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PSEUDO-RIEMANNIAN METRICS WITH PARALLEL SPINOR FIELDS AND VANISHING RICCI TENSOR

by

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Abstract. — I will discuss geometry and normal forms for pseudo-Riemannian metrics with parallel spinor fields in some interesting dimensions. I also discuss the interaction of these conditions for parallel spinor fields with the Einstein equations.

Résumé **(Métriques pseudo-riemanniennes admettant des spineurs parallèles et un tenseur de Ricci nul)**
Je discuterai la géométrie et les formes normales pour les métriques pseudo-riemanniennes qui ont des champs de spineurs parallèles en quelques dimensions intéressantes. Je discuterai aussi l'interaction de ces conditions pour les champs de spineurs parallèles avec les équations d'Einstein.

1. Introduction

1.1. Riemannian holonomy and parallel spinors. — The possible restricted holonomy groups of irreducible Riemannian manifolds have been known for some time now [**2, 6, 7**]. The list of holonomy-irreducible types in dimension n that have nonzero parallel spinor fields is quite short: The holonomy H of such a metric must be one of

- $H = \mathrm{SU}(m)$ (i.e., special Kähler metrics in dimension $n = 2m$);
- $H = \mathrm{Sp}(m)$ (i.e., hyper-Kähler metrics in dimensions $n = 4m$);
- $H = \mathrm{G}_2$ (when $n = 7$); or
- $H = \mathrm{Spin}(7)$ (when $n = 8$).

In Cartan's sense, the local generality [**6, 7**] of metrics with holonomy

- $H = \mathrm{SU}(m)$ $(n = 2m)$ is 2 functions of $2m-1$ variables,
- $H = \mathrm{Sp}(m)$ $(n = 4m)$ is $2m$ functions of $2m+1$ variables,
- $H = \mathrm{G}_2$ $(n = 7)$ is 6 functions of 6 variables, and

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– $H = \mathrm{Spin}(7)$ ($n = 8$) is 12 functions of 7 variables.

In each case, a metric with holonomy H has vanishing Ricci tensor.

1.2. Relations with physics. — The existence of parallel spinor fields seems to account for much of the interest in metrics with special holonomy in mathematical physics, since such spinor fields play a central role in supersymmetry. In the case of string theory, SU(3), and lately, with the advent of $\mathcal{M}$-theory, G_2 (and possibly even Spin(7)) seem to be of interest. I don't know much about these physical theories, so I will not attempt to discuss them.

1.3. Pseudo-Riemannian generalizations. — In the past few years, I have been asked by a number of physicists about the generality of pseudo-Riemannian metrics satisfying conditions having to do with parallel spinors and with solutions of the Einstein equations. (In contrast to the Riemannian case, an indecomposable pseudo-Riemannian metric can possess a parallel spinor field without being Einstein.)

For example, there seems to be some current interest in Lorentzian manifolds of type $(10, 1)$ having parallel spinor fields and perhaps also having vanishing Ricci curvature, about which I will have more to say later in the article.

Recall [**17, 5**] that in the pseudo-Riemannian case, there is a distinction to be made between a metric being holonomy-irreducible (no parallel subbundles of the tangent bundle), being holonomy-indecomposable (no parallel splitting of the tangent bundle), and being indecomposable (no local product decomposition of the metric). (In the Riemannian case, of course, these conditions are locally equivalent.) The classification of the holonomy-irreducible case proceeds much as in the positive definite case [**8**], but an indecomposable pseudo-Riemannian metric need not be holonomy irreducible. It is this difference that makes classifying the possible pseudo-Riemannian metrics having parallel spinor fields something of a challenge. For a general discussion of the differences, particularly the failure of the de Rham splitting theorem, see [**3, 4**]. Also, the results and examples in [**13, 14**] are particularly illuminating.

Now, quite a lot is known about the pseudo-Riemannian case when the holonomy acts irreducibly. For a general survey in this case, particularly regarding the existence of parallel spinor fields, see [**1**]. Note that, in all of these cases, the Ricci tensor vanishes. This is not so when the holonomy acts reducibly. Already in dimension 3, Lorentzian metrics can have parallel spinor fields without being Ricci-flat.

An intriguing relationship between the condition for having a parallel spinor and the Ricci equations came to my attention after a discussion during a 1997 summer conference in Edinburgh with Ines Kath. It had been known for a while [**6**] that the metrics in dimension 7 with holonomy G_2 depend locally on six functions of six variables (modulo diffeomorphism). Now, the condition of having holonomy in G_2 is equivalent to the condition of having a parallel spinor field. I had also shown that the $(4, 3)$-metrics with holonomy G_2^* depend locally on six functions of six variables, and the condition of having this holonomy in this group is the same as the condition that the $(4,3)$-metric admit a non-null parallel spinor field. Ines Kath had noticed that the structure equations of a $(4,3)$ metric with a null parallel spinor field did not seem to imply that the Ricci curvature vanished, and she wondered whether or not there existed examples in which it did not. After some analysis, I was able to show that there are indeed $(4,3)$-metrics with parallel spinor fields whose Ricci curvature is not zero and whose holonomy is equal to the full stabilizer of a null spinor. These metrics depend on three arbitrary functions of seven variables. However, a more intriguing result is that, when one combines the condition of having a parallel null spinor with the condition of being Ricci-flat, the $(4,3)$-metrics with this property depend on six functions of six variables, just as in the non-null case (where the vanishing of the Ricci tensor is automatic).

In any case, this and the questions from physicists motivates the general problem of determining the local generality of pseudo-Riemannian metrics with parallel spinors, with and without imposing the Ricci-flat condition. This article will attempt to describe some of what is known and give some new results, particularly in dimensions greater than 6.

Most of the normal forms that I describe for metrics with parallel spinor fields of various different algebraic types are already known in the literature, or have been derived independently by others. (In particular, Kath [**15**] has independently derived the normal forms for the split cases with a pure parallel spinor.) What I find the most interesting is that, in every known case, the system of PDE given by the Ricci-flat condition is either in involution (in Cartan's sense) with the system of PDE that describe the (p,q)-metrics with a parallel spinor of given algebraic type or else follows as a consequence (and so, in a manner of speaking, is trivially in involution with the parallel spinor field condition). I have no general proof that this is so in all cases, nor even a precise statement as to how general the solutions should be, since this seems to depend somewhat on the algebraic type of the parallel spinor. What does seem to be true in a large number of (though not all) cases, though, is that the local generality of the Ricci-flat (p,q)-metrics with a parallel spinor of a given algebraic type seems to be largely independent of the given algebraic type, echoing the situation for $(4,3)$-metrics mentioned above that first exhibited this phenomenon.

Since this article is mainly a discussion of cases, together with an explicit working out of the standard moving frame methods and applications of Cartan-Kähler theory, I cannot claim a great deal of originality for the results. Consequently, I do not state the results in the form of theorems, lemmas, and propositions, but instead discuss each case in turn. The most significant results are probably the descriptions of the generality of the Ricci-flat metrics with parallel spinors in the various cases. Another possibly significant result is the description of the $(10,1)$-metrics with a parallel null spinor field, since this seems to be of interest in physics [**11**].

2. Algebraic background on spinors

All of the material in this section is classical. I include it to fix notation and for the sake of easy reference for the next section. For more detail, the reader can consult [**12, 16**].

2.1. Notation. — The symbols $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$ denote, as usual, the rings of real numbers, complex numbers, quaternions, and octonions, respectively. When $\mathbb{F}$ is one of these rings, the notation $\mathbb{F}(n)$ means the ring of n-by-n matrices with entries in $\mathbb{F}$. The notation $\mathbb{F}^n$ will always denote the space of *column* vectors of height n with entries in $\mathbb{F}$. Vector spaces over $\mathbb{H}$ will always be regarded as having the scalar multiplication acting on the *right*. For an m-by-n matrix a with entries in $\mathbb{C}$ or $\mathbb{H}$, the notation a^* will denote its conjugate transpose. When a has entries in $\mathbb{R}$, a^* will simply denote the transpose of a.

The notation $\mathbb{R}^{p,q}$ denotes $\mathbb{R}^{p+q}$ endowed with an inner product of type (p,q). The notation $\mathbb{C}^{p,q}$ denotes $\mathbb{C}^{p+q}$ endowed with an Hermitian inner product of type (p,q), with a similar interpretation of $\mathbb{H}^{p,q}$, but the reader should keep in mind that a quaternion Hermitian inner product satisfies $\langle v, wq\rangle = \langle v,w\rangle q$ and $\langle vq, w\rangle = \bar{q}\langle v,w\rangle$ for $q \in \mathbb{H}$.

2.2. Clifford algebras. — The Clifford algebra $\mathrm{C}\ell(p,q)$ is the associative algebra generated by the elements of $\mathbb{R}^{p,q}$ subject to the relations $vw+wv = -2v{\cdot}w\,\mathbf{1}$. This is a $\mathbb{Z}_2$-graded algebra, with the even subalgebra $\mathrm{C}\ell^e(p,q)$ generated by the products vw for $v, w \in \mathbb{R}^{p,q}$.

Because of the following formulae, valid for $p, q \geq 0$ (see [**12, 16**]),

$$
\begin{aligned}
\mathrm{C}\ell^e(p{+}1,q) &\simeq \mathrm{C}\ell(p,q)\\
\mathrm{C}\ell(p{+}1,q{+}1) &\simeq \mathrm{C}\ell(p,q)\otimes \mathrm{C}\ell(1,1)\\
\mathrm{C}\ell(p{+}8,q) &\simeq \mathrm{C}\ell(p,q)\otimes \mathrm{C}\ell(8,0)\\
\mathrm{C}\ell(p,q{+}1) &\simeq \mathrm{C}\ell(q,p{+}1)
\end{aligned}
\tag{1}
$$

all these algebras can be worked out from the table

$$
\begin{aligned}
&\mathrm{C}\ell(0,1)\simeq \mathbb{R}\oplus\mathbb{R} &\qquad &\mathrm{C}\ell(1,1)\simeq \mathbb{R}(2)\\
&\mathrm{C}\ell(1,0)\simeq \mathbb{C} && \mathrm{C}\ell(2,0)\simeq \mathbb{H}\\
&\mathrm{C}\ell(3,0)\simeq \mathbb{H}\oplus\mathbb{H} && \mathrm{C}\ell(4,0)\simeq \mathbb{H}(2)\\
&\mathrm{C}\ell(5,0)\simeq \mathbb{C}(4) && \mathrm{C}\ell(6,0)\simeq \mathbb{R}(8)\\
&\mathrm{C}\ell(7,0)\simeq \mathbb{R}(8)\oplus\mathbb{R}(8) && \mathrm{C}\ell(8,0)\simeq \mathbb{R}(16).
\end{aligned}
\tag{2}
$$

For example, $\mathrm{C}\ell^e(p{+}1,p{+}1)\simeq \mathrm{C}\ell(p,p{+}1)\simeq \mathbb{R}(2^p)\oplus\mathbb{R}(2^p)$.

2.3. $\mathrm{Spin}(p,q)$ **and spinors.** — By the defining relations, if $v\cdot v \neq 0$, then $v \in \mathbb{R}^{p,q}$ is a unit in $\mathrm{C}\ell(p,q)$ and, moreover, the twisted conjugation $\rho(v) : \mathrm{C}\ell(p,q) \to \mathrm{C}\ell(p,q)$

defined on generators $w \in \mathbb{R}^{p,q}$ by $\rho(v)(w) = -vwv^{-1}$ preserves the generating subspace $\mathbb{R}^{p,q} \subset \mathrm{C}\ell(p,q)$, acting as reflection in the hyperplane $v^\perp \subset \mathbb{R}^{p,q}$.

The group $\mathrm{Pin}(p,q) \subset \mathrm{C}\ell(p,q)$ is the subgroup of the units in $\mathrm{C}\ell(p,q)$ generated by the elements v where $v \cdot v = \pm 1$ and the group $\mathrm{Spin}(p,q) = \mathrm{Pin}(p,q) \cap \mathrm{C}\ell^e(p,q)$ is the subgroup of the even Clifford algebra generated by the products vw, where $v{\cdot}v = w{\cdot}w = \pm 1$.

The map ρ defined above extends to a group homomorphism $\rho : \mathrm{Pin}(p,q) \to \mathrm{O}(p,q)$ that turns out to be a non-trivial double cover. The homomorphism $\rho : \mathrm{Spin}(p,q) \to \mathrm{SO}(p,q)$ is also a non-trivial double cover.

The space of spinors $\mathbb{S}^{p,q}$ is essentially an irreducible $\mathrm{C}\ell(p,q)$-module, considered as a representation of $\mathrm{Spin}(p,q)$.

When $p{-}q \equiv 3 \mod 4$, this definition is independent of which of the two possible irreducible $\mathrm{C}\ell(p,q)$ modules one uses in the construction.

When $p{-}q \equiv 0 \mod 4$, the space $\mathbb{S}^{p,q}$ is a reducible $\mathrm{Spin}(p,q)$-module, in fact, it can be written as a sum $\mathbb{S}^{p,q} = \mathbb{S}^{p,q}_+ \oplus \mathbb{S}^{p,q}_-$ where $\mathbb{S}^{p,q}_\pm$ are irreducible. Action by an element of $\mathrm{Pin}(p,q)$ not in $\mathrm{Spin}(p,q)$ exchanges these two summands.

When $p{-}q \equiv 1$ or $2 \mod 8$, the definition of $\mathbb{S}^{p,q}$ as given above turns out to be the sum of two equivalent representations of $\mathrm{Spin}(p,q)$. In this case, it is customary to redefine $\mathbb{S}^{p,q}$ to be one of these two summands, so I do this without comment in the rest of the article.

When $q = 0$, i.e., in the Euclidean case, I will usually simplify the notation by writing $\mathrm{C}\ell(p)$, $\mathrm{Spin}(p)$, and $\mathbb{S}^p$ instead of $\mathrm{C}\ell(p,0)$, $\mathrm{Spin}(p,0)$, and $\mathbb{S}^{p,0}$, respectively.

2.4. Orbits in the low dimensions. — I will now describe the $\mathrm{Spin}(p,q)$-orbit structure of $\mathbb{S}^{p,q}$ when $p{+}q \le 6$. This description made simpler by the fact that there are several 'exceptional isomorphisms' of Lie groups (as discovered by Cartan) that reduce the problem to a series of classical linear algebra problems.

When $p{+}q \le 1$, these groups are not particularly interesting and, since there is no holonomy in dimension 1 anyway, I will skip these cases.

2.4.1. Dimension 2. — Here there are two cases.

2.4.1.1. $\mathrm{Spin}(2) \simeq \mathrm{U}(1)$. — The action of $\mathrm{Spin}(2) = \mathrm{U}(1)$ on $\mathbb{S}^2 \simeq \mathbb{C}$ is the unit circle action

$$\lambda \cdot s = \lambda s\,. \tag{3}$$

The orbits of $\mathrm{Spin}(2)$ on $\mathbb{S}^2 = \mathbb{C}$ are simply the level sets of the squared norm, so all of the nonzero orbits have the same stabilizer, namely, the identity.

Identifying $\mathbb{R}^{2,0}$ with $\mathbb{C}$, the action of $\mathrm{Spin}(2)$ on $\mathbb{R}^{2,0}$ can be described as

$$\lambda \cdot v = \lambda^2\, v \tag{4}$$

and the inner product is $v \cdot v = |v|^2 = \bar{v}\, v$.

2.4.1.2. $\mathrm{Spin}(1,1) \simeq \mathbb{R}^*$. — The action of $\mathrm{Spin}(1,1)$ on $\mathbb{S}^{1,1} \simeq \mathbb{R} \oplus \mathbb{R}$ is

$$\lambda \cdot (s_+, s_-) = (\lambda\, s_+, \lambda^{-1}\, s_-). \tag{5}$$

There is an identification $\mathbb{R}^{1,1} \simeq \mathbb{R} \oplus \mathbb{R}$ for which the action of $\mathrm{Spin}(1,1)$ on $\mathbb{R}^{1,1}$ has the description

$$\lambda \cdot (u, v) = (\lambda^2\, u, \lambda^{-2}\, v). \tag{6}$$

and the inner product is $(u, v) \cdot (u, v) = uv$.

The nonzero orbits of $\mathrm{Spin}(1,1)$ on $\mathbb{S}^{1,1}$ are all of dimension 1 and have the same stabilizer, namely, the identity.

2.4.2. Dimension 3. — Again, there are two cases.

2.4.2.1. $\mathrm{Spin}(3) \simeq \mathrm{Sp}(1)$. — The action of $\mathrm{Spin}(3)$ on $\mathbb{S}^3 \simeq \mathbb{H}$ is as quaternion multiplication:

$$A \cdot v = Av, \tag{7}$$

where A and v are quaternions. There are only two types of orbits, classified according to their stabilizer types: Those of the point $(0,0)$ and those of the points $(r,0)$, where $r > 0$ is a real number. The stabilizer of each nonzero element is trivial.

Identify $\mathbb{R}^{3,0}$ with $\mathrm{Im}\,\mathbb{H}$, so that the representation of $\mathrm{Spin}(3)$ on $\mathbb{R}^{3,0}$ can be described as

$$A \cdot v = A\, v\, \overline{A}. \tag{8}$$

and the inner product is $v \cdot v = v\, \overline{v}$.

2.4.2.2. $\mathrm{Spin}(2,1) \simeq \mathrm{SL}(2,\mathbb{R})$. — The action of $\mathrm{Spin}(2,1)$ on $\mathbb{S}^{2,1} \simeq \mathbb{R}^2$ is as the usual matrix multiplication:

$$A \cdot s = A\, s. \tag{9}$$

There are two $\mathrm{Spin}(2,1)$-orbits in $\mathbb{S}^{2,1}$: The orbit of the zero vector and then everything else.

Identify $\mathbb{R}^{2,1}$ with the the space of symmetric 2-by-2 matrices, so that the representation of $\mathrm{Spin}(2,1)$ on $\mathbb{R}^{2,1}$ can be described as

$$A \cdot v = A\, v\, A^* \tag{10}$$

and the inner product is $v \cdot v = -\det(v)$.

There is an equivariant 'spinor squaring' mapping $\sigma : \mathbb{S}^{2,1} \to \mathbb{R}^{2,1}$ defined by $\sigma(s) = s\, s^*$. Its image is one nappe of the null cone in $\mathbb{R}^{2,1}$.

2.4.3. Dimension 4. — Now, there are three cases.

2.4.3.1. $\mathrm{Spin}(4) \simeq \mathrm{Sp}(1) \times \mathrm{Sp}(1)$. — The action of $\mathrm{Spin}(4)$ on $\mathbb{S}^4 \simeq \mathbb{H} \oplus \mathbb{H}$ is

$$(A, B) \cdot (s_+, s_-) = (As_+, Bs_-). \tag{11}$$

There are four types of spinor orbits (classified according to their stabilizer types), those of the points $(0,0)$, $(r_+,0)$, $(0,r_-)$, and (r_+,r_-), where $r_\pm > 0$ are real numbers. Note that the stabilizer of a 'generic' orbit (i.e., the fourth type) is trivial. Note that action by an element of $\mathrm{Pin}(4)$ not in $\mathrm{Spin}(4)$ exchanges the two summands and hence the two types of 3-dimensional orbits.

Under the identification $\mathbb{R}^{4,0} \simeq \mathbb{H}$, the action of $\mathrm{Spin}(4)$ can be described as

$$(A, B) \cdot v = A\, v\, \overline{B}. \tag{12}$$

and the inner product is $v \cdot v = v\,\overline{v}$.

2.4.3.2. $\mathrm{Spin}(3,1) \simeq \mathrm{SL}(2,\mathbb{C})$. — The action of $\mathrm{Spin}(3,1)$ on $\mathbb{S}^{3,1} \simeq \mathbb{C}^2$ is just

$$A \cdot s = As. \tag{13}$$

In this case, there are only two orbits, those of 0 and s, where $s \in \mathbb{C}^2$ is nonzero.

