistor spinors are also of interest in physics since they define infinitesimal isometries in semi-Riemannian supergeometry (see [**ACDS98**]). In Riemannian geometry the twistor equation first appeared as an integrability condition for the canonical almost complex structure of the twistor space of an oriented four-dimensional Riemannian manifold (see [**AHS78**]). In the second half of the 80's A.Lichnerowicz started the systematic investigation of twistor spinors on Riemannian spin manifolds from the view point of conformal differential geometry. Nowadays one has a lot of structure results and examples for manifolds with twistor spinors in the Riemannian setting (see e.g. [**Lic88b**], [**Lic88a**], [**Lic89**], [**Wan89**], [**Fri89**] [**Lic90**], [**BFGK91**], [**Hab90**], [**Bär93**], [**Hab94**], [**Hab96**], [**KR94**], [**KR96**], [**KR97b**], [**KR97a**], [**KR98**]).

An other special kind of spinor fields related to Killing vector fields and Killing tensors and therefore called Killing spinors is used in supergravity and superstring theories (see e.g. [**HPSW72**], [**DNP86**], [**FO99a**], [**AFOHS98**]). In mathematics the name Killing spinor is used (more restrictive than in physics literature) for those twistor spinors which are simultaneous eigenspinors of the Dirac operator. The interest of mathematicians in Killing spinors started with the observation of Th. Friedrich in 1980 that a special kind of Killing spinors realise the limit case in the eigenvalue estimate of the Dirac operator on compact Riemannian spin manifolds of positive scalar curvature. In the time after the Riemannian geometries admitting Killing spinors were intensively studied. They are now basically known and in low dimensions completely classified (see [**BFGK91**] [**Hij86**], [**Bär93**]). These results found applications also outside the spin geometry, for example as tool for proving rigidity theorems for asymptotically hyperbolic Riemannian manifolds (see [**AD98**], [**Her98**]). In the last years the investigation of special adapted spinorial field equations was extended to Kähler, quaternionic-Kähler and Weyl geometry (see e.g. [**MS96**], [**Mor99**], [**KSW98**], [**Buc00b**], [**Buc00a**]).

In opposite to the situation in the Riemannian setting, there is not much known about solutions of the twistor and Killing equation in the *pseudo*-Riemannian setting, where these equations originally came from. The general indefinite case was studied by Ines Kath in [**Kat00**], [**Kat98**], [**Katb**], [**Kata**], where one can find construction principles and examples for indefinite manifolds carrying Killing and parallel spinors. In the present paper we restrict ourselves to the Lorentzian case. We explain some results concerning the twistor and Killing equation in *Lorentzian* geometry, which we obtained in a common project with Ines Kath, Christoph Bohle, Felipe Leitner and Thomas Leistner.

## 2. Basic facts on twistor spinors

Let $(M^{n,k}, g)$ be a smooth semi-Riemannian spin manifold of index $k$ and dimension $n \geq 3$ with the spinor bundle $S$. There are two conformally covariant differential operators of first order acting on the spinor fields $\Gamma(S)$, the Dirac operator $D$ and the twistor operator (also called Penrose operator) $P$. The Dirac operator is defined as the composition of the spinor derivative $\nabla^S$ with the Clifford multiplication $\mu$

$$D : \Gamma(S) \xrightarrow{\nabla^S} \Gamma(T^*M \otimes S) \overset{g}{\approx} \Gamma(TM \otimes S) \xrightarrow{\mu} \Gamma(S),$$

whereas the twistor operator is the composition of the spinor derivative $\nabla^S$ with the projection $p$ onto the kernel of the Clifford multiplication $\mu$

$$P : \Gamma(S) \xrightarrow{\nabla^S} \Gamma(T^*M \otimes S) \overset{g}{\approx} \Gamma(TM \otimes S) \xrightarrow{p} \Gamma(\ker \mu).$$

The elements of the kernel of $P$ are called *twistor spinors*. A spinor field $\varphi$ is a twistor spinor if and only if it satisfies the *twistor equation*

$$\nabla^S_X \varphi + \frac{1}{n} X \cdot D\varphi = 0$$

for each vector field $X$. Special twistor spinors are the parallel and the Killing spinors, which satisfy simultaneous the Dirac equation. They are given by the spinorial field equation

$$\nabla^S_X \varphi = \lambda X \cdot \varphi, \quad \lambda \in \mathbb{C}.$$

The complex number $\lambda$ is called Killing number.

We are interested in the following geometric problems:

1. *Which semi-Riemannian (in particular Lorentzian) geometries admit solutions of the twistor equation?*
2. *How the properties of twistor spinors are related to the geometric structures where they can occur.*

The basic property of the twistor equation is that it is conformally covariant: Let $\tilde{g} = e^{2\sigma} g$ be a conformally equivalent metric to $g$ and let the spinor bundles of $(M, g)$ and $(M, \tilde{g})$ be identified in the standard way. Then for the twistor operators of $P$ and $\tilde{P}$ the relation

$$\tilde{P}\varphi = e^{-\frac{1}{2}\sigma} P(e^{-\frac{1}{2}\sigma}\varphi)$$

holds.

