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ON THE KATO INEQUALITY IN RIEMANNIAN GEOMETRY

by

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Abstract. — We describe recent works of the authors as well as of T. Branson on refined Kato inequalities for sections of vector bundles living in the kernel of natural first-order elliptic operators

***Résumé* (Sur l'inégalité de Kato en géométrie riemannienne).** — Nous faisons le point sur des travaux récents, dus aux auteurs et aussi à T. Branson, sur des raffinements de l'inégalité de Kato, valables pour des sections d'un fibré vectoriel annulées par un opérateur différentiel naturel et elliptique du premier ordre.

1. Introduction

The Kato inequality is a classical tool in Riemannian geometry. It stands as a useful way to relate vector-valued problems on vector bundles to scalar valued ones involving only functions. It says that for a smooth section ξ of a Riemannian vector bundle E equipped with a compatible connection ∇,

$$\big|d|\xi|\big| \;\leq\; |\nabla\xi|$$

outside the zero-set of ξ. This is an easy consequence of the Schwarz inequality.

More surprisingly, some authors noticed that refined Kato inequalities, of the type

$$\big|d|\xi|\big| \;\leq\; k\,|\nabla\xi| \quad \text{with} \quad k < 1\ ,$$

were true for ξ in the kernel of an elliptic first-order differential operator acting on sections of E. This remark was a crucial step in a number of problems involving either decay estimates at infinity of the norm of sections satisfying an elliptic equation (curvature of Einstein metrics on asymptotically flat manifolds, second form of minimal hypersurfaces in spaceforms, Yang-Mills fields on the flat four-space, etc...) or fine-tuned spectral problems.

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The constants k that were found depended strongly on the elliptic operators involved and it was observed that there should exist a systematic way to detect and compute them and that there should be a strong link between their values and representation-theoretic data of the given bundle.

At the time of the meeting in Marseille, we had devised a method leading to computations of optimal refined Kato inequalities in a few cases including all possible situations in dimensions 3 and 4 and a talk on that subject was delivered by the third author. The method was extended shortly after to a systematic one that computes almost all the possible constants and a large number of explicit values were then given [**6**]. During the same period, T. Branson independently found a different method to compute all of them [**5**], based on his earlier works on the spectrum of elliptic second-order differential operators on the round sphere [**4**]. We intend here to report on these two methods, and try to highlight their differences and their relationships. We shall also give a few examples of old and new uses of refined Kato inequalities.

We have tried to make this survey accessible for a reader not acquainted with slightly specialized tools of representation theory (all of which may however be found in the textbook [**8**]). This led us to be somehow imprecise or unspecific at some places in the main body of this text. We thought however that this could be useful for those that were interested rather in the results or the applications of refined Kato inequalities in global analysis on manifolds rather than in the precise course of the proofs. Appendices have been added at the end, containing more elaborate details and precise computations. We then hope that this text may serve as a reading guide before entering the two more technical papers [**5**] and [**6**].

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2. Basics: the classical Kato inequality

We consider from now on an oriented Riemannian manifold M endowed with a vector bundle E induced from a representation of the special orthogonal group $\mathrm{SO}(n)$ or the spin group $\mathrm{Spin}(n)$ (in which case M will be supposed to be spin). If ∇ is any metric connection on E and ξ is any section of E, then

$$2\,\big|d|\xi|\big|\,|\xi| \;=\; \big|d\left(|\xi|^2\right)\big| \;=\; 2\,\big|\langle\nabla\xi,\xi\rangle\big| \;\leq\; 2\,|\nabla\xi|\,|\xi|$$

(with the metric on $T^*M\otimes E$ given by the tensor product metric). Hence we get the classical Kato inequality

$$\big|d|\xi|\big| \;\leq\; |\nabla\xi| \tag{1}$$

outside the zero set of ξ. Moreover the equality case is achieved at a point if and only if there is a 1-form α such that

$$\nabla\xi \;=\; \alpha\otimes\xi.$$

Following J. P. Bourguignon [**3**], we now consider a section ξ lying in the kernel of a natural first-order operator P on E. Any such operator is the composition of the covariant derivative followed by projection Π on one (or more) irreducible components of the bundle $T^*M\otimes E$, and its symbol reads: $\sigma(P)=\sigma(\Pi\circ\nabla)=\Pi$. Now assume (1) is optimal at some point. The discussion above shows that $\nabla\xi \;=\; \alpha\otimes\xi$ at that point. But

$$0 \;=\; P\xi \;=\; \Pi\circ\nabla\,\xi=\Pi(\alpha\otimes\xi).$$

Thus, optimality is possible if and only if P is not an elliptic operator. Conversely, one might guess that any elliptic operator P gives rise, for any section ξ in its kernel, to a refined Kato inequality

$$|d\xi| \;\leq\; k_P\,|\nabla\xi| \tag{2}$$

with a constant k_P depending only on the operator P involved.

3. Background: conformal weights

We consider an irreducible natural vector bundle E over a Riemannian manifold (M,g) of dimension n, with scalar product $\langle\cdot,\cdot\rangle$ and a metric (not necessarily Levi-Civita) connection ∇. By assumption, E is associated to an irreducible representation λ of the group $\mathrm{SO}(n)$ (resp. $\mathrm{Spin}(n)$ if necessary). The tensor product of λ with the standard representation τ splits in irreducible components as $\tau\otimes\lambda=\oplus_{j=1}^N\mu_j$. Equivalently, and to set notations, we write

$$T^*M\otimes E=\oplus_{j=1}^N F_j.$$

Projection on the j-th summand will be denoted by Π_j. Apart from the exceptional case where $T^*M\otimes E$ contains two irreducible components for $\mathrm{SO}(n)$ whose sum is an irreducible representation for $\mathrm{O}(n)$, each F_j is an eigenspace for the endomorphism B of $T^*M\otimes E$ defined as

$$B(\alpha\otimes v)=\sum_{i=1}^n e_i\otimes(e_i\wedge\alpha)\cdot v$$

where the dot means the action of $\mathfrak{so}(n)$ on the representation space E. The endomorphism B plays an important role in conformal geometry [**9**]. Its eigenvalues are called the *conformal weights*, denoted w_j, and can be easily computed from representation-theoretic data : the Casimir numbers [**8**] of representations λ, τ and μ_j (normalized as to ensure $C(\mathfrak{so}(n),\tau)=n-1$, see Appendix A for more on this point). More precisely:

$$w_j=\frac{1}{2}\left(C(\mathfrak{so}(n),\mu_j)-C(\mathfrak{so}(n),\lambda)-C(\mathfrak{so}(n),\tau)\right).$$

We shall adopt here the convention *not to split* irreducible representations of $\mathrm{O}(n)$ inside $\tau\otimes\lambda$ into irreducibles for $\mathrm{SO}(n)$. This ensures the conformal weights are *always distinct*, henceforth F_j will always denote the eigenspace associated to w_j, and it corresponds to an irreducible summand of $\tau\otimes\lambda$ except in the exceptional case quoted above where it is a sum of two irreducibles. Moreover, irreducible components will be ordered from 1 to N (the number of distinct eigenspaces) in (strictly) decreasing order of conformal weights (see Appendix A for more details on the representation theory involved).