Under the identification $\mathbb{R}^{3,1} \simeq H_2(\mathbb{C})$, the Hermitian symmetric 2-by-2 complex matrices, the action of $\mathrm{Spin}(3,1)$ can be described as

$$A \cdot v = A\, v\, A^* \tag{14}$$

and the inner product is $v \cdot v = -\det(v)$.

There is an equivariant 'spinor squaring' mapping $\sigma : \mathbb{S}^{3,1} \to \mathbb{R}^{3,1}$ defined by $\sigma(s) = s\,s^*$. Its image is one nappe of the null cone in $\mathbb{R}^{3,1}$. In relativity, this is referred to as the 'forward light cone'.

2.4.3.3. $\mathrm{Spin}(2,2) \simeq \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$. — The action of $\mathrm{Spin}(2,2)$ on $\mathbb{S}^{2,2} \simeq \mathbb{R}^2 \oplus \mathbb{R}^2$ is

$$(A, B) \cdot (s_+, s_-) = (As_+, Bs_-). \tag{15}$$

There are four orbits of $\mathrm{Spin}(2,2)$ on $\mathbb{S}^{2,2}$, those of the points $(0,0)$, $(s,0)$, $(0,s)$, and (s,s), where s is any nonzero vector in $\mathbb{R}^2$. Note that action by an element of $\mathrm{Pin}(2,2)$ not in $\mathrm{Spin}(2,2)$ exchanges the two 2-dimensional orbits.

Under the identification $\mathbb{R}^{2,2} \simeq \mathbb{R}(2)$, the action of $\mathrm{Spin}(2,2)$ on $\mathbb{R}^{2,2}$ can be described as

$$(A, B) \cdot v = A\, v\, B^* \tag{16}$$

and the inner product is $v \cdot v = \det(v)$.

There is an equivariant 'spinor squaring' mapping $\sigma : \mathbb{S}^{2,2} \to \mathbb{R}^{2,2}$ defined by $\sigma(s_+, s_-) = s_+\, s_-^*$. Its image is the null cone in $\mathbb{R}^{2,2}$.

2.4.4. Dimension 5. — Again, there are three cases.

2.4.4.1. Spin(5) $\simeq$ Sp(2). — The action of Spin(5) on $\mathbb{S}^5 \simeq \mathbb{H}^2$ is

$$A \cdot s = A\, s. \tag{17}$$

The orbits are given by the level sets of $s \cdot s = s^*\, s$. Except for $s = 0$, these orbits all have the same stabilizer type, namely Sp(1).

Identify $\mathbb{R}^5$ with the space of traceless, quaternion Hermitian symmetric 2-by-2 matrices. Then the action of Spin(5) on $\mathbb{R}^5$ becomes

$$A \cdot m = AmA^*, \tag{18}$$

and the quadratic form is just $m \cdot m = \operatorname{tr}(m^* m)$.

There is an equivariant 'spinor squaring' mapping $\sigma : \mathbb{S}^5 \to \mathbb{R}^5$ defined by $\sigma(s) = s\, s^*$. Its image is all of $\mathbb{R}^5$.

2.4.4.2. Spin(4,1) $\simeq$ Sp(1,1). — Let $Q = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, so that $\mathrm{Spin}(4,1)$ is realized as the matrices $A \in \mathbb{H}(2)$ that satisfy $A^*QA = Q$. Here, $\mathbb{S}^{4,1} \simeq \mathbb{H}^2$ and the spinor action is matrix multiplication:

$$A \cdot s = A\, s\,. \tag{19}$$

The spinor orbits are essentially the level sets of the function $\nu : \mathbb{S}^{4,1} \to \mathbb{R}$ defined by $\nu(s) = s^*Qs$, with the one exception being the level set $\nu = 0$, which consists of two orbits, the zero vector and then everything else. The stabilizer of

$$s_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \text{is} \qquad G_0 = \left\{ \begin{pmatrix} 1+q & -q \\ -\bar q & 1+\bar q \end{pmatrix} \,\middle|\, q \in \operatorname{Im}\mathbb{H} \right\} \simeq \mathbb{R}^3, \tag{20}$$

while, for $r > 0$, the stabilizer of

$$s_{r^2} = \begin{pmatrix} r \\ 0 \end{pmatrix} \qquad \text{is} \qquad G_+ = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix} \,\middle|\, q \in \mathrm{Sp}(1) \right\} \simeq \mathrm{Sp}(1), \tag{21}$$

and the stabilizer of

$$s_{-r^2} = \begin{pmatrix} 0 \\ r \end{pmatrix} \qquad \text{is} \qquad G_- = \left\{ \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix} \,\middle|\, q \in \mathrm{Sp}(1) \right\} \simeq \mathrm{Sp}(1). \tag{22}$$

The two elements $s_{\pm r^2}$ are on the same Pin(4,1)-orbit, so for our purposes, they should be counted as the same.

Identify $\mathbb{R}^{4,1}$ with the space of quaternion Hermitian symmetric matrices m that satisfy $\operatorname{tr}(Qm) = 0$. Then the action of Spin(4,1) on this space is just

$$A \cdot m = AmA^*. \tag{23}$$

The invariant quadratic form is $m \cdot m = -\det(m)$, where, det is defined on the quaternion Hermitian symmetric 2-by-2 matrices by

$$\det \begin{pmatrix} a & b \\ \bar b & c \end{pmatrix} = ac - b\bar b, \qquad a, c \in \mathbb{R},\ b \in \mathbb{H}. \tag{24}$$

There is an equivariant 'spinor squaring' mapping $\sigma : \mathbb{S}^{4,1} \to \mathbb{R}^{4,1}$ defined by $\sigma(s) = s\,s^* - \frac{1}{2}\nu(s)\,Q$. Its image consists of half of the cone of elements m that satisfy $\det(m) \geq 0$. The image boundary, i.e., the 'forward light cone' is the image of the locus $\nu = 0$ in $\mathbb{S}^{4,1}$.

2.4.4.3. $\mathrm{Spin}(3,2) \simeq \mathrm{Sp}(2,\mathbb{R})$. — This classical isomorphism can be described as follows: Let $J = \begin{pmatrix} 0 & -\mathrm{I}_2 \\ \mathrm{I}_2 & 0 \end{pmatrix}$. Then $\mathrm{Sp}(2,\mathbb{R})$ is the subgroup of $\mathrm{GL}(4,\mathbb{R})$ consisting of those matrices A that satisfy $A^*JA = J$. This group is isomorphic to $\mathrm{Spin}(3,2)$ in such a way that $\mathbb{S}^{3,2}$ can be identified with $\mathbb{R}^4$ so that the spinor representation becomes the usual matrix multiplication:

$$A \cdot s = As. \tag{25}$$

There are only two $\mathrm{Sp}(2,\mathbb{R})$-orbits in this case: The zero orbit and everything else.

The vector representation is described as follows: Identify $\mathbb{R}^{3,2}$ with the space of skew-symmetric $v \in \mathbb{R}(4)$ that satisfy $\mathrm{tr}(vJ) = 0$. This space is preserved under the action $A \cdot v = A\,v\,A^*$. The inner product is $v \cdot v = \mathrm{Pf}(v)$. This is an irreducible representation and the inner product is seen to be of type $(3,2)$.

2.4.5. Dimension 6. — Now, there are four cases.

2.4.5.1. $\mathrm{Spin}(6) \simeq \mathrm{SU}(4)$. — The action of $\mathrm{Spin}(6)$ on $\mathbb{S}^6 \simeq \mathbb{C}^4$ is

$$A \cdot s = A\,s. \tag{26}$$

The orbits are given by the level sets of $s \cdot s = s^*\,s$. Except for $s = 0$, these orbits all have the same stabilizer type, namely $\mathrm{SU}(3)$.

To see the representation of $\mathrm{SU}(4)$ on $\mathbb{R}^{6,0}$, consider the space W of skewsymmetric $w \in \mathbb{C}(4)$. This is a complex vector space of dimension 6. The group $\mathrm{SL}(4,\mathbb{C})$ acts on W by the rule

$$A \cdot w = A\,w\,A^*. \tag{27}$$

Consider the complex inner product $(,)$ on W that satisfies $(w,w) = \mathrm{Pf}(w)$. This is a nondegenerate quadratic form that is invariant under $\mathrm{SL}(4,\mathbb{C})$ and hence under $\mathrm{SU}(4)$. There is also an Hermitian inner product on W defined by $\langle w, w\rangle = \frac{1}{4}\,\mathrm{tr}(ww^*)$ and it is easily seen to be invariant under $\mathrm{SU}(4)$ as well. It follows that there is an $\mathrm{SU}(4)$-invariant conjugate-linear map $c : W \to W$ so that $(cw, v) = \langle w, v\rangle$. This linear map satisfies $c^2 = \mathrm{I}$, so there is an $\mathrm{SU}(4)$-invariant splitting $W = W_+ \oplus W_-$ into the (real) eigenspaces of c, each of dimension 6. The spaces $W_\pm$ are each isomorphic to $\mathbb{R}^{6,0}$ with inner product $(,)$ and the action of $\mathrm{SU}(4)$ double covers to produce the standard $\mathrm{SO}(6)$ action.

2.4.5.2. $\mathrm{Spin}(5,1) \simeq \mathrm{SL}(2,\mathbb{H})$. — Here, $\mathbb{S}^{5,1} \simeq \mathbb{H}^2 \oplus \mathbb{H}^2$ and the spinor action is

$$A \cdot (s_+, s_-) = (A\,s\,, (A^*)^{-1}\,s_-). \tag{28}$$

There several different types of spinor orbits. First, there is the point $(0,0)$. Then there are the two orbits of dimension 7 of the points $(s_+,0)$ and $(0,s_-)$, where $s_\pm$ are nonzero. Third, there are the orbits that lie in the locus $s_-^*\,s_+=0$, but that have $s_\pm\neq 0$. These orbits all have dimension 11 and there is a 1-parameter family of them. In fact, for each positive real r, the orbit of

$$(29)\qquad s_r=\left(\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}r\\0\end{pmatrix}\right)\qquad\text{has stabilizer}\qquad G_0=\left\{\ \begin{pmatrix}1&q\\0&1\end{pmatrix}\ \middle|\ q\in\mathbb{H}\ \right\}\simeq\mathbb{H}.$$

Fourth, the remaining orbits have dimension 12. These are parametrized by $s_-^*\,s_+=\lambda\in\mathbb{H}^*$. This level set is the orbit of the element

$$(30)\qquad s_\lambda=\left(\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}\lambda\\0\end{pmatrix}\right)\qquad\text{with stabilizer}\qquad G_1=\left\{\ \begin{pmatrix}1&0\\0&q\end{pmatrix}\ \middle|\ q\in\mathrm{Sp}(1)\ \right\}\simeq\mathrm{Sp}(1).$$

Note that, because the centralizer of $\mathrm{Spin}(5,1)$ in $\mathrm{Aut}(\mathbb{S}^{5,1})$ is $\mathbb{H}^*\times\mathbb{H}^*$ (scalar multiplication (on the *right*) in each summand), the combined action of the centralizer and $\mathrm{Spin}(5,1)$ shows that all of the orbits of the third type should be regarded as essentially the same and that all of the orbits of the fourth type should be regarded as essentially the same. Thus, there are really only four distinct types of orbits to consider. Moreover, action by an element of $\mathrm{Pin}(5,1)$ not in $\mathrm{Spin}(5,1)$ exchanges the two 7-dimensional orbits, so they should really be regarded as belonging to the same type.

Identify $\mathbb{R}^{5,1}$ with the space of Hermitian symmetric 2-by-2 matrices with quaternion entries. The action of $\mathrm{Spin}(5,1)$ on this space can be be described as

$$(31)\qquad A\cdot a=A\ a\ A^*$$

and the inner product satisfies $a{\cdot}a=-\det(a)$, where the interpretation of determinant in this case is

$$(32)\qquad \det\begin{pmatrix}a&b\\\bar b&c\end{pmatrix}=ac-b\bar b$$

for $a,c\in\mathbb{R}$ and $b\in\mathbb{H}$. (That $\mathrm{SL}(2,\mathbb{H})$ does preserve this must be checked, since, normally, det is not defined for matrices with quaternion entries.)

There is an equivariant 'spinor squaring' mapping $\sigma_+:\mathbb{S}_+^{5,1}\to\mathbb{R}^{5,1}$ defined by $\sigma_+(s_+)=s_+\,s_+^*$. Its image consists of the 'forward light cone' in $\mathbb{R}^{5,1}$.

2.4.5.3. $\mathrm{Spin}(4,2)\simeq\mathrm{SU}(2,2)$. — The identification of $\mathrm{Spin}(4,2)$ with $\mathrm{SU}(2,2)$ is very similar with the identification of $\mathrm{Spin}(6)$ with $\mathrm{SU}(4)$ and can be seen as follows. Let $Q=\begin{pmatrix}\mathrm{I}_2&0\\0&-I_2\end{pmatrix}$ and recall that $\mathrm{SU}(2,2)$ is the group of matrices $A\in\mathrm{SL}(4,\mathbb{C})$ satisfying $A^*QA=Q$. It acts on $\mathbb{C}^{2,2}=\mathbb{C}^4$ preserving the Hermitian inner product defined by $\langle v,w\rangle=v^*Qw$. The orbits of this action are $0\in\mathbb{C}^4$ and the nonzero parts of the level sets of the Hermitian form $\langle v,w\rangle=v^*Qv$. Note that the stabilizer of a vector satisfying $v^*Qv = 0$ is not conjugate to the stabilizer of a vector satisfying $v^*Qv \neq 0$. Thus, it makes sense to say that there are essentially two distinct types of nonzero orbits, the null orbit and the non-null orbits (which form a single type).

To justify the identification of $\mathrm{Spin}(4,2)$ with $\mathrm{SU}(2,2)$, it will be necessary to construct a 6-dimensional real vector space V on which $\mathrm{SU}(2,2)$ acts as the identity component of the stabilizer of a quadratic form on V of type $(4,2)$. Here is how this can be done: Again, start with W being the space of skewsymmetric matrices $w \in \mathbb{C}(4)$, with the action of $\mathrm{SL}(4,\mathbb{C})$ being, as before, $A \cdot w = A\,w\,A^*$. Again define the complex inner product $(,)$ on W so that $(w,w) = \mathrm{Pf}(w)$. Now, consider the Hermitian inner product on W defined by $\langle w,v\rangle = \frac{1}{4}\,\mathrm{tr}(w^*Qv)$. This Hermitian inner product is invariant under $\mathrm{SU}(2,2)$, so there is an $\mathrm{SU}(2,2)$-invariant conjugate linear mapping $c : W \to W$ satisfying $(cw, v) = \langle w, v\rangle$. Again, c^2 is the identity, so that W can be split into real subspaces $W = W_+ \oplus W_-$ with $i\,W_\pm = W_\mp$. Then $\mathrm{SU}(2,2)$ acts on $V = W_+$ preserving $(,)$ and it is not difficult to see that the type of this quadratic form is $(4,2)$. Since $\mathrm{SU}(2,2)$ is simple and of dimension 15, the same dimension as $\mathrm{SO}(4,2)$, it follows that this representation of $\mathrm{SU}(2,2)$ must be onto the identity component of the stabilizer of this quadratic form, as desired. More detail about this representation will be supplied when it is needed in the next section.

2.4.5.4. $\mathrm{Spin}(3,3) \simeq \mathrm{SL}(4,\mathbb{R})$. — Here $\mathbb{S}^{3,3} \simeq \mathbb{R}^4 \oplus \mathbb{R}^4$ and the spinor action is

$$A \cdot (s_+, s_-) = (As_+,\, (A^*)^{-1}s_-). \tag{33}$$

There are several orbits of $\mathrm{Spin}(3,3)$ on $\mathbb{S}^{3,3}$: Those of the points $(0,0)$, $(s_+,0)$, $(0,s_-)$, and (s_+,s_-) where $s_-^* s_+ = \lambda$, where λ is any real number and $s_\pm$ are nonzero elements of $\mathbb{R}^4$. In this last family of orbits, there are two essentially different kinds. The orbit with $\lambda = 0$ has a different stabilizer type in $\mathrm{SL}(4,\mathbb{R})$ from those with $\lambda \neq 0$, even though it has the same dimension. This is accounted for by the fact that the centralizer of $\mathrm{Spin}(3,3)$ in $\mathrm{Aut}(\mathbb{S}^{3,3})$ is $\mathbb{R}^* \times \mathbb{R}^*$ (scalar multiplication in the fibers) and the combined action of the centralizer and $\mathrm{Spin}(3,3)$ makes all of the orbits with $\lambda \neq 0$ equivalent to each other. Moreover, action by an element of $\mathrm{Pin}(3,3)$ not in $\mathrm{Spin}(3,3)$ exchanges the two 4-dimensional orbits, so they should be regarded as belonging to the same orbit type.

Under the identification $\mathbb{R}^{3,3} \simeq A_4(\mathbb{R})$, the antisymmetric 4-by-4 matrices with real entries, the action of $\mathrm{Spin}(3,3)$ can be be described as

$$A \cdot a = A\;a\;A^* \tag{34}$$

and the inner product satisfies $a \cdot a = \mathrm{Pf}(a)$.

2.5. The split cases and pure spinors. — The orbit structure of $\mathrm{Spin}(p,q)$ grows increasingly complicated as $p+q$ increases. However, there are a few orbits that are easy to describe in the so-called 'split' cases, i.e., $\mathrm{Spin}(p{+}1,p)$ (the odd split case), and $\mathrm{Spin}(p,p)$ (the even split case).

When $p=q$ or $p=q{+}1$, the maximal dimension of a null plane $N\subset\mathbb{R}^{p,q}$ is q. Let $v_1,\ldots,v_q$ be a basis of such an N and let $[v]=v_1v_2\cdots v_q\in \mathrm{C}\ell(p,q)$. The element $[v]$ depends only on N and a choice of volume element for N. It is not hard to show that the left ideal $\mathrm{C}\ell(p,q)\cdot[v]\subset\mathrm{C}\ell(p,q)$ is minimal, and so is irreducible as a $\mathrm{C}\ell(p,q)$ module.

2.5.1. The odd case. — Now, according to the definitions in §2.3, when $p=q{+}1$, the odd case, $\mathrm{C}\ell(q{+}1,q)\cdot[v]$, when considered as a $\mathrm{Spin}(q{+}1,q)$-module, is two isomorphic copies of $\mathbb{S}^{q+1,q}$. Fix such a decomposition of $\mathrm{C}\ell(p,q)\cdot[v]$ and consider the image $\langle v\rangle$ of $[v]$ in one of these summands, henceforth denoted $\mathbb{S}^{q+1,q}$. The $\mathrm{Spin}(q{+}1,q)$-orbit of $\langle v\rangle$ is known as the space of *pure spinors*. This orbit is a cone and has dimension $\frac{1}{2}q(q{+}1)+1$, which turns out to be the lowest dimension possible for a nonzero orbit. The ρ-image of the stabilizer in $\mathrm{Spin}(q{+}1,q)$ of a pure spinor is the stabilizer in $\mathrm{SO}(q{+}1,q)$ of a corresponding null q-vector in $\mathbb{R}^{p,p}$.

2.5.1.1. Low values of q. — When $q\equiv 0,3 \mod 4$, $\mathrm{Spin}(q{+}1,q)$ preserves an inner product (of split type) on $\mathbb{S}^{q+1,q}$ while, when $q\equiv 1,2 \mod 4$, $\mathrm{Spin}(q{+}1,q)$ preserves a symplectic form on $\mathbb{S}^{q+1,q}$, see [**12**].