Let us denote by $R$ the scalar curvature and by Ric the Ricci curvature of $(M^{n,k}, g)$. $K$ denotes the Rho tensor

$$K = \frac{1}{n-2}\left\{\frac{R}{2(n-1)} g - \mathrm{Ric}\right\}.$$

We always identify $TM$ with $TM^*$ using the metric $g$. For a $(2,0)$-tensor field $B$ we denote by the same symbol $B$ the corresponding $(1,1)$-tensor field $B : TM \longrightarrow TM$,

$g(B(X),Y)=B(X,Y)$. Let $C$ be the (2,1)-Cotton-York tensor

$$C(X,Y)=(\nabla_X K)(Y)-(\nabla_Y K)(X).$$

Furthermore, let $W$ be the (4,0)-Weyl tensor of $(M,g)$ and let denote by the same symbol the corresponding (2,2)-tensor field $W:\Lambda^2M\longrightarrow\Lambda^2M$. Then we have the following integrability conditions for twistor spinors

***Proposition 2.1*** (**[BFGK91**, Th.1.3, Th.1.5]). — *Let $\varphi\in\Gamma(S)$ be a twistor spinor and $\eta=Y\wedge Z\in\Lambda^2M$ a two form. Then*

$$D^2\varphi=\frac{1}{4}\frac{n}{n-1}R\varphi \tag{1}$$

$$\nabla^S_X D\varphi=\frac{n}{2}K(X)\cdot\varphi \tag{2}$$

$$W(\eta)\cdot\varphi=0 \tag{3}$$

$$W(\eta)\cdot D\varphi=n\,C(Y,Z)\cdot\varphi \tag{4}$$

$$(\nabla_X W)(\eta)\cdot\varphi=X\cdot C(Y,Z)\cdot\varphi+\frac{2}{n}(X\lrcorner\, W(\eta))\cdot D\varphi \tag{5}$$

If $(M^n,g)$ admits Killing spinors the Ricci and the scalar curvature of $M$ satisfy in addition

***Proposition 2.2***. — *Let $\varphi\in\Gamma(S)$ be a Killing spinor with the Killing number $\lambda\in\mathbb{C}$. Then*

1. $(\mathrm{Ric}(X)-4\lambda^2(n-1)X)\cdot\varphi=0$ . *In particular, the image of the endomorphism* $\mathrm{Ric}-4\lambda^2(n-1)id_{TM}$ *is totally lightlike.*
2. *The scalar curvature is constant and given by $R=4n(n-1)\lambda^2$ . The Killing number $\lambda$ is real or purely imaginary.*

If the Killing number $\lambda$ is zero $(R=0)$, $\varphi$ is a parallel spinor, in case $\lambda$ is real and non-zero $(R>0)$, $\varphi$ is called real Killing spinor, and in case $\lambda$ is purely imaginary $(R<0)$, $\varphi$ is called imaginary Killing spinor.
We consider the following covariant derivative in the bundle $E=S\oplus S$

$$\nabla^E_X:=\begin{pmatrix}\nabla^S_X & \frac{1}{n}X\cdot\\ -\frac{n}{2}K(X) & \nabla^S_X\end{pmatrix}.$$

Using the integrability condition (2) of Proposition 2.1 one obtains the following

***Proposition 2.3*** (**[BFGK91**, Th.1.4]). — *For any twistor spinor $\varphi$ it holds $\nabla^E\binom{\varphi}{D\varphi}=0$. Conversely, if $\binom{\varphi}{\psi}$ is $\nabla^E$-parallel, then $\varphi$ is a twistor spinor and $\psi=D\varphi$.*

The calculation of the curvature of $\nabla^E$ and Proposition 2.3 yield

***Proposition 2.4***. — *The dimension of the space of twistor spinors is conformally invariant and bounded by*

$$\dim\ker P\leq 2^{[\frac{n}{2}]+1}=2\cdot\operatorname{rank}S=:d_n.$$

*For each simply connected, conformally flat semi-Riemannian spin manifold the dimension of the space of twistor spinors equals $d_n$. On the other hand, the maximal dimension $d_n$ can only occur if $(M,g)$ is conformally flat.*

Let $M^{n,k}$ be a conformally flat manifold with the universal covering $\tilde{M}^{n,k}$. The bundle $E$ is a tractor bundle associated to the conformal structure of $(M,g)$ and $\nabla^E$ is the covariant derivative on $E$ defined by the normal conformal Cartan connection. (For the definition of tractor bundles see for example [**CG99**]). Using this description one obtains a development of $\tilde{M}^{n,k}$ into a covering $\hat{C}^{n,k}$ of the (pseudo-) Möbius sphere. The corresponding holonomy representation

$$\rho : \pi_1(M) \longrightarrow \mathrm{O}(k+1, n-k+1)$$

of the fundamental group of $M$ characterizes conformally flat spin manifolds with twistor spinors.

***Proposition 2.5*** **([KR97a], [Lei00b])**. — *A conformally flat semi-Riemannian manifold is spin and admits twistor spinors iff the holonomy representation $\rho$ admits a lift*

$$\tilde{\rho} : \pi_1(M) \longrightarrow \mathrm{Spin}(k+1, n-k+1)$$

*and the the representation of $\pi_1(M)$ on the spinor module $\Delta_{k+1,n-k+1}$ has a proper trivial subrepresentation.*

If the scalar curvature $R$ of $(M^{n,k}, g)$ is constant and non-zero, the integrability conditions (1) and (2) of Proposition 2.1 show that the spinor fields

$$\psi_{\pm} := \frac{1}{2}\varphi \pm \sqrt{\frac{n-1}{nR}}\, D\varphi$$

are formal eigenspinors of the Dirac operator $D$ to the eigenvalue $\pm\frac{1}{2}\sqrt{\frac{nR}{n-1}}$.
For an Einstein space $(M^{n,k}, g)$ with constant scalar curvature $R \neq 0$ the spinor fields $\psi_{\pm}$ are Killing spinors to the Killing number $\lambda = \mp\frac{1}{2}\sqrt{\frac{R}{n(n-1)}}$. Hence, on this class of semi-Riemannian manifolds each twistor spinor is the sum of two Killing spinors.