Since they are easily computable, all the results that follow will be given in terms of the conformal weights, or more precisely in terms of the *modified conformal weights* $\widetilde{w}_j = w_j + (n-1)/2$, eigenvalues of the translated operator $\widetilde{B} = B + (n-1)/2$ id.

Natural first order differential are indexed by subsets I of $\{1,\ldots,N\}$. They all are of the following form:

$$P_I = \sum_{i\in I} a_i\,\Pi_i\circ\nabla\ ;$$

any such operator is said to be (injectively, or overdetermined) elliptic if its symbol $\Pi_I = \sum_{i\in I} a_i\Pi_i$ does not vanish on any decomposable element $\alpha\otimes v$ of $T^*M\otimes E$. The coefficients a_i can all be set to 1 without harm as lying in the kernel of the operator is equivalent to lying in the intersection of the kernels of all the elementary operators $P_i = \Pi_i\circ\nabla$ for i in I and being elliptic is equivalent to the fact that no decomposed tensor product lives in the intersection of the kernels of the Π_i.

Elliptic operators in this precise sense have been completely classified by T. Branson in [**4**]. Since any set J containing a subset I such that P_I is elliptic gives rise to an operator P_J which is also elliptic, it suffices to describe the set of *minimal elliptic operators*, *i.e.* the set of operators P_I such that P_J is not elliptic for any proper subset J of I. T. Branson's result provides an explicit description of this set (see Appendix B for more details). For example, the highest weight operator $P_{\{1\}}$ is always minimal elliptic. Moreover and quite surprisingly, sets of indices corresponding to minimal elliptic operators are always small: in fact they contain at most one or two elements.

Our guiding philosophy will now be to prove refined Kato inequalities for sections lying in the kernels of natural first-order elliptic operators on E, with the constants given in terms of the (modified) conformal weights. It is an intersting feature of the problem to note that two genuinely different methods lead to the results. Both end up with semi-explicit expressions of the constants, which can be obtained by solving a minimization problem over a finite set of real numbers. The results can then be made completely explicit in a large number of cases.

The first method, devised by the authors, can be considered as the *local method.* It relies on elaborate algebraic considerations on the conformal weights together with a "linear programming" problem. It is sharp and also provides an explicit description of the sections satisfying equality in the refined Kato inequality at each point. It has the unfortunate feature of being non-sharp for some small (precisely known) set of operators, hopefully seldom encountered in practice.

The second one, or the *global method*, is due to T. Branson. It gives a refined Kato inequality in every case, sharpness is also clear but the equality cases' description is less precise. The proofs rely on the spectral computations on the round sphere done in [**4**] using powerful techniques of harmonic analysis, together with a clever elementary lemma that relates the knowledge of the spectrum of an operator to information on its symbol.

4. Kato constants: linear programming method of computation

The local method finds its roots in the proof of the classical Kato inequality: it aims at obtaining a refined Schwarz inequality for

$$\left|\langle \nabla\xi, \xi\rangle\right|$$

when ξ is a section lying in the kernel of an elliptic first-order operator P_I.

4.1. Ansatz. — Consider Φ an element of $\ker \Pi_I$ at some point (as is $\nabla\xi$ at each point) and v an element of E at the same point (as is ξ). We let I a subset of $\{1,\dots,N\}$, denote by $\widehat{I}$ its complement in $\{1,\dots,N\}$ and compute

$$\begin{aligned}\sup_{|v|=1}\left|\langle\Phi, v\rangle\right| &\leq \sup_{|\alpha|=|v|=1}\left|\langle\Phi,\alpha\otimes v\rangle\right| = \sup_{|\alpha|=|v|=1}\left|\langle\Phi,\Pi_{\widehat{I}}(\alpha\otimes v)\rangle\right| \\ &\leq \left(\sup_{|\alpha|=|v|=1}|\Pi_{\widehat{I}}(\alpha\otimes v)|\right)|\Phi|. \end{aligned} \tag{3}$$

This gives a refined Kato inequality with $k_I = \sup_{|\alpha|=|v|=1}|\Pi_{\widehat{I}}(\alpha\otimes v)|$. Moreover, equality holds in it if and only if it holds in the refined Schwarz inequality with $v=\xi$ and $\Phi=\nabla\xi$. Hence it is algebraically sharp since the supremum is always attained by compactness. If equality holds, then $\nabla\xi = \Pi_{\widehat{I}}(\alpha\otimes\xi)$ for some $\alpha\otimes\xi$ such that $|\Pi_{\widehat{I}}(\alpha\otimes\xi)|$ is maximal among all $|\Pi_{\widehat{I}}(\alpha\otimes v)|$ with $|\alpha|=|v|=1$. Moreover such a situation can easily be achieved in the flat case with a suitable affine solution of $P_I\xi=0$.

4.2. Resolution of the problem. — We now follow the standard method of Lagrange interpolation. Each projection Π_j can be written as

$$\Pi_j = \prod_{k\neq j}\frac{\widetilde{B}-\widetilde{w}_k\,\mathrm{id}}{\widetilde{w}_j-\widetilde{w}_k} = \frac{\sum_{k=0}^{N}\widetilde{w}_j^{N-1-k}\left(\sum_{\ell=0}^{k}(-1)^{\ell}\sigma_\ell(w)\widetilde{B}^{k-\ell}\right)}{\prod_{k\neq j}(\widetilde{w}_j-\widetilde{w}_k)},$$

where $\sigma_i(w)$ denotes the i-th elementary symmetric function of the modified weights (as it will appear below, it is much easier to work with the modified rather than the original weights). Defining $\widetilde{A}_k$ as the operators

$$\widetilde{A}_k \;=\; \sum_{\ell=0}^{k}(-1)^{\ell}\sigma_{\ell}(w)\widetilde{B}^{k-\ell},$$

we end up with

$$|\Pi_j(\alpha\otimes v)|^2 = \big|\langle \Pi_j(\alpha\otimes v), \alpha\otimes v\rangle\big| = \frac{\sum_{k=0}^{N}\widetilde{w}_j^{N-1-k}\langle \widetilde{A}_k(\alpha\otimes v), \alpha\otimes v\rangle}{\prod_{k\neq j}(\widetilde{w}_j - \widetilde{w}_k)}.$$

