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UPDATE ON TORIC GEOMETRY

by

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Abstract. — This paper will survey some recent work on toric varieties. The goal is to help the reader understand how the papers in this volume relate to current trends in toric geometry.

Introduction

In recent years, toric varieties have been an active area of research in algebraic geometry. This article will give a partial overview of the work on toric geometry done since the 1995 survey paper [**90**]. One of our main goals is to help the reader understand the larger context of the eight papers in this volume:

[**74**] *Semigroup algebras and discrete geometry* by W. Bruns and J. Gubeladze.
[**93**] *How to calculate A-Hilb* $\mathbb{C}^3$ by A. Craw and M. Reid.
[**94**] *Crepant resolutions of Gorenstein toric singularities and upper bound theorem* by D. Dais.
[**96**] *Resolving* 3*-dimensional toric singularities* by D. Dais.
[**140**] *Producing good quotients by embedding into a toric variety* by J. Hausen.
[**159**] *Special McKay correspondence* by Y. Ito.
[**230**] *Lectures on height zeta functions of toric varieties* by Y. Tschinkel.
[**234**] *Toric Mori theory and Fano manifolds* by J. Wiśniewski.

These papers (and many others) were presented at the 2000 Summer School on the Geometry of Toric Varieties held at the Fourier Institute in Grenoble.

We will assume that the reader is familiar with basic facts about toric varieties. We will work over an algebraically closed field k and follow the notation used in Fulton [**121**] and Oda [**196**], except that we use Σ to denote a fan. Recall that one can

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think of a toric variety in many ways. First, we have the union of affine toric varieties presented by Fulton [**121**] and Oda [**196**]:

$$X_\Sigma = \bigcup_{\sigma\in\Sigma} X_\sigma, \quad X_\sigma = \operatorname{Spec}(k[\sigma^\vee \cap M]). \tag{0.1}$$

Second, when the support of Σ spans $N_\mathbb{R}$, we have the categorical quotient representation considered by Cox [**89**]:

$$X_\Sigma = \big(k^{\Sigma(1)} \smallsetminus \mathbf{V}(B)\big)/G, \quad G = \operatorname{Hom}(A_{n-1}(X_\Sigma), k^*), \tag{0.2}$$

where $B = \langle x^{\hat\sigma} : \sigma\in\Sigma\rangle$ and $x^{\hat\sigma} = \prod_{\rho\notin\sigma(1)} x_\rho$. We call $S = k[x_\rho : \rho\in\Sigma(1)]$ the *homogeneous coordinate ring* of X_Σ, which is graded by $A_{n-1}(X_\Sigma)$. The representation (0.2) is a geometric quotient if and only if Σ is simplicial.

Finally, $\mathcal{A} = \{m_1,\dots,m_\ell\} \subset \mathbb{Z}^n$ gives the semigroup algebra $k[t^{m_1},\dots,t^{m_\ell}] \subset k[t_1^{\pm1},\dots,t_n^{\pm1}]$. Then we have the (possibly non-normal) affine toric variety discussed by Sturmfels [**223, 224**]:

$$X_\mathcal{A} = \operatorname{Spec}(k[t^{m_1},\dots,t^{m_\ell}]). \tag{0.3}$$

The map $x_i \mapsto t^{m_i}$ gives a surjection $k[x_1,\dots,x_\ell] \to k[t^{m_1},\dots,t^{m_\ell}]$ whose kernel

$$I_\mathcal{A} = \ker(k[x_1,\dots,x_\ell] \to k[t^{m_1},\dots,t^{m_\ell}]) \tag{0.4}$$

is the toric ideal of $\mathcal{A}$. This ideal is generated by binomials and is the defining ideal of $X_\mathcal{A} \subset k^\ell$. If $I_\mathcal{A}$ is homogeneous, then $X_\mathcal{A}$ is the affine cone over the (possibly non-normal) projective toric variety $Y_\mathcal{A} \subset \mathbb{P}^{\ell-1}$.

This survey concentrates on work done since our earlier survey [**90**]. Hence most of the papers we discuss appeared in 1996 or later. We caution the reader in advance that our survey is not complete, partly for lack of space and partly for ignorance on our part. We apologize for the many fine papers not mentioned below.

1. The Minimal Model Program and Fano Toric Varieties

The paper [**234**] by Jarosław Wiśniewski discusses toric Mori theory and Fano varieties. The main goal of the paper is to illustrate aspects of the minimal model program using toric varieties. As Wiśniewski points out, toric varieties are rational and hence trivial from the point of view of the minimal model program. Nevertheless, many hard results about minimal models can be proved without difficulty in the toric case. It makes for an excellent introduction to the subject.

An important feature of the minimal model program is that singularities are unavoidable in higher dimensions. In our discussion of Wiśniewski's lectures, we will assume that X is a normal projective variety such that K_X is $\mathbb{Q}$-Cartier (meaning that some positive integer multiple of K_X is a Cartier divisor). Such a variety is called $\mathbb{Q}$-*Gorenstein.* Given a resolution of singularities $\pi : Y \to X$, we can write

$$K_Y = \pi^*(K_X) + \textstyle\sum_i d_i E_i$$

where the exceptional set $E = \bigcup_i E_i$ is a divisor with normal crossings. We call $\sum_i d_i E_i$ the *discrepancy divisor*. Then we say that the singularities X are:

$$
\begin{gathered}
\textit{terminal} \text{ if } d_i > 0 \text{ for all } i; \\
\textit{canonical} \text{ if } d_i \geqslant 0 \text{ for all } i; \\
\textit{log-terminal} \text{ if } d_i > -1 \text{ for all } i; \text{ and} \\
\textit{log-canonical} \text{ if } d_i \geqslant -1 \text{ for all } i.
\end{gathered} \tag{1.1}
$$

Furthermore, $\pi : Y \to X$ is *crepant* if the discrepancy is zero, i.e., $d_i = 0$ for all i or, equivalently, $K_Y = \pi^*(K_X)$. In Section 2.2, we will explain what these singularities mean in the toric case.

1.1. Extremal Rays, Contractions, and Flips. — The first three lectures in Wiśniewski's article [**234**] are based primarily on Reid [**209**] and discuss aspects of the minimal model program related to the Mori cone $NE(X)$, which is the cone of $H_2(X, \mathbb{R})$ generated by homology classes of irreducible curves on X. For a simplicial toric variety, $NE(X)$ is generated by the torus-invariant curves in X (which correspond to codimension 1 cones of the fan of X). In [**234**, Lec. 1], Wiśniewski describes in detail how this relates to Mori's move-bend-break strategy.

When X is projective, the 1-dimensional faces of $NE(X)$ are *extremal rays*. In the toric case, it follows that each extremal ray is the class of a torus-invariant curve in X. Wiśniewski contrasts this with the Cone Theorem of Mori and Kawamata, which for a general variety X gives only a partial description of $NE(X)$.

Extremal rays are important in the minimal model program because of the Contraction Theorem of Kawamata and Shokurov, which asserts that if a projective variety X has terminal singularities, then every Mori ray R (= an extremal ray with $R \cdot K_X < 0$) gives a contraction

$$\varphi_R : X \longrightarrow X_R$$

with connected fibers such that X_R is normal and projective and a curve in X is contracted to a point if and only if its class lies in R.

For an extremal ray R on a simplicial projective toric variety of dimension n, Wiśniewski gives Reid's construction [**209**] of the corresponding contraction. Here is a brief summary. Given R, define α and β to be

$$
\begin{aligned}
\alpha &= |\{D_\rho : D_\rho \cdot R < 0\}| \\
\beta &= n + 1 - |\{D_\rho : D_\rho \cdot R > 0\}|,
\end{aligned}
$$

where the D_ρ are the torus-invariant divisors of X. These will be important invariants of the contraction φ_R. The formulas given in [**234**, Lec. 2] show that α and β are easy to compute in practice.

Now let ω be a codimension 1 cone in the fan Σ of X such that the corresponding curve lies in R. Then ω is a face of two top-dimensional cones δ, δ' in Σ. One can show that the sum $\delta + \delta'$ is again a convex cone. Then consider the "fan" Σ_R^* obtained from Σ by removing all such ω's and, for each such ω, replacing the corresponding δ, δ' with $\delta + \delta'$. We put "fan" in parenthesis because of the following result.

Lemma 1.1. — *If $\alpha > 0$, then Σ_R^* is a fan, but if $\alpha = 0$, then there is a subspace $\mu(R)$ of dimension $n - \beta$ such that $\sigma \cap -\sigma = \mu(R)$ for every cone $\sigma \in \Sigma_R^*$.*

The extremal ray R then gives the desired contraction $\varphi_R : X \to X_R$ as follows:

- When $\alpha = 0$, Σ_R^* is a *degenerate fan.* Then $\Sigma_R^*/\mu(R)$ becomes a fan in $N_{\mathbb{R}}/\mu(R)$. Furthermore, if X_R is the toric variety of $\Sigma_R^*/\mu(R)$, then X_R has dimension β and φ_R is a toric fibration whose fibers are weighted projective spaces.
- When $\alpha > 0$, then Σ_R^* is a fan, and if X_R is the toric variety of Σ_R^*, then φ_R is birational. Furthermore:
  - If $\alpha = 1$, then φ_R is the blow-up of a subset of X_R of dimension $\beta - 1$. Thus the exceptional set is a divisor. Also, X_R is terminal if X is.
  - If $\alpha > 1$, then the exceptional set of φ_R has codimension > 1, so that φ_R is an isomorphism in codimension 2. We say that R is a *small ray.*

Notice how degenerate fans arise naturally in this context.

In terms of the minimal model program, the cases when $\alpha = 0$ or 1 work nicely, since in these cases we can replace X with X_R. But $\alpha > 1$ causes problems because in this case, the cones $\delta + \delta'$ are not simplicial, so that X_R has bad singularities from the minimal model point of view. This is where the next big result of the minimal model program comes into play, the Flip Theorem. This is more properly called the Flip Conjecture, since for general varieties, it has been proved only for dimension $\leqslant 3$ (by Mori). However, it is true for all dimensions in the toric case.

The rough idea is that when R is a small ray, X_R isn't suitable, so instead we "flip" R to $-R$ on a birational model X_1 and then replace X with X_1. More precisely, the Toric Flip Theorem, as stated in [**234**, Lec. 3], constructs a fan Σ_1 with toric variety X_1 and a birational map

$$\psi : X_1 \dashrightarrow X$$

with the following properties:

- If X is terminal with $K_X \cdot R < 0$ (i.e., R is a Mori ray), then X_1 is terminal.
- ψ is an isomorphism in codimension 1.
- $R_1 = -\psi^*(R)$ is an extremal ray for X_1 and $\varphi_1 = \varphi_R \circ \psi : X_1 \to X_R$ is the corresponding contraction of R_1.

Furthermore, Σ_1 is easy to construct: using the natural decomposition of $\delta + \delta'$ into simplices described in [**234**, Lec. 3], one simply replaces each cone $\delta + \delta' \in \Sigma_R^*$ with these simplices.