To each spinor field $\varphi$ we associate a vector field $V_\varphi$ (Dirac current) by the formula

$$g(V_\varphi, X) := i^{k+1}\langle X \cdot \varphi, \varphi\rangle\,, \qquad X \in \Gamma(TM).$$

***Proposition 2.6***. — *Let $\varphi \in \Gamma(S)$ be a twistor spinor. Then $V_\varphi$ is a conformal vector field with the divergence*

$$\mathrm{div}(V_\varphi) = -2(-1)^{[\frac{k}{2}]}\, h(\langle D\varphi, \varphi\rangle)\,,$$

*where $h(f)$ denotes the real part of $f$ if the index $k$ of $g$ is odd and the imaginary part of $f$, if the index $k$ of $g$ is even.*

From now on we restrict our consideration to the case of *Lorentzian* manifolds $(M^{n,1}, g)$. Then for each spinor field the vector field $V_\varphi$ is causal: $g(V_\varphi, V_\varphi) \leq 0$. Let denote by $\mathrm{Zero}(\varphi)$ and $\mathrm{Zero}(V_\varphi)$ the zero sets of the spinor and the associated vector field, respectively. In the Lorentzian setting we have the following special feature of these zero sets

***Proposition 2.7*** **([Lei00c])**. — *For each spinor field $\varphi$ on a Lorentzian manifold the zero sets $\mathrm{Zero}(\varphi)$ and $\mathrm{Zero}(V_\varphi)$ coincide. If $\varphi$ is a twistor spinor with zero, then $V_\varphi$ is an essential conformal field satisfying $\nabla V_\varphi(p) = 0$ for each $p \in \mathrm{Zero}(V_\varphi)$. The zero set of $\varphi$ is the union of isolated points and isolated lightlike geodesics. Furthermore, the Weyl tensor vanishes on the zero set of $\varphi$.*

## 3. Twistor spinors on 4-dimensional spacetimes

Let us first collect some results in the 4-dimensional case.

***Proposition 3.1***. — *Let $(M, g)$ be a 4-dimensional Lorentzian spin manifold and let $\varphi \in \Gamma(S^{\pm})$ be a half spinor. Then $V_\varphi \cdot \varphi = 0$. In particular, the vector field $V_\varphi$ is lightlike. In case $\varphi$ is a twistor spinor we have $V_\varphi \lrcorner W = 0$.*

From the Propositions 2.7 and 3.1 it follows that a 4-dimensional spacetime with nontrivial twistor spinors is in each point of Petrov type $N$ or 0.
There is a standard model for 4-dimensional spacetimes admitting parallel spinors, known by physicists for a long time, the so-called pp-manifolds

$$\mathbb{R}^{4,1}, g_f := -2dx_1dx_2 + f(x_2, x_3, x_4)dx_2^2 + dx_3^2 + dx_4^2,$$

where $f$ denotes a smooth function.

***Proposition 3.2*** **([Ehl62])**. — *Each 4-dimensional spacetime admitting parallel spinors is locally isometric to a standard pp-manifold $(\mathbb{R}^{4,1}, g_f)$.*

***Proposition 3.3*** **([Boh98])**. — *Each 4-dimensional spacetime admitting real Killing spinors has constant positive sectional curvature. If a 4-dimensional spacetime admits 2 linearly independent imaginary Killing spinors, then it has constant negative sectional curvature.*

The following spacetime has exactly 1 imaginary Killing spinor:

$$\big(\mathbb{R}^4, h_f := e^{2x_4}(-2dx_1dx_2 + f(x_2, x_3)dx_2^2 + dx_3^2) + dx_4^2\big).$$

If $\partial^2 f/\partial x_3^2 \neq 0$, then $(\mathbb{R}^4, h_f)$ is neither conformally flat nor Einstein.

One kind of spacetimes of Petrov type $N$ are the so-called Fefferman spaces which are known in CR-geometry. In 1991 J. Lewandowski proved the following

***Proposition 3.4*** **([Lew91])**. — *Let $\varphi$ be a twistor half spinor without zeros on a 4-dimensional spacetime $(M^{4,1}, g)$.*

1. *If $V_\varphi$ is hypersurface orthogonal, then $(M^{4,1})$ is locally conformal equivalent to a pp-manifold.*
2. *If the rotation* $\mathrm{rot}(V_\varphi)$ *of $V_\varphi$ is nondegenerate on $V_\varphi^\perp/V_\varphi$, then $(M^{4,1},g)$ is locally conformal equivalent to a Fefferman space.*

*On the other hand, there exist local solutions of the twistor equation on each* 4-*dimensional Fefferman space and each pp-manifold.*

As in the Riemannian situation there is a twistor space of each 4-dimensional (real) Lorentzian manifold. The structure of this twistor space was studied for example in [**Nur96**], [**Nur97**], [**MS94**], [**Lei98**], [**Lei**]. In [**Lei98**] it is shown, that similarly to the Riemannian situation a twistor spinor on a 4-dimensional spacetime can be considered as holomorphic section (with respect to an optical structure) in the canonical line bundle over the twistor space of the spacetime.