Since $\mathbb{S}^{q+1,q}$ is a real vector space of dimension 2^q, as q increases, the pure spinors become a relatively small $\mathrm{Spin}(q{+}1,q)$-orbit in $\mathbb{S}^{q+1,q}$.

However, for low values of q, the situation is different. When $q=1$ or 2, every spinor is pure.

When $q=3$, dimension count shows that the pure spinors are a hypersurface in $\mathbb{S}^{4,3}$. Since they form a cone, they must constitute the null cone in $\mathbb{S}^{4,3}\simeq\mathbb{R}^8$ of the $\mathrm{Spin}(4,3)$-invariant quadratic form on $\mathbb{S}^{4,3}$. Moreover, the other nonzero $\mathrm{Spin}(4,3)$-orbits in $\mathbb{S}^{4,3}$ are the nonzero level sets of this quadratic form, and so are also of dimension 7. The stabilizer of a non-null element $v\in\mathbb{S}^{4,3}$ is isomorphic to $\mathrm{G}_2^*\subset\mathrm{Spin}(4,3)$, the split form of type G_2.

When $q=4$, the pure spinors constitute an 11-dimensional cone in $\mathbb{S}^{5,4}\simeq\mathbb{R}^{16}$, which must therefore lie in the null cone of the $\mathrm{Spin}(5,4)$-invariant quadratic form on $\mathbb{S}^{5,4}$. It is an interesting fact that each of the nonzero level sets of this quadratic form constitutes a single $\mathrm{Spin}(5,4)$-orbit. (This is because, as can be seen in [**9**], $\mathrm{Spin}(9)$ acts transitively on the unit spheres in $\mathbb{S}^9\simeq\mathbb{R}^{16}$. The existence of hypersurface orbits in the compact case implies the existence of hypersurface orbits in the complexification, which implies the existence of hypersurface orbits in the split form, i.e., $\mathrm{Spin}(5,4)$.) Thus, although the null cone is the limit of hypersurface orbits, it does not constitute a single orbit, but must contain at least two orbits (besides the zero orbit). One of those orbits is the 11-dimensional space of pure spinors, but I do not know whether the complement of the pure spinors in the null spinors constitutes a single orbit or not.

2.5.2. The even case. — According to the definitions in §2.3, when $p=q$, the relation $\mathbb{S}^{p,p}_+ \oplus \mathbb{S}^{p,p}_- = \mathbb{S}^{p,p} \simeq \mathrm{C}\ell(p,p)\cdot[v]$ holds. It turns out that $[v]$ lies in one of the two summands (which one depends on the orientation of $\mathbb{R}^{p,p}$, since this decides which one is $\mathbb{S}^{p,p}_+$). This corresponds to the well-known fact that the space of maximal null p-planes in $\mathbb{R}^{p,p}$ consists of two components. By this construction, each component of the space of null p-planes endowed with a choice of volume form in $\mathbb{R}^{p,p}$ is double covered by a $\mathrm{Spin}(p,p)$ orbit (in fact, a closed cone) in $\mathbb{S}^{p,p}_\pm$. The elements of these two orbits are the pure spinors. Each forms a minimal (i.e., maximally degenerate) orbit in $\mathbb{S}^{p,p}$. The dimension of each of these orbits is $\frac{1}{2}p(p-1)+1$. The ρ-image of the stabilizer in $\mathrm{Spin}(p,p)$ of a pure spinor maps onto the stabilizer of a null p-vector in $\mathbb{R}^{p,p}$.

2.5.2.1. Low values of p. — When $p \equiv 1 \mod 2$, the spaces $\mathbb{S}^{p,p}_+$ and $\mathbb{S}^{p,p}_-$ are naturally dual as $\mathrm{Spin}(p,p)$-modules. When $p\equiv 2 \mod 4$, each of $\mathbb{S}^{p,p}_\pm$ is a symplectic representation of $\mathrm{Spin}(p,p)$. When $p\equiv 0 \mod 4$, each of $\mathbb{S}^{p,p}_\pm$ is an orthogonal representation of $\mathrm{Spin}(p,p)$. Again, see [**12**] for proofs of these facts.

Since $\mathbb{S}^{p,p}$ is a sum of two $\mathrm{Spin}(p,p)$-irreducible real vector spaces of dimension 2^{p-1}, as p increases, the pure spinors become a vanishingly small $\mathrm{Spin}(p,p)$-orbit in $\mathbb{S}^{p,p}$.

However, for low values of p, the situation is different. When $p=1$, 2, or 3, every spinor in $\mathbb{S}^{p,p}_\pm$ is pure.

When $p=4$ (the famous case of triality), $\mathrm{Spin}(4,4)$ acts on each of $\mathbb{S}^{4,4}_\pm \simeq \mathbb{R}^{4,4}$ as the full group of linear transformations preserving the spinor inner product. In particular, the nonzero orbits are just the level sets of the invariant quadratic form. Thus, the pure spinors in each space constitute the null cone (minus the origin) of the quadratic form. Using this description, it is not difficult completely to describe the orbits of $\mathrm{Spin}(4,4)$ on $\mathbb{S}^{4,4}$. I will go into more detail as necessary in what follows.

When $p=5$, the situation is more subtle. $\mathrm{Spin}(5,5)$ acts on each of $\mathbb{S}^{5,5}_\pm \simeq \mathbb{R}^{16}$ with open orbits. The cone of pure spinors in each summand has dimension 11. In fact, in the direct sum action on $\mathbb{S}^{5,5}$, the group $\mathrm{Spin}(5,5)$ preserves the quadratic form that is the dual pairing on the two factors and a nontrivial quartic form. The generic orbits of $\mathrm{Spin}(5,5)$ on $\mathbb{S}^{5,5}$ are simultaneous level sets of these two polynomials and so have dimension 30. I do not know the full orbit structure.

2.6. The octonions and $\mathrm{Spin}(10,1)$**.** — In this section, I will develop just enough of the necessary algebra to discuss the geometry of one higher dimensional case, that of parallel spinors in a metric of type $(10,1)$. The reason for considering this case is that there is some interest in it for physical reasons, see [**11**].

2.6.1. Octonions. — A few background facts about the octonions will be needed. For proofs, see [**12**].

As usual, let $\mathbb{O}$ denote the ring of octonions. Elements of $\mathbb{O}$ will be denoted by bold letters, such as $\boldsymbol{x}$, $\boldsymbol{y}$, etc. Thus, $\mathbb{O}$ is the unique $\mathbb{R}$-algebra of dimension 8 with unit $\mathbf{1} \in \mathbb{O}$ endowed with a positive definite inner product $\langle , \rangle$ satisfying $\langle \boldsymbol{xy}, \boldsymbol{xy} \rangle = \langle \boldsymbol{x}, \boldsymbol{x} \rangle \langle \boldsymbol{y}, \boldsymbol{y} \rangle$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{O}$. As usual, the norm of an element $\boldsymbol{x} \in \mathbb{O}$ is denoted $|\boldsymbol{x}|$ and defined as the square root of $\langle \boldsymbol{x}, \boldsymbol{x} \rangle$. Left and right multiplication by $\boldsymbol{x} \in \mathbb{O}$ define maps $L_{\boldsymbol{x}}, R_{\boldsymbol{x}} : \mathbb{O} \to \mathbb{O}$ that are isometries when $|\boldsymbol{x}| = 1$.

The conjugate of $\boldsymbol{x} \in \mathbb{O}$, denoted $\overline{\boldsymbol{x}}$, is defined to be $\overline{\boldsymbol{x}} = 2\langle \boldsymbol{x}, \mathbf{1} \rangle \, \mathbf{1} - \boldsymbol{x}$. When a symbol is needed, the map of conjugation will be denoted $C : \mathbb{O} \to \mathbb{O}$. The identity $\boldsymbol{x}\,\overline{\boldsymbol{x}} = |\boldsymbol{x}|^2$ holds, as well as the conjugation identity $\overline{\boldsymbol{xy}} = \overline{\boldsymbol{y}}\,\overline{\boldsymbol{x}}$. In particular, this implies the useful identities $C\,L_{\boldsymbol{x}}\,C = R_{\overline{\boldsymbol{x}}}$ and $C\,R_{\boldsymbol{x}}\,C = L_{\overline{\boldsymbol{x}}}$.

The algebra $\mathbb{O}$ is not commutative or associative. However, any subalgebra of $\mathbb{O}$ that is generated by two elements is associative. It follows that $\boldsymbol{x}\,\big(\overline{\boldsymbol{x}}\boldsymbol{y}\big) = |\boldsymbol{x}|^2\,\boldsymbol{y}$ and that $(\boldsymbol{xy})\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{yx})$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{O}$. Thus, $R_{\boldsymbol{x}}\,L_{\boldsymbol{x}} = L_{\boldsymbol{x}}\,R_{\boldsymbol{x}}$ (though, of course, $R_{\boldsymbol{x}}\,L_{\boldsymbol{y}} \neq L_{\boldsymbol{y}}\,R_{\boldsymbol{x}}$ in general). In particular, the expression $\boldsymbol{xyx}$ is unambiguously defined. In addition, there are the *Moufang Identities*

$$
\begin{aligned}
(\boldsymbol{xyx})\boldsymbol{z} &= \boldsymbol{x}\big(\boldsymbol{y}(\boldsymbol{xz})\big),\\
\boldsymbol{z}(\boldsymbol{xyx}) &= \big((\boldsymbol{zx})\boldsymbol{y}\big)\boldsymbol{x},\\
\boldsymbol{x}(\boldsymbol{yz})\boldsymbol{x} &= (\boldsymbol{xy})(\boldsymbol{zx}),
\end{aligned} \tag{35}
$$

which will be useful below.

2.6.2. Spin(8). — For $\boldsymbol{x} \in \mathbb{O}$, define the linear map $m_{\boldsymbol{x}} : \mathbb{O} \oplus \mathbb{O} \to \mathbb{O} \oplus \mathbb{O}$ by the formula

$$
m_{\boldsymbol{x}} = \begin{bmatrix} 0 & C\,R_{\boldsymbol{x}} \\ -C\,L_{\boldsymbol{x}} & 0 \end{bmatrix}. \tag{36}
$$

By the above identities, it follows that $(m_{\boldsymbol{x}})^2 = -|\boldsymbol{x}|^2$ and hence this map induces a representation on the vector space $\mathbb{O} \oplus \mathbb{O}$ of the Clifford algebra generated by $\mathbb{O}$ with its standard quadratic form. This Clifford algebra is known to be isomorphic to $M_{16}(\mathbb{R})$, the algebra of 16-by-16 matrices with real entries, so this representation must be faithful. By dimension count, this establishes the isomorphism $\mathrm{C}\ell\big(\mathbb{O}, \langle , \rangle\big) = \mathrm{End}_{\mathbb{R}}\big(\mathbb{O} \oplus \mathbb{O}\big)$.

The group $\mathrm{Spin}(8) \subset \mathrm{GL}_{\mathbb{R}}(\mathbb{O}\oplus\mathbb{O})$ is defined as the subgroup generated by products of the form $m_{\boldsymbol{x}}\,m_{\boldsymbol{y}}$ where $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{O}$ satisfy $|\boldsymbol{x}| = |\boldsymbol{y}| = 1$. Such endomorphisms preserve the splitting of $\mathbb{O} \oplus \mathbb{O}$ into the two given summands since

$$
m_{\boldsymbol{x}}\,m_{\boldsymbol{y}} = \begin{bmatrix} -L_{\overline{\boldsymbol{x}}}\,L_{\boldsymbol{y}} & 0 \\ 0 & -R_{\overline{\boldsymbol{x}}}\,R_{\boldsymbol{y}} \end{bmatrix}. \tag{37}
$$

In fact, setting $\boldsymbol{x} = -\mathbf{1}$ in this formula shows that endomorphisms of the form

$$
\begin{bmatrix} L_{\boldsymbol{u}} & 0 \\ 0 & R_{\boldsymbol{u}} \end{bmatrix}, \qquad \text{with } |\boldsymbol{u}| = 1 \tag{38}
$$

lie in Spin(8). In fact, they generate Spin(8), since $m_{\boldsymbol{x}}\,m_{\boldsymbol{y}}$ is clearly a product of two of these when $|\boldsymbol{x}| = |\boldsymbol{y}| = 1$.

Fixing an identification $\mathbb{O} \simeq \mathbb{R}^8$ defines an embedding $\mathrm{Spin}(8) \subset \mathrm{SO}(8) \times \mathrm{SO}(8)$, and the projections onto either of the factors is a group homomorphism. Since neither of these projections is trivial, since the Lie algebra $\mathfrak{so}(8)$ is simple, and since $\mathrm{SO}(8)$ is connected, it follows that each of these projections is a surjective homomorphism. Since $\mathrm{Spin}(8)$ is simply connected and since the fundamental group of $\mathrm{SO}(8)$ is $\mathbb{Z}_2$, it follows that that each of these homomorphisms is a non-trivial double cover of $\mathrm{SO}(8)$. Moreover, it follows that the subsets $\{\ L_{\boldsymbol{u}} \mid |\boldsymbol{u}| = 1\ \}$ and $\{\ R_{\boldsymbol{u}} \mid |\boldsymbol{u}| = 1\ \}$ of $\mathrm{SO}(8)$ each suffice to generate $\mathrm{SO}(8)$.

Let $H \subset \bigl(\mathrm{SO}(8)\bigr)^3$ be the set of triples $(g_1, g_2, g_3) \in \bigl(\mathrm{SO}(8)\bigr)^3$ for which

$$g_2(\boldsymbol{x}\boldsymbol{y}) = g_1(\boldsymbol{x})\, g_3(\boldsymbol{y}) \tag{39}$$

for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{O}$. The set H is closed and is evidently closed under multiplication and inverse. Hence it is a compact Lie group.

By the third Moufang identity, H contains the subset

$$\Sigma = \{\ (L_{\boldsymbol{u}},\, L_{\boldsymbol{u}}\, R_{\boldsymbol{u}},\, R_{\boldsymbol{u}}) \mid |\boldsymbol{u}| = 1\}\,. \tag{40}$$

Let $K \subset H$ be the subgroup generated by Σ, and for $i = 1, 2, 3$, let $\rho_i : H \to \mathrm{SO}(8)$ be the homomorphism that is projection onto the i-th factor. Since $\rho_1(K)$ contains $\{\ L_{\boldsymbol{u}} \mid |\boldsymbol{u}| = 1\ \}$, it follows that $\rho_1(K) = \mathrm{SO}(8)$, so, *a fortiori*, $\rho_1(H) = \mathrm{SO}(8)$. Similarly, $\rho_3(H) = \mathrm{SO}(8)$.

The kernel of ρ_1 consists of elements (I_8, g_2, g_3) that satisfy $g_2(\boldsymbol{x}\boldsymbol{y}) = \boldsymbol{x}\, g_3(\boldsymbol{y})$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{O}$. Setting $\boldsymbol{x} = \boldsymbol{1}$ in this equation yields $g_2 = g_3$, so that $g_2(\boldsymbol{x}\boldsymbol{y}) = \boldsymbol{x}\, g_2(\boldsymbol{y})$. Setting $\boldsymbol{y} = \boldsymbol{1}$ in this equation yields $g_2(\boldsymbol{x}) = \boldsymbol{x}\, g_2(\boldsymbol{1})$, i.e., $g_2 = R_{\boldsymbol{u}}$ for $\boldsymbol{u} = g_2(\boldsymbol{1})$. Thus, the elements in the kernel of ρ_1 are of the form $(1, R_{\boldsymbol{u}}, R_{\boldsymbol{u}})$ for some $\boldsymbol{u}$ with $|\boldsymbol{u}| = 1$. However, any such $\boldsymbol{u}$ would, by definition, satisfy $(\boldsymbol{x}\boldsymbol{y})\boldsymbol{u} = \boldsymbol{x}(\boldsymbol{y}\boldsymbol{u})$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{O}$, which is impossible unless $\boldsymbol{u} = \pm\boldsymbol{1}$. Thus, the kernel of ρ_1 is $\bigl\{(I_8, \pm I_8, \pm I_8)\bigr\} \simeq \mathbb{Z}_2$, so that ρ_1 is a 2-to-1 homomorphism of H onto $\mathrm{SO}(8)$. Similarly, ρ_3 is a 2-to-1 homomorphism of H onto $\mathrm{SO}(8)$, with kernel $\bigl\{(\pm I_8, \pm I_8, I_8)\bigr\}$. Thus, H is either connected and isomorphic to $\mathrm{Spin}(8)$ or else disconnected, with two components.

Now K is a connected subgroup of H and the kernel of ρ_1 intersected with K is either trivial or $\mathbb{Z}_2$. Moreover, the product homomorphism $\rho_1{\times}\rho_3 : K \to \mathrm{SO}(8){\times}\,\mathrm{SO}(8)$ maps the generator $\Sigma \subset K$ into generators of $\mathrm{Spin}(8) \subset \mathrm{SO}(8){\times}\,\mathrm{SO}(8)$. It follows that $\rho_1{\times}\rho_3(K) = \mathrm{Spin}(8)$ and hence that ρ_1 and ρ_3 must be non-trivial double covers of $\mathrm{Spin}(8)$ when restricted to K. In particular, it follows that K must be all of H and, moreover, that the homomorphism $\rho_1{\times}\rho_3 : H \to \mathrm{Spin}(8)$ must be an isomorphism. It also follows that the homomorphism $\rho_2 : H \to \mathrm{SO}(8)$ must be a double cover of $\mathrm{SO}(8)$ as well.

Henceforth, H will be identified with $\mathrm{Spin}(8)$ via the isomorphism $\rho_1{\times}\rho_3$. Note that the center of H consists of the elements $(\varepsilon_1\, I_8, \varepsilon_2\, I_8, \varepsilon_3\, I_8)$ where ${\varepsilon_i}^2 = \varepsilon_1\varepsilon_2\varepsilon_3 = 1$ and is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

2.6.2.1. Triality. — For $(g_1, g_2, g_3) \in H$, the identity $g_2(\boldsymbol{xy}) = g_1(\boldsymbol{x})\, g_3(\boldsymbol{y})$ can be conjugated, giving

$$Cg_2C(\boldsymbol{xy}) = \overline{g_2(\overline{\boldsymbol{y}}\,\overline{\boldsymbol{x}})} = \overline{g_1(\overline{\boldsymbol{y}})\, g_3(\overline{\boldsymbol{x}})} = \overline{g_3(\overline{\boldsymbol{x}})}\,\overline{g_1(\overline{\boldsymbol{y}})}. \tag{41}$$

This implies that $\big(Cg_3C, Cg_2C, Cg_1C\big)$ also lies in H. Also, replacing $\boldsymbol{x}$ by $\boldsymbol{z}\overline{\boldsymbol{y}}$ in the original formula and multiplying on the right by $\overline{g_3(\boldsymbol{y})}$ shows that

$$g_2(\boldsymbol{z})\overline{g_3(\boldsymbol{y})} = g_1(\boldsymbol{z}\overline{\boldsymbol{y}}), \tag{42}$$

implying that $\big(g_2, g_1, Cg_3C\big)$ lies in H as well. In fact, the two maps $\alpha, \beta : H \to H$ defined by

$$\alpha(g_1, g_2, g_3) = \big(Cg_3C, Cg_2C, Cg_1C\big), \quad \text{and} \quad \beta(g_1, g_2, g_3) = \big(g_2, g_1, Cg_3C\big) \tag{43}$$

are outer automorphisms (since they act nontrivially on the center of H) and generate a group of automorphisms isomorphic to S_3, the symmetric group on three letters. The automorphism $\tau = \alpha\beta$ is known as the triality automorphism.

