Séminaires & Congrès
6, 2002, p. 213–225

SPECIAL MCKAY CORRESPONDENCE

by

Yukari Ito

Dedicated to Professor Riemenschneider for his 60th birthday

Abstract. — There are many generalizations of the McKay correspondence for higher dimensional Gorenstein quotient singularities and there are some applications to compute the topological invariants today. But some of the invariants are completely different from the classical invariants, in particular for non-Gorenstein cases. In this paper, we would like to discuss the McKay correspondence for 2-dimensional quotient singularities via "special" representations which gives the classical topological invariants and give a new characterization of the special representations for cyclic quotient singularities in terms of combinatorics.

Contents

1. McKay correspondence

The McKay correspondence is originally a correspondence between the topology of the minimal resolution of a 2-dimensional rational double point, which is a quotient singularity by a finite subgroup G of $\mathrm{SL}(2,\mathbb{C})$, and the representation theory (irreducible representations or conjugacy classes) of the group G. We can see the

2000 ***Mathematics Subject Classification.*** — 14C05, 14E15.
Key words and phrases. — McKay correspondence, quotient singularity, group representation, Hilbert scheme, Young diagram.

The author is partially supported by JSPS, the Grant-in-aid for Scientific Research (No.13740019).

correspondence via Dynkin diagrams, which came from McKay's observation in 1979 ([**16**]).

Let G be a finite subgroup of $\mathrm{SL}(2,\mathbb{C})$, then the quotient space $X := \mathbb{C}^2/G$ has a rational double point at the origin. As there exists the minimal resolution $\widetilde{X}$ of the singularity, we have the exceptional divisors E_i. The dual graph of the configuration of the exceptional divisors is just the Dynkin diagram of type A_n, D_n, E_6, E_7 or E_8.

On the other hand, we have the set of the irreducible representations ρ_i of the group G up to isomorphism and let ρ be the natural representation in $\mathrm{SL}(2,\mathbb{C})$. The tensor product of these representations

$$\rho_i \otimes \rho = \bigoplus_{j=0}^{r} a_{ij}\rho_j,$$

where ρ_0 is the trivial representation and r is the number of the non-trivial irreducible representations, gives a set of integers a_{ij} and it determines the Cartan matrix which defines the Dynkin diagram. (1)

Then we have a one-to-one numerical correspondence between non-trivial irreducible representations $\{\rho_i\}$ and irreducible exceptional curves $\{E_i\}$, that is, the intersection matrix of the exceptional divisors is the opposite of the Cartan matrix.

This phenomenon was explained geometrically in terms of vector bundles on the minimal resolution by Gonzalez-Sprinberg and Verdier ([**8**]) (2) by case-by-case computations in 1983. In 1985, Artin and Verdier ([**1**]) proved this more generally with reflexive modules and this theory was developed by Esnault and Knörrer ([**5**], [**6**]) for more general quotient surface singularities. After Wunram ([**21**]) constructed a nice generalized McKay correspondence for any quotient surface singularities in 1986 in his dissertation, Riemenschneider introduced the notion of "special representation etc." and made propaganda for the more generalized McKay correspondence (cf. [**18**]). (3)

In dimension three, we have several "McKay correspondences" but they are just bijections between two sets: Let X be the quotient singularity $\mathbb{C}^3/G$ where G is a finite subgroup of $\mathrm{SL}(3,\mathbb{C})$. Then X has a Gorenstein canonical singularity of index 1 but not a terminal singularity. It is known that there exist crepant resolutions $\widetilde{X}$ of this singularity. The crepant resolution is a minimal resolution and preserves the triviality of the canonical bundle in this case.

As for the McKay correspondence, the followings are known:

(1) (Ito-Reid [**12**]) There exists a base of cohomology group $H^{2i}(\widetilde{X},\mathbb{Q})$, indexed by the conjugacy classes of "age" i in G.

(1) More precisely, the Cartan matrix is defined as the matrix $2E - A$, where E is the $r \times r$ identity matrix and $A = \{a_{ij}\}$ $(i, j \neq 0)$.

(2) They gave the name *McKay correspondence* (in French, *la correspondance de McKay*) in this paper!

(3) Similar generalization for $G \subset \mathrm{GL}(2,\mathbb{C})$ was obtained by Gonzalez-Sprinberg and the related topics were discussed in [**7**].