## 4. Lorentzian twistor spinors, CR geometry and Fefferman spaces

In this section we want to explain how the result of Lewandowski can be generalised to arbitrary even dimensions. Detailed proofs of the statements can be found in [**Bau99a**]. First we recall some notions from CR-geometry which are necessary to define the Fefferman spaces.

Let $N^{2m+1}$ be a smooth oriented manifold of odd dimension $2m+1$. A CR-structure on $N$ is a pair $(H,J)$, where

1. $H\subset TM$ is a real $2m$-dimensional subbundle,
2. $J:H\longrightarrow H$ is an almost complex structure on $H:\ J^2=-\mathrm{id}$,
3. If $X,Y\in\Gamma(H)$, then $[JX,Y]+[X,JY]\in\Gamma(H)$ and $N_J(X,Y):=J([JX,Y]+[X,JY])-[JX,JY]+[X,Y]\equiv 0$ (integrability condition).

Let us fix in addition a contact form $\theta\in\Omega^1(N)$ such that $\theta|_H\equiv 0$ and let us denote by $T$ the Reeb vector field of $\theta$ which is defined by $\theta(T)=1,\ T\lrcorner\ d\theta=0$. In the following we suppose that the Leviform $L_\theta: H\times H\longrightarrow\mathbb{R}$

$$L_\theta(X,Y):=d\theta(X,JY)$$

is positive definite. In this case $(N,H,J,\theta)$ is called a strictly pseudoconvex manifold. The tensor $g_\theta:=L_\theta+\theta\circ\theta$ defines a Riemannian metric on $N$. There is a special metric covariant derivative on a strictly pseudoconvex manifold, the Tanaka-Webster connection $\nabla^W:\Gamma(TN)\longrightarrow\Gamma(TN^*\otimes TN)$ given by the conditions

$$\begin{aligned}\nabla^W g_\theta &= 0\\ Tor^W(X,Y) &= L_\theta(JX,Y)\cdot T\\ Tor^W(T,X) &= -\frac{1}{2}([T,X]+J[T,JX])\end{aligned}$$

for $X, Y \in \Gamma(H)$. This connection satisfies $\nabla^W J = 0$ and $\nabla^W T = 0$ (see [**Tan75**], [**Web78**]). Let us denote by $T_{10} \subset TN^{\mathbb{C}}$ the eigenspace of the complex extension of $J$ on $H^{\mathbb{C}}$ to the eigenvalue $i$. Then $L_\theta$ extends to a hermitian form on $T_{10}$ by $L_\theta(U,V) := -id\theta(U, \overline{V})$, $U, V \in T_{10}$. For a complex 2-form $\omega \in \Lambda^2 N^{\mathbb{C}}$ we denote by $Tr_\theta \omega$ the $\theta$-trace of $\omega$:

$$Tr_\theta \omega := \sum_{\alpha=1}^{m} \omega(Z_\alpha, \overline{Z}_\alpha)\,,$$

where $(Z_1, \ldots, Z_m)$ is an unitary basis of $(T_{10}, L_\theta)$. Let $\mathfrak{R}^W$ be the (4,0)-curvature tensor of the Tanaka-Webster connection $\nabla^W$ on the complexified tangent bundle of $N$

$$\mathfrak{R}^W(X,Y,Z,V) := g_\theta(([\nabla^W_X, \nabla^W_Y] - \nabla^W_{[X,Y]})Z, \overline{V}).$$

and let us denote by

$$\mathrm{Ric}^W := \mathrm{Trace}^{(3,4)}_\theta := \sum_{\alpha=1}^{m} \mathfrak{R}^W(\cdot,\cdot, Z_\alpha, \overline{Z}_\alpha)$$

the *Tanaka-Webster-Ricci-curvature* and by $R^W := \mathrm{Trace}_\theta \mathrm{Ric}^W$ the *Tanaka-Webster-scalar curvature*. Then $\mathrm{Ric}^W$ is a $(1,1)$-form on $N$ with $\mathrm{Ric}^W(X,Y) \in i\mathbb{R}$ for real vectors $X, Y \in TN$ and $R^W$ is a real function.

Now, let us suppose, that $(N^{2m+1}, H, J, \theta)$ is a strictly pseudoconvex spin manifold. The spin structure of $(N, g_\theta)$ defines a square root $\sqrt{\Lambda^{m+1,0}N}$ of the canonical line bundle

$$\Lambda^{m+1,0}N := \{\omega \in \Lambda^{m+1}N^{\mathbb{C}} \mid V \lrcorner\ \omega = 0 \quad \forall V \in \overline{T_{10}}\}.$$