To emphasize the group action, denote $\mathbb{O} \simeq \mathbb{R}^8$ by V_i when regarding it as a representation space of Spin(8) via the representation ρ_i. Thus, octonion multiplication induces a Spin(8)-equivariant projection

$$V_1 \otimes V_3 \longrightarrow V_2\,. \tag{44}$$

In the standard notation, it is traditional to identify V_1 with $\mathbb{S}^8_-$ and V_3 with $\mathbb{S}^8_+$ and to refer to V_2 as the 'vector representation' $\mathbb{R}^8$. Let $\rho_i' : \mathfrak{spin}(8) \to \mathfrak{so}(8)$ denote the corresponding Lie algebra homomorphisms, which are, in fact, isomorphisms. For simplicity of notation, for any $a \in \mathfrak{spin}(8)$, the element $\rho_i'(a) \in \mathfrak{so}(8)$ will be denoted by a_i when no confusion can arise.

2.6.3. Spin(10,1). — I will now go directly to the construction of Spin(10,1) and its usual spinor representation. For more detail and for justification of some of the statements, the reader can consult [**9**], although there are, of course, many classical sources for this material.

It is convenient to identify $\mathbb{C} \otimes \mathbb{O}^2$ with $\mathbb{O}^4$ explicitly via the identification

$$\boldsymbol{z} = \begin{pmatrix} \boldsymbol{x}_1 + i\,\boldsymbol{x}_2 \\ \boldsymbol{y}_1 + i\,\boldsymbol{y}_2 \end{pmatrix} = \begin{pmatrix} \boldsymbol{x}_1 \\ \boldsymbol{y}_1 \\ \boldsymbol{x}_2 \\ \boldsymbol{y}_2 \end{pmatrix}. \tag{45}$$

Via this identification, $\mathfrak{spin}(10)$ can be identified with the subspace

$$\mathfrak{spin}(10) = \left\{ \begin{pmatrix} a_1 & C\,R_{\boldsymbol{x}} & -r\,I_8 & -C\,R_{\boldsymbol{y}} \\ -C\,L_{\boldsymbol{x}} & a_3 & -C\,L_{\boldsymbol{y}} & r\,I_8 \\ r\,I_8 & C\,R_{\boldsymbol{y}} & a_1 & C\,R_{\boldsymbol{x}} \\ C\,L_{\boldsymbol{y}} & -r\,I_8 & -C\,L_{\boldsymbol{x}} & a_3 \end{pmatrix} \;\middle|\; \begin{matrix} r \in \mathbb{R}, \\ \boldsymbol{x}, \boldsymbol{y} \in \mathbb{O}, \\ a \in \mathfrak{spin}(8) \end{matrix} \right\}. \tag{46}$$

Consider the one-parameter subgroup $\boldsymbol{R} \subset \mathrm{SL}_{\mathbb{R}}(\mathbb{O}^4)$ defined by

$$\boldsymbol{R} = \left\{ \begin{pmatrix} t\, I_{16} & 0 \\ 0 & t^{-1}\, I_{16} \end{pmatrix} \,\middle|\, t \in \mathbb{R}^+ \right\}. \tag{47}$$

It has a Lie algebra $\mathfrak{r} \subset \mathfrak{sl}(\mathbb{O}^4)$. Evidently, the the subspace $[\mathfrak{spin}(10), \mathfrak{r}]$ consists of matrices of the form

$$\begin{pmatrix} 0_8 & 0_8 & r\, I_8 & C\, R_{\boldsymbol{y}} \\ 0_8 & 0_8 & C\, L_{\boldsymbol{y}} & -r\, I_8 \\ r\, I_8 & C\, R_{\boldsymbol{y}} & 0_8 & 0_8 \\ C\, L_{\boldsymbol{y}} & -r\, I_8 & 0_8 & 0_8 \end{pmatrix}, \qquad r \in \mathbb{R},\ \boldsymbol{y} \in \mathbb{O}. \tag{48}$$

Let $\mathfrak{g} = \mathfrak{spin}(10) \oplus \mathfrak{r} \oplus [\mathfrak{spin}(10), \mathfrak{r}]$. Explicitly,

$$\mathfrak{g} = \left\{ \begin{pmatrix} a_1 + x\, I_8 & C\, R_{\boldsymbol{x}} & y\, I_8 & C\, R_{\boldsymbol{y}} \\ -C\, L_{\boldsymbol{x}} & a_3 + x\, I_8 & C\, L_{\boldsymbol{y}} & -y\, I_8 \\ z\, I_8 & C\, R_{\boldsymbol{z}} & a_1 - x\, I_8 & C\, R_{\boldsymbol{x}} \\ C\, L_{\boldsymbol{z}} & -z\, I_8 & -C\, L_{\boldsymbol{x}} & a_3 - x\, I_8 \end{pmatrix} \,\middle|\, \begin{matrix} x, y, z \in \mathbb{R}, \\ \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathbb{O}, \\ a \in \mathfrak{spin}(8) \end{matrix} \right\}. \tag{49}$$

One can show that $\mathfrak{g}$ is isomorphic to $\mathfrak{so}(10,1)$ and hence is the Lie algebra of a representation of $\mathrm{Spin}(10,1)$. It is not hard to argue that this representation on $\mathbb{O}^4 \simeq \mathbb{R}^{32}$ must be equivalent to the representation $\mathbb{S}^{10,1}$.

Thus, define $\mathrm{Spin}(10,1)$ to be the (connected) subgroup of $\mathrm{SL}_{\mathbb{R}}(\mathbb{O}^4)$ that is generated by $\mathrm{Spin}(10)$ and the subgroup $\boldsymbol{R}$. Its Lie algebra $\mathfrak{g}$ will henceforth be written as $\mathfrak{spin}(10,1)$.

Consider the polynomial

$$p(\boldsymbol{z}) = |\boldsymbol{x}_1|^2 |\boldsymbol{x}_2|^2 + |\boldsymbol{y}_1|^2 |\boldsymbol{y}_2|^2 - (\boldsymbol{x}_1 \cdot \boldsymbol{x}_2 + \boldsymbol{y}_1 \cdot \boldsymbol{y}_2)^2 + 2\, (\boldsymbol{x}_1 \boldsymbol{y}_1) \cdot (\boldsymbol{x}_2 \boldsymbol{y}_2)\,. \tag{50}$$

It is not difficult to show that p is nonnegative and is also invariant under the action of $\mathrm{Spin}(10,1)$. Moreover, the orbits of $\mathrm{Spin}(10,1)$ are the positive level sets of this polynomial and the zero level set minus the origin. The positive level sets are smooth and have dimension 31, while the zero level set is smooth away from the origin and has dimension 25.

In fact, p has the following interpretation: Consider the squaring map $\sigma : \mathbb{O}^4 \to \mathbb{R}^{2,1} \oplus \mathbb{O} = \mathbb{R}^{10,1}$ that takes spinors for $\mathrm{Spin}(10,1)$ to vectors. This map σ is defined as follows:

$$\sigma\left(\begin{pmatrix} \boldsymbol{x}_1 \\ \boldsymbol{y}_1 \\ \boldsymbol{x}_2 \\ \boldsymbol{y}_2 \end{pmatrix} \right) = \begin{pmatrix} |\boldsymbol{x}_1|^2 + |\boldsymbol{y}_1|^2 \\ 2\, (\boldsymbol{x}_1 \cdot \boldsymbol{x}_2 - \boldsymbol{y}_1 \cdot \boldsymbol{y}_2) \\ |\boldsymbol{x}_2|^2 + |\boldsymbol{y}_2|^2 \\ 2\, (\boldsymbol{x}_1\, \boldsymbol{y}_2 + \boldsymbol{x}_2\, \boldsymbol{y}_1) \end{pmatrix}. \tag{51}$$

Define the inner product on vectors in $\mathbb{R}^{2,1} \oplus \mathbb{O} = \mathbb{R}^{10,1}$ by the rule

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ \boldsymbol{x} \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \boldsymbol{y} \end{pmatrix} = -2(a_1 b_3 + a_3 b_1) + a_2 b_2 + \boldsymbol{x} \cdot \boldsymbol{y} \tag{52}$$

and let $\mathrm{SO}(10,1)$ denote the subgroup of $\mathrm{SL}(\mathbb{R}^{2,1} \oplus \mathbb{O})$ that preserves this inner product. This group still has two components of course, but only the identity component $\mathrm{SO}^{\uparrow}(10,1)$ will be of interest here. Let $\rho : \mathrm{Spin}(10,1) \to \mathrm{SO}^{\uparrow}(10,1)$ be the homomorphism whose induced map on Lie algebras is given by the isomorphism

$$\rho'\left(\begin{pmatrix} a_1 + x\, I_8 & C\, R_{\boldsymbol{x}} & y\, I_8 & C\, R_{\boldsymbol{y}} \\ -C\, L_{\boldsymbol{x}} & a_3 + x\, I_8 & C\, L_{\boldsymbol{y}} & -y\, I_8 \\ z\, I_8 & C\, R_{\boldsymbol{z}} & a_1 - x\, I_8 & C\, R_{\boldsymbol{x}} \\ C\, L_{\boldsymbol{z}} & -z\, I_8 & -C\, L_{\boldsymbol{x}} & a_3 - x\, I_8 \end{pmatrix}\right) = \begin{pmatrix} 2x & y & 0 & \overline{\boldsymbol{y}}^* \\ 2z & 0 & 2y & 2\,\overline{\boldsymbol{x}}^* \\ 0 & z & -2x & \overline{\boldsymbol{z}}^* \\ 2\,\overline{\boldsymbol{z}} & -2\,\overline{\boldsymbol{x}} & 2\,\overline{\boldsymbol{y}} & a_2 \end{pmatrix}. \tag{53}$$

The map σ has the equivariance $\sigma\big(g\,\boldsymbol{z}\big) = \rho(g)\,\big(\sigma(\boldsymbol{z})\big)$ for $g \in \mathrm{Spin}(10,1)$ and $\boldsymbol{z} \in \mathbb{O}^4$.

With these definitions, the polynomial p has the expression $p(\boldsymbol{z}) = -\frac{1}{4}\,\sigma(\boldsymbol{z}) \cdot \sigma(\boldsymbol{z})$, from which its invariance is immediate. Moreover, it follows from this that σ carries the orbits of $\mathrm{Spin}(10,1)$ to the orbits of $\mathrm{SO}^{\uparrow}(10,1)$ and that the image of σ is the union of the origin, the forward light cone, and the future-directed time-like vectors.

In particular, a spinor $\boldsymbol{z}$ that satisfies $p(\boldsymbol{z}) > 0$ defines a non-zero time-like vector $\sigma(\boldsymbol{z}) \in \mathbb{R}^{10,1}$. Using this fact, it follows without difficulty that the stabilizer of such a $\boldsymbol{z}$ is a conjugate of $\mathrm{SU}(5) \subset \mathrm{Spin}(10) \subset \mathrm{Spin}(10,1)$. On the other hand, the Lie algebra $\mathfrak{h}$ of the stabilizer for the null spinor

$$\boldsymbol{z}_0 = \begin{pmatrix} \mathbf{1} \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \text{is} \qquad \mathfrak{h} = \left\{ \begin{pmatrix} a_1 & 0 & y\, I_8 & C\, R_{\boldsymbol{y}} \\ 0 & a_3 & C\, L_{\boldsymbol{y}} & -y\, I_8 \\ 0 & 0 & a_1 & 0 \\ 0 & 0 & 0 & a_3 \end{pmatrix} \,\middle|\, \begin{matrix} y \in \mathbb{R}, \\ \boldsymbol{y} \in \mathbb{O}, \\ a \in \mathfrak{k}_1 \end{matrix} \right\}, \tag{54}$$

where $\mathfrak{k}_1$ is the Lie algebra of $K_1 \subset \mathrm{Spin}(8)$. Thus, the stabilizer is a semi-direct product of $\mathrm{Spin}(7)$ with a copy of $\mathbb{R}^9$, and so has dimension $30 = 55 - 25$, as desired.

In conclusion, there are essentially two distinct types of $\mathrm{Spin}(10,1)$ orbits in $\mathbb{S}^{10,1}$, those of the positive level sets of p and the nonzero elements in the zero level set of p.

3. Metrics with Parallel Spinor Fields

In this section, I will describe some of the normal forms and methods for obtaining them for metrics that have parallel spinor fields.

3.1. Dimension 3. — As a warmup, consider the case of metrics in dimension 3.

3.1.1. Type $(3,0)$. — Recall that $\mathrm{Spin}(3) \simeq \mathrm{Sp}(1)$, with $\mathbb{S}^{3,0} \simeq \mathbb{H}$. Thus, the $\mathrm{Spin}(3)$-stabilizer of any nonzero element of $\mathbb{S}^{3,0}$ is trivial. Consequently, if (M^3, g) has a nonzero parallel spinor field, its holonomy is trivial and the metric is flat.

3.1.2. Type $(2,1)$. — Since $\mathrm{Spin}(2,1)$ is isomorphic to $\mathrm{SL}(2,\mathbb{R})$, with $\mathbb{S}^{2,1} \simeq \mathbb{R}^2$, all of the nonzero spinors constitute a single orbit. In particular, the stabilizers of these are all conjugate to the one-dimensional unipotent upper triangular matrices in $\mathrm{SL}(2,\mathbb{R})$. Thus, take the structure equations for coframes $\omega_{ij} = \omega_{ji}$ so that

$$g = \omega_{11}\,\omega_2 - \omega_{21}\,\omega_{12} = \omega_{11}\,\omega_2 - {\omega_{21}}^2 \tag{55}$$

to have the form

$$d\begin{pmatrix}\omega_{11} & \omega_{12}\\ \omega_{21} & \omega_{22}\end{pmatrix} = -\begin{pmatrix}0 & \alpha\\ 0 & 0\end{pmatrix} \wedge \begin{pmatrix}\omega_{11} & \omega_{12}\\ \omega_{21} & \omega_{22}\end{pmatrix} + \begin{pmatrix}\omega_{11} & \omega_{12}\\ \omega_{21} & \omega_{22}\end{pmatrix} \wedge \begin{pmatrix}0 & 0\\ \alpha & 0\end{pmatrix}. \tag{56}$$

Since $d\omega_{22} = 0$, I can write $\omega_{22} = dx_{22}$ for some function x_{22}. Since $d\omega_{21} = \omega_{22}{\wedge}\alpha$, there exists locally a coordinate x_{21} so that $\omega_{21} = dx_{21} - p\,dx_{22}$. This makes $\alpha = dp + q\,dx_{22}$ for some function q. Reducing frames to make $p = 0$ (which can clearly be done) makes $\alpha = q\,dx_{22}$ and

$$d\omega_{11} = -2\,\alpha \wedge \omega_{21} = 2\,q\,dx_{21} \wedge dx_{22}\,, \tag{57}$$

so that there must be a function f on an open set in $\mathbb{R}^2$ so that

$$2\,q\,dx_{21} \wedge dx_{22} = d\big(f(x_{21}, x_{22})\,dx_{22}\big). \tag{58}$$

Thus, there is an $\mathbb{R}$-valued coordinate x_{11} so that $\omega_{11} = dx_{11} + f(x_{21}, x_{22})\,dx_{22}$. In particular, the metric g is locally of the form

$$g = dx_{11}{\circ}dx_{22} - dx_{21}{\circ}dx_{12} + f(x_{21}, x_{22})\,(dx_{2\bar{2}})^2. \tag{59}$$

Conversely, via this formula, any function f of two variables will produce a $(2,1)$-metric with a parallel spinor field. Note that g will be flat if and only if the curvature 2-form

$$F = d\alpha = d\left(\frac{1}{2}\,\frac{\partial f}{\partial x_{21}}\;dx_{22}\right) \tag{60}$$

vanishes. Of course, imposing the Einstein condition makes the curvature vanish identically.

Since the ambiguity in the choice of coordinates x_{22}, x_{21}, x_{11} involved only choosing arbitrary functions of one variable, it makes sense to say that the general metric of type $(2,1)$ that has a parallel spinor field depends on one function of two variables.

3.2. Dimension 4. — In this subsection, I will review the well-known classification of pseudo-Riemannian metrics with parallel spinors in dimension 4.

3.2.1. Type $(4,0)$. — Since $\mathrm{Spin}(4) \simeq \mathrm{Sp}(1) \times \mathrm{Sp}(1)$ and there are only two orbit types (up to orientation), there are only two possibilities:

3.2.1.1. Generic. — If (M^4, g) has a parallel spinor of generic type, then its holonomy is a subgroup of the stabilizer of the generic type, i.e., it is trivial, so (M^4, g) is flat.

3.2.1.2. Special. — If (M^4, g) has a nonzero parallel spinor of the special type, i.e., a parallel half-spinor, this reduces its holonomy to $\mathrm{Sp}(1) \simeq \mathrm{SU}(2) \subset \mathrm{SO}(4)$. Of course, this implies that (M^4, g) can be regarded as a Ricci-flat Kähler metric (in a 2-parameter family of ways, in fact). These metrics are locally in one-to-one correspondence with solutions of the complex Monge-Ampere equation in two complex variables. This has the local generality of two functions of three variables. The solutions are all real-analytic.

3.2.2. Type $(3, 1)$. — Suppose $(M^{3,1}, g)$ has a nonzero parallel spinor. Since there is only one nonzero $\mathrm{Spin}(3, 1)$-orbit in $\mathbb{S}^{3,1} \simeq \mathbb{C}^2$, there is only one possible algebraic type of parallel spinor. I can now apply the moving frame analysis to the coframe bundle adapted to a single nonzero element in $\mathbb{S}^{3,1}$.

Since the stabilizer subgroup of a nonzero vector in $\mathbb{C}^2$ under the action of $\mathrm{SL}(2, \mathbb{C})$ is conjugate to the unipotent upper triangular matrices, take the structure equations for coframes $\omega_{i\bar{j}} = \overline{\omega_{j\bar{\imath}}}$ so that

$$g = \omega_{1\bar{1}} \circ \omega_{2\bar{2}} - \omega_{2\bar{1}} \circ \omega_{1\bar{2}} \tag{61}$$

to have the form

$$d\begin{pmatrix} \omega_{1\bar{1}} & \omega_{1\bar{2}} \\ \omega_{2\bar{1}} & \omega_{2\bar{2}} \end{pmatrix} = -\begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix} \wedge \begin{pmatrix} \omega_{1\bar{1}} & \omega_{1\bar{2}} \\ \omega_{2\bar{1}} & \omega_{2\bar{2}} \end{pmatrix} + \begin{pmatrix} \omega_{1\bar{1}} & \omega_{1\bar{2}} \\ \omega_{2\bar{1}} & \omega_{2\bar{2}} \end{pmatrix} \wedge \begin{pmatrix} 0 & 0 \\ \bar{\alpha} & 0 \end{pmatrix}. \tag{62}$$

Since $d\omega_{2\bar{2}} = 0$, write $\omega_{2\bar{2}} = dx_{2\bar{2}}$ for some $\mathbb{R}$-valued function $x_{2\bar{2}}$. Since $d\omega_{2\bar{1}} = \omega_{2\bar{2}} \wedge \bar{\alpha}$, there exists locally a $\mathbb{C}$-valued coordinate $x_{2\bar{1}}$ so that $\omega_{2\bar{1}} = dx_{2\bar{1}} - \bar{p}\, dx_{2\bar{2}}$. This forces $\alpha = dp + q\, dx_{2\bar{2}}$. Reducing frames to make $p = 0$ makes $\alpha = q\, dx_{2\bar{2}}$ and

$$d\omega_{1\bar{1}} = -\alpha \wedge \omega_{2\bar{1}} + \omega_{1\bar{2}} \wedge \bar{\alpha} = (\bar{q}\, dx_{1\bar{2}} + q\, dx_{2\bar{1}}) \wedge dx_{2\bar{2}}\,, \tag{63}$$

so that there must be an $\mathbb{R}$-valued function f on an open set in $\mathbb{C} \times \mathbb{R}$ so that

$$(\bar{q}\, dx_{1\bar{2}} + q\, dx_{2\bar{1}}) \wedge dx_{2\bar{2}} = d\big(f(x_{1\bar{2}}, x_{2\bar{2}})\, dx_{2\bar{2}}\big). \tag{64}$$

Thus, there is an $\mathbb{R}$-valued coordinate $x_{1\bar{1}}$ so that $\omega_{1\bar{1}} = dx_{1\bar{1}} + f(x_{1\bar{2}}, x_{2\bar{2}})\, dx_{2\bar{2}}$. In particular, the metric g is locally of the form

$$g = dx_{1\bar{1}} \circ dx_{2\bar{2}} - dx_{2\bar{1}} \circ dx_{1\bar{2}} + f(x_{1\bar{2}}, x_{2\bar{2}})\, (dx_{2\bar{2}})^2. \tag{65}$$

Conversely, via this formula, any function of 3 variables will produce a $(3, 1)$-metric with a parallel spinor field. Note that g will be flat if and only if the ($\mathbb{C}$-valued) curvature 2-form

$$F = d\alpha = d\left(\frac{\partial f}{\partial x_{2\bar{1}}}\; dx_{2\bar{2}}\right) \tag{66}$$

vanishes, i.e., f is linear in $x_{2\bar{1}}$ and $x_{1\bar{2}}$. Moreover, g is Ricci-flat if and only if f is harmonic in the complex variable $x_{2\bar{1}}$, which does not imply flatness.