There are some recent papers related to these topics. First, concerning extremal rays, Bonavero [**47**] observes that if X is a projective toric variety and $\pi : X \to X'$ is a smooth toric blow-down, then X' is projective if and only if a line contained in a non-trivial fiber of π is an extremal ray. He then uses this to classify certain smooth blow-downs to non-projective varieties. Second, concerning minimal models, if $Y \subset X$ is a hypersurface in a complete toric variety such that the intersection of Y with every orbit is either empty or transverse of codimension 1, then S. Ishii [**157**] uses the toric framework described above to show that minimal model program works for Y, as described in the introduction to [**234**]. See also Ishii's paper [**156**].

Returning to the lectures [**234**], Wiśniewski points out that when X is toric and projective, any face of $NE(X)$ can be contracted, not just edges (= extremal rays). This is not true for general projective varieties. Then [**234**, Lec. 3] ends with a discussion of toric flips from the point of view of Morelli-Włodarczyk corbodisms, which is based on the work of Morelli [**189**] and Włodarczyk [**236**]. In [**234**, Lec. 4], Wiśniewski defines terminal and canonical singularities as in (1.1) and explains how these relate to the toric versions of the Contraction Theorem and Flip Theorem. He also describes the Euler sequence of a smooth toric variety.

1.2. Fano Varieties. — In [**234**, Lec. 5], Wiśniewski discusses Fano varieties. In general, a normal variety X is Fano when some multiple of $-K_X$ is an ample Cartier divisor. As explained in the introduction to [**234**], part of the minimal model program includes Fano-Mori fibrations, whose fibers are Fano varieties. Wiśniewski focuses on the case of toric Fano manifolds for simplicity.

Results of Batyrev show that in any given dimension, there are at most finitely many toric Fano manifolds (up to isomorphism). In dimension 2, it is easy to see that there are only five: $\mathbb{P}^1 \times \mathbb{P}^1$ together with the blow-up of $\mathbb{P}^2$ at $0, 1, 2$ or 3 fixed points of the torus action. In dimension 3, Wiśniewski sketches the proof that there are precisely 18 smooth toric Fano 3-folds. He also discusses the classification of non-toric Fano manifolds, where the situation is considerably more complicated.

In dimension 4, Batyrev [**28**] recently published a classification of smooth toric Fano 4-folds. As noted by Sato [**220**], Batyrev missed one, so that Batyrev's list of 123 is now a list of 124 smooth toric Fano 4-folds. The key point is that toric Fano manifolds of dimension n correspond to n-dimensional lattice polytopes $P \subset N_{\mathbb{R}} \simeq \mathbb{R}^n$ with the origin as an interior point such that the vertices of every facet are a basis of N. (Given such a P, the cones over the faces of P give a fan whose toric variety is a Fano manifold.) Hence the proof reduces to classifying the possible polytopes.

One can generalize the polytopes of the previous paragraph to the idea of a *Fano polytope*. This is an n-dimensional lattice polytope $P \subset N_{\mathbb{R}} \simeq \mathbb{R}^n$ with the property that 0 is the unique lattice point in the interior of P. In this case, taking cones over faces as above gives a Fano toric variety X. Furthermore, the singularities of S can be read off from the polytope. For example, Section 2.2 below implies that:

- If the only lattice points in P are 0 plus the vertices, then X has terminal singularities.
- If every facet of P is defined by an equation of the form $\langle m, u\rangle = 1$ for some $m \in M$, then X is Gorenstein.

In the latter case, we say that P is *reflexive*. These polytopes play an important role in mirror symmetry (see the book [**91**] by Cox and Katz) and are classified in dimensions 3 and 4 by Kreuzer and Skarke [**169, 170**]. As noted by A. Borisov [**50**], there are interesting similarities between the classification of toric Fano varieties and the classification of toric singularities.

Other work on toric Fano varieties includes the paper [**48**], where Bonavero studies toric varieties whose blow-up at one point is Fano. (This has been generalized to the non-toric case by Bonavero, Campana and Wiśniweski [**49**].) Also, Bonavero's paper [**47**] mentioned earlier contains results about toric Fano varieties. Birational maps between toric Fano 4-folds are studied by Casagrande in [**85**], and forthcoming papers of Casagrande [**86**] will generalize some of the results of [**47**]. In another direction, Einstein-Kähler metrics and the Futaki invariant have been studied by Batyrev and Selivanova [**34**] for symmetric toric Fano manifolds and by Yotov [**240**] for almost Fano toric varieties. Finally, there has been a *lot* of work on non-toric Fano manifolds. As a small hint, the reader might want to consult the 1994 paper [**233**], where Wiśniewski surveys Fano manifolds X such that $b_2(X) \geqslant 2$ and K_X is divisible by $\dim(X)/2$ in $\mathrm{Pic}(X)$. There is also the 2000 book [**88**] on the birational geometry of 3-folds, which includes several papers on Fano 3-folds.

2. Singularities of Toric Varieties

The articles [**94, 96**] by Dimitrios Dais study the singularities of toric varieties. The paper [**96**] surveys the problem of resolving toric singularities, with an emphasis on dimension 3, while [**94**] studies crepant resolutions of Gorenstein toric singularities.

2.1. Singularities in Dimensions 2 and 3. — Our purpose here is to give a introduction to Dais' article [**96**]. In [**96**, Sec. 1] Dais defines various types of singularities encountered in algebraic geometry, including local complete intersections and rational and elliptic singularities. Dais also defines crepant resolutions and terminal, canonical, log-terminal and log-canonical singularities as we did (1.1), and he discusses several general properties of these singularities.

Then [**96**, Sec. 2] summarizes facts about singularities in dimension $\leqslant 3$. For surfaces, this includes a careful statement of the classic classification of ADE singularities (also called Kleinian or Du Val), as well as the following nice result.

Theorem 2.1. — *Let (X,x) be a normal surface singularity. Then:*

$$\begin{array}{ccc} x \text{ is terminal} & \iff & x \text{ is a smooth point of } X \\ x \text{ is canonical} & \iff & \begin{array}{c}(X,x) \simeq (\mathbb{C}^2/G,0) \text{ with } G \\ \text{a finite subgroup of } \mathrm{SL}(2,\mathbb{C})\end{array} \\ x \text{ is log-terminal} & \iff & \begin{array}{c}(X,x) \simeq (\mathbb{C}^2/G,0) \text{ with } G \\ \text{a finite subgroup of } \mathrm{GL}(2,\mathbb{C}).\end{array} \end{array}$$

(The version of this theorem in [**96**] also includes the case when x is log-canonical, which is a bit more complicated to state). In the 3-dimensional case, Dais recalls the definition of compound Du Val singularity (cDV for short) and gives a weak analog (due to Reid) of Theorem 2.1 for terminal and canonical singularities. He also explains Reid's four-step strategy for studying canonical singularities.

2.2. The Toric Case. — In [**96**, Sec. 3], Dais deals with toric singularities. After a review of toric geometry, Dais explains what various types of singularities mean in toric terms. Given an n-dimensional rational polyhedral cone $\sigma \subset N_{\mathbb{R}} \simeq \mathbb{R}^n$, we let X_σ be the corresponding affine toric variety. Also let $e_1, \ldots, e_s$ be the minimal generators of σ. Then one easily sees that

– X_σ is $\mathbb{Q}$-Gorenstein $\iff$ there is $m \in M_{\mathbb{Q}}$ such that $\langle m, e_i \rangle = 1$ for all i.

If we write the affine hyperplane as $\langle \widetilde{m}, u \rangle = r$ where $\widetilde{m} \in M$ and $r \in \mathbb{Z}_+$ is minimal, then we call r the *index* of the singularity. It is the smallest positive integer such that rK_{X_σ} is Cartier. Thus X_σ is Gorenstein $\iff$ it is has index 1.

Furthermore, when X_σ is $\mathbb{Q}$-Gorenstein, let $m \in M_{\mathbb{Q}}$ be as above. Then:

– X_σ is terminal $\iff$ $\sigma \cap \{u \in N : \langle m, u \rangle \leqslant 1\} = \{0, e_1, \ldots, e_s\}$.
– X_σ is canonical $\iff$ $\sigma \cap \{u \in N : \langle m, u \rangle < 1\} = \{0\}$.

Nice pictures of terminal and canonical cones can be found in Reid's article [**209**]. Dais also points out the following easy implications among these singularities:

– X_σ is $\mathbb{Q}$-Gorenstein $\Longrightarrow$ X_σ is log-terminal.
– X_σ is Gorenstein $\Longrightarrow$ X_σ is canonical.

In the Gorenstein case, the convex hull of $\{e_1, \ldots, e_s\}$ is a lattice polytope P of dimension $n-1$. By changing coordinates in N, we can assume that

$$\sigma \text{ is the cone over } \{1\} \times P \subset \mathbb{R} \times \mathbb{R}^{n-1}. \tag{2.1}$$

As Dais notes in [**96**, Rem. 3.15], it follows that n-dimensional Gorenstein terminal singularities correspond to $(n-1)$-dimensional *elementary polytopes*, which are lattice polytopes whose only lattices points are vertices. In general, there is a strong relation between Gorenstein singularities and lattice polytopes. Numerous references are given, to which we would add the paper [**50**] of A. Borisov discussed earlier.

Note also that [**96**, Sec. 3] contains a characterization of when a Gorenstein X_σ is a local complete intersection. The result involves Nakajima polytopes, which are defined in [**96**, Def. 3.10].

In [**96**, Sec. 4], Dais explains how to resolve toric singularities in dimensions 2 and 3. To resolve a singularity in dimension 2, we can use the *Hilbert basis* of $\sigma \cap N$, which is the set of elements of $\sigma \cap N$ not expressible as the sum of two or more nonzero elements of the semigroup. Then subdividing σ using rays through the points of its Hilbert basis gives the minimal resolution of X_σ.

The situation in dimension 3 is more complicated since minimal resolutions no longer exist. So instead the goal is to find a resolution which is "canonical" in some sense. For example, one could try to mimic the 2-dimensional case by using the Hilbert basis of $\sigma \cap N$. As Dais points out, this has been done by Bouvier and González-Sprinberg [**54**] and Aguzzoli and Mundici [**14**], but in both cases the resolution is not unique. Another approach deals with the special case when X_σ is simplicial and Gorenstein. Since σ has dimension 3, this implies that $X_\sigma = \mathbb{C}^3/G$, where $G \subset \mathrm{SL}(3,\mathbb{C})$ is a finite Abelian subgroup. As we will see in the paper of Ito [**159**] to be discussed in Section 3, the G-Hilbert scheme of $\mathbb{C}^3$ gives a canonical crepant resolution of X_σ in this case.

The paper [**96**] concludes with a description of a new approach to resolving X_σ (still in dimension 3) which was inspired by the strategy of Reid mentioned above. According to [**96**, Thm. 4.1], this is done in five stages:

(i) Subdivide to make the singularities canonical.
(ii) Change the lattice to make them canonical of index 1, i.e., Gorenstein.
(iii) By working with lattice polygons and blowing up points, reduce to certain cDV singularities.
(iv) Blow up certain 1-dimensional loci to make the singularities terminal.
(v) Finally, add diagonals to get a crepant resolution.