This formula for the N quantities $|\Pi_j(\alpha\otimes v)|^2$ in terms of the other N quantities $Q_k = \langle \widetilde{A}_k(\alpha\otimes v), \alpha\otimes v\rangle$ is the heart of our method. But the crucial step comes from a careful and quite technical analysis of the modified conformal weights. As is shown in [**6**], they are intimately related with important representation-theoretic data of the Lie algebra $\mathfrak{so}(n)$ called *higher order casimir operators*. These are elements of the centre of the universal enveloping algebra $\mathcal{U}(\mathfrak{so}(n))$, which thus act on every irreducible representation of $\mathrm{SO}(n)$ homothetically. These operators (or more precisely the value of the ratio of the homothety on each representation λ) can be explicitely computed from the knowledge of the operator $\widetilde{B}$ associated to λ (this provides an alternative proof of old results due to Perelomov and Popov [**13**]). This leads in turn to a precise expression for the traces of the operators $\widetilde{B}^k$, hence for those of $\widetilde{A}_k$.

A result due to T. Diemer and G. Weingart (private communication) proves that each family of polynomials in $\widetilde{B}$ satisfying some special recurrence formula involving their traces has nice symmetry properties. The preceding computations show that a simple function of the $\widetilde{A}_k$ satisfies the recurrence formula. The output of this technical analysis is:

Lemma 1. — *If N is odd, then $Q_{2j+1} = 0$ for every j. If N is even, then $2\,Q_{2j+1} + Q_{2j} = 0$ for every $j \geq 1$.*

This enables us to eliminate approximately half of the Q_k's in the expressions of $|\Pi_j(\alpha\otimes v)|^2$ given above. This Lemma stands as the main reason for using the modified weights rather than the original weights (see Appendix C for more details).

Each quantity $|\Pi_j(\alpha\otimes v)|^2$ is then given as an affine function in the remaining variables $Q_k = \langle \widetilde{A}_k(\alpha\otimes v), \alpha\otimes v\rangle$. To avoid confusion, we now denote by $p_j(Q)$ this affine function of the Q_k's it defines. Following the Ansatz above, our main goal is now to find a supremum of the affine function

$$\sum_{i\in\widehat{I}} p_i = 1 - \sum_{i\in I} p_i$$

over some subset of the points $Q = (Q_k)$: precisely those such that there exists unit α and v such that $Q_k = \langle \widetilde{A}_k(\alpha\otimes v), \alpha\otimes v\rangle$ for each k (we shall call these points *admissible* points).

Fortunately, this subset turns out to be contained in a compact convex polyhedron in the Q-space: this comes from noticing that each norm $|\Pi_i(\alpha\otimes v)|^2 = p_i(Q)$ is non-negative (and also no larger than 1, but this is a redundant information) if Q is an admissible point.

Consider then an elliptic operator P_I. We will now estimate the supremum of the affine function

$$\sum_{i\in\widehat{I}} p_i = 1 - \sum_{i\in I} p_i$$

over the polyhedron. Its extremal values are then achieved when the family of affine hyperplanes it is associated with in the space of the variables (Q_k) touches for the first or last time the convex polyhedron. Hence, they are surely achieved at some vertex of the polyhedron and we are now reduced to maximize the affine function over the vertices of the polyhedron.

The next step relies on the fact that the vertices of the polyhedron are easy to describe: they are points where a maximal number of functions $p_j = |\Pi_j(\alpha\otimes v)|^2$ vanish. Among them are certainly the following admissible points: if J is the index set of a non-elliptic operator of maximal length (*i.e.* involving a maximal number of projections), there is α and v such that $\Pi_j(\alpha\otimes v) = 0$ for each j in J. In other words, the point q which is uniquely determined by the equations

$$p_j(q) = 0, \quad \forall j\in J, \quad \text{with} \quad P_J \quad \text{maximal non-elliptic},$$

is both an admissible point and a vertex. Non-elliptic operators of maximal length are easy to determine from T. Branson's work [**4**] (a complete list of these is given in [**6**]), and it turns out that, in almost all cases, one can show that these form exactly the set of vertices. In all these cases the sought supremum on the polyhedron is attained at some vertex and since each of these is admissible, there exists a decomposed tensor $\alpha\otimes v$ such that $\Pi_J(\alpha\otimes v) = 0$ (the set J corresponding to the vertex is non-elliptic). As a result the estimate is sharp and equality case is achieved if and only if $\nabla\xi = \alpha\otimes\xi$ with $\Pi_{I\cup J}(\alpha\otimes\xi) = 0$. However, here comes the main problem of our method: there are some special, seldom encountered, circumstances where the vertices *do contain* a few points corresponding to elliptic operators. In this case, if the supremum is achieved at such a "bad point" (corresponding to index set J), it will not be sharp since there does not exist any decomposed tensor such that $\Pi_J(\alpha\otimes v) = 0$ and the infimum of $\sum_I p_i$ on the polyhedron is smaller than the infimum of $|\Pi_I(\alpha\otimes v)|^2$ on all unit α and v.

The explicit values of the norms $|\Pi_i(\alpha\otimes v)|^2$ at each vertex turn out to be easily expressible and we are now in a position to state our main formula. We denote by $\mathcal{NE}$ the set of (sets of indices corresponding to) vertices (a complete list of these is given in [**6**], see also Appendix B) and we let, for J a subset of $\{1,\dots,N\}$ and i an element of $\{1,\dots,N\}$, $\varepsilon_i(J)$ be 0 if i belongs to J and 1 if not. Then we can state :

Theorem 1. — *Let I a subset of $\{1,\ldots,N\}$ corresponding to an elliptic operator P_I acting on E. Then a refined Kato inequality $\big|d|\xi|\big| \leq k_I|\nabla\xi|$ holds for any section ξ in the kernel of P_I, outside the zero set of ξ.*

If N is odd, then

$$k_I^2 = 1 \;-\; \inf_{J\in\mathcal{NE}}\left(\sum_{i\in I}\frac{\prod_{j\in J\setminus\{i\}}(\widetilde{w}_i+\widetilde{w}_j)}{\prod_{j\in\widehat{J}\setminus\{i\}}(\widetilde{w}_i-\widetilde{w}_j)}\varepsilon_i(J)\right). \tag{4}$$

These results are sharp except if n is odd where some "bad cases" may appear.