The conclusion is that the local Ricci-flat examples with a parallel spinor field depend on two (real) functions of two (real) variables. (The coordinate ambiguity is functions of one variable.) Of course, this normal form is well-known in general relativity.

3.2.3. Type $(2,2)$. — The most interesting 4-dimensional case, from my point of view, is that of $(M^{2,2}, g)$ and the different possibilities for a parallel spinor. Recall from 2.4.3.3 that $\mathrm{Spin}(2,2)$ has one open orbit in $\mathbb{S}^{2,2}$ and two degenerate orbits, which form a single $\mathrm{Pin}(2,2)$ orbit. Thus, there are two subcases:

3.2.3.1. Generic type. — The case of a parallel spinor field in the open orbit is very much like that just treated. Take the model spinor to be

$$s = (s_+, s_-) = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right). \tag{67}$$

Then the tautological form ω takes values in $\mathbb{R}^{2,2} = \mathbb{R}(2)$ and satisfies $d\omega = -\alpha{\wedge}\omega - \omega{\wedge}\beta$ where α and β take values in the Lie algebra of the stabilizer of $s_\pm$, i.e.,

$$\alpha = \begin{pmatrix} 0 & \alpha^1_2 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad \beta = \begin{pmatrix} 0 & 0 \\ \beta^2_1 & 0 \end{pmatrix}. \tag{68}$$

The structure equations then become

$$d \begin{pmatrix} \omega^1_1 & \omega^1_2 \\ \omega^2_1 & \omega^2_2 \end{pmatrix} = - \begin{pmatrix} 0 & \alpha^1_2 \\ 0 & 0 \end{pmatrix} \wedge \begin{pmatrix} \omega^1_1 & \omega^1_2 \\ \omega^2_1 & \omega^2_2 \end{pmatrix} - \begin{pmatrix} \omega^1_1 & \omega^1_2 \\ \omega^2_1 & \omega^2_2 \end{pmatrix} \wedge \begin{pmatrix} 0 & 0 \\ \beta^2_1 & 0 \end{pmatrix}. \tag{69}$$

Thus $d\omega^2_2 = 0$, so there exists a function x^2_2, unique up to an additive constant, so that $\omega^2_2 = dx^2_2$. The equation $d\omega^2_1 = \beta^2_1{\wedge}\omega^2_2$ then implies that there exist functions x^2_1 and b on the frame bundle, with x^2_1 unique up to the addition of a function of x^2_2, so that $\omega^2_1 = dx^2_1 + b\,dx^2_2$. Similarly, there exist functions x^1_2 and a on the frame bundle, with x^1_2 unique up to the addition of a function of x^2_2, so that $\omega^1_2 = dx^1_2 - a\,dx^2_2$. Reducing frames so that $a = b = 0$ yields $\omega^1_2 = dx^1_2$ and $\omega^2_1 = dx^2_1$ and the structure equations now imply that $\beta^2_1{\wedge}dx^2_2 = \alpha^1_2{\wedge}dx^2_2 = 0$, so that there must exist functions p and q so that $\alpha^1_2 = p\,dx^2_2$ and $\beta^2_1 = -q\,dx^2_2$. The structure equation

$$d\omega^1_1 = -\alpha^1_2 \wedge \omega^2_1 + \beta^2_1 \wedge \omega^1_2 = (p\,dx^2_1 + q\,dx^1_2) \wedge dx^2_2 \tag{70}$$

now implies that there must exist functions x^1_1 and f, with x^1_1 unique up to the addition of a function of x^2_2 so that $\omega^1_1 = dx^1_1 + f\,dx^2_2$. Going back to the $d\omega^1_1$ structure equation, this implies that the function f satisfies

$$\frac{\partial f}{\partial x^1_1} = 0, \quad \frac{\partial f}{\partial x^2_1} = p, \quad \text{and} \quad \frac{\partial f}{\partial x^1_2} = q. \tag{71}$$

This analysis shows that there exist local coordinates $x^1_1, x^2_2, x^1_2, x^2_1$ and a function f on an open set in $\mathbb{R}^3$ so that

$$g = dx^1_1\,dx^2_2 - dx^2_1\,dx^1_2 + f(x^1_2, x^2_1, x^2_2)\,(dx^2_2)^2. \tag{72}$$

Moreover, these coordinates are canonical up to functions of one variable. This metric is flat if and only if the curvature forms

$$d\alpha^1_2 = d\left(\frac{\partial f}{\partial x^2_1}\right) \wedge dx^2_2 \qquad \text{and} \qquad d\beta^2_1 = -d\left(\frac{\partial f}{\partial x^1_2}\right) \wedge dx^2_2 \tag{73}$$

both vanish, which can only happen if f is linear in x^2_1 and x^1_2.

This metric is Ricci-flat if and only if f satisfies

$$\frac{\partial^2 f}{\partial x^2_1 \partial x^1_2} = 0, \tag{74}$$

so the Ricci-flat metrics with a generic parallel spinor depend on two functions of two variables.

3.2.3.2. Degenerate type. — Finally, consider the degenerate case, i.e., where the metric has a parallel spinor field whose corresponding $\mathrm{Spin}(2,2)$-orbit is 3-dimensional. Then, on the adapted frame bundle, the tautological form ω takes values in $\mathbb{R}^{2,2} = \mathbb{R}(2)$ and satisfies $d\omega = -\alpha{\wedge}\omega - \omega{\wedge}\beta$ where α and β take values in the Lie algebra of the stabilizer of s_+, i.e.,

$$\alpha = \begin{pmatrix} 0 & \alpha^1_2 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad \beta = \begin{pmatrix} \beta^1_1 & \beta^1_2 \\ \beta^2_1 & -\beta^1_1 \end{pmatrix}. \tag{75}$$

The structure equations then become

$$d\begin{pmatrix} \omega^1_1 & \omega^1_2 \\ \omega^2_1 & \omega^2_2 \end{pmatrix} = -\begin{pmatrix} 0 & \alpha^1_2 \\ 0 & 0 \end{pmatrix} \wedge \begin{pmatrix} \omega^1_1 & \omega^1_2 \\ \omega^2_1 & \omega^2_2 \end{pmatrix} - \begin{pmatrix} \omega^1_1 & \omega^1_2 \\ \omega^2_1 & \omega^2_2 \end{pmatrix} \wedge \begin{pmatrix} \beta^1_1 & \beta^1_2 \\ \beta^2_1 & -\beta^1_1 \end{pmatrix}. \tag{76}$$

This implies that the form $\omega^2_1{\wedge}\omega^2_2$ is parallel, and, in particular, closed. Thus, each point of M has an open neighborhood U on which there exist functions $x = (x_1, x_2)$ so that $\omega^2_1{\wedge}\omega^2_2 = dx_1{\wedge}dx_2$. One can then do a bundle reduction over U so that $\omega^2_1 = dx_1$ and $\omega^2_2 = dx_2$. The structure equations for $d\omega^2_i$ then imply that

$$0 = dx_1 \wedge \beta^1_1 + dx_2 \wedge \beta^2_1 = dx_1 \wedge \beta^1_2 - dx_2 \wedge \beta^1_1\,. \tag{77}$$

By Cartan's Lemma, it follows that there exist functions $q_1, \ldots, q_4$ so that

$$\begin{pmatrix} -\beta^1_2 \\ \beta^1_1 \\ \beta^2_1 \end{pmatrix} = \begin{pmatrix} q_1 & q_2 \\ q_2 & q_3 \\ q_3 & q_4 \end{pmatrix} \begin{pmatrix} dx_1 \\ dx_2 \end{pmatrix}. \tag{78}$$

Using this, it follows from the structure equations that

$$d\omega^1_1 \equiv d\omega^1_2 \equiv 0 \mod dx_1, dx_2\,. \tag{79}$$

Consequently, each point of U has an open neighborhood $V \subset U$ on which there exist functions $y = (y_1, y_2)$ for which $\omega^1_i \equiv dy_i \mod dx_1, dx_2$. Obviously, the functions (x_1, x_2, y_1, y_2) are independent on V, so by shrinking V if necessary, one can assume that they form a cubic coordinate system on V. The congruences above show that

$$g = \omega_1^1\,\omega_2^2 - \omega_2^1\,\omega_1^2 = dy_1\,dx_2 - dy_2\,dx_1 + s^{ij}(x,y)\,dx_i\,dx_j \tag{80}$$

for some functions $s^{ij} = s^{ji}$ on the range of the coordinate chart $(x,y) : V \to \mathbb{R}^4$. By a final reduction of the bundle structure over V, one can arrange

$$\omega_1^1 = dy_1 + s^{12}\,dx_1 + s^{22}\,dx_2\,, \qquad \omega_2^1 = dy_2 - s^{11}\,dx_1 - s^{12}\,dx_2\,. \tag{81}$$

On this bundle, $\alpha_2^1 = p^1\,dx_1 + p^2\,dx_2 + r^1\,dy_1 + r^2\,dy_2$ for some functions p^1, p^2, r^1, r^2.

Now, going back to the structure equations, one finds that they force

$$\frac{\partial s^{11}}{\partial y_2} = \frac{\partial s^{12}}{\partial y_1} \qquad \text{and} \qquad \frac{\partial s^{12}}{\partial y_2} = \frac{\partial s^{22}}{\partial y_1}\,, \tag{82}$$

implying that there must be a function f on the hypercube $(x,y)(V) \subset \mathbb{R}^4$ so that

$$s^{ij} = \frac{\partial^2 f}{\partial y_i \partial y_j}\,. \tag{83}$$

Conversely, given any smooth function f on a domain $D \subset \mathbb{R}^4$, one can define s^{ij} by the above formulae and then the structure equations above can be solved uniquely for the quantities p, q and r (it turns out that $r \equiv 0$ anyway). Consequently, the metric

$$g = dy_2\,dx_1 - dy_1\,dx_2 + \frac{\partial^2 f}{\partial y_i \partial y_j}(x,y)\,dx^i\,dx^j \tag{84}$$

always has a parallel spinor field of degenerate type. Thus, these metrics depend on one arbitrary function of four variables. (The ambiguities in the choice of coordinates are easily seen to depend on three functions of two variables.) By examining the curvature of this metric for 'generic' f, one sees that the generic such metric does not have more than one parallel spinor field. In fact, the holonomy group of the generic example is equal to the full stabilizer of a degenerate spinor, the maximum possible.

Now, about the Einstein equations: Using the derived formulae for β_j^i and α_2^1, one computes that

$$d\alpha_2^1 = S(f)\,dx_1 \wedge dx_2 + R^{ij}(f)\,dy_i \wedge dx_j \tag{85}$$

for certain fourth order differential operators S and $R^{ij} = R^{ji}$ $(1 \le i,j \le 2)$. The Ricci tensor of g turns out (apart from an overall constant factor) to be

$$\mathrm{Ric}(g) = R^{ij}(f)\,dx_i\,dx_j\,. \tag{86}$$

Thus, the metric is Ricci-flat if and only if f satisfies a system of three fourth order quasilinear PDE. Although I will not give details here (anyway, a more interesting example of this sort of calculation will be presented later during the 7-dimensional discussion), this system turns out to be involutive, with the general solution depending on two arbitrary functions of three variables, *the same generality as in the positive definite case.* Moreover, the generic Ricci-flat $(2,2)$-metric with a degenerate parallel spinor field has holonomy equal to the full stabilizer of a degenerate spinor, again, the maximum possible.

3.3. Dimension 5. — Now we move into slightly less familiar territory.

3.3.1. Type $(5,0)$. — In the Riemannian case, $\mathrm{Spin}(5) = \mathrm{Sp}(2)$ acts transitively on the unit sphere in $\mathbb{S}^{5,0} \simeq \mathbb{H}^2$, so there is only one kind of spinor, having stabilizer subgroup $\mathrm{Sp}(1)$. This $\mathrm{Sp}(1)$ maps into $\mathrm{SO}(5)$ faithfully and so lies in a copy of $\mathrm{SO}(4) \subset \mathrm{SO}(5)$. Thus, a Riemannian 5-manifold with a parallel spinor is locally the product of the metric on a line and a Ricci-flat Kähler metric, which reduces our problem to the 4-dimensional case.

3.3.2. Type $(4,1)$. — This case is considerably more interesting. Now, $\mathrm{Spin}(4,1) = \mathrm{Sp}(1,1)$ acts transitively on the level sets of the spinor 'norm' $\nu(s) = s^*Qs$ in $\mathbb{S}^{4,1} \simeq \mathbb{H}^{1,1}$ minus $0 \in \mathbb{H}^{1,1}$. Thus, as explained earlier, there are two essentially different kinds of orbits: The first corresponding to the nonzero level sets of ν, and the second corresponding to the zero level sets of ν.

3.3.2.1. Generic type. — If the parallel spinor field has nonzero spinor norm, then it corresponds to a spinor in $\mathbb{S}^{4,1}$ whose stabilizer subgroup is $\mathrm{Sp}(1)$. Looking at the spinor squaring map, this $\mathrm{Sp}(1)$ maps into $\mathrm{SO}(4,1)$ faithfully and so lies in a copy of $\mathrm{SO}(4) \subset \mathrm{SO}(4,1)$. Thus a metric g of this type is locally of the form $g = -dt^2 + \bar{g}$, where $\bar{g}$ is a Ricci-flat Kähler metric on a 4-manifold, which again reduces our problem to the 4-dimensional case.

3.3.2.2. Degenerate type. — If the parallel spinor field has vanishing spinor norm, then it corresponds to a spinor in $\mathbb{S}^{4,1}$ whose stabilizer subgroup is $G_0 \simeq \mathbb{R}^3$. I can now apply the moving frame analysis to the coframe bundle adapted to a such a spinor, which can be assumed to be s_0, as defined in §2.4.4.2.

Since the stabilizer subgroup of s_0 is G_0, take the structure equations for coframes

$$\omega = \begin{pmatrix} \omega_1 & \omega_2 \\ \overline{\omega_2} & \omega_1 \end{pmatrix} \quad \text{where } \omega_1 = \overline{\omega_1}\,, \text{ and} \quad \alpha = \begin{pmatrix} \phi & -\phi \\ \phi & -\phi \end{pmatrix} \quad \text{where } \phi = -\overline{\phi}\,, \tag{87}$$

with $d\omega = -\alpha{\wedge}\omega + \omega{\wedge}\alpha^*$. It simplifies the calculations to set $\omega_1 = \rho+\xi$ and $\omega_2 = \rho+\sigma$ where ρ and ξ are $\mathbb{R}$-valued while σ is $\mathrm{Im}\,\mathbb{H}$-valued. Then the structure equations are expressed as

$$d\xi = 0, \qquad d\sigma = -2\xi\wedge\phi, \qquad d\rho = -\phi\wedge\sigma + \sigma\wedge\phi\,. \tag{88}$$

Now, by the first equation, there must exist a local coordinate x, unique up to an additive constant, so that $\xi = dx$. By the second equation $d\sigma = 2\phi{\wedge}dx$, so, locally, there exist functions s and h with values in $\mathrm{Im}\,\mathbb{H}$ so that $\sigma = ds + 2h\,dx$. The function s is unique up to the addition of an $\mathrm{Im}\,\mathbb{H}$-valued function of x. The second equation now implies that $\phi = dh + p\,dx$ for some unique $\mathrm{Im}\,\mathbb{H}$-valued function p. Now reduce frames to make $h = 0$ (which can clearly be done). Then the structure equations so far say that $\xi = dx$, $\sigma = ds$, and $\phi = p\,dx$. The third structure equation now reads

$$d\rho = -\phi \wedge \sigma + \sigma \wedge \phi = (p\,ds + ds\,p) \wedge dx \tag{89}$$

from which it follows that there exist $\mathbb{R}$-valued functions r and f so that $\rho = dr + f\,dx$, where r is unique up to the addition of a function of x. The third structure equation then further implies that

$$df \equiv p\,ds + ds\,p \mod dx, \tag{90}$$

so that f is a function of x and s (and, moreover, that p is essentially one-half the gradient of f in the s variables).

Thus, the calculations so far have shown that any metric of type $(4,1)$ with a null parallel spinor field has local coordinate charts $(x,s,r) : U \to \mathbb{R} \times \operatorname{Im}\mathbb{H} \times \mathbb{R}$ in which the metric can be written in the form

$$g = d\bar{s}\,ds - 2dr\,dx - (1 + 2f(x,s))\,dx^2 \tag{91}$$

where f is an arbitrary function of four variables. Conversely, for any sufficiently differentiable function f of four variables, the above formula defines a metric that has a parallel null spinor field, since, setting $\xi = dx$, $\sigma = ds$, $\rho = dr + f\,dx$, the structure equations above will be satisfied by taking $\phi = p\,dx$ where p is the unique solution of the equation $df \equiv p\,ds + ds\,p \mod dx$.

Since coordinate charts of the above form are determined by the metric up to a choice of functions of one variable, the type $(4,1)$ metrics possessing a parallel null spinor field depend on one arbitrary function of four variables.

The metric g will be flat if and only if the connection form $\phi = p\,dx$ is closed, which is the same thing as saying that f is linear in s. Computation shows that the Ricci curvature of g vanishes if and only if f is harmonic in the s-variables. Consequently, the Ricci-flat metrics of this type depend on two functions of three variables up to diffeomorphism, *exactly as in the positive definite case.*

3.3.3. Type $(3,2)$. — Since $\mathrm{Spin}(3,2) \simeq \mathrm{Sp}(2,\mathbb{R})$ with $\mathbb{S}^{3,2} \simeq \mathbb{R}^4$, the standard representation of $\mathrm{Sp}(2,\mathbb{R})$, it follows that all of the nonzero elements of $\mathbb{S}^{3,2}$ belong to a single $\mathrm{Spin}(3,2)$-orbit. Thus, there is only one type of parallel spinor for $(3,2)$-metrics. Since this is a 'split' case, this orbit must be the pure spinor orbit. Consequently, this case is treated in §3.5.1, so I will not consider it further here.

3.4. Dimension 6. — In this section, I will describe the less well-known classification of metrics with parallel spinors in dimension 6 and types $(6,0)$, $(5,1)$, and $(3,3)$.

3.4.1. Type $(6,0)$. — In the Riemannian case, $\mathrm{Spin}(6) = \mathrm{SU}(4)$ acts transitively on the unit sphere in $\mathbb{S}^{6,0} \simeq \mathbb{C}^4$, so there is only one kind of spinor, having stabilizer subgroup $\mathrm{SU}(3)$. This $\mathrm{SU}(3)$ maps into $\mathrm{SO}(6)$ as the standard representation, so a

Riemannian 6-manifold with a parallel spinor is a Ricci-flat Kähler manifold. As is well-known, these are determined by a convex solution of the complex Monge-Ampere equation and so depend on two functions of five variables.