We denote by $(F, \pi, N)$ the $S^1$-principal bundle associated to $\sqrt{\Lambda^{m+1,0}N}$.
If one fixes a connection form $A$ on $F$ and the corresponding decomposition of the tangent bundle $TF = ThF \oplus TvF = H^* \oplus \mathbb{R}T^* \oplus TvF$ into the horizontal and vertical part, then a Lorentzian metric $h$ is defined by

$$h := \pi^* L_\theta - ic\pi^*\theta \circ A,$$

where $c$ is a non-zero real number.
The Fefferman metric arrises from a special choice of $A$ and $c$ done in such a way that the conformal class $[h]$ of $h$ does not depend on the pseudohermitian form $\theta$. Such a choice can be made with the connection

$$A_\theta := A^W - \frac{i}{4(m+1)} R^W \cdot \theta,$$

where $A^W$ is the connection form on $F$ defined by the Tanaka-Webster connection $\nabla^W$. The curvature form of $A^W$ is $\Omega^{A^W} = -\frac{1}{2}\mathrm{Ric}^W$. Then

$$h_\theta := \pi^* L_\theta - i\frac{8}{m+2}\pi^*\theta \circ A_\theta$$

is a Lorentzian metric such that the conformal class $[h_\theta]$ is an invariant of the CR-structure $(N, H, J)$. The metric $h_\theta$ is $S^1$-invariant, the fibres of the $S^1$-bundle are

lightlike. We call $(F^{2m+2}, h_\theta)$ with its canonically induced spin structure *Fefferman space of the strictly pseudoconvex spin manifold* $(N, H, J, \theta)$.
The Fefferman metric was first discovered by C. Fefferman for the case of strictly pseudoconvex hypersurfaces $N \subset \mathbb{C}^{m+1}$ ([**Fef76**]), who showed that $N \times S^1$ carries a Lorentzian metric whose conformal class is induced by biholomorphisms.The considerations of Fefferman were extended by Burns, Diederich and Snider ([**BDS77**]) and by Lee ([**Lee86**]) to the case of abstract (not necessarily embedded) CR-manifolds. A geometric characterisation of Fefferman metrics was given by Sparling (see [**Spa85**], [**Gra87**]).

The spin structure of $(N, g_\theta)$ induces a spin structure of the vector bundle $(H, L_\theta)$. We denote the corresponding spinor bundle on $N$ by $S_H$. Then we can prove the following

***Proposition 4.1*** **([Bau99a**, Prop.22**])**. — *Let $(N, H, J, \theta)$ be a strictly pseudoconvex spin manifold with the Fefferman space $(F, h_\theta)$ and the spinor bundle $S_H$. Then*

1. *The 2-form $d\theta$ acts by Clifford multiplication as endomorphism on the spinor bundle $S_H$ and has an eigenspace decomposition of the form*
$$S_H = S_{-ni} \oplus S_{-ni+2i} \oplus S_{-ni+4i} \oplus \cdots \oplus S_{ni-2i} \oplus S_{ni},$$
*where the subbundles $S_{ki}$ are the eigenspaces of $d\theta$ to the eigenvalue $ki$ which have the rang $\binom{n}{(n+k)/2}$.*
2. *The lifts of the two line bundles $S_{-ni}$ and $S_{ni}$ over $N$ to the Fefferman space $F$ are trivial bundles.*
3. *The spinor bundle $S_F$ of the Fefferman space can be identified with two copies of the lifted bundle $S_H$ : $S_F = \pi^* S_H \oplus \pi^* S_H$.*
4. *There exist global non-projectable sections $\psi_\pm$ in the trivial line bundles $\pi^* S_{\pm ni}$ such that the spinor fields*
$$\phi_\pm = (\psi_\pm, 0)$$
*are twistor spinors on the Fefferman space $(F, h_\theta)$.*

Studying the properties of the spinor fields $\phi_\pm$ we obtain the following twistorial characterisation of Fefferman spaces

***Proposition 4.2*** **([Bau99a**, Theorems 1 and 2**])**. — *Let $(N^{2m+1}, H, J, \theta)$ be a strictly pseudoconvex spin manifold and let $(F, h_\theta)$ be its Fefferman space. Then there exist two linearly independent twistor spinors $\varphi$ on $(F, h_\theta)$ with the following properties:*

1. *$V_\varphi$ is a regular, lightlike Killing field.*
2. *$V_\varphi \cdot \varphi = 0$.*
3. *$\nabla^S_{V_\varphi} \varphi = i\, c\, \varphi, \qquad$ where $c \in \mathbb{R} \setminus \{0\}$.*

*Conversely, let $(B^{2m+2}, h)$ be a Lorentzian spin manifold which admits a nontrivial twistor spinor satisfying the conditions 1., 2. and 3., then there exists a strictly* *pseudoconvex spin manifold* $(N^{2m+1}, H, J, \theta)$ *such that* $(B, h)$ *is locally isometric to the Fefferman space* $(F, h_\theta)$ *of* $(N, H, J, \theta)$.

The proof of Proposition 4.2 is based on the following characterisation of Fefferman spaces given by Sparling and Graham ([**Spa85**], [**Gra87**]):
Let $(B^n, h)$ be a Lorentzian manifold and let us denote by $R$ the scalar curvature, by Ric the Ricci-curvature, by $W$ the (4,0)-Weyl tensor, by $K$ the Rho tensor

$$K := \frac{1}{n-2}\left\{\frac{1}{2(n-1)}R \cdot h - \mathrm{Ric}\right\},$$

and by $C$ the (3,0)-Cotton-York-tensor

$$C(X,Y,Z) := h\Big(X, (\nabla_Y K)(Z) - (\nabla_Z K)(Y)\Big)$$

of $(B, h)$. If $V$ is a regular lightlike Killing field on $(B, h)$ such that

– $V \lrcorner W = 0$ ,
– $V \lrcorner C = 0$ and
– $K(V, V) = \mathrm{const} < 0$ ,

then there exists a strictly pseudoconvex manifold $(N, H, J, \theta)$ such that $(B, h)$ is locally isometric to the Fefferman space $(F, h_\theta)$ of $(N, H, J, \theta)$.
The integrability conditions (2), (3), and (4) of Proposition 2.1 imply that for each twistor spinor $\varphi$ the equation $V_\varphi \lrcorner C = 0$ holds. Using in addition the assumptions of Proposition 4.2 we obtain $V_\varphi \lrcorner W = 0$ and $K(V_\varphi, V_\varphi) = -c^2 < 0$ .