If N is even, then

$$k_I^2 = 1 \;-\; \inf_{J\in\mathcal{NE}}\left(\sum_{i\in I}\left(\widetilde{w}_i-\frac{1}{2}\right)\frac{\prod_{j\in J\setminus\{i\}}(\widetilde{w}_i+\widetilde{w}_j)}{\prod_{j\in\widehat{J}\setminus\{i\}}(\widetilde{w}_i-\widetilde{w}_j)}\varepsilon_i(J)\right). \tag{5}$$

This result is always sharp.

For sake of simplicity, we have not reproduced here the precise characterization of the "bad set" of operators. Moreover, this theorem is a slightly simplified and weakened version of the main result of [**6**], since, in the *a priori* non-sharp cases singled out above, there are still a lot of operators where one can show that the infimum is not achieved at a "bad vertex". Hence our approach leads to sharp estimates. The interested reader should find in Appendix A and B or in [**6**] all the details needed to understand the "bad set" of operators.

There are a lot of cases where the constant k_I can be more explicitely derived. The idea is always to guess which is the vertex of the polyhedron where the minimum of the function $\sum_I p_i$ is achieved. Such a work can be done without too much effort for a set of indices I (or, equivalently, operators) containing, among others, the highest weight set $\{1\}$, its complement, and all minimal elliptic operators (except one in the "bad" case). It should be noticed that both the method and the results are straightforward and simple-minded if the number N of components of $T^*M\otimes E$ is small, *e.g.* if $N\leq 4$, which is the general case in applications.

We shall give here the values of the constants for the highest weight (or twistor) elliptic operator P_1 and we refer to [**6**] or [**5**] for more complete expressions. In each case, the value is optimal and the pointwise equality case may be studied precisely, following the guiding philosophy given in the Ansatz 4.1.

Theorem 2. — *Let $\rho = E(N/2)$. If N is odd, then*

$$k_{\{1\}}^2 = 1 - \frac{\prod_{k=\rho+2}^{N}(\widetilde{w}_1+\widetilde{w}_k)}{\prod_{k=2}^{\rho+1}(\widetilde{w}_1-\widetilde{w}_k)}. \tag{6}$$

If N is even, then

$$k_{\{1\}}^2 = 1 - (\widetilde{w}_1-\frac{1}{2})\frac{\prod_{k=\rho+2}^{N}(\widetilde{w}_1+\widetilde{w}_k)}{\prod_{k=2}^{\rho+1}(\widetilde{w}_1-\widetilde{w}_k)}. \tag{7}$$

5. Kato constants: spectral method

The method devised by T. Branson relies on his explicit description of the spectrum of operators of type $P_I^* P_I$ on the standard sphere S^n, obtained by harmonic analysis techniques [**4**]. The link with refined Kato inequalities is provided by the following Lemma, which turns global estimates (on the spectrum of an elliptic second order operator) into pointwise infinitesimal ones (on the symbol of the operator).

Lemma 2 **([5])**. — *Suppose D is a (self-adjoint) second-order differential operator on E. If there is a constant η and a positive real number ε such that, for any smooth compactly supported section φ,*

$$\langle D\varphi, \varphi\rangle_{L^2} \ \geq\ \varepsilon \langle \nabla^*\nabla\varphi, \varphi\rangle_{L^2} - \eta \langle \varphi, \varphi\rangle_{L^2}, \tag{8}$$

then the symbol $\sigma_\alpha(D) - \varepsilon|\alpha|^2 \,\mathrm{id}$ is a nonnegative symmetric endomorphism of E for any 1-form α.

Proof. — Let φ, ψ be smooth functions on M. Then apply the estimate (8) above to $h = e^{it\varphi}\psi$, divide by t^2 and let t go to infinity. We get

$$\langle \sigma_{d\varphi}(D)\psi, \psi\rangle_{L^2} \ \geq\ \varepsilon\, |d\varphi|^2 \,\langle \psi, \psi\rangle_{L^2}.$$

Taking ψ as a member of a family of cut-off functions whose supports converge to a single point gives the pointwise estimate on the symbol. □

If we could apply the lemma to $D = P_I^* P_I$ (keeping the same notations as in the previous sections), we would get that the symbol

$$\sigma_\alpha(P_I^* P_I) - \varepsilon|\alpha|^2 \,\mathrm{id}$$

is a nonnegative map of E. Hence for any section ξ, and any 1-form α,

$$\begin{aligned} 0 \ &\leq\ \langle \sigma_\alpha(P_I^* P_I)\xi, \xi\rangle \ -\ \varepsilon\,|\alpha|^2\,|\xi|^2 \\ &=\ |\Pi_I(\alpha\otimes\xi)|^2 \ -\ \varepsilon\,|\alpha|^2\,|\xi|^2 \ . \end{aligned}$$

This can of course be rewritten as

$$|\Pi_{\hat{I}}(\alpha\otimes\xi)|^2 \ \leq\ (1-\varepsilon)\,|\alpha|^2\,|\xi|^2 \tag{9}$$

and we recognise here the desired inequality of Ansatz 4.1.

This reduces the problem of finding refined Kato inequalities to the problem of comparing spectra of second order differential operators to that of the rough Laplacian of E. This can be done on the sphere $S^n = \mathrm{SO}(n+1)/\mathrm{SO}(n)$, where all operators involved may be described completely algebraically. In [**4**], T. Branson computed the spectrum and eigenspaces of any operator of type $P_i^* P_i$ (i in $\{1, \ldots, N\}$) on the sphere: if λ is the representation attached to E, all of them are diagonalized by the decomposition of the space of L^2-sections into the Hilbert sum of

$$\mathcal{V}(\chi, \lambda) \ = \chi \otimes \mathrm{Hom}^{\mathfrak{so}(n)}(\chi, \lambda).$$

where χ runs over all representations for $\mathfrak{so}(n+1)$ such that λ appears in the decomposition of χ into $\mathfrak{so}(n)$-irreducible components. The values of the eigenvalues are given fairly explicitely in terms of the weights λ and χ and the modified conformal weights but the work requires extensive use of powerful techniques of harmonic analysis (see Appendix D for details and the exact values of the eigenvalues).

It is then possible to find spectral estimates of the type (8) for the pair of operators $P_I^* P_I$ and $\nabla^*\nabla$ (note that the constant η appears there to take into account the fact that non-parallel sections may exist in the kernel of P_I). One obtains this way the constants appearing in the refined Kato inequalities in a semi-explicit form as the solution of a minimizing problem. Whereas the local method leads to minimization over a set of vertices, *i.e.* is based on the selection rule that gives the irreducible components of the tensor $\tau \otimes \lambda$, T. Branson's methods leads to a minimizing process based on the branching rule, *i.e.* the rule that gives the components of the representation λ seen as a module for the smaller Lie algebra $\mathfrak{so}(n-1)$ (further details and explicit formulas are given in Appendix D).