3.4.2. Type $(5,1)$. — In the Lorentzian case, $\mathrm{Spin}(5,1) = \mathrm{SL}(2,\mathbb{H})$ acting on $\mathbb{S}^{5,1} \simeq \mathbb{H}^2 \oplus \mathbb{H}^2$, has several types of orbits, as laid out in §2.4.5.2. Each of these will be treated in turn.

3.4.2.1. Generic type. — Suppose that the metric has a parallel spinor field whose associated orbit in $\mathbb{S}^{5,1}$ has dimension 12. Then the stabilizer of an element of this orbit is isomorphic to $\mathrm{Sp}(1)$ and is hence compact. Moreover, examining the vector representation of $\mathrm{Spin}(5,1)$ on $\mathbb{R}^{5,1}$, one sees that this $\mathrm{Sp}(1)$ gets mapped into a copy of an $\mathrm{SU}(2) \subset \mathrm{SO}(4)$ fixing an orthogonal 2-plane of type $(1,1)$. It follows from the generalized de Rham splitting theorem then that the metric is a local product of flat $R^{1,1}$ with a 4-dimensional Ricci-flat Kähler metric.

3.4.2.2. Null type. — Suppose next that the metric has a parallel spinor field whose associated orbit in $\mathbb{S}^{5,1}$ is the 11-dimensional null orbit. This case is more interesting. The stabilizer is now four dimensional and abelian, as was described in §2.4.5.2.

This case is formally very much like the cases treated in §3.1.2, §3.2.2, and §3.3.2.2, so I will not go into details, but just give the results.

One shows that a $(5,1)$-metric with a parallel spinor field of this type always has local coordinates $x = (x_{1\bar{1}}, x_{1\bar{2}}, x_{2\bar{2}}) : U \to \mathbb{R} \times \mathbb{H} \times \mathbb{R})$ in which the metric can be written in the form

$$g = -dx_{1\bar{1}}\, dx_{2\bar{2}} + |dx_{1\bar{2}}|^2 - g(x_{1\bar{2}}, x_{2\bar{2}})\, {dx_{2\bar{2}}}^2 \tag{92}$$

where g is a smooth function on the open set $(x_{1\bar{2}}, x_{2\bar{2}})(U) \subset \mathbb{H}\times\mathbb{R}$. These coordinates are unique up to a choice of arbitrary functions of one variable. Thus, metrics of this type depend on one arbitrary function of five variables.

The Ricci tensor of such a metric vanishes if and only if g is harmonic in the $x_{1\bar{2}}$ variables. Thus, the Ricci-flat metrics of this kind depend locally on two arbitrary functions of four variables.

3.4.2.3. Degenerate type. — Finally, suppose that the metric has a parallel spinor field whose associated orbit in $\mathbb{S}^{5,1}$ is one of the two 7-dimensional degenerate orbits, i.e., the spinor field is either of positive chirality or negative chirality. By switching orientations, it can be assumed that the spinor is of positive chirality, so I will do this for the rest of the discussion.

Suppose, then, that $(M^{5,1}, g)$ is a pseudo-Riemannian manifold with a degenerate, positive chirality parallel spinor field. The structure equations of the adapted coframe bundle in this case, where $\omega = \omega^*$ takes values in quaternion Hermitian 2-by-2 matrices and α takes values in the Lie algebra of the stabilizer of the standard first basis element of $\mathbb{S}_+^{5,1} = \mathbb{H}^2$ are $d\omega = -\alpha{\wedge}\omega + \omega{\wedge}\alpha^*$, where

$$\omega = \begin{pmatrix} \omega_{1\bar{1}} & \omega_{1\bar{2}} \\ \omega_{2\bar{1}} & \omega_{2\bar{2}} \end{pmatrix} \qquad \text{and} \qquad \alpha = \begin{pmatrix} 0 & \alpha_2^1 \\ 0 & \alpha_2^2 \end{pmatrix}. \tag{93}$$

and α_2^2 takes values in $\operatorname{Im}\mathbb{H}$.

It follows from the structure equations $\omega_{2\bar{2}}$ is well-defined on the manifold and is a parallel null 1-form. In particular, it is closed, so that, locally, one can introduce a $\mathbb{R}$-valued function $x_{2\bar{2}}$, unique up to an additive constant, so that $\omega_{2\bar{2}} = dx_{2\bar{2}}$.

The g-dual vector field (also null) will be denoted $E_{1\bar{1}}$. The structure equations then give

$$d\omega_{1\bar{2}} = -\alpha_2^1 \wedge dx_{2\bar{2}} - \omega_{1\bar{2}} \wedge \alpha_2^2\,. \tag{94}$$

This equation has an interesting interpretation. It says that, on each (5-dimensional) leaf of $dx_{2\bar{2}} = 0$, the metric g pulls back to be the positive semidefinite quadratic form $|\omega_{1\bar{2}}|^2$ and that this restricted quadratic form is constant along its null curves, i.e., the integral curves of $E_{1\bar{1}}$. Thus, this quadratic form is well-defined on the quotient of the leaf by the (parallel) family of null geodesics defined by $E_{1\bar{1}}$. Moreover, the quotient metric on each leaf has holonomy $\mathrm{Sp}(1) \subset \mathrm{SO}(4)$, i.e., it defines a Ricci-flat Kähler structure on the 4-dimensional quotient space of each $dx_{2\bar{2}}$-leaf. Geometrically, if one considers the quotient $\bar{M}$ by the $E_{1\bar{1}}$ curves, it locally fibers over $\mathbb{R}$ canonically (up to translation) in the form $x_{2\bar{2}} : \bar{M} \to \mathbb{R}$ where the 4-dimensional fibers are Ricci-flat Kähler manifolds.

Pursuing the structure equations further, the equation

$$d\omega_{1\bar{1}} = -\alpha_2^1 \wedge \omega_{2\bar{1}} + \omega_{1\bar{2}} \wedge \overline{\alpha_2^1} \equiv 0 \mod \omega_{2\bar{1}}, \omega_{1\bar{2}}, dx_{2\bar{2}} \tag{95}$$

implies that there exists a function $x_{1\bar{1}}$, locally defined on M so that $\omega_{1\bar{1}} \equiv dx_{1\bar{1}} \mod \omega_{2\bar{1}}, \omega_{1\bar{2}}, dx_{2\bar{2}}$. This function is unique up to the addition of a function constant along the integral curves of $E_{1\bar{1}}$, i.e., a function on $\bar{M}$ (i.e., a function of five variables).

Once $x_{1\bar{1}}$ has been chosen, there is a unique reduction of the structure bundle for which $\omega_{1\bar{1}} = dx_{1\bar{1}} + f\,dx_{2\bar{2}}$ for some $\mathbb{R}$-valued function f. This implies

$$-\alpha_2^1 \wedge \omega_{2\bar{1}} + \omega_{1\bar{2}} \wedge \overline{\alpha_2^1} \equiv d\omega_{1\bar{1}} \equiv 0 \mod dx_{2\bar{2}}. \tag{96}$$

In particular, this implies that $\alpha_2^1 \equiv 0 \mod \omega_{2\bar{1}}, \omega_{1\bar{2}}, dx_{2\bar{2}}$, so that

$$df \wedge dx_{2\bar{2}} = -\alpha_2^1 \wedge \omega_{2\bar{1}} + \omega_{1\bar{2}} \wedge \overline{\alpha_2^1} \tag{97}$$

implies that $df \equiv 0 \mod \omega_{2\bar{1}}, \omega_{1\bar{2}}, dx_{2\bar{2}}$, i.e., that f is constant along the $E_{1\bar{1}}$ curves and so is a function on $\bar{M}$.

Conversely, starting with a 1-parameter family of Ricci-flat manifolds $x : \bar{M} \to \mathbb{R}$, one can attempt to reconstruct a $(5,1)$ metric as follows: Locally, choose an $\mathbb{H}$-valued 1-form η on $\bar{M}$ so that, on each x-fiber, it is a section of the associated $\mathrm{SU}(2) = \mathrm{Sp}(1)$ coframe bundle, i.e., so that the metric on each x-fiber is given by $|\eta|^2$ and the three parallel self-dual 2-forms are the components of the $\operatorname{Im}\mathbb{H}$-valued 2-form $\eta{\wedge}\bar{\eta}$. This determines η modulo dx up to right multiplication by a function with values in the unit quaternions, i.e., Sp(1).

There is then an $\mathrm{Im}\,\mathbb{H}$-valued 1-form ϕ, unique modulo dx, so that $d\eta \equiv -\eta\wedge\phi$ mod dx. In other words, there exists a $\mathbb{H}$-valued 1-form ψ so that

$$d\eta = -\psi \wedge dx - \eta \wedge \phi. \tag{98}$$

Consider the effect of different choices. Let $\eta' = \eta + p\,dx$ where p is a function with values in $\mathbb{H}$, and let $\phi' = \phi + q\,dx$ where q takes values in $\mathrm{Im}\,\mathbb{H}$. Then

$$d\eta' = -\psi'\,dx - \eta' \wedge \phi' \tag{99}$$

where $\psi' = dp - p\,\phi + \eta\,q + \psi + r\,dx$ for some $\mathbb{H}$-valued function r. If this system is to satisfy the structure equations above, then it will have to satisfy

$$-\psi' \wedge \overline{\eta'} + \eta' \wedge \overline{\psi'} \equiv 0 \mod dx, \tag{100}$$

i.e., it must be possible to choose p and q so that

$$\mathrm{Re}\left((dp - p\,\phi + \eta\,q + \psi) \wedge \eta\right) \equiv 0 \mod dx. \tag{101}$$

Rewriting this slightly, this becomes

$$d\left(\mathrm{Re}(p\eta)\right) \equiv -\,\mathrm{Re}(\psi \wedge \eta) - \mathrm{Re}(\eta\,q\,\eta) \mod dx. \tag{102}$$

The term $\mathrm{Re}(p\eta)$ represents a 1-form on each x-fiber and the term $\mathrm{Re}(\eta\,q\,\eta)$ represents an arbitrary anti-self dual 2-form on each fiber. In other words, the above equation represents determining the 1-form $\mathrm{Re}(p\eta)$ by specifying the self-dual part of its exterior derivative. This is, of course, an underdetermined elliptic equation and so can always be solved locally.

Suppose that such a solution has been found. (Actually, it is a 1-parameter family of such solutions, varying with x.) Once this has been done, the equation $\mathrm{Re}(\psi'\wedge\overline{\eta'}) \equiv 0$ mod dx is satisfied and, then, by choosing r appropriately, one can arrange that $\mathrm{Re}(\psi'\wedge\overline{\eta'}) = 0$ (not just modulo dx.

For notational clarity, drop the primes and assume that $\mathrm{Re}(\psi\wedge\overline{\eta}) = 0$. Then the metric

$$g = -dy \circ dx + |\eta|^2 + f\,dx^2 \tag{103}$$

where f is an arbitrary function on $\bar{M}$, will satisfy the structure equations necessary to be a metric of the desired type. A count of the ambiguity in the construction shows that the solutions depend on two arbitrary functions of five variables. (One is f and the other is the arbitrariness in the choice of p.)

Thus, the conclusion is that these metrics depend locally on two arbitrary functions of five variables.

I have not completed the analysis of the Einstein equations in this case, but hope to return to it in the future.

3.4.3. Type $(4,2)$. — In this case, as explained in §2.4.5.3, there are two kinds of orbits.

3.4.3.1. Generic type. — The generic orbits in $\mathbb{S}^{4,2} \simeq \mathbb{C}^{2,2}$ are the ones on which the spinor norm is nonzero. Each of these orbits is a hypersurface and the stabilizer of a point in such a hypersurface is a subgroup of $\mathrm{SU}(2,2)$ that is conjugate to $\mathrm{SU}(2,1)$. Moreover, in the spinor double cover, this subgroup is represented faithfully as a subgroup of $\mathrm{SO}(4,2)$ that is conjugate to the standard $\mathrm{SU}(2,1)$. Consequently, these metrics are simply the Ricci-flat pseudo-Kähler metrics of type $(2,1)$. In this respect, their analysis is essentially the same as the analysis in the positive definite case. The local metrics of this kind depend on two functions of five variables.

3.4.3.2. Null type. — However, the situation changes when the spinor field is null. Now the subgroup of $\mathrm{SO}(4,2)$ is not semi-simple, even though it is also of dimension 8. I have not completed the analysis of this case, so I will leave it for later.

3.4.4. Type $(3,3)$. — Now consider the split case, where $\mathrm{Spin}(3,3) \simeq \mathrm{SL}(4,\mathbb{R})$ acts anti-diagonally on the sum of the two half-spinor subspaces $\mathbb{S}^{3,3}_{\pm}$.

3.4.4.1. Generic type. — For the generic spinor orbit, the stabilizer subgroup is a copy of $\mathrm{SL}(3,\mathbb{R}) \subset \mathrm{SL}(4,\mathbb{R})$ and its action on $\mathbb{R}^{3,3}$ is reducible, as $\mathbb{R}^{3,3} = \mathbb{R}^3 \oplus \mathbb{R}^3$, where the two subspaces are null. In fact, the action of $\mathrm{SL}(3,\mathbb{R})$ and the quadratic form are just

$$a \cdot (v_+, v_-) = (a\, v_+,\ (a^*)^{-1} v_-) \tag{104}$$

$$Q(v_+, v_-) = v_-^* \, v_+ \,, \tag{105}$$

for $a \in \mathrm{SL}(3,\mathbb{R})$ and $v_\pm \in \mathbb{R}^3$.

Consequently, it is not difficult to show that a metric with this holonomy must have local coordinates (x^i, y_j) in which it can be expressed in the form

$$g = \frac{\partial^2 f}{\partial x^i \partial y_j}\, dx^i \circ dy_j \tag{106}$$

where f satisfies the real Monge-Ampere equation

$$\det\left(\frac{\partial^2 f}{\partial x^i \partial y_j}\right) = 1. \tag{107}$$

Thus, the $(3,3)$-metrics with a generic parallel spinor depend on two functions of five variables, just as in the $(6,0)$ case. Moreover, these metrics are all Ricci-flat, just as in the $(6,0)$ case.

3.4.4.2. Null type. — On the other hand, if the spinor is on the null orbit, the situation is rather different. Now the stabilizer subgroup of $\mathrm{Spin}(3,3)$ is a conjugate of the subgroup G consisting of matrices of the form

$$\begin{pmatrix} 1 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{108}$$

The character of these solutions will be somewhat different. In the interest of time, let me just state the result, whose proof is quite similar to the previous proofs. One shows that a $(3,3)$ metric with a null parallel spinor of this kind always has local coordinates $(x^1, x^2, x^3, y_1, y_2, y_3)$ in which the metric has the form

$$g = dy^i\,dx_i + f_{11}(x,y)\,(dx^1)^2 + 2\,f_{12}(x,y)\,dx^1\,dx^2 + f_{22}(x,y)\,(dx^2)^2 \tag{109}$$

where the functions f_{11}, $f_{12} = f_{21}$, and f_{22} satisfy the two *constraint equations*

$$\frac{\partial f_{11}}{\partial y_1} + \frac{\partial f_{21}}{\partial y_2} = \frac{\partial f_{12}}{\partial y_1} + \frac{\partial f_{22}}{\partial y_2} = 0, \tag{110}$$

and that, conversely, every metric of this form has a parallel spinor field of this kind. Moreover, these coordinates are unique up to choices that depend on five arbitrary functions of two variables. It follows that metrics satisfying these conditions essentially depend on one arbitrary function of six variables.

The calculation of the Ricci tensor follows from the calculations to be done below in §3.5, so I will not redo them here. Instead, I will simply report that the general metric of this kind is not Ricci-flat, but that, when one imposes the Ricci-flat condition as a system of equations, the resulting system is in involution and the general solution depends on two arbitrary functions of five variables, *exactly as for the case of a non-null parallel spinor field.*

3.4.4.3. Degenerate type. — Finally, consider the case where the parallel spinor field is associated to one of the most degenerate orbits, either $\mathbb{S}^{3,3}_+$ or $\mathbb{S}^{3,3}_-$ (minus the origin, of course). Now, this is the split case and each of these orbits constitute the pure spinors. Thus, this is a special case of the treatment in §3.5.2, so I will not consider it further here, except to mention that, as in the previous two cases, the Ricci-flat solutions depend on two arbitrary functions of five variables.

3.5. Parallel pure spinor fields. — As was pointed out in §2.5, the most degenerate orbits in the split cases $\mathrm{Spin}(p{+}1, p$ and $\mathrm{Spin}(p,p)$ are the so-called 'pure' spinors. The stabilizer of a pure spinor in either case maps under the double covering to the stabilizer of a maximal null p-vector in $\mathbb{R}^{p+1,p}$ or $\mathbb{R}^{p,p}$, respectively. Thus, having a parallel pure spinor field (i.e., of the most degenerate type) is the same as having a parallel null p-plane field. From that point of view, the metrics with this property are easily analyzed. Equivalent normal forms to the ones derived below have been derived independently by Ines Kath [**15**]. My main interest is in how this condition interacts with the Einstein condition, which I explain at some length.

3.5.1. The odd case. — Suppose that $(M^{p+1,p}, g)$ is a metric with a parallel null p-plane field. Consider the bundle of coframes of the form

$$\omega = \begin{pmatrix} \zeta \\ \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \zeta \\ \xi^i \\ \eta_i \end{pmatrix} \tag{111}$$

(where lower case Latin indices range from 1 to p and the summation convention will be in force) with the property that $g = \zeta^2 + 2\eta^i\,\xi_i$ and the parallel null p-form is $\xi = \xi^1{\wedge}\cdots\xi^p$. The hypothesis that ξ is parallel implies that the Levi-Civita connection 1-form α associated to ω will have the form

$$\alpha = \begin{pmatrix} 0 & -\tau^* & 0 \\ 0 & \phi & 0 \\ \tau & \sigma & -\phi^* \end{pmatrix} = \begin{pmatrix} 0 & -\tau_j & 0 \\ 0 & \phi^i_j & 0 \\ \tau_i & \sigma_{ij} & \phi^j_i \end{pmatrix}, \tag{112}$$

where $\operatorname{tr}\phi = 0$ and $\sigma + \sigma^* = 0$.

The first structure equation is $d\omega = -\alpha{\wedge}\omega$. In particular, this implies that $d\xi = -\phi{\wedge}\xi$, so that there exists (locally) a submersion $x : U(\subset M) \to \mathbb{R}^p$ so that $\xi = f^{-1}\,dx$ where $f : U \to \mathrm{SL}(p,\mathbb{R})$ is some smooth mapping. By an allowable change of coframe, it can be assumed that $f \equiv \mathrm{I}_p$, so do this. Thus, $\xi^i = dx^i$, implying that

$$0 = d\xi^i = -\phi^i_j \wedge \xi^j = -\phi^i_j \wedge dx^j. \tag{113}$$

By Cartan's Lemma, this implies that there exist functions $f^i_{jk} = f^i_{kj}$ on U so that $\phi^i_j = f^i_{jk}\,dx^k$. Since ϕ has trace equal to zero, it follows that $f^i_{ij} = 0$.

Now, the first structure equation gives $d\zeta = \tau_i{\wedge}\xi^i \equiv 0 \mod dx^1,\dots,dx^p$. Consequently, there exists a function z on U (shrunken, if necessary) so that $\zeta = dz + t_i\,dx^i$. By an allowable change of coframe, it can be assumed that the t_i are all zero, so do this. This now implies that $\zeta = dz$, so

$$0 = d\zeta = \tau_i \wedge dx^i, \tag{114}$$

implying, again, by Cartan's Lemma, that there exist functions $t_{ij} = t_{ji}$ so that $\tau_i = t_{ij}\,dx^j$.