## 5. Lorentzian manifolds with parallel spinors

From Riemannian geometry it is known that the existence of Killing spinors on a Riemannian manifold $M$ is strongly related to the existence of parallel spinors on a certain Riemannian manifold $\hat{M}$ associated to $M$ (see [**Bär93**], [**Bau89**]). In [**BK99**] we studied the relation between parallel spinors and the holonomy of pseudo-Riemannian manifolds. Generalising a result of McK. Wang ([**Wan89**]) we showed

***Proposition 5.1***. — *Let* $(M, g)$ *be a simply connected, non locally symmetric, irreducible semi-Riemannian spin manifold of dimension* $n = p + q$ *and signature* $(p, q)$. *Let* $N$ *denote the dimension of the space of parallel spinor fields on* $M$. *Then* $N > 0$ *if and only if the holonomy representation* $H$ *of* $(M.g)$ *is (up to conjugacy in the full orthogonal group) on of the groups listed in Table 1.*

This list shows that there is no irreducible Lorentzian manifold with parallel spinors. A special class of non-irreducible Lorentzian manifold with parallel spinors is the following generalisation of pp-manifolds. Let $(F, h)$ be a Riemannian manifold

| $H$ | $p$ | $q$ | $N$ |
|---|---|---|---|
| $\mathrm{SU}(r,s) \subset \mathrm{SO}(2r,2s)$ | $2r$ | $2s$ | 2 |
| $\mathrm{Sp}(r,s) \subset \mathrm{SO}(4r,4s)$ | $4r$ | $4s$ | $r+s+1$ |
| $\mathrm{G}_2 \subset \mathrm{SO}(7)$ | 0 | 7 | 1 |
| $\mathrm{G}^*_{2(2)} \subset \mathrm{SO}(4,3)$ | 4 | 3 | 1 |
| $\mathrm{G}_2^{\mathbb{C}} \subset \mathrm{SO}(7,7)$ | 7 | 7 | 2 |
| $\mathrm{Spin}(7) \subset \mathrm{SO}(8)$ | 0 | 8 | 1 |
| $\mathrm{Spin}^+(4,3) \subset \mathrm{SO}(4,4)$ | 4 | 4 | 1 |
| $\mathrm{Spin}(7)^{\mathbb{C}} \subset \mathrm{SO}(8,8)$ | 8 | 8 | 1 |

TABLE 1

with holonomy in $\mathrm{SU}(m)$ (Ricci flat Kähler), $\mathrm{Sp}(m)$ (hyperKähler), $\mathrm{G}_2$ or $\mathrm{Spin}(7)$ and let $f : \mathbb{R} \times F :\longrightarrow \mathbb{R}$ be a smooth function. Then the Lorentzian manifold

$$M := \mathbb{R}^2 \times F \ , \ g_{(t,s,x)} := -2dtds + f(s,x)ds^2 + h_x$$

has parallel spinors. $(M,h)$ is Ricci-flat iff the functions $f(s,\cdot) : F \longrightarrow \mathbb{R}$ are harmonic for all $s \in \mathbb{R}$.

Low dimensional Lorentzian manifolds with parallel spinors and their holonomy were studied in [**FO99a**], [**FO99b**], [**Bry99**] and [**Bry00**]. R. Bryant obtained the local normal form of all 11-dimensional Lorentzian manifolds with parallel lightlike spinors and maximal holonomy (now called Bryant-metrics). In [**Lei00a**] indecomposable, reducible Lorentzian manifolds with a special kind of holonomy and parallel spinors are discussed.

It is known that an even-dimensional Riemannian manifold admits pure parallel spinors iff it is Ricci-flat and Kähler. In [**Kata**] this fact is generalised to the pseudo-Riemannian situation. The existence of a pure parallel spinor on a pseudo-Riemannian manifold can be characterised by curvature properties of the associated optical structure.

Each homogeneous Riemannian manifold with parallel spinors is flat. The situation changes in the pseudo-Riemannian situation. In [**Bau99b**] we describe all twistor spinors on the Lorentzian symmetric spaces explicitly. In particular, we prove that each non conformally-flat simply connected Lorentzian symmetric space admits parallel spinors. These Lorentzian symmetric spaces have solvable transvection group and are special pp-manifolds.