This should not come as a surprise: to increase intuition on this phenomenon, let us recall that ellipticity of an operator P_I can be reinterpreted in terms of representations of $\mathfrak{so}(n-1)$. Indeed, by naturality (equivariance under the group $\mathrm{SO}(n)$ or $\mathrm{Spin}(n)$) and the transitive action of $\mathrm{SO}(n)$ on the round sphere in $\mathbb{R}^n$, the symbol Π_I of P_I never vanishes on the decomposed tensors $\alpha \otimes v$ if and only if the $\mathrm{SO}(n-1)$-equivariant homomorphism form V to $\mathbb{R}^n \otimes V$ defined by $v \longmapsto \Pi_I(e \otimes v)$ (where e is an arbitrary unit vector in $\mathbb{R}^n$) is an injective map. Hence, $\mathrm{SO}(n-1)$-representations naturally enter investigations of ellipticity of natural first-order operators and related questions.

Although appearing in a different form, the constants that arise this way are the same as the ones found by the direct method. This is likely to confirm the intuition gained in the previous sections that the inequalities find their equality cases in the flat (or in the conformally equivalent round sphere) case. As the local method, the global one is sharp. This relies on the following remarkable fact: if $k' = 1 - \varepsilon'$ –in the notation of (9)– was a better Kato constant than $k = 1 - \varepsilon$ found by the above procedure, formula (9) would imply that the operator $P_I^* P_I - \frac{\varepsilon + \varepsilon'}{2} \nabla^*\nabla$ is elliptic with positive definite symbol. It should thus have only a finite number of negative eigenvalues. But the explicit computations of **[4]** show that this is not the case on the round sphere S^n as soon as $\varepsilon < \varepsilon'$.

Once again the semi-explicit expressions can be made fully explicit in a number of cases (it does not come as a surprise to notice that these are more or less the same as the ones that could be handled completely by the local method). As above we shall give one explicit value, this time for the Rarita-Schwinger operator in odd dimensions **[12]** (if $n = 3$ this is an example of an operator to which the local method does not apply, the local method however works for every other odd dimension but needs an *ad* *hoc* substitute for dimension 3) and we refer to [**5**] or [**6**] for all other explicit values and computations.

Theorem 3. — *Let E be the twistor bundle of an odd-dimensional spin manifold M^n. Then, for any section ψ of E in the kernel of the Rarita-Schwinger operator, we have the refined Kato inequality*

$$\big|d|\psi|\big| \;\leq\; \frac{n-2}{n(n+2)}\,|\nabla\psi|$$

outside the zero-set of ψ.

6. Epilogue: some old and new uses of Kato refined inequalities

We collect here a few uses of refined Kato inequalities. As it is easily understood from the previous proofs, the inequalities show up whenever one uses a metric connection, but *not necessarily the Levi-Civita connection*, and the value of the constant only depends on the principal symbol of the operator involved. Hence they apply to a very large number of operators and admit a wide range of applications.

For brevity's sake, we have not tried to establish a full list of occurrences of such refined inequalities, but have rather tried to detail three very different circumstances where they already happened to be useful. It would certainly be desirable to find new ones. From the beginning, we have restricted ourselves to a purely Riemannian setting, *i.e.* $\mathrm{SO}(n)$-equivariant operators, but there is little doubt that analogous Kato inequalities could be found with special holonomy reductions. This may open up further opportunities of applications.

6.1. Subelliptic estimates. — Consider a (usually complete, non-compact) manifold M and a section ξ of a bundle E lying in the kernel of some natural first-order elliptic operator P. We moreover assume that P is part of a Weitzenböck formula:

$$P^*P = \nabla^*\nabla + \mathcal{R} \tag{10}$$

where $\mathcal{R}$ is a curvature term. Standard computations then show that

$$\langle \xi, \Delta\xi\rangle - |\nabla\xi|^2 = \frac{1}{2}\Delta|\xi|^2 = |\xi|\,\Delta|\xi| - \big|d|\xi|\big|^2. \tag{11}$$

Substracting the latter from the former and taking into account the Weitzenböck formula (10) and the classical Kato inequality yields the socalled *subelliptic estimate*

$$\Delta|\xi| \;\leq\; |\mathcal{R}|\,|\xi| \quad \text{outside } \{\xi = 0\}. \tag{12}$$

If the manifold has a non-zero isoperimetric constant, the Moser iteration scheme shows that $|\xi|$ behaves at infinity (with respect to the geodesic distance r to a fixed point) as $O(r^{-2})$. Now a refined Kato inequality of the type

$$\big|d|\xi|\big| \leq k|\nabla\xi| \quad \text{with } k<1,$$

leads to a substantially improved version of (12):

$$\Delta\left(|\xi|^{2-1/k^2}\right) \;\le\; \left(2-\frac{1}{k^2}\right)|\mathcal{R}|\,|\xi|^{2-\frac{1}{k^2}} \quad \text{outside } \{\xi=0\} \tag{13}$$

and Moser iteration procedures produces better decay estimates, for instance such as $\xi = O(r^{-2-\varepsilon})$ with $\varepsilon > 0$ around infinity.

Though history may be difficult to trace back, it seems that the subharmonicity property (in case the curvature term $\mathcal{R}$ vanishes) was first remarked in the foundational paper of E. Stein and G. Weiss [**17**] (see alo [**16**]). The full argument has been used successfully in a number of cases, for example in S. Bando, A. Kasue and H. Nakajima's study of Ricci-flat maximal volume growth complete Riemannian manifolds (applied to the Weyl curvature, closed and co-closed if the metric is Einstein) [**1**], in R. Schoen, L. Simon and S.-T. Yau's work on the Bernstein problem (applied to the second fundamental form of a minimal immersion in flat space) [**15**], in S.-T. Yau's proof of the Calabi conjecture [**18**], in J. Råde's study of Yang-Mills fields on flat four-space [**14**], and in P. Feehan study of PU(2)-monopoles and harmonic spinors for the Spin^c-Dirac operator [**7**] (notice that this is an example where the connection is not Levi-Civita but where our computations still apply), etc.

6.2. Spectral problems. — The refined Kato inequality for spinors in the kernel of the Dirac operator leads to a new proof of the well-known *Hijazi inequality* relating the first eigenvalue of the Dirac operator to the first eigenvalue of the conformally covariant Yamabe operator. We thank C. Bär and A. Moroianu (private communication) who suggested this application and kindly accepted to let it be reproduced here.