Now, the structure equations imply that

$$d\eta = -\tau \wedge \zeta - \sigma \wedge dx + \phi^* \wedge \eta \equiv 0 \mod dx^1,\dots,dx^p \tag{115}$$

so it follows that, after shrinking U if necessary, there is a function $y : U \to \mathbb{R}^n$ so that $\eta \equiv dy \mod dx^1,\dots,dx^p$. I.e., there exist functions f_{ij} on U so that

$$\eta_i = dy_i + f_{ij}\,dx^j. \tag{116}$$

Applying an allowable coframe change, I can arrange that $f_{ij} = f_{ji}$, so assume this from now on. Substituting this formula back into the structure equation for $d\eta$ and using the skewsymmetry of σ and the trace-free property of ϕ, it follows that the functions f_{ij} must satisfy the p first order equations

$$\frac{\partial f_{ij}}{\partial y_j} = 0. \tag{117}$$

Thus, it has been shown that a $(p{+}1,p)$-metric that possesses a parallel pure spinor field has local coordinate charts $(x,y,z) : U \to \mathbb{R}^{2p+1}$ in which the metric can be expressed as

$$g = dz^2 + 2\,dy_i\,dx^i + 2f_{ij}(x,y,z)\,dx^i\,dx^j \tag{118}$$

where the functions $f_{ij} = f_{ji}$ satisfy (117).

Conversely, I claim that a metric that can be written in this form does possess a parallel pure spinor field. To see this, it suffices to take the coframing

$$\zeta = dz, \qquad \xi^i = dx^i, \qquad \eta_i = dy_i + f_{ij}\,dx^j \tag{119}$$

and verify that setting

$$\begin{aligned} \phi^i_j &= -\frac{\partial f_{jk}}{\partial y_i}\,dx^k, \qquad \tau_i = \frac{\partial f_{ik}}{\partial z}\,dx^k, \qquad \text{and} \\ \sigma_{ij} &= \left(\frac{\partial f_{ik}}{\partial x^j} - \frac{\partial f_{jk}}{\partial x^i} + f_{il}\frac{\partial f_{jk}}{\partial y_l} - f_{jl}\frac{\partial f_{ik}}{\partial y_l}\right)dx^k, \end{aligned} \tag{120}$$

satisfies the structure equations. (Note that (117) is needed in order for ϕ to be trace-free.)

Thus, the $(p{+}1,p)$-metrics with a parallel pure spinor field depend essentially on $\frac{1}{2}p(p{+}1) - p = \frac{1}{2}p(p{-}1)$ arbitrary functions of $2p{+}1$ variables. (The ambiguity in the choice of these coordinates is measured in functions of p variables, which is negligible.)

3.5.1.1. Curvature and holonomy. — I am now going to show that the metrics of this type do not, generally have any more parallel spinors by showing that the holonomy group of the generic metric of this kind is equal to the full stabilizer of a null p-vector. This will be done by examining the curvature of such a metric.

The components of the curvature 2-form $\Theta = d\alpha + \alpha{\wedge}\alpha$ are

$$\begin{aligned} \Phi^i_j &= d\phi^i_j + \phi^i_k \wedge \phi^k_j\,, \\ T_i &= d\tau_i - \phi^j_i \wedge \tau_j\,, \\ \Sigma_{ij} &= d\sigma_{ij} - \phi^k_i \wedge \sigma_{kj} + \sigma_{ik} \wedge \phi^k_j - \tau_i \wedge \tau_j\,. \end{aligned} \tag{121}$$

Note that the expressions (120) for the components ϕ^i_j, τ_i, and σ_{ij} are all linear combinations of the ξ^i, i.e., of $dx^i, \ldots, dx^p$. One consequence of this fact is that the curvature 2-forms must all lie in the ideal $\mathcal{X}$ generated by $dx^i, \ldots, dx^p$.

Now, let $\mathfrak{g} \subset \mathfrak{so}(p{+}1,p)$ be the Lie algebra of the stabilizer of the null p-vector as described above. By the Ambrose-Singer holonomy theorem, the Lie algebra $\mathfrak{h} \subset \mathfrak{g}$ of the holonomy group at $0 \in \mathbb{R}^{2p+1}$ is spanned by the matrices of the form

$$P_\gamma^{-1}\,\Theta(v,w)\,P_\gamma \tag{122}$$

where $\gamma : [0,1] \to \mathbb{R}^{2p+1}$ is a differentiable curve with $\gamma(0) = 0$ and v and w are tangent vectors to $\mathbb{R}^{2p+1}$ at $\gamma(1)$. In particular, $\mathfrak{h}$ contains the subspace $\mathfrak{p}$ that is spanned by matrices of the form $\Theta(v,w)$ where v and w are tangent vectors to $\mathbb{R}^{2p+1}$ at 0. Thus, to show that $\mathfrak{h} = \mathfrak{g}$, it suffices to show that $\mathfrak{p}$ generates $\mathfrak{g}$ as a Lie algebra.

Now, let $\mathcal{X}{\wedge}\mathcal{X}$ denote the span of the 2-forms $\{dx^i{\wedge}dx^j \mid 1 \le i,j \le p\}$. Using the given expressions for the components of θ, the components of Θ satisfy congruences modulo $\mathcal{X}{\wedge}\mathcal{X}$ of the form

$$
\begin{aligned}
\Phi^i_j &\equiv \frac{\partial^2 f_{jk}}{\partial z\,\partial y_i}\,dx^k\wedge dz + \frac{\partial^2 f_{jk}}{\partial y_l\partial y_i}\,dx^k\wedge dy_l\,,\\
T_i &\equiv -\frac{\partial^2 f_{ik}}{\partial z^2}\,dx^k\wedge dz - \frac{\partial^2 f_{ik}}{\partial z\,\partial y_l}\,dx^k\wedge dy_l\,,\\
\Sigma_{ij} &\equiv -\bigg(\frac{\partial^2 f_{ik}}{\partial z\,\partial x^j} - \frac{\partial^2 f_{jk}}{\partial z\,\partial x^i} + f_{pi}\,\frac{\partial^2 f_{jk}}{\partial z\,\partial y_p} - f_{pj}\,\frac{\partial^2 f_{ik}}{\partial z\,\partial y_p}\\
&\qquad\qquad + \frac{\partial f_{pi}}{\partial z}\,\frac{\partial f_{jk}}{\partial y_p} - \frac{\partial f_{pj}}{\partial z}\,\frac{\partial f_{ik}}{\partial y_p}\bigg)\,dx^k\wedge dz\\
&\quad -\bigg(\frac{\partial^2 f_{ik}}{\partial y_l\,\partial x^j} - \frac{\partial^2 f_{jk}}{\partial y_l\,\partial x^i} + f_{pi}\,\frac{\partial^2 f_{jk}}{\partial y_l\,\partial y_p} - f_{pj}\,\frac{\partial^2 f_{ik}}{\partial y_l\,\partial y_p}\\
&\qquad\qquad + \frac{\partial f_{pi}}{\partial y_l}\,\frac{\partial f_{jk}}{\partial y_p} - \frac{\partial f_{pj}}{\partial y_l}\,\frac{\partial f_{ik}}{\partial y_p}\bigg)\,dx^k\wedge dy_l
\end{aligned}
\tag{123}
$$

Consider now a particular solution of the form

$$f_{ij} = \tfrac12\,h^{kl}_{ij}\,y_k y_l + \tfrac12\,h_{ij}\,z^2 \tag{124}$$

where $h_{ij} = h_{ji}$ and $h^{kl}_{ij} = h^{lk}_{ij} = h^{kl}_{jk}$ are constants satisfying the condition $h^{kl}_{kj} = 0$. This choice satisfies the constraint equations (117) and, moreover, satisfies

$$\Phi^i_j \equiv h^{il}_{jk}\,dx^k\wedge dy_l\,, \qquad \text{and} \qquad T_i \equiv -h_{ik}\,dx^k\wedge dw\,, \tag{125}$$

the congruences being taken modulo $\mathcal{X}{\wedge}\mathcal{X}$. Moreover, the 2-forms Σ_{ij} vanish to order at least 2 at the origin $x = y = z = 0$.

It follows that, when the constants h_{ij} and h^{ij}_{kl} are taken sufficiently generically, the space $\mathfrak{p}$ (and hence $\mathfrak{h}$) contains all the matrices of the form

$$\begin{pmatrix} 0 & -r^* & 0\\ 0 & q & 0\\ r & 0 & -q^*\end{pmatrix} \tag{126}$$

with $r \in \mathbb{R}^p$ and $q \in \mathfrak{sl}(p,\mathbb{R})$. However, the space of such matrices generates $\mathfrak{g}$. It follows that the holonomy group is equal to the full stabilizer $G \subset \mathrm{SO}(p{+}1,p)$, as was desired.

It follows, moreover, that there is an open, dense condition on the 2-jet of the functions f_{ij} whose satisfaction will imply that the corresponding metric g will have holonomy equal to G. In particular, such a metric will have exactly one parallel spinor, which will moreover, be pure.

3.5.1.2. The Ricci tensor. — Finally, I want to examine the conditions for such a metric to be Ricci-flat. A calculation shows that the formula for the Ricci tensor of the metric g defined by (118) is

$$\mathrm{Ric}(g) = 2\left(\frac{\partial^2 f_{jl}}{\partial z^2} - \frac{\partial^2 f_{jl}}{\partial x^k\,\partial y_k} + f_{mk}\,\frac{\partial^2 f_{jl}}{\partial y_m\,\partial y_k} - \frac{\partial f_{mj}}{\partial y_k}\,\frac{\partial f_{kl}}{\partial y_m}\right) dx^j\,dx^l. \tag{127}$$

Thus, the generic such metric is not Ricci-flat.

There remains the question of how many $(p{+}1,p)$-metrics there are that both have a parallel pure spinor field and are Ricci-flat. By the above formula, this is, locally, the same as asking for the simultaneous solutions to the overdetermined system:

$$\begin{aligned} \frac{\partial f_{ij}}{\partial y_j} &= 0,\\ \frac{\partial^2 f_{jl}}{\partial z^2} - \frac{\partial^2 f_{jl}}{\partial x^k\,\partial y_k} + f_{mk}\,\frac{\partial^2 f_{jl}}{\partial y_m\,\partial y_k} - \frac{\partial f_{mj}}{\partial y_k}\,\frac{\partial f_{kl}}{\partial y_m} &= 0. \end{aligned} \tag{128}$$

Fortunately, this system is involutive in Cartan's sense, so that local solutions are guaranteed to exist, at least in the real-analytic category. (See **[10]** for a discussion of what this means.)

In fact, though, it is not necessary to invoke the Cartan-Kähler theory in this case, as a direct proof can be given for the existence of solutions to the Cauchy problem. Here is how this can be done: Consider functions $a_{ij} = a_{ji}$ and $b_{ij} = b_{ji}$ on $\mathbb{R}^{2p}$ with coordinates x^i, y_j and suppose that these functions satisfy the *constraint equations*

$$\frac{\partial a_{ij}}{\partial y_j} = \frac{\partial b_{ij}}{\partial y_j} = 0. \tag{129}$$

Now consider the nonlinear initial value problem

$$\begin{aligned} \frac{\partial^2 f_{jl}}{\partial z^2} &= \frac{\partial^2 f_{jl}}{\partial x^k\,\partial y_k} - f_{mk}\,\frac{\partial^2 f_{jl}}{\partial y_m\,\partial y_k} + \frac{\partial f_{mj}}{\partial y_k}\,\frac{\partial f_{kl}}{\partial y_m}\,,\\ f_{jl}(0,x,y) &= a_{jl}(x,y)\,,\\ \frac{\partial f_{jl}}{\partial z}(0,x,y) &= b_{jl}(x,y)\,. \end{aligned} \tag{130}$$

If a_{ij} and b_{ij} are real-analytic, then the Cauchy-Kowalewski theorem implies that there is a unique real-analytic solution f_{jl} to this problem on an open neighborhood of $\mathbb{R}^{2p}\times\{0\}$ in $\mathbb{R}^{2p}\times\{0\} = \mathbb{R}^{2p+1}$. It must now be shown that the resulting functions f_{jl} satisfy the constraint equations

$$\frac{\partial f_{ij}}{\partial y_j} = 0. \tag{131}$$

in order to know that they satisfy the system (128).

To show this, consider the real-analytic quantities

$$A_l = \frac{\partial f_{jl}}{\partial y_j}\,. \tag{132}$$

Using the fact that f_{jl} satisfies (130), one computes that

$$\begin{aligned}\frac{\partial^2 A_l}{\partial z^2} &= \frac{\partial^3 f_{jl}}{\partial z^2\,\partial y_j} = \frac{\partial}{\partial y_j}\left(\frac{\partial^2 f_{jl}}{\partial z^2}\right)\\ &= \frac{\partial}{\partial y_j}\left(\frac{\partial^2 f_{jl}}{\partial x^k\,\partial y_k} - f_{mk}\,\frac{\partial^2 f_{jl}}{\partial y_m\,\partial y_k} + \frac{\partial f_{mj}}{\partial y_k}\,\frac{\partial f_{kl}}{\partial y_m}\right)\\ &= \frac{\partial^2 A_l}{\partial x^k\,\partial y_k} - f_{mk}\,\frac{\partial^2 A_l}{\partial y_m\,\partial y_k} + \frac{\partial f_{kl}}{\partial y_m}\,\frac{\partial A_m}{\partial y_k}\,.\end{aligned} \tag{133}$$

(Note the very fortunate circumstance that, in expanding this last step, the terms that appear that cannot be expressed in terms of the A_l cancel. It is this cancellation that ensures that the constraint equations are compatible with the Ricci equations.) Thus, the A_l satisfy a linear second order system of PDE in Cauchy-Kowalewski form. Moreover, A_l satisfies the initial conditions

$$A_l(0,x,y) = \frac{\partial a_{ij}}{\partial y_j}(x,y) = 0\,, \qquad \text{and} \qquad \frac{\partial A_l}{\partial w}(0,x,y) = \frac{\partial b_{ij}}{\partial y_j}(x,y) = 0\,. \tag{134}$$

Thus, by the uniqueness of real-analytic solutions to the initial value problem, it follows that $A_l(z,x,y) = 0$, as was to be shown.

In conclusion, it follows that the Ricci-flat $(p{+}1,p)$-metrics that possess a parallel pure spinor depend on $p(p{-}1)$ functions of $2p$ variables, locally. Moreover, examining the discussion of curvature and holonomy of solutions in §3.5.1.1, one sees that it is possible to choose the initial data for the Cauchy problem in such a way as to construct Ricci-flat solutions with the full stabilizer group as holonomy. Details are left to the reader.

3.5.1.3. The case $p = 3$. — This analysis is particularly interesting in the case $p = 3$, as I shall now explain. The above argument shows that the Ricci-flat $(4,3)$-metrics with a parallel pure spinor field depend locally on six arbitrary functions of six variables. This is the same generality as that for $(4,3)$-metrics with a parallel spinor field that is not null, since these are precisely the $(4,3)$-metrics whose holonomy lies in G_2^*, the stabilizer of any non-null spinor in $\mathbb{S}^{4,3}$, see **[6]**. It is interesting that, even though the orbits of the null spinors and the non-null spinors have the same dimension, the condition to have a null parallel spinor field is weaker than that of having a non-null parallel spinor field. However, adding in the Ricci-flat condition (which is automatic for metrics with a non-null parallel spinor field) restores equality between the two cases, as far as local generality goes.

3.5.1.4. The case $p = 4$. — The case $p = 4$ is also worth mentioning for comparing the case of a non-null parallel spinor field with that of a pure spinor field. Recall from the discussion in §2.5.1.1 that the generic $\mathrm{Spin}(5,4)$-orbit in $\mathbb{S}^{5,4} \simeq \mathbb{R}^{16}$ is a quadratic hypersurface. The stabilizer of a spinor on such an orbit is isomorphic to $\mathrm{Spin}(4,3)$ and this maps to a copy of $\mathrm{Spin}(4,3) \subset \mathrm{SO}(4,4) \subset \mathrm{SO}(5,4)$ and so stabilizes a non-null vector in $\mathbb{R}^{5,4}$. In particular, a metric with a non-null parallel spinor must locally be a product of a 1-dimensional factor with a metric on an 8-manifold with holonomy in $\mathrm{Spin}(4,3)$. In particular, such metrics are Ricci-flat and depend locally on 12 arbitrary functions of 7 variables [**6**].

In contrast, a $(5,4)$-metric with a parallel pure spinor field does not necessarily factor and need not be Ricci-flat. Moreover, even if one imposes the Ricci-flat condition, the local generality of such metrics is still 12 functions of 8 variables.

3.5.2. The even case. — The even case is very similar to the odd case, so I will just state the results and leave the arguments to the reader.

First of all, one shows that a (p,p)-metric g that possesses a parallel pure spinor field has local coordinate charts $(x,y) : U \to \mathbb{R}^{2p}$ in which the metric can be expressed as

$$g = dy_i\, dx^i + f_{ij}(x,y)\, dx^i\, dx^j \tag{135}$$

where the functions $f_{ij} = f_{ji}$ satisfy (117).

A calculation shows that the formula for the Ricci tensor of the metric g defined by (135) is

$$\mathrm{Ric}(g) = -2\left(\frac{\partial^2 f_{jl}}{\partial x^k\, \partial y_k} - f_{mk}\, \frac{\partial^2 f_{jl}}{\partial y_m\, \partial y_k} + \frac{\partial f_{mj}}{\partial y_k}\, \frac{\partial f_{kl}}{\partial y_m}\right) dx^j\, dx^l. \tag{136}$$

Thus, the generic such metric is not Ricci-flat. An examination of the curvature of this metric shows that the generic such metric has holonomy equal to the stabilizer of a null p-vector (and hence has only one parallel spinor field).

Finally, the combination of the constraint equations (117) and $\mathrm{Ric}(g) = 0$ forms an involutive system, whose general solution depends $p(p-1)$ arbitrary functions of $2p-1$ variables. Moreover, the general solution has holonomy equal to the stabilizer of a null p-vector (and hence has only one parallel spinor field).

3.6. $(10,1)$**-metrics with a parallel null spinor field.** — In this final section, I will show that there are $(10,1)$-metrics with parallel null spinor fields whose holonomy group is the maximum possible, namely that of the group $H \subset \mathrm{SO}(10,1)$ of dimension 30 that stabilizes a null spinor in $\mathbb{S}^{10,1}$. The notation of §2.6.3 will be continued in this section. By the analysis there, the image group $\rho(H) \subset \mathrm{SO}^{\uparrow}(10,1)$ has Lie algebra

$$\rho'(\mathfrak{h}) = \left\{ \begin{pmatrix} 0 & y & 0 & \overline{\boldsymbol{y}}^* \\ 0 & 0 & 2y & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2\,\overline{\boldsymbol{y}} & a_2 \end{pmatrix} \;\middle|\; \begin{matrix} y \in \mathbb{R}, \\ \boldsymbol{y} \in \mathbb{O}, \\ a \in \mathfrak{k}_1 \end{matrix} \right\}. \tag{137}$$

Thus, the problem devolves on understanding the structure equations of a torsion-free $\rho(H)$-structure $B \to M^{10,1}$ of the form

$$\begin{pmatrix} d\omega_1 \\ d\omega_2 \\ d\omega_3 \\ d\boldsymbol{\omega} \end{pmatrix} = - \begin{pmatrix} 0 & \psi & 0 & {}^t\boldsymbol{\phi} \\ 0 & 0 & 2\,\psi & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2\,\boldsymbol{\phi} & \boldsymbol{\theta} \end{pmatrix} \wedge \begin{pmatrix} d\omega_1 \\ d\omega_2 \\ d\omega_3 \\ d\boldsymbol{\omega} \end{pmatrix} \tag{138}$$

where $\boldsymbol{\omega}$ and $\boldsymbol{\phi}$ take values in $\mathbb{O}$ and $\boldsymbol{\theta}$ takes values in the subalgebra $\mathfrak{spin}(7) \subset \mathfrak{gl}(\mathbb{O})$ that consists of the elements of the form a_2 with $a \in \mathfrak{k}_1$. For such a $\rho(H)$-structure, the Lorentzian metric $g = -4\,\omega_1\,\omega_3 + {\omega_2}^2 + \boldsymbol{\omega} \cdot \boldsymbol{\omega}$ has a parallel null spinor and B represents the structure reduction afforded by this parallel structure. Note that the null 1-form ω_3 is parallel and well-defined on M. It (or, more properly, its metric dual vector field) is the square of the parallel null spinor field.