Steps (i)–(iv) are unique, while step (v) involves $2^{\#\text{diagonals}}$ choices. Dais gives an example of this construction and notes that details may be found in the forthcoming paper [**102**] of Dais, Henk and Ziegler.

2.3. Crepant Resolutions. — There are many situations in algebraic geometry where one is interested in a crepant resolution of a singular $\mathbb{Q}$-Gorenstein variety X. For example:

- When X is an orbifold (i.e., has finite quotient singularities), the Euler characteristic of a crepant resolution of X is an intrinsic invariant of X called the stringy (or physicists) Euler number.
- When X is Calabi-Yau, its canonical divisor is trivial. If we want a resolution $\pi : \widetilde{X} \to X$ such that $\widetilde{X}$ is also Calabi-Yau, then π must be crepant.

We will discuss "stringy" matters briefly in Section 7.9, but for now we will concentrate on the question of crepant resolutions of toric singularities. This is the main subject of Dais' second article [**94**] in this volume.

In Section 2.2, we saw that the affine toric variety X_σ of a n-dimensional cone $\sigma \subset N_\mathbb{R} \simeq \mathbb{R}^n$ is Gorenstein if and only if the minimal generators lie on an affine hyperplane $\langle m, u\rangle = 1$ for some $m \in M$. As in (2.1), we can change coordinates so that σ becomes the cone over $\{1\} \times P$. If T is a lattice triangulation of P (so the vertices of each simplex in T are lattice points), then taking cones over these simplices gives a subdivision of σ. This gives a birational map $X_T \to X_\sigma$. We will be interested in the following two kinds of lattice triangulations T:

– T is *maximal* if every simplex in T is elementary. As in the discussion following (2.1), this means that the vertices of every simplex are its only lattice points.
– T is *basic* (or *unimodular*) if every simplex in T is *basic* (or *unimodular*). This means that the vertices of every top-dimensional simplex form a basis of N.

Every unimodular triangulation is maximal, though the converse is true only in dimension 2. Furthermore, maximal triangulations always exist, but there are polytopes which have no unimodular triangulations.

In terms of the singularities of X_σ, Dais [**94**, Sec. 1] considers the following three possibilities:

(A) P is an elementary polytope, which means X_σ is terminal. The key point is that when a singular variety has terminal singularities, then no crepant resolution exists. This is why the name "terminal" is used for such singularities.
(B) P has no basic triangulation. Thus, if we pick a maximal triangulation T, then X_T is singular with terminal singularities. Hence $X_T \to X$ is the closest we can get to a crepant resolution.
(C) P has a basic triangulation. In this case, a crepant resolution exists.

In order to solve **(A)**, one needs to classify elementary polytopes up to lattice isomorphism. The more general problem of classifying polytopes with few lattice points is discussed by A. Borisov in [**50**]. For **(C)**, there has been a lot of work finding interesting examples of Gorenstein toric singularities which have crepant resolutions. For example:

– Ito [**158**], Markushevich [**175**] and Roan [**214**] proved that all 3-dimensional Gorenstein quotient singularities have crepant resolutions. (Such a singularity is toric in the Abelian case.)
– Dais, Henk and Zeigler [**101**] showed that in any dimension, Abelian quotient local complete intersections have crepant resolutions. This was generalized to toric local complete intersections by Dais, Haase and Ziegler in [**99**].
– Dais and Henk [**97**] and Dais, Haus and Henk [**100**] show that certain infinite families of Gorenstein cyclic quotient singularities (which are not local complete intersections) have crepant resolutions.

This leaves **(B)**, which leads to the question of *finding a combinatorial characterization of those polytopes which don't have a basic triangulation.* In [**94**, Sec. 3], Dais explains how the Upper Bound Theorem leads to a necessary condition for a polytope to have a basic triangulation. For this purpose, recall that the kth *cyclic polytope* $\mathrm{CycP}_n(k)$ is the convex hull of k distinct points on the monomial curve $t \mapsto (t, \dots, t^n) \in \mathbb{R}^n$. McMullen's Upper Bound Theorem asserts that if a polytope $Q \subset \mathbb{R}^n$ has k vertices and dimension n, then

$$\mathfrak{f}_i(Q) \leqslant \mathfrak{f}_i\big(\mathrm{CycP}_n(k)\big), \quad 0 \leqslant i \leqslant n-1,$$

where as usual $\mathfrak{f}_i(Q)$ denotes the number of i-dimensional faces of Q. Using this and other facts about $\mathfrak{f}$-vectors and Ehrhart polynomials, Dais proves the following result [**94**, Thm. 3.1]:

Theorem 2.2. — *Let the n-dimensional cone σ come from the $(n-1)$-dimensional P as in* (2.1). *If X_σ has a crepant resolution, then the normalized volume* $\mathrm{Vol}_{n-1}(P)$ *satisfies the inequality*

$$\mathrm{Vol}_{n-1}(P) \leqslant \mathfrak{f}_{n-1}\big(\mathrm{CycP}_n(|P\cap M|)\big) - |(\partial P \cap M)| + n - 1.$$

Dais also mentions current work with Henk and Ziegler [**103**] to improve the bound in Theorem 2.2. It follows that if P violates the inequality of this theorem, it cannot have a basic triangulation and hence lies in **(B)**. The challenge is to find other combinatorial conditions which lead to not only necessary but also sufficient conditions for the existence of a basic triangulation.

2.4. Other Work on Toric Singularities. — Finally, we want to briefly mention some other papers on toric singularities. In our 1996 survey [**90**], we reported on the work of Altmann. He also has a paper [**16**] which reviews his work up to 1996. Altmann's basic objects of study are $T^1_{X_\sigma}$ and $T^2_{X_\sigma}$, which determine the infinitesimal deformations and obstructions to lifting deformations respectively. (As usual, X_σ is the affine toric variety of σ.) The main goals of his paper [**15**] are to compute the graded pieces of $T^1_{X_\sigma}$ and, for the case of 3-dimensional Gorenstein singularities, to determine for exactly which degrees the graded piece is nonzero. Also, the paper [**20**] by Altmann and Sletsjøe determines the André-Quillen cohomology groups $T^p_{X_\sigma}$ for all p when X_σ has an isolated singularity. In [**21**], Altmann and van Straten relate $T^p_{X_\sigma}$ to invariants defined by Brion in [**63**] and prove a vanishing theorem for polytopes arising from quivers. (We will discuss Brion's paper [**63**] in Section 7.11 below.)

Matsushita [**180**] studies maps $\pi : Y \to X_\sigma$ where X_σ has canonical singularities, Y has $\mathbb{Q}$-factorial singularities, and $K_Y = \pi^* K_{X_\sigma} + \sum_i a_i E_i$, $a_i \leqslant 0$. These are classified by radicals of certain initial ideals. He also considers the case when X_σ is Gorenstein. In [**181**], Matsushita studies simultaneous terminalizations of Gorenstein homogeneous toric deformations $F : X \to \mathbb{C}^m$ (as defined by Altmann). He proves that simultaneous terminalizations exist when X has a crepant resolution and gives examples to show that they do not exist in general.

Toric methods also play an interesting role in recent work on the resolution of arbitrary singularities. We will discuss this in Section 7.3 below.

3. The McKay Correspondence and G-Hilbert Schemes

In 1979, McKay [**186**] observed that the irreducible representations of a finite group $G \subset \mathrm{SL}(2,\mathbb{C})$ correspond naturally to the vertices of an (extended) Dynkin diagram of type ADE. Since the Dynkin diagram is the dual graph of the exceptional fiber of the minimal resolution of singularities $\mathbb{C}^2/G$, we get a correspondence between components of the exceptional fiber and the (nontrivial) irreducible representations of the group. Following Ito and Nakajima [**160**], we get the following table:

finite subgroup G of $\mathrm{SL}(2,\mathbb{C})$	(nontrivial) irreducible representations	decompositions of tensor products
simple Lie algebra of type ADE	simple roots	(extended) Cartan matrix
minimal resolution $\widetilde{X} \to \mathbb{C}^2/G$	irreducible components of the exceptional set (= a basis of $H_2(\widetilde{X},\mathbb{Z})$)	intersection matrix

In the more general setting of a finite subgroup $G \subset \mathrm{GL}(n,\mathbb{C})$, this has led to the problem of finding relations between the group theory of G (representations, conjugacy classes, etc.) and a resolution of singularities of $\mathbb{C}^n/G$ (exceptional fiber, cohomology, derived category, etc.). These relations—many of which are still conjectural—are collectively called the *McKay correspondence*. Surveys of the McKay correspondence can be found in Reid's Bourbaki talk [**210**] and Kinosaki lectures [**211**].

The papers in this volume by Yukari Ito [**159**] and Alastair Craw and Miles Reid [**93**] touch on aspects of the McKay correspondence which use toric geometry. Ito's paper [**159**] also includes a brief introduction to the McKay correspondence.

3.1. Resolutions of $\mathbb{C}^n/G$. — For a finite subgroup $G \subset \mathrm{SL}(n,\mathbb{C})$, one problem with extending the McKay correspondence for $n > 2$ is the lack of a unique minimal resolution of singularities of $\mathbb{C}^n/G$. The best one can hope for is a *crepant* resolution of $\mathbb{C}^n/G$, as defined in the discussion following (1.1). Here, $G \subset \mathrm{SL}(n,\mathbb{C})$ implies that the dualizing sheaf of $\mathbb{C}^n/G$ is trivial (hence $\mathbb{C}^n/G$ is Gorenstein), so that a resolution $\widetilde{X} \to \mathbb{C}^n/G$ is crepant if and only if $\omega_{\widetilde{X}} \simeq \mathcal{O}_{\widetilde{X}}$. Crepant resolutions exist when $n = 2$ (classical) and $n = 3$ (see Section 2.3) but may fail to exist for larger n.

One attempt to avoid this non-uniqueness is the paper [**162**] of Ito and Reid, which shows that the crepant divisors in any resolution (this has to be defined carefully) correspond to junior conjugacy classes of G. We define *junior* as follows. Fix a primitive rth root of unity ε, where r is divisible by the order of every element of G. If $g \in G$ is conjugate to a diagonal matrix whose ith diagonal entry is ε^{a_i}, then the *age* of g is $\frac{1}{r}(a_1 + \cdots + a_n)$, which is an integer since $G \subset \mathrm{SL}(n,\mathbb{C})$. The junior elements of G are those of age 1.

A more recent method to cope with non-uniqueness is Nakamura's idea of using the G-Hilbert scheme to resolve $\mathrm{SL}(n,\mathbb{C})$. Roughly speaking, G-Hilb $\mathbb{C}^n$ is the moduli space of all G-invariant 0-dimensional subschemes $Z \subset \mathbb{C}^n$ such that the action of G on $H^0(Z,\mathcal{O}_Z)$ is the regular representation. As explained by Craw and Reid [**93**], two ways of making this precise can be found in the literature, which fortunately agree at least when $n = 2$ or 3.