***Theorem 4* (Hijazi [11]).** — *Let (M,g) be a compact spin Riemannian manifold of dimension $n \ge 3$. Then the first eigenvalue λ_1 of the Dirac operator and the first eigenvalue μ_1 of the conformal Laplacian $4\frac{n-1}{n-2}\Delta + \mathrm{scal}$ satisfy:*

$$\lambda_1^2 \ge \frac{n}{4(n-1)}\,\mu_1. \tag{14}$$

Proof. — If ψ is an eigenspinor with eigenvalue λ, then ψ lies in the kernel of the Dirac operator given by the Friedrich connection $\tilde\nabla_X\psi = \nabla_X\psi + (\lambda/n)X\cdot\psi$, which is a metric connection on spinors. Hence we have the following refined Kato inequality for ψ, wherever it is nonzero:

$$\left|d|\psi|\right|^2 \le \frac{n-1}{n}|\tilde\nabla\psi|^2. \tag{15}$$

We next consider the conformal Laplacian of $|\psi|^{2\alpha}$ where $\alpha = n-2/2(n-1)$: the conformal Laplacian is invariant on scalars of weight $2-n/2$ and so this power is natural in view of the conformal weight $1-n/2$ for the Dirac operator. Using the

Lichnerowicz formula, the elementary identity

$$d^*d(f^\alpha) = \alpha f^{\alpha-1} d^*df - \alpha(\alpha-1) f^{\alpha-2}|df|^2$$

with $f = |\psi|^2$ and $|\tilde{\nabla}\psi|^2 = |\nabla\psi|^2 - \frac{1}{n}\lambda^2|\psi|^2$, we obtain the following equality on the open set where ψ is nonzero:

$$\tfrac{1}{2\alpha} d^*d\big(|\psi|^{2\alpha}\big) + \tfrac{1}{4}\,\mathrm{scal}\,|\psi|^{2\alpha} - \tfrac{n-1}{n}\lambda^2|\psi|^{2\alpha} = |\psi|^{2\alpha-2}\Big(\tfrac{n}{n-1}\big|d|\psi|\big|^2 - |\tilde{\nabla}\psi|^2\Big).$$

This is nonpositive by (15). In order to globalize, we consider the Rayleigh quotient for the first eigenvalue μ_1 of the conformal Laplacian with test-function $\varphi = |\psi|^{2\alpha}$ on the open set where ψ is nonzero, take $\lambda = \lambda_1$ and integrate over $\{x, |\psi|(x) \geq \epsilon\}$. Letting $\epsilon \to 0$ easily gives (14). □

6.3. Special properties of Einstein metrics. — Building on the computations done in section 6.1 above, one may derive from refined Kato inequalities some powerful integral estimates on the curvature of Einstein metrics. Following M. Gursky and C. LeBrun [**10**], the refined Kato inequality for the co-closed positive half Weyl tensor of an Einstein four-dimensional manifold (M,g) (outside its zero set):

$$\big|d|W^+|\big| \leq \sqrt{\frac{3}{5}}\,|\nabla W^+|, \tag{16}$$

shows that the function $u = |W^+|^{1/3}$ satisfies:

$$6\Delta u + (\mathrm{scal}_g - 2\sqrt{6}|W^+_g|)u \;\; \leq \;\; 0.$$

Hence there exists a metric $\widehat{g}$ in the conformal class of g such that

$$\int_M \Big(\mathrm{scal}_{\widehat{g}} - 2\sqrt{6}|W^+_{\widehat{g}}|_{\widehat{g}}\Big) \leq 0$$

and one may conclude from this that the curvature of every four-dimensional Einstein manifold (M,g) satisfies the following remarkable inequality:

$$\int_M |W^+_g|^2 \geq \frac{1}{24}\int_M \mathrm{scal}_g^2\,.$$

Appendix A: more representation theory

We review here the basic concepts of representation theory which are necessary to state completely and precisely all the results of [**5**] and [**6**]. All the facts quoted in this appendix may be found in the book [**8**] or in analogous textbooks.

Finite dimensional irreducible representations of the Lie algebra $\mathfrak{so}(n)$ are classified by elements of the dual of a Cartan subalgebra of $\mathfrak{so}(n)$ called *dominant weights*. If $m = E(n/2)$, these are encoded by m-tuplets $(\lambda_1, \ldots, \lambda_m)$, all integers or all properly half-integers, satisfying the *dominance* conditions:

$$\lambda_1 \geq \ldots \geq \lambda_{m-1} \geq |\lambda_m|, \quad \text{if } n = 2m,$$
$$\lambda_1 \geq \ldots \geq \lambda_m \geq 0, \quad \text{if } n = 2m+1.$$

In this notation, the standard representation is given by $\tau = (1, 0, \dots, 0)$ and we shall hereafter identify any irreducible representation with its dominant weight.

The decomposition of the tensor product $\tau \otimes \lambda$ into irreducibles obeys the following *selection rule*: an irreducible representation μ appears in the decomposition iff.

1. $\mu \pm \varepsilon_j$ for some j (where (ε_j) is the standard basis of $\mathbb{R}^m$) or, if n is odd, $\mu = \lambda$, and
2. μ is a dominant weight.

For each component μ, its conformal weight w is given by the rule

$$w = \frac{1}{2} \left(C(\mathfrak{so}(n), \mu) - C(\mathfrak{so}(n), \lambda) - C(\mathfrak{so}(n), \tau) \right)$$

where the notation $C(.)$ denotes a Casimir operator: letting $\langle ., . \rangle$ be the standard scalar product on $\mathbb{R}^m$ and ${}^n\delta$ be the half-sum of the roots of the Lie algebra $\mathfrak{so}(n)$, given in coordinates by ${}^n\delta_i = (n - 2i)/2$, the Casimir operator of a representation μ is the number $C(\mathfrak{so}(n), \mu) = \langle \lambda, \lambda \rangle + 2 \langle \lambda, {}^n\delta \rangle$.

A careful examination of the selection rule above shows that, if ν is the number of different (absolute values of) integers or half-integers appearing as coordinates of a given weight λ, the number N of irreducible components (according to the convention of distinctness of conformal weights) in $\tau \otimes \lambda$ is $N = 2\nu - 1$ if $\lambda_m = 0$, $N = 2\nu + 1$ if $n = 2m + 1$ and $\lambda_m > 1/2$ and $N = 2\nu$ in all other cases.