(2) (Ito-Nakajima [**10**]) There exists a base of Grothendieck group $K(\widetilde{X})$, indexed by the irreducible representations of G, when G is a finite abelian subgroup.

(3) (Bridgeland-King-Reid [**3**]) There exists an equivalence between the derived category $D(\widetilde{X})$ and the equivariant dereived category $D^G(\mathbb{C}^3)$ for any finite subgroup.

Remark 1.1. — In (1), the **age** of $g \in G$ is defined as follows: After diagonalization, if $g^r = 1$, we obtain $g' = \operatorname{diag}(\varepsilon^a, \varepsilon^b, \varepsilon^c)$ where ε is a primitive r-th root of unity. Then $\operatorname{age}(g) := (a+b+c)/r$. For the identity element *id*, we define age(*id*)= 0 and all ages are integers if $G \subset \mathrm{SL}(3, \mathbb{C})$.

The correspondence (2) can be included in (3), but note that the 2-dimensional numerical McKay correspondence can be explained very clearly as a corollary of the result in [**10**].

As a generalization of the first McKay correspondence (1), we have a precise correspondence for each $2i$-th cohomology with conjugacy classes of age i for any $i = 1, \ldots, n-1$ in dimension n which was given by Batyrev and Kontsevich via "motivic integral" under the assumption of the existence of a crepant resolution, and this idea was developed to "string theoretic cohomology" for all quotient singularities (cf. [**2**]).

And we can see that the string theoretic Euler number of the resolution is the same as the order of the acting group G in case $G \subset \mathrm{GL}(n, \mathbb{C})$, but it is different from the usual topological Euler number of the minimal resolution. Of course, it is very interesting to consider the geometrical meaning of these new invariants.

By the way, in (2) we don't have such a difference among representations as *age*. But the author is interested in the relation between the group theory and the classical topological invariants. Then we would like to remind the reader of the notion of special representations which gives some differences between irreducible representations. The special representations were defined by Riemenschneider and Wunram ([**18**]); each of the special irreducible respresentations corresponds to an exceptional divisor of the minimal resolution of a 2-dimensional quotient singularity.

In particular, we would like to discuss special representations and the minimal resolution for quotient surface singularities from now on. Around 1996, Nakamura and the author showed another way to the McKay correspondence with the help of the G-Hilbert scheme, which is a 2-dimensional G-fixed set of the usual Hilbert scheme of $|G|$-points on $\mathbb{C}^2$ and isomorphic to the minimal resolution. Kidoh ([**14**]) proved that the G-Hilbert scheme for general cyclic surface singularities is the minimal resolution. Then Riemenschneider checked the cyclic case and conjectured that the representations which are given by the Ito-Nakamura type McKay correspondence via G-Hilbert scheme are just special representations in 1999 ([**19**]) and this conjecture was proved by A. Ishii recently ([**9**]). In this paper, we will give another characterization of the special representations by combinatorics for the cyclic quotient case, using results on the G-Hilbert schemes.

As a colorful introduction to the McKay correspondence, the author would like to recommend a paper presented at the Bourbaki seminar by Reid ([**17**]) and also on the Web page (http://www.maths.warwick.ac.uk/ ∼miles/McKay), one can find some recent papers related to the McKay correspondence.

This paper is organized as follows: In this section, we already gave a brief history of the McKay correspondence and we will discuss the special representations and the generalized McKay correspondence in the following section. In section three, we treat G-Hilbert schemes as a resolution of singularities, consider the relation with the toric resolution in the cyclic case, and show how to find the special representations by combinatorics. In the final section, we will discuss an example and related topics.

Acknowledgements. — Most of the contents of this paper are based on the author's talk in the summer school on toric geometry at Fourier Institute in Grenoble, France in July 2000, and she would like to thank the organizers for their hospitarity and the participants for the nice atmosphere. She would like to express her gratitude to Professor Riemenschneider for giving her a chance to consider the special representations via G-Hilbert schemes and for the various comments and useful suggestions on her first draft.

2. Special representations

In this section, we will discuss the special representations. Let G be a finite small subgroup of $\mathrm{GL}(2,\mathbb{C})$, that is, the action of the group G is free outside the origin, and ρ be a representation of G on V. G acts on $\mathbb{C}^2\times V$ and the quotient is a vector bundle on $(\mathbb{C}^2\smallsetminus\{0\})/G$ which can be extended to a reflexive sheaf $\mathcal{F}$ on $X\colon=\mathbb{C}^2/G$.