## 6. Lorentzian Einstein-Sasaki structures and imaginary Killing spinors

It is easy to check that a Lorentzian manifold $(M, g)$ has imaginary Killing spinors to the Killing number $i\lambda$ iff the cone over $M$ with timelike cone axis

$$C^{-}_{2\lambda}(M) := (M \times \mathbb{R}\ ,\ g_C := (2\lambda t)^2 g - dt^2)$$

has parallel spinors. We describe here the case of irreducible cone $C^{-}(M)$. Proposition 5.1 shows that the only irreducible restricted holonomy representation of a non locally-symmetric pseudo-Riemannian manifold of index 2 with parallel spinors is $\mathrm{SU}(1, m)$. This leads to Lorentzian Einstein-Sasaki structures on $M$.
A Lorentzian Sasaki manifold is a tripel $(M, g, \xi)$, where

1. $g$ is a Lorentzian metric.
2. $\xi$ is a timelike Killing vector field with $g(\xi, \xi) = -1$.
3. $J := -\nabla\xi : TM \longrightarrow TM$ satisfies

$$J^2(X) = -X - g(X, \xi)\xi \quad \text{and} \quad (\nabla_X J)(Y) = -g(X, Y)\xi + g(Y, \xi)X$$

Lorentzian Sasaki structures are related to Kähler structures by the following

***Proposition 6.1***

1. *$(M^{2m+1}, g)$ has a Lorentzian Sasaki structure iff the cone $C^{-}_{1}(M)$ has a (pseudo-Riemannian) Kähler structure.*
2. *$(M^{2m+1}, g)$ is a Einstein space of negative scalar curvature $R = -2m(2m + 1)$ iff the cone $C^{-}_{1}(M)$ is Ricci-flat.*

This Proposition shows that the cone $C^{-}_{1}(M)$ has holonomy in $\mathrm{SU}(1, m)$ if and only if $(M^{2m+1}, g)$ is a Lorentzian Einstein-Sasaki manifold. Then we can prove a twistorial characterisation of the Lorentzian Einstein-Sasaki geometry, similar to that of Fefferman spaces in Proposition 4.2.

***Proposition 6.2***. — *Let $(M^{2m+1}, g, \xi)$ be a simply connected Lorentzian Einstein-Sasaki manifold. Then $(M, g)$ is a spin manifold and there exists a twistor spinor $\varphi \in \Gamma(S)$ such that*

1. *$V_\varphi$ is a timelike Killing vector field with $g(V_\varphi, V_\varphi) = -1$.*
2. *$V_\varphi \cdot \varphi = -\varphi$.*
3. *$\nabla^S_{V_\varphi}\varphi = -\frac{1}{2}\, i\, \varphi$.*

*In particular, $\varphi$ is an imaginary Killing spinor and $V_\varphi = \xi$. Conversely, let $(M^{2m+1}, g)$ be a Lorentzian spin manifold with a twistor spinor satisfying 1., 2. and 3., then $(M, g, \xi = V_\varphi)$ is a Lorentzian Einstein-Sasaki manifold.*

If we proceed in the same way as above in the case of strictly pseudoconvex spin manifolds but starting with Kähler manifolds we end up with Lorentzian Einstein-Sasaki manifolds admitting imaginary Killing spinors:
Let $(X^{2m}, h, J)$ be a Kähler-Einstein spin manifold of negative scalar curvature $R_X < 0$. Let us denote by $(M, \pi, X)$ the $S^1$-principal bundle associated to the square root of the canonical line bundle $K := \Lambda^{m,0}X$ defined by the spin structure of $(X, h)$ and let $A$ be the connection form on $M$ defined by the Levi-Civita connection of $(X, h)$. We consider the Lorentzian metric

$$g := \pi^* h - \frac{16m}{R_X(m+1)} A \circ A.$$

The manifold $(M, g)$ is a Lorentzian Einstein-Sasaki spin manifold. The spinor bundle $S_X$ of $(X, h, J)$ decomposes into the eigenspaces $S_{ki}$ of the Kähler form $\omega$ to the eigenvalues $ki$:

$$S_X = S_{-im} \oplus S_{-im+2i} \oplus S_{-mi+4i} \oplus \cdots \oplus S_{mi-2i} \oplus S_{mi}.$$

The spinor bundle $S_M$ of $(M, g)$ is isomorphic to the lift $\pi^* S_X$. There exist global sections $\psi_\varepsilon$ in the line bundles $\pi^* S_{\varepsilon mi} \subset S_M$ which are imaginary Killing spinors to the Killing number $\lambda_\varepsilon := (-1)^m \varepsilon^{m+1} \sqrt{\frac{-R_X}{16m(m+1)}}\, i \;\; , \varepsilon = \pm 1$.

The above described construction is a special case of an investigation of I.Kath in the general pseudo-Riemannian situation (see [**Kata**]), which extends the results of Ch. Bär ([**Bär93**]) concerning the Riemannian case. If $M$ is a simply connected pseudo-Riemannian manifold such that the holonomy group of the cone of $M$ is contained in one of the groups $H$ listed in Table 1 or in some of the other non-compact real forms corresponding to these groups, then $M$ admits Killing spinors and the special geometry of the cone, defined by the holonomy, defines a special geometry on $M$.

Finally, let us give an example of a Lorentzian manifold with imaginary Killing spinors, which is non-Einstein: Let $(F, h)$ be a Riemannian manifold with holonomy in $\mathrm{SU}(m)$, $\mathrm{Sp}(m)$, $\mathrm{G}_2$ or $\mathrm{Spin}(7)$ and let $f : F \times \mathbb{R} \longrightarrow \mathbb{R}$ be a smooth function. We consider the manifold $M = \mathbb{R}^3 \times F$ with the metric

$$g_{u,s,t,x} = e^{2u}(-2dsdt + f(s,x)ds^2 + h_x) + du^2.$$

Then $(M, g)$ is a Lorentzian manifold with imaginary Killing spinors which is Einstein if and only if the functions $f(s, \cdot) : F \to \mathbb{R}$ are harmonic for all $s$.