In view of appendix D below, it will also be useful to know which are the irreducible factors which appear when decomposing an irreducible representation χ of $\mathfrak{so}(n + 1)$ under the action of $\mathfrak{so}(n)$: this is known as the *branching rule* and an irreducible representation of $\mathfrak{so}(n)$ given by its dominant weight λ appears in χ iff.

$$\begin{aligned} \chi_1 \geq \lambda_1 \geq \dots \geq \chi_m \geq |\lambda_m| \quad & \text{if } \ n = 2m, \\ \chi_1 \geq \lambda_1 \geq \dots \geq \lambda_m \geq |\chi_{m+1}|, \quad & \text{if } n = 2m + 1. \end{aligned}$$

Following T. Branson [**5**], we shall consider, for a given dominant weight λ, the set $\mathcal{X}(\lambda)$ of dominant weights χ for $\mathfrak{so}(n+1)$ such that the interlacing inequalities above are satisfied. We denote by $\mathcal{T}(\lambda)$ the set of indices i such that the squared i-th coordinate of $(\chi + {}^{n+1}\delta)$ takes at least two different values when χ runs among all elements of $\mathcal{X}(\lambda)$. Last, we denote by $\mathcal{Y}(\lambda)$ the set of dominant weights β for $\mathfrak{so}(n-1)$ such that β appears as an irreducible factor of λ when it is restricted to the smaller Lie algebra $\mathfrak{so}(n - 1)$. At this point, it is important to remark that a weight $\alpha = (\alpha_1, \beta)$ belongs to $\mathcal{X}(\lambda)$ iff. α_1 is in $\lambda_1 + \mathbb{N}$ and β is in $\mathcal{Y}(\lambda)$.

Appendix B: elliptic and non-elliptic first-order operators

T. Branson describes in [**4**] the set of *minimal elliptic* first-order operators acting on a bundle E. Following the notation given in Appendix A and the convention that irreducible components are given in strictly decreasing order of conformal weights, its elements are enumerated as follows:

1. the operator P_1;
2. the operator $P_{\nu+1}$ if $N = 2\nu$ or if $N = 2\nu + 1$ and λ is properly half-integral;
3. the operators $P_{j,N+2-j}$ for $j = 2, \ldots, \nu$;
4. the operator $P_{\nu+1,\nu+2}$ if $N = 2\nu + 1$ and λ is integral.

We then notice the following remarkable facts : on the one hand, minimal elliptic operators have small targets, but on the other hand, it is possible to find non-elliptic operators with relatively large targets.

One may identify the set of *maximal non-elliptic operators*, which is build the following way : one picks exactly one index in each of the $\nu - 1$ sets $\{j, N + 2 - j\}$ (for $j = 2, \ldots, \nu$) and take the associated operator if $N = 2\nu - 1$ or $N = 2\nu$. If $N = 2\nu + 1$, one must either add the index $\nu + 2$ if λ is properly half-integral or add any of the two indices $\nu + 1, \nu + 2$ if not.

The set $\mathcal{NE}$ of vertices of the polyhedron built in the course of the linear programming method of computation is exactly the set of maximal non-elliptic operators, except in the case $N = 2\nu + 1$ and λ properly half-integral where we define $\mathcal{NE}$ as in the integral case. The reason for this is the following: recall the polyhedron is defined as an intersection of half-spaces $H_i = p_i^{-1}([0, +\infty[)$, indexed by elements in $1, \ldots, N$. Its vertices lie among the larger set of points defined by (a number equal to the dimension of the Q-space) of equations $p_i = 0$. This latter set corresponds to all subsets of $\{1, \ldots, N\}$ of that precise size. It is then possible to show with a few algebraic manipulations (see **[6]**) that each would-be vertex corresponding to a subset J such that P_J is elliptic lies outside the polyhedron (hence is not a vertex at all) in almost all cases, except if $N = 2\nu + 1$, λ properly half-integral when J contains the index $\nu + 1$ and no other minimal elliptic subset. This explains the occurrence of the "bad" vertices in the computation in that case. With our extra notations, it is now possible to strenghten a bit Theorem 1 as follows: in the "bad" case ($N = 2\nu + 1$, λ properly half-integral), the value found at the end of the minimization procedure is indeed sharp provided that it is achieved at a vertex whose associated subset does not contain the index $\nu + 1$.

Appendix C: higher order Casimir operators

It is an easy consequence of its definition that the trace of B^2 on $T^*M \otimes E$ is related to the Casimir number $C(\mathfrak{so}(n), \lambda)$ already defined. Equivalently, the partial trace on the $\mathbb{R}^n$-factor

$$\mathrm{ptr}B^2 : v \in E \longmapsto \sum_{i=1} n\langle e_i, B^2(e_i \otimes v)\rangle \in E$$

is twice the Casimir operator of λ. For every $k > 2$, the partial traces $\mathrm{ptr}B^k$ are, similarly, higher order Casimir operators, *i.e* elements (and more precisely a basis) of the center of the universal enveloping algebra $\mathcal{U}(\mathfrak{so}(n))$. They act homothetically on each irreducible representation and it is the aim of this section to give a few explicit expressions for them.

As it already appeared (see also below), it is more natural to work with the modified operator $\widetilde{B}$ whose eigenvalues are the modified conformal weights introduced in section 3. This together with the Weyl dimension formula easily produces the following generating series:

$$1+\sum_{\ell\geq 0}\mathrm{ptr}\widetilde{B}^{\ell}\,t^{\ell+1}=\frac{t}{2}+\Big(1-(-1)^N\frac{t}{2}\Big)\prod_{j=1}^{N}\frac{1+\widetilde{w}_j t}{1-\widetilde{w}_j t}.$$

This result may be modified in two different ways. Firstly, we may wish to express the partial traces of the operators $\widetilde{A}_k$ introduced in section 4. One gets

$$\mathrm{ptr}\widetilde{A}_j=\big(1+(-1)^j\big)\sigma_{j+1}(\widetilde{w})+\frac{1}{2}\big((-1)^j-(-1)^N\big)\sigma_j(\widetilde{w}).$$

where $\sigma_k(\widetilde{w})$ denotes the k-th elementary symmetric function of the weights $\widetilde{w}_j$ ($1\leq j\leq N$). Inspired by a work of T. Diemer and G. Weingart (private communication), this led us to a simple recurrence formula for the family of operators $\widetilde{C}_k=\widetilde{A}_k+\frac{1}{4}\left((-1)^N-(-1)^k\right)\widetilde{A}_{k-1}$: for $j\geq 0$,

$$\begin{aligned}\widetilde{C}_{j+1}=&\Big(\widetilde{B}+\frac{(-1)^j}{2}\,\mathrm{id}\Big)\circ\widetilde{C}_j+\tfrac{1}{8}\big(1-(-1)^{N+j}\big)\widetilde{C}_{j-1}-\tfrac{1}{2}\mathrm{ptr}\widetilde{C}_j\\&+\tfrac{1}{2}\big(1-(-1)^j\big)\Big(\sigma_{j+1}(\widetilde{w})-\tfrac{1}{2}\big(1-(-1)^N\big)\sigma_j(\widetilde{w})\Big)\,\mathrm{id}\,.\end{aligned}$$

The work of T. Diemer and G. Weingart already alluded to shows that any family having a recurrence definition of this type has nice symmetry properties and this result is a crucial step in the derivation of the explicit values of the refined constants through the linear programming method.