Let $\widetilde{X} = G\text{-Hilb}\,\mathbb{C}^n$. Then there is a well-defined morphism $\widetilde{X} \to \mathbb{C}^n/G$. The amazing fact is that this is a crepant resolution when $n = 2$ (Ito and Nakamura [**161**]) or $n = 3$ (Nakamura [**193**] for G Abelian, Bridgeland, King and Reid [**59**] for G general). Hence, in these cases, we have a canonical choice of crepant resolution. Furthermore, the authors of [**59**] also show that for $n = 2$ or 3, the Mukai transform induces an equivalence of categories between the derived category $D(\widetilde{X})$ and the equivariant derived category $D^G(\mathbb{C}^n)$. Hence we have a very sophisticated version of the McKay correspondence in this case. (We should mention the paper [**160**] where Ito and Nakajima study the McKay correspondence for $n = 3$ from the point of view of K-theory. Batyrev and Dais also consider the McKay correspondence in [**33**].)

3.2. The Special McKay Correspondence. — In [**159**], Ito studies the McKay correspondence for the cyclic group

$$C_{r,a} = \left\langle \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^a \end{pmatrix} \right\rangle \subset \mathrm{GL}(2,\mathbb{C}), \tag{3.1}$$

where ε is a primitive rth root of unity. If $a \equiv -1 \bmod r$, then $C_{r,a} \subset \mathrm{SL}(2,\mathbb{C})$, which allows us to use the McKay correspondence described above. But when $a \not\equiv -1 \bmod r$, there are more nontrivial irreducible representations than components of the exceptional fiber of the resolution $\widetilde{X} \to \mathbb{C}^2/G$. In 1988, Wunram [**239**] solved this problem by using certain *special* representations of G, which gave rise to vector bundles on $\widetilde{X}$ whose first Chern classes are dual to the components of the exceptional fiber. See Ito's paper [**159**] for details. Ito also describes recent work of A. Ishii [**155**] which explains how to interpret Wunram's special representations in terms of the $C_{r,a}$-Hilbert scheme of $\mathbb{C}^2$.

However, since $C_{r,a}$ is Abelian, the quotient $\mathbb{C}^2/G$ has a natural structure of a toric variety, and, as described in Section 2.2, so does its minimal resolution $\widetilde{X}$. In [**159**], Ito shows how to explicitly recover the special representations in this case. As a preview of what she does, note that each monomial $x^i y^j$ is an eigenvector for the $C_{r,a}$ action since the generator of $C_{r,a}$ displayed in (3.1) acts on $x^i y^j$ via

$$x^i y^j \longmapsto (\varepsilon x)^i (\varepsilon^a y)^j = \varepsilon^{i+aj} x^i y^j.$$

In particular, you can read the character from the monomial. Hence the search for special characters reduces to a search for certain special monomials, which is explained in [**159**, Thm. 3.7]. Ito's paper also includes explicit details for the group $C_{7,3}$.

3.3. The A-Hilbert Scheme of $\mathbb{C}^3$. — If $A \subset \mathrm{SL}(3,\mathbb{C})$ is Abelian, we can assume that $A \subset (\mathbb{C}^*)^3$. As with the case just considered, $\mathbb{C}^3/A$ is a toric variety and hence has toric resolutions (which are now non-unique). In the paper [**93**] in this volume, Craw and Reid show that one of these toric resolutions is A-Hilb $\mathbb{C}^3$ and they give an explicit algorithm for computing it.

As we did above, fix a primitive rth root of unity ε, where r is divisible by the order of every element of A. Then $g \in A$ is a diagonal matrix with diagonal entries $\varepsilon^{a_1}, \varepsilon^{a_2}, \varepsilon^{a_3}$, where $0 \leqslant a_i \leqslant r-1$. Then let L be the lattice generated by $\mathbb{Z}^3$ together with the rational vectors $\frac{1}{r}(a_1, a_2, a_3)$ for all $g \in A$. The junior elements of A are those for which $\frac{1}{r}(a_1 + a_2 + a_3) = 1$. It follows that the junior elements give lattice points of L which lie in the triangle $\Delta = (1,0,0), (0,1,0), (0,0,1)$. In [**93**], Craw and Reid call this the *junior simplex*.

The first main result of [**93**] is the description of an explicit set of triangles (called *regular triangles*) which partition the junior simplex Δ. A nice example of this construction can be found in Reid's survey [**211**, Ex. 2.2]. Then the second main result of Craw and Reid is as follows.

Theorem 3.1. — *Let Σ denote the toric fan obtained by taking the regular tesselation of all regular triangles in the junior simplex Δ. The associated toric variety X_Σ is Nakamura's A-Hilbert scheme A-Hilb $\mathbb{C}^3$.*

This toric fan is smooth by construction, and furthermore, since the lattice L was generated by junior elements, standard discrepancy calculations (as explained in Reid's Bowdoin article [**212**]) imply that we get a crepant resolution. Thus the above theorem shows that A-Hilb $\mathbb{C}^3$ gives a crepant resolution of $\mathbb{C}^3/A$.

Finally, we should also mention the paper [**92**], where Craw draws on [**93**] to give an explicit version of the McKay correspondence for Abelian subgroups of $\mathrm{SL}(3, \mathbb{C})$.

4. Polytopal Algebra

In [**74**], Winfried Bruns and Joseph Gubeladze introduce the reader to *polytopal linear algebra*, which is an ambitious program to understand the category of polytopal semigroup algebras. To define such an algebra, let $P \subset M_\mathbb{R} \simeq \mathbb{R}^n$ be a lattice polytope (so all vertices of P lie in M). This gives the *polytopal semigroup algebra*

$$k[P] = k[t^m : m \in \mathcal{A}], \quad \mathcal{A} = \{1\} \times (P \cap M) \subset \mathbb{Z} \times M \simeq \mathbb{Z}^{n+1}.$$

The factor of $\{1\}$ means that the corresponding toric ideal $I_\mathcal{A}$ is homogeneous, so that $k[P]$ has a natural grading such that monomials of degree 1 correspond to lattice points of P, monomials of degree 2 correspond to those lattice points of $2P$ which are the sum of two lattice points of P, and so on. In particular, $k[P]$ is generated by its elements of degree 1.

One sees easily that $\mathrm{Spec}(k[P])$ is the (possibly non-normal) affine toric variety $X_\mathcal{A}$ defined in (0.3) and that $\mathrm{Proj}(k[P])$ is the corresponding (possibly non-normal) projective toric variety $Y_\mathcal{A}$. To relate these to the more usual toric varieties, let $\sigma \subset \mathbb{R} \times M_\mathbb{R} \simeq \mathbb{R}^{n+1}$ be the cone over $\{1\} \times P$ as in (2.1). Then the semigroup algebra $k[\sigma \cap M]$ is the normalization of $k[P]$. This implies in particular that the normalization of $Y_{\mathcal{A}}$ is the projective toric variety determined by the polytope P. Notice also that $k[P]$ agrees with its normalization in degree 1.

It follows that polytopal semigroup algebras are algebraic objects which in some sense "remember" their geometric origin. This is emphasized by a theorem of Gubeladze [**134**], which states that two polytopal semigroup algebras $k[P]$ and $k[Q]$ are isomorphic as k-algebras if and only if the corresponding lattice polytopes P and Q are integrally-affine equivalent (Bruns and Gubeladze discuss this in [**74**, Rem. 2.2.2]).

Polytopal semigroup algebras were introduced in the paper [**75**] by Bruns, Gubeladze and Trung. This paper also considers *normal polytopes*, which are those lattice polytopes for which $k[P]$ is normal. One of the main results of [**75**] is that if P is a lattice polytope, then cP is normal for any integer $c \geqslant \dim P - 1$. This relates nicely to the result of Ewald and Wessels [**116**] that for an ample divisor D on a complete toric variety X, cD is very ample for any integer $c \geqslant \dim X - 1$.

4.1. Triangulations and Coverings. — A strongly convex rational polyhedral cone $\sigma \subset N_{\mathbb{R}} \simeq \mathbb{R}^2$ of dimension 2 gives an affine toric surface X_σ with a unique singular point (= the fixed point of the torus action). To resolve this singularity, we noted in Section 2.2 that one can do this using the Hilbert basis of $\sigma \cap M$, since subdividing σ using rays through the points of its Hilbert basis gives the minimal resolution of X_σ.

For a polytopal semigroup algebra $k[P]$, the Hilbert basis of the semigroup can be identified with the lattice points of P. Hence, to generalize the above paragaph, we could use a unimodular (or basic) lattice triangulation, as defined in Section 2.3. If such a triangulation exists, it automatically implies that the polytope is normal. However, we noted in Section 2.3 that such triangulations don't always exist. Fortunately, for normality, we don't need the unimodular simplices to triangulate P—if P is simply a union of unimodular simplices, then P is normal. In this case, we say that P is *covered* by unimodular lattice simplices. This leads to the question *can all normal polytopes be covered by unimodular lattice simplices?*

Bruns and Gubeladze use this question to introduce the material of [**74**, Sec. 3], which studies the relation between covering and normality in detail. One of the high points is the description (based on the paper [**69**] of Bruns and Gubeladze) of a counterexample to the existence of unimodular coverings. They also consider some variants of the unimodular covering property.

4.2. Automorphisms and Retractions. — In [**72**], Bruns and Gubeladze study the graded automorphisms of a polytopal semigroup algebra $k[P]$. For them, the motivating example is the standard $(n-1)$-simplex $\mathrm{Conv}(e_1, \ldots, e_n)$. The corresponding polytopal semigroup ring is $k[x_1, \ldots, x_n]$, which has $\mathrm{GL}(n, k)$ as its group of graded automorphisms. In beginning linear algebra, one learns that an element of $\mathrm{GL}(n, k)$ is a product of elementary matrices, which include:

– permutations matrices (coming from symmetries of the $(n-1)$-simplex);
– diagonal matrices (coming from the torus); and
– elementary matrices which add a multiple of one row to another.

The paper [**72**] explains how this generalizes to a polytopal semigroup algebra $k[P]$.

The reader should note that algebra automorphisms arise naturally in the theory of toric varieties. For example, when X is simplicial, its automorphism group $\mathrm{Aut}(X)$ is related to algebra automorphisms as follows. Let S be the homogeneous coordinate ring of X and let $X = (k^{\Sigma(1)} \smallsetminus \mathbf{V}(B))/G$ be the quotient presentation (0.2). Then the group $\mathrm{Aut}_g(S)$ of graded automorphisms of S contains G is a normal subgroup, and Cox [**89**] shows that $\mathrm{Aut}_g(S)/G$ is naturally isomorphic to the connected component of the identity of $\mathrm{Aut}(X)$. Then one gets the full automorphism group using symmetries of the fan of X. (We should mention that Demazure's description of the automorphism group of a smooth complete toric variety X was extended by Cox [**89**] to the simplicial case and by Bühler [**80**] to the general case.)

In [**74**, Sec. 5], Bruns and Gubeladze describe the automorphisms of polytopal semigroup algebras and explain the relation to automorphisms of toric varieties of their results. The proofs use the divisor theory from [**74**, Sec. 4], which first appeared in their paper [**72**] (with further developments in [**67**]).