Differentiating the Cartan structure equations yields the first Bianchi identities:

$$0 = \begin{pmatrix} 0 & \Psi & 0 & {}^t\boldsymbol{\Phi} \\ 0 & 0 & 2\,\Psi & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2\,\boldsymbol{\Phi} & \boldsymbol{\Theta} \end{pmatrix} \wedge \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \boldsymbol{\omega} \end{pmatrix}. \tag{139}$$

where $\Psi = d\psi$, $\boldsymbol{\Phi} = d\boldsymbol{\phi} + \boldsymbol{\theta}{\wedge}\boldsymbol{\phi}$, and $\boldsymbol{\Theta} = d\boldsymbol{\theta} + \boldsymbol{\theta}{\wedge}\boldsymbol{\theta}$.

By the second line of this system, $\Psi{\wedge}\omega_3 = 0$, while the first line implies that $\Psi{\wedge}\omega_2 \equiv 0 \bmod \boldsymbol{\omega}$, so there must be functions p and $\boldsymbol{q}$, with values in $\mathbb{R}$ and $\mathbb{O}$ respectively, so that

$$\Psi = (p\,\omega_2 + \boldsymbol{q} \cdot \boldsymbol{\omega}) \wedge \omega_3\,. \tag{140}$$

Substituting this into the first line of the system yields

$${}^t\big(\boldsymbol{\Phi} - \boldsymbol{q}\,\omega_2 \wedge \omega_3\big) \wedge \boldsymbol{\omega} = 0, \tag{141}$$

so it follows that

$$\boldsymbol{\Phi} = \boldsymbol{q}\,\omega_2 \wedge \omega_3 + \boldsymbol{\sigma} \wedge \boldsymbol{\omega}\,, \tag{142}$$

where $\boldsymbol{\sigma} = {}^t\boldsymbol{\sigma}$ is some 1-form with values in the symmetric part of $\mathfrak{gl}(\mathbb{O})$, which will be denoted $S^2(\mathbb{O})$ from now on. Substituting this last equation into the last line of the Bianchi identities, yields

$$2\,\boldsymbol{\sigma} \wedge \boldsymbol{\omega} \wedge \omega_3 + \boldsymbol{\Theta} \wedge \boldsymbol{\omega} = \mathbf{0}. \tag{143}$$

In particular, this implies that $\boldsymbol{\Theta}{\wedge}\boldsymbol{\omega} = \mathbf{0} \bmod \omega_3$, so that $\boldsymbol{\Theta} \equiv \boldsymbol{R}\big(\boldsymbol{\omega}{\wedge}\boldsymbol{\omega}\big) \bmod \omega_3$ where $\boldsymbol{R}$ is a function on B with values in $\mathcal{K}\big(\mathfrak{spin}(7)\big)$, which is the irreducible Spin(7) module of highest weight $(0,2,0)$ and of (real) dimension 168. (This uses the usual calculation of the curvature tensor of Spin(7)-manifolds.) Thus, set

$$\boldsymbol{\Theta} = \boldsymbol{R}\big(\boldsymbol{\omega} \wedge \boldsymbol{\omega}\big) + 2\,\boldsymbol{\alpha} \wedge \omega_3\,, \tag{144}$$

where $\boldsymbol{\alpha}$ is a 1-form with values in $\mathfrak{spin}(7)$ whose entries can be assumed, without loss of generality, to be linear combinations of ω_1, ω_2, and the components of $\boldsymbol{\omega}$. Substituting this last relation into the last line of the Bianchi identities now yields

$$2\,\boldsymbol{\sigma}\wedge\boldsymbol{\omega}\wedge\omega_3 + 2\,(\boldsymbol{\alpha}\wedge\omega_3)\wedge\boldsymbol{\omega} = \mathbf{0}, \tag{145}$$

which is equivalent to the condition

$$\boldsymbol{\sigma}\wedge\boldsymbol{\omega} \equiv \boldsymbol{\alpha}\wedge\boldsymbol{\omega} \mod \omega_3. \tag{146}$$

In particular, this implies that $\boldsymbol{\sigma}-\boldsymbol{\alpha} \equiv 0 \mod \omega_3, \boldsymbol{\omega}$. Since $\boldsymbol{\sigma}$ and $\boldsymbol{\alpha}$ take values in $S^2(\mathbb{O})$ and $\mathfrak{spin}(7)$ respectively, which are disjoint subspaces of $\mathfrak{gl}(\mathbb{O})$, it follows that $\boldsymbol{\sigma}\equiv\boldsymbol{\alpha}\equiv 0 \mod \omega_3, \boldsymbol{\omega}$. In particular, neither ω_1 nor ω_2 appear in the expressions for $\boldsymbol{\sigma}$ and $\boldsymbol{\alpha}$. Recall that, by definition, ω_3 does not appear in the expression for $\boldsymbol{\alpha}$, so $\boldsymbol{\alpha}$ must be a linear combination of the components of $\boldsymbol{\omega}$ alone. Now, from the above equation, it follows that

$$\boldsymbol{\sigma}\wedge\boldsymbol{\omega} = \boldsymbol{\alpha}\wedge\boldsymbol{\omega} + \boldsymbol{s}\,\omega_3\wedge\boldsymbol{\omega} \tag{147}$$

where $\boldsymbol{s}$ takes values in $S^2(\mathbb{O})$. Finally, the first line of the Bianchi identities show that ${}^t\boldsymbol{\omega}\wedge\boldsymbol{\alpha}\wedge\boldsymbol{\omega} = 0$, so it follows that $\boldsymbol{\alpha} = \boldsymbol{a}(\boldsymbol{\omega})$ where $\boldsymbol{a}$ is a function on B that takes values in a subspace of $\operatorname{Hom}\big(\mathbb{O}, \mathfrak{spin}(7)\big)$ that is of dimension $8\cdot 21 - 56 = 112$. By the usual weights and roots calculation, it follows that this subspace is irreducible, with highest weight $(0,1,1)$.

To summarize, the Bianchi identities show that the curvature of a torsion-free $\rho(H)$-structure B must have the form

$$\begin{aligned}
\Psi &= (p\,\omega_2 + \boldsymbol{q}\cdot\boldsymbol{\omega})\wedge\omega_3\,,\\
\boldsymbol{\Phi} &= \boldsymbol{q}\,\omega_2\wedge\omega_3 + \boldsymbol{s}\,\omega_3\wedge\boldsymbol{\omega} + \boldsymbol{a}(\boldsymbol{\omega})\wedge\boldsymbol{\omega}\\
\boldsymbol{\Theta} &= \boldsymbol{R}\big(\boldsymbol{\omega}\wedge\boldsymbol{\omega}\big) + 2\,\boldsymbol{a}(\boldsymbol{\omega})\wedge\omega_3
\end{aligned} \tag{148}$$

where $\boldsymbol{R}$ takes values in $\mathcal{K}\big(\mathfrak{spin}(7)\big)$, the irreducible Spin(7)-representation of highest weight $(0,2,0)$ (of dimension 168), $\boldsymbol{a}$ takes values in the irreducible Spin(7)-representation of highest weight $(0,1,1)$ (of dimension 112), $\boldsymbol{s}$ takes values in $S^2(\mathbb{O})$ (the sum of a trivial representation with an irreducible one of highest weight $(0,0,2)$ and of dimension 35), $\boldsymbol{q}$ takes values in $\mathbb{O}$, and p takes values in $\mathbb{R}$. Thus, the curvature space $\mathcal{K}\big(\rho'(\mathfrak{h})\big)$ has dimension 325. By inspection, this curvature space passes Berger's first test (i.e., the generic element has the full $\rho'(\mathfrak{h})$ as its range). Thus, a structure with the full holonomy is not ruled out by this method.

To go further in the analysis, it will be useful to integrate the structure equations, at least locally. This will be done by a series of observations.

To begin, notice that, since $d\omega_3 = 0$, there exists, locally, a function x_3 on M so that $\omega_3 = dx_3$. This function is determined up to an additive constant, and can be defined on any simply connected open subset $U_0 \subset M$.

Since $d\omega_2 = -2\,\psi\wedge\omega_3 = -2\,\psi\wedge dx_3$, it follows that any point of U_0 has an open neighborhood $U_1 \subset U_0$ on which there exists a function x_2 for which $\omega_2\wedge\omega_3 = dx_2\wedge dx_3$.

The function x_2 is determined up to the addition of an arbitrary function of x_3. In consequence, there exists a function r on $B_1 = \pi^{-1}(U_1)$ so that $\omega_2 = dx_2 - 2r\,dx_3$. It now follows from the structure equation for $d\omega_2$ that $\psi{\wedge}\omega_3 = dr{\wedge}dx_3$. Consequently, there is a function f on B_1 so that $\psi = dr + f\,dx_3$. Since $\Psi = d\psi$ is π-basic, it follows that $df{\wedge}dx_3$ is well-defined on U_1. Consequently, f is well-defined on U_1 up to the addition of an arbitrary function of x_3.

Now, since

$$d\omega_1 = -\psi\wedge\omega_2 - {}^t\boldsymbol{\phi}\wedge\boldsymbol{\omega} = -(dr + f\,dx_3)\wedge(dx_2 - 2r\,dx_3) - {}^t\boldsymbol{\phi}\wedge\boldsymbol{\omega}, \tag{149}$$

it follows that

$$d(\omega_1 + r\,dx_2 - r^2\,dx_3) = f\,dx_2\wedge dx_3 - {}^t\boldsymbol{\phi}\wedge\boldsymbol{\omega}. \tag{150}$$

The fact that the 2-form on the right hand side is closed, together with the fact that the system I of dimension 9 spanned by dx_3 and the components of $\boldsymbol{\omega}$ is integrable (which follows from the structure equations), implies that there are functions G and $\boldsymbol{F}$ on B so that

$$d(\omega_1 + r\,dx_2 - r^2\,dx_3) = d(G\,dx_3 - {}^t\boldsymbol{F}\,\boldsymbol{\omega}), \tag{151}$$

from which it follows that there is a function x_1 on B so that

$$\omega_1 = dx_1 - r\,dx_2 + r^2\,dx_3 + G\,dx_3 - {}^t\boldsymbol{F}\,\boldsymbol{\omega}\,. \tag{152}$$

The function x_1 is determined (once the choices of x_3 and x_2 are made) up to an additive function that is constant on the leaves of the system I, i.e., up to the addition of an (arbitrary) function of 9 variables. Expanding $d(G\,dx_3 - {}^t\boldsymbol{F}\,\boldsymbol{\omega}) = f\,dx_2{\wedge}dx_3 - {}^t\boldsymbol{\phi}{\wedge}\boldsymbol{\omega}$ via the structure equations and reducing modulo dx_3 yields

$${}^t\big(d\boldsymbol{F} + \boldsymbol{\theta}\,\boldsymbol{F}\big)\wedge\boldsymbol{\omega} \equiv {}^t\boldsymbol{\phi}\wedge\boldsymbol{\omega} \mod dx_3\,. \tag{153}$$

so that there must exist functions $\boldsymbol{H}$ and $\boldsymbol{u} = {}^t\boldsymbol{u}$ so that

$$\boldsymbol{\phi} = d\boldsymbol{F} + \boldsymbol{\theta}\,\boldsymbol{F} + \boldsymbol{H}\,dx_3 + \boldsymbol{u}\,\boldsymbol{\omega}\,. \tag{154}$$

Substituting this back into the relation $d(G\,dx_3 - {}^t\boldsymbol{F}\,\boldsymbol{\omega}) = f\,dx_2{\wedge}dx_3 - {}^t\boldsymbol{\phi}{\wedge}\boldsymbol{\omega}$ yields

$$dG + 2\,{}^t\boldsymbol{F}\,d\boldsymbol{F} - {}^t\big(\boldsymbol{H} - 2\boldsymbol{u}\,\boldsymbol{F}\big)\,\boldsymbol{\omega} \equiv f\,dx_2 \mod dx_3\,. \tag{155}$$

Setting $G = g - \boldsymbol{F}\cdot\boldsymbol{F}$ and $\boldsymbol{h} = \boldsymbol{H} - 2\boldsymbol{u}\,\boldsymbol{F}$, this becomes

$$dg \equiv f\,dx_2 + {}^t\boldsymbol{h}\,\boldsymbol{\omega} \mod dx_3\,, \tag{156}$$

with the formulae

$$\begin{aligned} \omega_1 &= dx_1 - r\,dx_2 + r^2\,dx_3 + (g - \boldsymbol{F}\cdot\boldsymbol{F})\,dx_3 - {}^t\boldsymbol{F}\,\boldsymbol{\omega}\,,\\ \boldsymbol{\phi} &= d\boldsymbol{F} + \boldsymbol{\theta}\,\boldsymbol{F} + (\boldsymbol{h} + 2\boldsymbol{u}\,\boldsymbol{F})\,dx_3 + \boldsymbol{u}\,\boldsymbol{\omega}\,. \end{aligned} \tag{157}$$

Now the final structure equation becomes

$$d\boldsymbol{\omega} = -2\big(d\boldsymbol{F} + \boldsymbol{\theta}\,\boldsymbol{F} + \boldsymbol{u}\,\boldsymbol{\omega}\big)\wedge dx_3 - \boldsymbol{\theta}\wedge\boldsymbol{\omega} \tag{158}$$

which can be rearranged to give

$$d\big(\boldsymbol{\omega} + 2\boldsymbol{F}\,dx_3\big) = -\big(\boldsymbol{\theta} - 2\boldsymbol{u}\,dx_3\big)\wedge\big(\boldsymbol{\omega} + 2\boldsymbol{F}\,dx_3\big)\,. \tag{159}$$

This suggests setting $\boldsymbol{\eta} = \boldsymbol{\omega} + 2\boldsymbol{F}\,dx_3$ and writing the formulae found so far as

$$\begin{aligned}
\omega_1 &= dx_1 - r\,dx_2 + r^2\,dx_3 + (g{+}\boldsymbol{F}\cdot\boldsymbol{F})\,dx_3 - {}^t\boldsymbol{F}\,\boldsymbol{\eta}\,,\\
\omega_2 &= dx_2 - 2r\,dx_3\,,\\
\omega_3 &= dx_3\,,\\
\boldsymbol{\omega} &= -2\boldsymbol{F}\,dx_3 + \boldsymbol{\eta}\,,\\
\psi &= dr + f\,dx_3\,,\\
\boldsymbol{\phi} &= d\boldsymbol{F} + \boldsymbol{\theta}\,\boldsymbol{F} + \boldsymbol{h}\,dx_3 + \boldsymbol{u}\,\boldsymbol{\eta}\,,\\
dg &\equiv f\,dx_2 + {}^t\boldsymbol{h}\,\boldsymbol{\eta} \mod dx_3\,,\\
d\boldsymbol{\eta} &= -\big(\boldsymbol{\theta} - 2\boldsymbol{u}\,dx_3\big)\wedge\boldsymbol{\eta}\,.
\end{aligned} \tag{160}$$

where, in these equations, $\boldsymbol{\theta}$ takes values in $\mathfrak{spin}(7)$ and $\boldsymbol{u} = {}^t\boldsymbol{u}$. Note that

$$-4\,\omega_1\,\omega_3 + {\omega_2}^2 + \boldsymbol{\omega}\cdot\boldsymbol{\omega} = -4\,dx_1\,dx_3 + {dx_2}^2 - 4g\,{dx_3}^2 + \boldsymbol{\eta}\cdot\boldsymbol{\eta}. \tag{161}$$

I now want to describe how these formulae give a recipe for writing down all of the solutions to our problem.

By the last of the structure equations, the eight components of $\boldsymbol{\eta}$ describe an integrable system of rank 8 that is (locally) defined on the original 11-manifold. Let us restrict to a neighborhood where the leaf space of $\boldsymbol{\eta}$ is simple, i.e., is a smooth manifold K^8. The equation $d\boldsymbol{\eta} = -\big(\boldsymbol{\theta} - 2\boldsymbol{u}\,dx_3\big)\wedge\boldsymbol{\eta}$ shows that on $\mathbb{R}\times K^8$, with coordinate x_3 on the first factor, there is a $\{1\}\times\mathrm{Spin}(7)$-structure, which can be thought of as a 1-parameter family of torsion-free $\mathrm{Spin}(7)$-structures on K^8 (the parameter is x_3, of course).

This 1-parameter family is not arbitrary because the matrix $\boldsymbol{u}$ is symmetric. This condition is equivalent to saying that if Φ is the canonical $\mathrm{Spin}(7)$-invariant 4-form (depending on x_3, of course) then

$$\frac{\partial\Phi}{\partial x_3} = \lambda\,\Phi + \Upsilon \tag{162}$$

for some function λ on $\mathbb{R}\times K^8$ and Υ is an anti-self dual 4-form (via the x_3-dependent metric on the fibers of $\mathbb{R}\times K\to\mathbb{R}$, of course). It is not hard to see that this is seven equations on the variation of torsion-free $\mathrm{Spin}(7)$-structures and that, moreover, given any 1-parameter variation of torsion-free $\mathrm{Spin}(7)$-structures, one can (locally) gauge this family by diffeomorphisms preserving the fibers of $\mathbb{R}\times K\to\mathbb{R}$ so that it satisfies these equations. (In fact, if K is compact and the cohomology class of Φ in $H^4(K,\mathbb{R})$ is independent of x_3 then this can be done globally.) Call such a variation *conformally anti-self dual.*

Now from the above calculations, this process can be reversed: One starts with any conformally anti-self dual variation of Spin(7)-structures on K^8, then on $\mathbb{R}^3 \times K$ one forms the Lorentzian metric

$$ds^2 = -4\,dx_1\,dx_3 + {dx_2}^2 - 4g\,{dx_3}^2 + \boldsymbol{\eta}\cdot\boldsymbol{\eta} \tag{163}$$

where g is any function on $\mathbb{R}^3 \times K$ that satisfies $\partial g/\partial x_1 = 0$ and $\boldsymbol{\eta}\cdot\boldsymbol{\eta}$ is the x_3-dependent metric associated to the variation of Spin(7)-structures. Then this Lorentzian metric has a parallel null spinor. For generic choice of the variation of Spin(7)-structures and the function g, this will yield a Lorentzian metric whose holonomy is the desired stabilizer group of dimension 30. This can be seen by combining the standard generality result **[6]** for Spin(7)-metrics on 8-manifolds, which shows that for generic choices as above the curvature tensor has range equal to the full $\rho'(\mathfrak{h})$ at the generic point, with the Ambrose-Singer holonomy theorem, which implies that such a metric will have its holonomy equal to the full group of dimension 30.

In particular, it follows that, up to diffeomorphism, the local solutions to this problem depend on one arbitrary function of 10 variables. One can show that such a solution is not, in general, Ricci-flat, in contrast to the case where a $(10,1)$-metric has a non-null parallel spinor field.

Note, by the way, that the 4-form Φ will not generally be closed, let alone parallel. However, the 5-form $dx_3{\wedge}\Phi$ will be closed and parallel. The other non-trivial parallel forms are the 1-form dx_3, the 2-form $dx_2{\wedge}dx_3$, and the 6-, 9-, and 10-forms that are the duals of these.

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