Secondly, one may wish to express the partial traces of the operators $\widetilde{B}^k$ directly in terms of the dominant weight λ rather than in terms of the conformal weights. This relies on the following elementary (but useful !) property: let us define the *virtual modified conformal weight* $\widetilde{w}^{r,\pm}=\frac{1}{2}\pm(\lambda_r+{}^n\delta_r)$; such a weight is said to be *effective* iff. $\mu=\lambda+\varepsilon_r$ does appear as an irreducible component in the tensor product $\tau\otimes\lambda$, and it indeed equals the eigenvalue of $\widetilde{B}$ on this factor. It is moreover easily seen that if $\lambda_r=\lambda_{r+1}$, then $\widetilde{w}^{r,+}+\widetilde{w}^{r+1,-}=0$. In other words, noneffective virtual weights cancel pairwise. This trick allows to reintroduce all virtual non-effective weights in the generating series quoted above. One gets at the end an explicit expression of the partial traces of $\widetilde{B}^k$ in terms of the original weight λ. Our approach then provides a different proof of the computations done by Perelomov and Popov **[13]** of the higher order Casimir operators of the orthogonal groups. In our notation, the results are

$$1+\sum_{\ell\geq 0} \operatorname{ptr}\widetilde{B}^{\ell}\, t^{\ell+1} =$$

$$\begin{cases} \dfrac{t}{2}+(1-\dfrac{t}{2})\displaystyle\prod_{i=1}^{m}\dfrac{(1+(\frac{1}{2}+x_i)\,t)\,(1+(\frac{1}{2}-x_i)\,t)}{(1-(\frac{1}{2}+x_i)\,t)\,(1-(\frac{1}{2}-x_i)\,t)} & \text{if } n \text{ is even,}\\ \dfrac{t}{2}+(1+\dfrac{t}{2})\displaystyle\prod_{i=1}^{m}\dfrac{(1+(\frac{1}{2}+x_i)\,t)\,(1+(\frac{1}{2}-x_i)\,t)}{(1-(\frac{1}{2}+x_i)\,t)\,(1-(\frac{1}{2}-x_i)\,t)} & \text{if } n \text{ is odd,}\end{cases}$$

where we have denoted $x=\lambda+{}^{n}\delta$.

Appendix D: spectra of natural second order differential operators and refined Kato inequalities

The well known Peter-Weyl theorem asserts that the rough laplacian $\nabla^*\nabla$ on E provides a Hilbert sum splitting of the L^2 sections of E into its eigenspaces. Using the notation introduced in Appendix A,

$$L^2(E)=\overline{\oplus_{\chi\in\mathcal{X}(\lambda)}\ \mathcal{V}(\chi,\lambda)}\ =\overline{\oplus_{\mathcal{X}(\lambda)}\ \chi\otimes \operatorname{Hom}^{\mathfrak{so}(n)}(\chi,\lambda)},$$

where $\mathcal{X}(\lambda)$ has been defined earlier in Appendix A. As quoted in the main body of the text, deep techniques of harmonic analysis, such as $(\mathfrak{g},K)$-modules and Knapp-Stein intertwining operators were used by T. Branson to show in [**4**] that each elementary second-order operator $P_i^*P_i$ is also diagonalized in the same splitting. Eigenvalues are given by the following formulae:

$$\begin{aligned} &\operatorname{eig}(\nabla^*\nabla,\mathcal{V}(\chi,\lambda))\ =C(\mathfrak{so}(n+1),\chi)\ -\ C(\mathfrak{so}(n),\lambda),\\ &\operatorname{eig}(P_i^*P_i,\mathcal{V}(\chi,\lambda))\ =c_i(\lambda)\prod_{r\in\mathcal{T}(\lambda)}\left(\left(\chi_r+{}^{n+1}\delta_r\right)^2-(\widetilde{w}_i)^2\right). \end{aligned} \tag{17}$$

where $\mathcal{T}(\lambda)$ has also been defined in Appendix A. To explicit completely the last eigenvalues, we only have to express the normalization constant $c_i(\lambda)$:

$$c_i(\lambda)=\begin{cases}(-1)^{\rho+2}\left(\prod_{j\neq i}(\widetilde{w}_i-\widetilde{w}_j)\right)^{-1} & \text{if } N \text{ is odd,}\\ (-1)^{\rho+1}(\widetilde{w}_i-\frac{1}{2})\left(\prod_{j\neq i}(\widetilde{w}_i-\widetilde{w}_j)\right)^{-1} & \text{if } N \text{ is even,}\end{cases} \tag{18}$$

unless we are in the exceptional case already mentioned in section 3 where the i-th component of $T^*M\otimes E$ is an irreducible component for the full orthogonal group $\mathrm{O}(n)$ which splits into two irreducible components for $\mathrm{SO}(n)$ (this corresponds to n even, $\lambda_m=0\neq\lambda_{m-1}$ and $|\mu_m|=1$). In this case,

$$c_i(\lambda)=(-1)^{\rho+2}\left(\prod_{j\neq i}(\frac{1}{2}-\widetilde{w}_j)\right)^{-1}. \tag{19}$$

It is interesting to remark that the computations leading to the values of the normalization constants involve VanderMonde systems and a Lagrange interpolation procedure that is very similar (although not identical) to the one appearing in the linear programming method.

We can now describe the precise contents of T. Branson's minimization formula for the refined Kato constants of all possible elliptic operators P_I [**5**].

Theorem 5. — *Let I a subset of $\{1,\ldots,N\}$ corresponding to an elliptic operator P_I acting on E. Then a refined Kato inequality $\big|d|\xi|\big| \leq k_I|\nabla\xi|$ holds for any section ξ in the kernel of P_I, outside the zero set of ξ. Moreover,*

$$k_I^2 = 1 \; - \inf_{\beta\in\mathcal{Y}(\lambda)} \sum_{i\in I} c_i(\lambda) \prod_{r\in\mathcal{T}(\lambda), r\neq 1} \big((\beta_{r-1} + {}^{n-1}\delta_{r-1})^2 - \widetilde{w}_i^2\big)\,. \tag{20}$$

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