Another topic of [**74**, Sec. 5] concerns *retractions*, which are graded algebra endomorphisms $\varphi : k[P] \to k[P]$ with the property that $\varphi^2 = \varphi$. In linear algebra, such an endomorphism $\varphi : V \to V$ of a vector space induces a decomposition

$$V = \ker(\varphi) \oplus \mathrm{im}(\varphi). \tag{4.1}$$

Is the same true for a retraction $\varphi : k[P] \to k[P]$? Consider the following example. Suppose that $P \subset \mathbb{R}^n$ and $Q \subset \mathbb{R}^m$ are lattice polytopes, and let $\overline{PQ} \subset \mathbb{R}^{n+m+1}$ be their join (so $\overline{PQ}$ is the union of all line segments joining a point of P to a point of Q). In this situation, one easily sees that

$$k[\overline{PQ}] \simeq k[P] \otimes_k k[Q]. \tag{4.2}$$

Then tensoring the obvious maps $k[P] \to k \to k[P]$ with the identity on $k[Q]$ gives a retraction $\varphi : k[\overline{PQ}] \to k[\overline{PQ}]$ such that the analog of (4.1) is (4.2). To see the analogy, remember the natural isomorphism of symmetric algebras

$$\mathrm{Sym}(V_1 \oplus V_2) \simeq \mathrm{Sym}(V_1) \otimes_k \mathrm{Sym}(V_2).$$

Retractions are studied by Bruns and Gubeladze in [**73**], where they present two conjectures about the structure of retractions, together with supporting evidence in special cases. All of this is covered in [**74**, Sec. 5].

The final topic of [**74**, Sec. 5] concerns the structure of graded k-algebra homorphisms between polytopal semigroup algebras. This material is based on the authors' paper [**71**], which discusses a general conjecture for the structure of these homomorphisms. The results for automorphisms and retractions mentioned can be viewed as confirmation of a refined version of special cases of this conjecture.

We should also mention that in [**68, 70**], Bruns and Gubeladze apply these ideas to K-theory to define what the authors call *higher polyhedral K-groups*. Also, in [**133**], Gubeladze studies the usual higher K-groups of various semigroup algebras.

5. Quotients and Embeddings

The paper in this volume by Jürgen Hausen [**140**] brings together ideas dealing with quotients of toric varieties and embeddings into toric varieties. We begin by discussing these topics separately. In this section we will work over $k = \mathbb{C}$.

5.1. Quotients of Toric Varieties. — Given a subtorus H of the torus T of a toric variety X, one can ask for the quotient $X/\!\!/H$. The most basic notion of quotient is that of *categorical quotient* $\pi : X \to X/\!\!/H$, meaning that any morphism $X \to Y$ which is constant on H-orbits factors through π. On the other hand, if $\pi : X \to X/\!\!/H$ is affine and satisfies $\mathcal{O}_{X/\!\!/H} \simeq (\pi_*\mathcal{O}_X)^H$, then we call π a *good quotient*. These definitions come from Mumford's Geometric Invariant Theory (GIT), which is where the modern study of quotients began. GIT seeks to construct projective good quotients and, failing this, to describe maximal open subsets where such quotients exist. In general, the existence of quotients is quite subtle.

In the toric situation described above, A'Campo-Neuen and Hausen [**11**] study the existence of good quotients by first constructing a toric quotient, which is a categorical quotient in the category of toric varieties and toric morphisms. This toric quotient need not be a good quotient, but the authors construct an H-equivariant toric morphism $X \to \overline{X}$ such that $\overline{X}/\!\!/H$ is a good quotient and coincides with the toric quotient $X/\!\!/H$.

The quotients of greatest interest are often projective or quasi-projective. When X is quasi-projective, the same need not be true for the toric quotient $X/\!\!/H$. A'Campo-Neuen and Hausen define in [**9**] the quasi-projective reduction Y^r of a toric variety Y (for example, the 3-dimensional complete non-projective toric variety described in [**121**, p. 71] has trivial quasi-projective reduction). Then the authors show that X has a quotient by H in the category of quasi-projective varieties if and only if the composed map $X \to X/\!\!/H \to (X/\!\!/H)^r$ is surjective, in which case $(X/\!\!/H)^r$ is the quotient.

In a related paper [**7**], A'Campo-Neuen studies when the toric quotient $X/\!\!/H$ is a categorical quotient (for all varieties). She shows that if every curve in $X/\!\!/H$ is the image of a curve in X and $\dim X/\!\!/H = \dim X - \dim H$, then $X/\!\!/H$ is a categorical quotient. Furthermore, if the fan of X has convex support, then she shows that $X/\!\!/H$ is a categorical quotient.

One of the tools used in [**7**] is the notion of a *toric prevariety*, which is a non-separated toric variety. Toric prevarieties were used by Włodarczyk in 1993, but their systematic study began with the paper [**12**] by A'Campo-Neuen and Hausen. In the non-separated case, one has a finite index set I and a collection of fans Σ_{ij} for $i, j \in I$ which satisfy the following two properties for all indices:

– $\Sigma_{ij} = \Sigma_{ji}$.
– $\Sigma_{ij} \cap \Sigma_{jk}$ is a subfan of Σ_{ik}.

The second item implies that Σ_{ij} is a subfan of both Σ_{ii} and Σ_{jj}. Then we get a toric prevariety by gluing together $X_{\Sigma_{ii}}$ and $X_{\Sigma_{jj}}$ along the open subvariety $X_{\Sigma_{ij}} = X_{\Sigma_{ji}}$.

In [**12**] the authors also study the notion of a good prequotient and give necessary and sufficient conditions for the existence of a good prequotient. We should also mention the related paper [**8**] by the same authors, which gives several examples to illustrate the existence and non-existence of various sorts of quotients. In particular, they obtain an example of a toric variety acted on by a subtorus with a good prequotient (as a toric prevariety) but without categorical quotient.

The paper [**10**] by A'Campo-Neuen and Hausen studies subtorus actions on divisorial toric varieties. A toric variety X is *divisorial* if for every $x \in X$ there is an effective Cartier divisor D such that $X \smallsetminus \mathrm{Supp}(D)$ is an affine neighborhood of x. One can show that this condition is equivalent to assuming that X has enough invariant effective Cartier divisors, as defined by Kajiwara [**164**]. When a subtorus H acts on a divisorial toric variety X, the toric quotient $X/\!\!/H$ need not be divisorial. The authors construct its divisorial reduction $(X/\!\!/H)^{dr}$ and show that X has a quotient by H in the category of divisorial varieties if and only if the composed map $X \to X/\!\!/H \to (X/\!\!/H)^{dr}$ is surjective, in which case $(X/\!\!/H)^{dr}$ is the quotient.

Good quotients of subtorus actions have been studied by other authors as well. For example, Hamm [**137**] and Święcicka [**227**] independently discovered necessary and sufficient conditions for the existence of a good quotient. An ambitious study of toric quotients, which pays careful attention to the combinatorial aspects of the situation, is due to Hu [**149**].

We should also mention that torus quotients play an important role in the study of quotients by a reductive group G. Białynicki-Birula and Święcicka [**42**] show that for a normal variety X with an action by G, a good quotient $X/\!\!/G$ exists if and only if there is a good quotient $X/\!\!/H$ for every 1-dimensional torus $H \subset G$. Also, in a series of papers [**41, 43, 226**], these authors consider study G-actions where the goal is to find maximal open subsets on which a good quotient exists. One of their ideas is to restrict to the maximal torus. Note that quotients of affine or projective spaces by tori are toric varieties. This work is used in the results discussed in Section 5.3.

5.2. Embeddings into Toric Varieties. — It is well known that a variety X can be embedded into projective space if and only if every finite subset of X lies in an affine open. In 1993, Włodarczyk [**237**] proved the surprising result that any normal variety X can be embedded into a toric variety if and only if every two-element subset of X lies in an affine open. A variety satisfying the latter condition is said to be A_2. Włodarczyk also showed that if we drop the A_2 condition, then every normal variety can be embedded into a toric prevariety. This is the context in which toric prevarieties were first introduced.

In [**139**], Hausen gives an $\mathbb{C}^*$-equivariant version of the embedding theorem into prevarieties. He also shows that if the normal variety is $\mathbb{Q}$-factorial, then the toric prevariety can be chosen to be simplicial and of affine intersection. (The latter condition means that the intersection of two affine open subsets is affine.) Furthermore, Hausen and Schröer [**141**] show that there are normal surfaces with 2 non-$\mathbb{Q}$-factorial points which are neither embeddable into a simplicial toric prevariety nor into a toric prevariety of affine intersection.

Hausen's paper [**138**] next studies what happens if one drops the normality hypothesis. One of the main results is that an irreducible variety X is divisorial if and only if X can be embedded into a smooth toric prevariety of affine intersection. Then define X to be 2-divisorial if for every $x, y \in X$ there is an effective Cartier divisor D such that $X \smallsetminus \mathrm{Supp}(D)$ is an affine open subset containing x and y. In this situation, Hausen proves that an irreducible variety is 2-divisorial if and only if X can be embedded into a smooth toric variety. He also provides equivariant versions of these results for actions by connected linear algebraic groups.

5.3. Quotients of Embeddings. — When we combine the ideas of quotients by tori and embeddings into larger toric varieties, we get the question of whether a quotient can be extended to an embedding. Here is the situation studied in [**140**]: we have a $\mathbb{Q}$-factorial A_2-variety X with an effective action by a torus H. A good quotient $X/\!\!/H$ need not exist, but there are always nonempty open H-invariant subsets $U \subset X$ such that we have a good quotient $U/\!\!/H$. On the other hand, one way to obtain a good quotient would be to find an equivariant embedding $X \to Z$ where Z is a toric variety and H becomes a subtorus of the torus of Z. Then, given any open H-invariant subset $W \subset Z$ for which a good quotient exists, it follows automatically that $W \cap X$ is an open subset of X for which a good quotient also exists. Hence it makes sense to ask if all open $U \subset X$ as above arise in this way. The following result of [**140**, Cor. 2.6] answers this question.

Theorem 5.1. — *Given H and X as above, there is a H-equivariant embedding into a smooth toric variety Z on which H acts as a subtorus of the torus of Z such that every maximal open set $U \subset X$ having a good A_2 quotient $U/\!\!/H$ is of the form $U = W \cap X$ for some toric open set $W \subset Z$ with good quotient $W/\!\!/H$.*

We also note that [**140**, Sec. 1] is a useful review of good quotients of toric varieties and [**140**, Appendix] is a nice survey of embedding theorems.

6. Heights on Toric Varieties

The study of rational points on a variety X defined over a number field K is an important part of Diophantine geometry. The paper in this volume by Yuri Tschinkel [**230**] discusses how some of these ideas apply to toric varieties. The basic object of interest is

$$N(X,\mathcal{L},B) = |\{x \in X(K) : H_{\mathcal{L}}(x) \leqslant B\}|,$$

which counts the number of K-rational points of height at most B. Here, $\mathcal{L}$ is a (metrized) line bundle on X and $H_{\mathcal{L}}$ is the height function described in [**230**]. The main question of interest concerns the asymptotic behavior of $N(X,\mathcal{L},B)$ as $B \to \infty$.