## 7. Lorentzian manifolds with real Killing spinors

Lorentzian manifolds with real Killing spinors were studied by Ch. Bohle in [**Boh**]. Similarly to the case of imaginary Killing spinors Lorentzian manifolds with real

Killing spinors can be obtained by warped product constructions out of Riemannian ones: It is easy to check that the warped product

$$F \times_\sigma I := (F \times I, g = \sigma^2 h + \varepsilon dt^2)$$

has real Killing spinors to the Killing number $\lambda$ iff (up to coordinate transformations) one of the cases of the following Table 2 occur.

| *case* | $(F,h)$ | $I$ | $\sigma$ | $\varepsilon$ |
|---|---|---|---|---|
| 1 | Riemannian manifold with real Killing spinor to the Killing number $\lambda$ | $\mathbb{R}$ | $\cosh 2\lambda t$ | 1 |
| 2 | Riemannian manifold with parallel spinor | $\mathbb{R}$ | $e^{2\lambda t}$ | 1 |
| 3 | Riemannian manifold with imaginary Killing spinor to the Killing number $i\lambda$ | $(0,\infty)$ | $\sinh 2\lambda t$ | 1 |
| 4 | Lorentzian manifold with real Killing spinor to the Killing number $\lambda$ | $(\frac{-\pi}{4\lambda}, \frac{\pi}{4\lambda})$ | $\cos \lambda t$ | $-1$ |

TABLE 2

On the other hand, each Lorentzian manifold with real Killing spinors has locally such a warped product structure.
Let us denote by $u := \langle\varphi,\varphi\rangle \in C^\infty(M)$ the length function of a spinor field $\varphi$ and by $Q_\varphi$ the function

$$Q_\varphi = u^2 + g(V_\varphi, V_\varphi).$$

Now, let $\varphi$ be a real Killing spinor. Then $V_\varphi$ is a closed conformal vector field and $\mathrm{grad}(u) = -2\lambda V_\varphi \neq 0$. Hence, the level sets of $u$ define a foliation of $M$ into submanifolds of codimension 1. Furthermore, the function $Q_\varphi$ is constant on $M$. Since $g(V_\varphi, V_\varphi) \leq 0$ we have $Q_\varphi \leq u^2$. All level sets with $u^2 > Q_\varphi$ are timelike submanifolds, those with $u^2 = Q_\varphi$ are degenerate. Let $p \in M$ be a point where $V_\varphi(p)$ is timelike, then around the point $p$ the manifold $(M,g)$ is locally isometric to the following warped product

- $Q_\varphi < 0$: case 1 of Table 2
- $Q_\varphi = 0$: case 2 of Table 2
- $Q_\varphi > 0$: case 3 of Table 2

In particular, $(M,g)$ is an Einstein manifold.
For a complete Lorentzian manifold one can prove, that the length function $u : M \to \mathbb{R}$ is surjective. Hence, on a complete Lorentzian manifold the first integral $Q_\varphi$ is

nonpositive. Using the results about parallel and Killing spinors in the Riemannian situation ([**BFGK91**], [**Bär93**], [**Wan89**], we obtain the following Splitting Theorem for complete Lorentzian manifolds in the presence of Killing spinors

***Proposition 7.1.*** — *Let $(M^n, g)$ be a complete, connected Lorentzian manifold carrying a real Killing spinor $\varphi$ to the Killing number $\lambda$.*

1. $Q_\varphi < 0$. *Then $(M, g)$ is of constant sectional curvature or is (up to a rescaling of the metric) globally isometric to the warped product*
$$(F \times \mathbb{R}, (\cosh t)^2\, h - dt^2),$$
*where $(F, h)$ is a complete Riemannian manifold which is covered by a simply connected Einstein-Sasaki manifold ($n = 2k$), 3-Sasaki manifold ($n = 4k$), nearly Kähler, non-Kähler manifold ($n = 7$) or a manifold admitting a nearly parallel* $\mathrm{G}_2$*-structure ($n = 8$).*
2. $Q_\varphi = 0$. *Then $\{u = 0\}$ is a degenerate hypersurface. $(M, g)$ is of constant sectional curvature or $M \setminus \{u = 0\}$ is globally isometric to the disjoint union of warped products*
$$(F_1 \times \mathbb{R}, e^{2\lambda t} h_1 - dt^2) \cup (F_2 \times \mathbb{R}, e^{2\lambda t} h_2 - dt^2),$$
*where $(F_1, h_1)$ and $(F_2, h_2)$ are complete Riemannian manifolds which are covered by products of simply connected manifolds with holonomy* $\mathrm{SU}(m)$, $\mathrm{Sp}(m)$, $\mathrm{G}_2$, $\mathrm{Spin}(7)$ *or* $\{1\}$.

We conjecture that the first integral $Q_\varphi = 0$ can only occur on manifolds with constant sectional curvature. For example, each spinor field $\varphi$ on the 3-dimensional spaceform $S^{3,1}$ of sectional curvature 1 has the first integral $Q_\varphi = 0$.

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