A first observation is that the canonical divisor K_X plays an important role. For curves, the Mordell conjecture (proved by Faltings) says that a smooth curve of genus $g > 1$ has at most finitely many rational points over a number field. Since the canonical divisor of a curve C has degree $2g-2$, the inequality $g > 1$ is equivalent to the ampleness of K_C. In general, if you want a good supply of rational points on a variety X, then the canonical divisor K_X should be far from ample.

A second observation is that some subsets of X may have too many rational points. This happens, for example, if you blow up a rational point on a variety. The exceptional fiber will be a projective space and hence will have lots of rational points. So to best reflect what's happening "in general" on X, one studies the asymptotic behavior of $N(U,\mathcal{L},B)$ for sufficiently small Zariski open subsets $U \subset X$.

6.1. Asymptotic Formulas. — One case of interest is a smooth Fano variety X, which as in Section 1.2 means that $-K_X$ is an ample divisor. If we consider the height function $H_{\mathcal{L}}$ constructed using $\mathcal{L} = \mathcal{O}(-K_X)$, then Manin conjectured that

$$N(U,\mathcal{L},B) \sim cB(\log B)^{r-1}, \tag{6.1}$$

where c is a constant, r is the rank of $\mathrm{Pic}(U)$, and $U \subset X$ is a suitably small Zariski open. This conjecture was verified for for generalized flag manifolds G/P by Franke, Manin and Tschinkel [**120**]. Their proof uses the *height zeta function*

$$Z(s) = \sum_{x \in G/P(K)} H_{\mathcal{L}}(x)^{-s}.$$

The authors of [**120**] identify this with a Langlands-Eisenstein series for G/P, which gives knowledge about the analytic continuation and poles of $Z(s)$. From here, adelic harmonic analysis and Tauberian theorems imply the desired asymptotic estimates. In [**230**], Tschinkel explains how this strategy (minus the Langlands-Eisenstein part) is now standard.

If one uses other line bundles besides $\mathcal{L} = \mathcal{O}(-K_X)$, one gets different asymptotic results. The main theorem proved in [**230**] goes as follows.

Theorem 6.1. — *Let $\mathcal{L}$ be a line bundle on a smooth toric variety X. If the class $L = [\mathcal{L}] \in \mathrm{Pic}(X)$ is in the interior of the cone of effective divisors, then for a suitable*

Zariski open subset $U \subset X$, there are constants $\Theta(U,\mathcal{L})$, $a(L)$ and $b(L)$ such that

$$N(U,\mathcal{L},B) \sim \frac{\Theta(U,\mathcal{L})}{a(L)(b(L)-1)!}\, B^{a(L)}(\log B)^{b(L)-1}.$$

This theorem says that $a(L)$ and $b(L)$ depend only on the divisor class of $\mathcal{L}$ and are independent of U. When $\mathcal{L}$ is given by the anticanonical divisor, the theorem was first proved by Batyrev and Tschinkel in [**35**]. Note that we do not assume that X is Fano. However, the standard formula $K_X = -\sum_\rho D_\rho$ for the canonical divisor of a toric variety shows that the anticanonical class is in the interior of the cone of effective divisors. In the terminology of Peyre [**202**], this means that X is almost Fano. (Note that [**202**] contains some detailed examples of asymptotic formulas.)

6.2. Tamagawa Numbers. — When $\mathcal{L} = \mathcal{O}(-K_V)$, the constant $\Theta(U,\mathcal{L})$ is very interesting. As conjectured by Peyre, it is related to the Tamagawa number $\tau(X)$ of X defined by Peyre in [**203**]. More precisely, when U is the torus of X, then in [**35**], Batyrev and Tschinkel give the formula

$$\Theta(U,\mathcal{L}) = \alpha(X)\beta(X)\tau(X), \tag{6.2}$$

where $\tau(X)$ is the above Tamagawa number, $\alpha(X)$ depends only on the geometry of the cone of effective divisors, and $\beta(X)$ is the cardinality of a certain Galois cohomology group (to be described below).

Motivated by (6.2) and Peyre's paper [**203**], Salberger [**218**] realized that one could explain the factor $\beta(X)\tau(X)$ in terms of the Tamagawa number of the *universal torseur* of the toric variety X. Salberger worked out this theory in great generality, not just for toric varieties. Peyre independently defined Tamagawa numbers for universal torseurs in [**204**]. We should also mention the paper [**36**] of Batyrev and Tschinkel which defines Tamagawa numbers for a broad class of varieties (even for certain singular ones) and discusses the relation to the minimal model program.

Finally, we should note that over an algebraically closed field k, we've already seen the universal torseur of a smooth toric variety X. In general, if G is an algebraic group, then (roughly speaking) a *torseur* is a morphism $\mathcal{T} \to X$ of varieties such that G acts freely on $\mathcal{T}$ with X as quotient, and it is *universal* if a certain classifying map is the identity. (Careful definitions can be found in [**218**, Sec. 3 and 5].) If X is a smooth toric variety over k, then the quotient representation (0.2) can be written

$$X = \big(k^{\Sigma(1)} \smallsetminus \mathbf{V}(B)\big)/G, \quad G = \mathrm{Hom}(\mathrm{Pic}(X), k^*)$$

since $A_{n-1}(X) = \mathrm{Pic}(X)$ in the smooth case. Then one can show that the projection map $k^{\Sigma(1)} \smallsetminus \mathbf{V}(B) \to X$ is the universal G-torseur. See [**218**, Sec. 8] for a proof.

6.3. Toric Varieties over Number Fields. — In earlier sections, we always worked over an algebraically closed field k. Given the above discussion, we should say a few words about toric varieties over a number field K. Details can be found in the papers of Salberger [**218**] and Tschinkel [**230**].

We begin by describing a torus over K. Given a lattice $M \simeq \mathbb{Z}^n$, we get the split torus $(\mathbb{G}_{m,K})^n = \mathrm{Spec}(K[M])$. Then any other torus T over K comes from a Galois representation $G \to \mathrm{GL}(M)$, where $G = \mathrm{Gal}(E/K)$ for some finite extension $K \subset E$. In this situation, $T \times_K E \simeq (\mathbb{G}_{m,E})^n$.

Then a toric variety over K is determined by the following data. The lattices M and N are now G-modules, where $G = \mathrm{Gal}(E/K)$ as above. If the fan Σ is G-invariant (meaning that $\gamma \in G$ and $\sigma \in \Sigma$ implies $\gamma(\sigma) \in \Sigma$), then there is a variety X defined over K containing T such that the T-action extends to X and

$$X \times_K E \simeq \bigcup_{\sigma \in \Sigma} X_{\sigma,E}, \quad X_{\sigma,E} = \mathrm{Spec}(E[\sigma^\vee \cap M]).$$

In this notation, the constant $\beta(X)$ discussed earlier is the cardinality of the cohomology group $H^1(G, \mathrm{Pic}(X \times_K E))$. Also, when X is smooth, the construction of the universal torseur mentioned above also works over the number field K, with some obvious modifications (see [**218**, Sec. 8]).

7. Further Developments

Besides the papers mentioned above, there has been a lot of other interesting work on toric varieties since our earlier survey [**90**] appeared 1996. We will now discuss some of this work. For reasons of brevity, we will not mention the many interesting papers dealing with:

- Mirror symmetry and Lagrangian torus fibrations.
- Gromov-Witten invariants and quantum cohomology.
- GKZ hypergeometric functions.
- Resultants, residues, and solutions of polynomial equations.
- Symplectic geometry and toric varieties.

The first two items are discussed in the book by Cox and Katz [**91**] and the third is covered in the book by Saito, Sturmfels and Takayama [**217**]. Unfortunately, we are not aware of any survey of current work on the last two items.

7.1. Toric Ideals. — In [**223**], Sturmfels surveys work up to 1995 on toric ideals $I_{\mathcal{A}}$. One area of recent study concerns invariants related to the free resolution of $I_{\mathcal{A}}$. For example, Hibi and Ohsugi [**143**] give a criterion for when the toric ideal of a graph is generated by quadratic binomials. Syzygies of toric ideals have been studied by Campillo and Gimenez [**82**] and Pisón-Casares and Vigneron-Tenorio [**205**], and the regularity of a toric ideal has been computed—without knowing the free resolution—by Briales-Morales and Pisón-Casares [**58**]. A recent paper in this area is [**57**] by Briales-Morales, Campillo and Pisón-Casares.

Given a sublattice $L \subset \mathbb{Z}^n$, the corresponding lattice ideal $I_L \subset k[x_1, \ldots, x_n]$ is $I_L = \langle x^a - x^b : a, b \in \mathbb{N}^n, a - b \in L\rangle$. Every toric ideal is a lattice ideal, but not conversely. The minimal free resolution of a generic lattice ideal (suitably defined) is described by Peeva and Sturmfels in [**200**], and in [**201**], the same authors study the minimal free resolution of a codimension 2 lattice ideal. (There is also related work by Gasharov and Peeva in codimension 2 [**125**] and in dimension 2 [**124**].)

Eisenbud and Sturmfels [**114**] proved (among many other things) that the primary components of a binomial ideal are binomial. For lattice ideals, the primary decomposition has been studied further by Hoşten and Shapiro [**147**], and the associated primes of their initial ideals have been studied by Hoşten and Thomas [**148**] and Altmann [**17**].

Monomial ideals also play an important role here. For example, the resolution described in [**200**] is constructed using the Scarf resolution of a reverse lexicographic initial ideal of the lattice ideal. Other papers dealing with resolutions of monomial ideals are [**37, 39, 188, 225**]. Also, the book [**232**] by Villarreal discusses monomial ideals and their relation to affine toric varieties. An interesting monomial ideal introduced by Hoşten and Maclagan [**146**] encodes the vertices of all fibers of a lattice.

We should also mention one special type of lattice ideal called a Lawrence ideal. For example, if X is a toric variety with homogeneous coordinate ring S, then the ideal of the diagonal of $X \times_k X$ is a Lawrence ideal in $S \otimes_k S$. The minimal free resolution of a unimodular Lawrence ideal is described in [**38**].

Finally, given a toric ideal $I_{\mathcal{A}}$, one can study the set of all ideals which have the same multigraded Hilbert function as $I_{\mathcal{A}}$. This leads to the notion of the toric Hilbert scheme $\mathcal{H}_{\mathcal{A}}$, first introduced by Peeva and Stillman [**199**] (though inspired by earlier work of Arnold, Sturmfels and others—see [**199**] for references). As a scheme, $\mathcal{H}_{\mathcal{A}}$ is a union of irreducible components, each of which is a toric variety. When $I_{\mathcal{A}}$ has codimension 2, the results of Gasharov and Peeva [**125**] imply that $\mathcal{H}_{\mathcal{A}}$ is irreducible. In general, it is an open question whether $\mathcal{H}_{\mathcal{A}}$ is connected. This question is studied by Maclagan and Thomas in [**174**], and *Macaulay2* algorithms for computing $\mathcal{H}_{\mathcal{A}}$ are described by Stillman, Sturmfels and Thomas in [**222**]. Also, the paper [**198**] by Peeva and Stillman gives local equations for $\mathcal{H}_{\mathcal{A}}$.

7.2. Generalizations of homogeneous coordinates. — The quotient representation (0.2) shows that any toric variety is a categorical quotient of an $(k^*)^{\Sigma(1)}$-stable open subset of $k^{\Sigma(1)}$. This has been generalized in a variety of ways. For example, Hamm [**137**] shows that any toric variety is a very good quotient of such an open subset. More generally, A'Campo-Neuen, Hausen and Schröer [**13**] study quotient representations of toric varieties, which are affine surjective toric morphisms $\hat{X} \to X$ such that $\hat{X}$ is quasiaffine and $\hat{X}$ and X have the same invariant Weil divisors. Also, Hu and Keel [**151**] consider a "Mori dream space", which is a projective variety X with the property that under Mori equivalence, the cone of effective divisors on X

decomposes into polyhedral chambers in a suitably nice way. This leads to a representation of X as a GIT quotient of an affine variety by a torus, which reduces to (0.2) for a simplicial projective toric variety. Every rational contraction of a Mori dream space comes from GIT and all possible factorizations of a rational contraction can be read off from the chamber decomposition.

7.3. Alterations, Weak Resolutions, and Semi-Stable Reductions. — In 1995, de Jong [**105**] proved that every variety X has a *smooth alteration* $\pi : \widetilde{X} \to X$, meaning that $\widetilde{X}$ is smooth and π is proper, surjective, and generically finite. The important feature of this result is that it applies in *arbitrary* characteristic, in contrast to Hironaka's resolution of singularities, which was proved only in characteristic 0. As reported in our earlier survey [**90**], this quickly led to the work of Abramovich and de Jong [**1**] and Bogomolov and Pantev [**44**] on *weak resolution of singularities* in characteristic 0. A weak resolution of X is a proper birational morphism $\pi : \widetilde{X} \to X$ such that $\widetilde{X}$ is smooth and the inverse image of X_{sing} is a divisor with normal crossings, though π need not be an isomorphism over $X \smallsetminus X_{\mathrm{sing}}$. The interesting feature of [**1, 44**] is that in addition to de Jong's results, both papers use the toroidal embeddings of Kempf, Knudsen, Mumford and Saint Donat [**167**].

The recent survey [**3**] by Abramovich and Oort includes numerous references and sketches the proofs of the major results in this area. Their paper appears in the 2000 book [**142**], which includes other interesting applications of alterations. This volume also contains the paper [**126**] by Goldin and Teissier, which uses toric morphisms to resolve singularities of plane analytic branches. A generalization of their method to the case of quasi-ordinary singularities appears in the thesis of González Pérez [**129**]. Part of this has been published in [**130**].

There has also been interesting work on semi-stable reduction, which was proved for a one-dimensional base in [**167**]. In [**105**], de Jong does semi-stable reduction for families of curves, which plays an important role in his work on alterations as well as in the papers on weak resolution cited above. Going beyond this, Abramovich and Karu [**2**] prove the existence of weak semi-stable reduction in characteristic 0. Furthermore, the analysis of triangulations done by Abramovich and Rojas in [**4**] implies that the reduction morphism constructed in [**2**] is semi-stable in codimension 1. We should also note that Karu [**166**] has proved semi-stable reduction in characteristic 0 for families of surfaces and 3-folds.

7.4. Factorization of Birational Maps. — In the middle 1990s, Morelli [**189**] and Włodarczyk [**235**] proved that a proper equivariant birational map between smooth toric varieties factors into a sequence of smooth toric blow-ups and blow-downs. This is called a *weak factorization* since the blowups and blowdowns can occur in any order. Morelli also claimed *strong factorization*, where all of the blowups occur first.

Using a toroidal version of Morelli's arguments, this was extended by Włodarczyrk [**236, 238**] and Abramovich, Karu, Matsuki and Włodarczyrk [**6**] to weak factorization of birational maps between smooth varieties in characteristic 0. (The paper [**236**] did the case of quasi-smooth centers and introduced the Morelli-Włodarczyrk cobordisms which were mentioned in Wiśniewski's lectures [**234**, Lec. 3] at the end of Section 1.1.) However, gaps were noticed in Morelli's proof of π-desingularization, which were filled by Abramovich, Matsuki and Rashid in [**5**]. Unfortunately, as noted by Matsuki in [**179**], the strong factorization in the toric case claimed in [**5**] is still an open problem, even in dimension 3. For more details, we refer the reader to Bonavero's Seminar Bourbaki lecture [**46**] and the very complete lecture notes of Matsuki [**178**].

In related work, Hu and Keel [**150**] have a different proof of Włodarczyrk's version of weak factorization with quasi-smooth centers. Also, the paper [**85**] by Casagrande mentioned in Section 1.2 concerns factorization of birational maps between toric Fano 4-folds.

7.5. Toric Clusters. — In the early 20th century, Enriques and Chisini studied plane curves passing through collections of infinitely near points in the plane with assigned multiplicities. Zariski later recast this as the theory of complete ideals in the local ring of a point in the plane. In higher dimensions, one gets a *constellation*, which is a finite sequence of maps $\cdots \to X_{i+1} \to X_i \to \cdots \to X_0$ such that X_0 is smooth, X_1 is the blow-up of $Q_0 \in X_0$, X_2 is the blow-up of Q_1 in the exceptional locus of X_1, and so on. If in addition we specify a multiplicity $m_i \geqslant 0$ for each Q_i, then we have a *cluster*. In [**84**], Campillo, González-Sprinberg and Lejeune-Jalabert encode the combinatorics of the constellation in an oriented graph which generalizes what Enriques and Chisini did in the 2-dimensional case. They also define *idealistic* clusters and show that they are related to certain complete ideals. In the toric case, [**84**] contains a combinatorial description of toric clusters and a characterization of toric idealistic clusters. In [**132**], González-Sprinberg and Pereyra extend the characterization to all toric clusters, not just the idealistic ones. The survey paper [**131**] by González-Sprinberg covers these and other topics, along with further references.

7.6. Cohomology of Toric Varieties. — Eisenbud, Mustaţă and Stillman consider the cohomology of coherent sheaves on a toric variety X in [**113**]. Using the homogeneous coordinate ring of X, the problem reduces to computing local cohomology with supports in a monomial ideal, which in turn is a direct limit of Ext groups. By using Mustaţă's paper [**190**] and working with graded pieces, the authors reduce the computation to finding a graded piece of a single Ext group. In [**191**], Mustaţă uses these results to prove refined versions of the Kawamata-Viehweg vanishing theorem and Fujita's conjecture in the toric case. The latter was proved earlier (also in the toric case) by Laterveer in [**172**]. We should also mention the paper [**194**], where Nikbakht-Tehrani considers the cohomology of toric varieties.

7.7. Toric Nakai Criterion. — In [**196**, p. 86], Oda proves the toric Nakai criterion, which states that a Cartier divisor D on a smooth complete toric variety is ample if and only if $D \cdot C > 0$ for every torus-invariant curve $C \subset X$. While many experts knew that this criterion applied more generally, only recently did Mavlyutov [**184**] and Mustaţă [**191**] independently publish proofs that the toric Nakai criterion holds for *all* complete toric varieties, not just smooth ones.

7.8. Cohomology of Toric Hypersurfaces. — The cohomology of an ample toric hypersurface in a simplicial toric variety was studied by Batyrev and Cox [**31**]. These results were generalized to the semiample case by Mavlyutov [**184, 185**], who also worked out formulas for cup product. This led to a generalization of the monomial-divisor mirror map in mirror symmetry. The papers [**184, 185**] also include a careful study of semiample divisors. In a related paper [**183**], Mavlyutov describes the chiral ring of a Calabi-Yau toric hypersurface.

7.9. Stringy Hodge Numbers and Orbifold Cohomology. — String-theoretic Hodge numbers were introduced by Batyrev and Dais in [**33**] to give the desired equality of Hodge numbers for certain singular mirror pairs in mirror symmetry. Batyrev [**30**] gave a different definition, calling them *stringy Hodge numbers* $h^{p,q}_{\mathrm{st}}$. The introductions to these papers explain why the usual Hodge numbers $h^{p,q}$ don't work. For complete intersections in toric varieties, these numbers were computed by Batyrev and L. Borisov in [**32**].

Stringy Hodge numbers are defined when X has finite quotient singularities or Gorenstein toric singularities. In [**30**], Batyrev shows that if such an X has a crepant resolution $\widetilde{X} \to X$, then

$$h^{p,q}(\widetilde{X}) = h^{p,q}_{\mathrm{st}}(X). \tag{7.1}$$

These numbers also arise when considering the *stringy* (or *physicists*) Euler number. If a finite group G acts on a smooth variety M such that K_M is G-invariant, then $X = M/G$ is Gorenstein, and its stringy Euler number is

$$e_{\mathrm{st}}(X) = \frac{1}{|G|} \sum_{gh=hg} e(M^g \cap M^h),$$

where e denotes the usual Euler number and M^g is the fixed point locus of $g \in G$. This number arose in string theory in the 1980s (see Reid [**210**] for references). A key result is that if $\widetilde{X} \to X$ is a crepant resolution, then $e(\widetilde{X}) = e_{\mathrm{st}}(X)$. This follows by combining (7.1) with the result of Batyrev and Dais [**33**] that $e_{\mathrm{st}}(X)$ can be computed using stringy Hodge numbers.

As explained by Batyrev and Dais [**33**] and Reid [**210**], stringy Euler numbers are related to the McKay correspondence discussed in Section 3. Also, Dais and Roczen [**98**] and Dais [**95**] compute the stringy Euler numbers of various singularities, and Batyrev [**29**] defines stringy Euler numbers for log terminal pairs.

A recent development is the definition of orbifold cohomology due to Chen and Ruan [**87**]. The idea is to create cohomology groups whose Hodge numbers will be the stringy Hodge numbers. A good survey can be found in Ruan [**215**], which includes references and relations with quantum cohomology and mirror symmetry. In [**207**], Poddar computes $h^{1,1}_{\mathrm{orb}}$ and $h^{n-2,1}_{\mathrm{orb}}$ for a hypersurface in a n-dimensional symplicial Fano toric variety. Also, L. Borisov and Mavlyutov [**53**] propose a definition of stringy cohomology of a semiample anticanonical hypersurface in a simplicial toric variety. Their definition depends on a parameter, which for a special value gives the definition of Chen and Ruan. The paper [**53**] also discusses conditions under which the stringy Hodge numbers defined in [**33**] and [**30**] coincide.

7.10. Intersection Cohomology. — The intersection cohomology of toric varieties was described in 1991. In [**154**], Ishida describes the intersection complex of a toric variety (for any perversity) in terms of its fan, and for middle perversity, the author derives a decomposition which allows him to give a new proof of McMullen's conjecture for the h-vector of a simplicial polytope (as described by Fulton in [**121**, Ch. 5]). There is also the work of Timorin [**229**], who studies an analog of Hard Lefschetz for polytopes such that each facet contains at most one nonsimple vertex and each edge is incident to exactly $d-1$ facets, where d is the dimension of the polytope. In a different direction, a face F of a rational polytope P gives toric varieties X_F and X_P. In [**55**], Braden and MacPherson study the relation between the intersection homologies of X_F and X_P. They use this to prove a combinatorial conjecture of Kalai for rational polytopes.

The equivariant intersection cohomology of a toric variety is described by Barthel, Brasselet, Fiesler and Kaup in [**23**]. The authors use sheaves on a finite topological space determined by the fan, which allows them to introduce "virtual" intersection cohomology for equivariant non-rational fans. In [**24**], the same authors prove the Kalai conjecture mentioned above as well as Hard Lefschetz for the combinatorial intersection cohomology of a polytopal fan which satisfies a certain vanishing condition. A somewhat similar approach to combinatorial intersection cohomology is due to Bressler and Lunts [**56**], drawing on ideas introduced by Bernstein and Lunts in [**40**]. In both [**24**] and [**56**], the major open problem is whether Hard Lefschetz holds for all non-rational fans. There is also the work of Fine [**117, 118, 119**] who defines new intersection homology groups ("local-global" intersection homology) and the 1996 paper [**25**] of Barthel and Fiesler which investigates which Betti numbers of a non-simplicial toric variety are combinatorial invariants of the fan.

7.11. The Polytope Algebra and Equivariant Chow Groups. — The polytope algebra over an ordered field K was introduced by McMullen [**187**] and related to toric varieties by Fulton and Sturmfels [**122**] when $K = \mathbb{Q}$. In [**61**], Brion introduces a new approach to studying the polytope algebra over $\mathbb{Q}$ which relates it to the equivariant cohomology of toric varieties, and in [**63**] he proves a structure theorem for the polytope algebra when K is a subfield of $\mathbb{R}$.

The equivariant approach is also used by Brion and Vergne in their paper [**64**] on the equivariant Riemann-Roch theorem for complete simplicial toric varieties. They use this to extend previous results on counting lattice points in polytopes, including a version of the Euler-Maclaurin formula for lattice polytopes. In [**60**], Brion discusses the algebraic equivariant Chow groups of Edidin and Graham [**111**] for torus actions and computes these groups for simplicial toric varieties. The papers [**65, 66**] of Brion and Vergne use non-toric methods to study lattice points in integer and rational polytopes. For a survey of toric and non-toric methods of counting lattice points in polytopes, the reader should consult the paper [**26**] by Barvinok and Pommersheim. Pommersheim's paper [**206**] is also relevant.

7.12. K-theory and Topology. — An open question concerns the two flavors of K-theory, one computed using vector bundles and the other using coherent sheaves. These coincide for quasi-projective smooth varieties and are conjectured to be the same for quasi-projective orbifolds after tensoring with $\mathbb{Q}$. Edidin and Laterveer [**112**] claim to prove this for simplicial quasi-projective toric varieties, though it appears that their argument has a gap. Also, the equivariant K-theory of toric varieties is studied by Vezzosi and Vistoli in [**231**] and their KO-theory is computed by Bahri and Bendersky in [**22**].

In topology, there is a generalization of smooth toric varieties called toric manifolds due to Davis and Januszkiewicz [**104**]. In [**79**], Buchstaber and Ray show that toric manifolds generate the complex cobordism ring. In [**77, 78**], Buchstaber and Panov study toric manifolds, and in [**197**], Panov computes the χ_y-genus of a toric manifold. Also, Battaglia and Prato [**27**] study complex quasifolds, which are (possibly non-Hausdorff) topological spaces associated to simple polytopes. Finally, a unitary generalization of toric varieties is considered by Masuda [**176**].

7.13. Toric Fibrations. — In toric geometry, there is a well-defined notion of a equivariant toric fibration—see, for example, [**196**, p. 58] in Oda. These have been used in classify toric varieties of low dimension [**196**, p. 59]. However, in mirror symmetry, one encounters slightly different notions of what a toric fibration means. For example, Kreuzer and Skarke [**168, 171**] have studied the classification of low dimension toric fibrations coming from reflexive polytopes. For them, a toric fibration means having a subfan $\Sigma' \subset \Sigma$ whose support is a subspace $N'_{\mathbb{R}} \subset N_{\mathbb{R}}$ such that the image of Σ in $(N/N')_{\mathbb{R}}$ is a fan. In a slightly different direction, Hu, Liu and Yau [**152**] define a toric fibration to be a surjective morphism of toric varieties $X \to X'$ coming from a map of fans with the property that all components of all fibers have the same dimension. This paper also includes a careful study of toric morphisms.

7.14. Degenerations to Toric Varieties. — In [**91**, 12.2.9], Cox and Katz survey work about how a Grassmannian and other flag varieties can degenerate to a toric variety. This has implications for mirror symmetry. One paper not mentioned in [**91**] is [**128**] by Gonciulea and Lakshmibai.

In an earlier paper [**127**], the same authors show that Kempf varieties in $\mathrm{SL}(n,\mathbb{C})/B$ and Schubert varieties in a minuscule G/P also degenerate to toric varieties. This line of thought was pursued by Dehy and Yu [**106, 107**] who show that many other Schubert varieties in $\mathrm{SL}(n,\mathbb{C})/B$ degenerate to toric varieties. The most general result in this area is due to Caldero [**81**], who proves that if G is a semisimple algebraic group over $\mathbb{C}$, then every Schubert variety in G/B degenerates to a toric variety.

7.15. Characteristic *p*. — Using the fact the Frobenius morphism of a toric variety in characteristic p lifts to p^2, Buch, Thomsen, Lauritzen and Mehta [**76**] prove the Bott vanishing theorem for all toric varieties (previously known only in the simplicial case) and the degeneration of the Danilov spectral sequence. Also, if L is a line bundle on a smooth toric variety over an algebraically closed field of characteristic p, then F_*L, where F is the Frobenius morphism, is a direct sum of line bundles. This was proved independently by Bøgvad [**45**] and Thomsen [**228**].

7.16. Toric Varieties and Modular Forms. — In [**52**], L. Borisov and Gunnells show how to construct modular forms (with character) of level ℓ using a piecewise linear function on the cones of a complete rational polyhedral fan. The resulting modular forms are stable under the action of Hecke operators and the Fricke involution and are related to products of logarithmic derivatives of theta functions with characteristic. In [**51**], the authors show that modulo Eisenstein series, the weight two toric forms coincide exactly with the vector space generated by all cusp eigenforms f such that $L(f,1) \neq 0$. A survey of more recent work of L. Borisov, Gunnells and Popescu can be found in [**135**].

7.17. Shokurov's Conjecture. — In 1997, Shokurov [**221**] conjectured that if we have a projective log variety (X,D), $D = \sum d_i D_i$, with $K_X + D$ numerically trivial and at worst log canonical singularities, then $\sum d_i \leqslant \operatorname{rank} NS(X) + \dim(X)$. Furthermore, equality should hold if and only if X is a toric variety and the D_i are the torus-invariant divisors on X. Shokurov proved this for surfaces and then Prohkorov [**208**] proved a special case in dimension 3.

7.18. Other Results of Interest. — Here is a selection of some of the many interesting papers dealing with toric varieties:

- In [**18**], Altmann computes the torsion submodule of Ω^1_Y, where Y is any affine toric variety.

– In [**62**], Brion characterizes rationally smooth points on a variety with a torus action. Although not mentioned in [**62**], this implies the "folklore" result that a toric variety is an orbifold (i.e., has at worst finite quotient singularities) if and only if its fan is simplicial.
– In [**83**], Campillo, Grabowski and Müller study when a non-normal affine toric variety is determined by the Lie algebra of derivations of its coordinate ring.
– In [**108**], DiRocco studies k-jet ample line bundles on smooth toric varieties. The key tool is the notion of a k-convex support function.
– In [**109**], DiRocco and Sommese obtain strong lower bounds for the Chern numbers of ample vector bundles $\mathcal{E}$ on smooth projective toric surfaces.
– In [**110**], Druel shows that the only toric varieties with a contact structure (hence of odd dimension) are $\mathbb{P}^{2n+1}$ and the projectivized tangent bundle of $(\mathbb{P}^1)^{n+1}$.
– In [**115**], Elizondo shows that for a Cartier divisor on a complete toric variety, the ring of global sections of multiples of the line bundle associated to the divisor is finitely generated.
– In [**123**], Garoufalidis and Pommersheim relate special values of zeta functions to invariants of toric varieties and generalized Dedekind sums. The Todd class of a toric variety is used to give new formulas for the zeta function of a real quadratic field at nonpositive integers.
– In [**136**], Halic describes a compactification of the space of morphisms from a smooth projective curve to a smooth projective toric variety representing a fixed homology class.
– In [**144**], Hille shows that certain moduli spaces of quivers are toric varieties and describes the fan explicitly. In a follow-up paper [**19**], Hille and Altmann study the universal bundle over this moduli space.
– In [**145**], Hoşten verifies a conjecture of Batyrev by constructing a smooth polytope whose normal fan has a very large number of primitive collections.
– In [**153**], Huber and Thomas describe an algorithm for computing the Gröbner fan of a toric ideal.
– In [**165**], Kajiwara and Nakayama show that for an r-dimensional complete toric variety over a finite field k, the l-adic cohomology group $H^m(X \otimes_k \overline{k}, \mathbb{Q}_l)$ is of pure weight if $m = 0, 1, 2, 3, 2r-3, 2r-2, 2r-1, 2r$.
– In [**173**], Liu and Yau study the splitting type of equivariant vector bundles on smooth toric varieties. They show by example that the tangent bundle does not always have a splitting type.
– In [**177**], Materov computes the global sections of the sheaf $\Omega_X^p(D)$ of Zariski p-forms twisted by an ample divisor D on a complete simplicial toric variety X. The answer involves a generalization of the Ehrhart polynomial.

– In [**182**], Mavlyutov uses the Cayley trick to study the cohomology of complete intersections in toric varieties.
– In [**192**], Mustaţă, Smith, Tsai and Walther study D-modules on smooth toric varieties algebraically using the ring of differential operators on the homogeneous coordinate ring of the toric variety.
– In [**195**], Occhetta and Wiśniewski show that if we have a surjective map $X \to Y$ where X is a complete toric variety and Y is a smooth projective variety with Picard number one, then $Y \simeq \mathbb{P}^n$.
– In [**213**], Reyes, Villarreal and Zárate study when $k^n \to \mathbf{V}(I_{\mathcal{A}}) \subset k^\ell$ is onto, where $I_{\mathcal{A}}$ is the toric ideal (0.4) and k is an arbitrary field.
– In [**216**], Russell studies certain toric varieties which arise naturally when studying the subscheme of $\mathrm{Hilb}^d(k[[x_1,\dots,x_n]])$ parametrizing subschemes isomorphic to $k[[x_1,\dots,x_n]]/I$, where I is a fixed monomial ideal of colength d.
– In [**219**], Sankaran investigate the possibility of embedding minimal abelian surfaces in smooth toric 4-folds with Picard number two. See also [**163**].

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