For any reflexive sheaf $\mathcal{F}$ on a rational surface singularity X and the minimal resolution $\pi\colon\widetilde{X}\to X$, we define a sheaf $\widetilde{\mathcal{F}}\colon=\pi^*\mathcal{F}/\text{torsion}$.

Definition 2.1 ([**5**]). — The sheaf $\widetilde{\mathcal{F}}$ is called a *full sheaf* on $\widetilde{X}$.

Theorem 2.2 ([**5**]). — *A sheaf $\widetilde{\mathcal{F}}$ on $\widetilde{X}$ is a full sheaf if the following conditions are fulfilled:*

(1) *$\widetilde{\mathcal{F}}$ is locally free,*
(2) *$\widetilde{\mathcal{F}}$ is generated by global sections,*
(3) *$H^1(\widetilde{X},\widetilde{\mathcal{F}}^\vee\otimes\omega_{\widetilde{X}})=0$, where $\vee$ means the dual.*

Note that a sheaf $\widetilde{\mathcal{F}}$ is indecomposable if and only if the corresponding representation ρ is irreducible. Therefore we obtain an indecomposable full sheaf $\widetilde{\mathcal{F}}_i$ on $\widetilde{X}$ for each irreducible representation ρ_i, but in general, the number of the irreducible representations is larger than that of irreducible exceptional components. Therefore Wunram and Riemenschneider introduced the notion of speciality for full sheaves:

Definition 2.3 (**[18]**). — A full sheaf is called *special* if and only if

$$H^1(\widetilde{X}, \widetilde{\mathcal{F}}^\vee) = 0.$$

A reflexive sheaf $\mathcal{F}$ on X is *special* if $\widetilde{\mathcal{F}}$ is so.

A representation ρ is *special* if the associated reflexive sheaf $\mathcal{F}$ on X is special.

With these definitions, the following equivalent conditions for the speciality hold:

Theorem 2.4 (**[18]**, **[21]**)

(1) *$\widetilde{\mathcal{F}}$ is special $\Longleftrightarrow$ $\widetilde{\mathcal{F}} \otimes \omega_{\widetilde{X}} \to [(\mathcal{F} \otimes \omega_{\widetilde{X}})^{\vee\vee}]^{\sim}$ is an isomorphism,*

(2) *$\mathcal{F}$ is special $\Longleftrightarrow$ $\mathcal{F} \otimes \omega_{\widetilde{X}}/$torsion is reflexive,*

(3) *ρ is a special representation $\Longleftrightarrow$ the map $(\Omega^2_{\mathbb{C}^2})^G \otimes (\mathcal{O}_{\mathbb{C}^2} \otimes V)^G \to (\Omega^2_{\mathbb{C}^2} \otimes V)^G$ is surjective.*

Then we have the following nice generalized McKay correspondence for quotient surface singularities:

Theorem 2.5 (**[21]**). — *There is a bijection between the set of special non-trivial indecomposable reflexive modules $\mathcal{F}_i$ and the set of irreducible components E_i via $c_1(\widetilde{\mathcal{F}}_i)E_j = \delta_{ij}$ where c_1 is the first Chern class, and also a one-to-one correspondence with the set of special non-trivial irreducible representations.*

As a corollary of this theorem, we get back the original McKay correspondence for finite subgroups of $\mathrm{SL}(2, \mathbb{C})$ because in this case all irreducible representations are special.

3. G-Hilbert schemes and combinatorics

In this section, we will discuss G-Hilbert schemes and a new way to find the special representations for cyclic quotient singularities by combinatorics.

The Hilbert scheme of n points on $\mathbb{C}^2$ can be described as a set of ideals:

$$\mathrm{Hilb}^n(\mathbb{C}^2) = \{I \subset \mathbb{C}[x, y] \mid I \text{ ideal}, \dim \mathbb{C}[x, y]/I = n\}.$$

It is a $2n$-dimensional smooth quasi–projective variety. The G-Hilbert scheme $\mathrm{Hilb}^G(\mathbb{C}^2)$ was introduced in the paper by Nakamura and the author ([**11**]) as follows:

$$\mathrm{Hilb}^G(\mathbb{C}^2) = \{I \subset \mathbb{C}[x, y] \mid I\ G\text{-invariant ideal}, \mathbb{C}[x, y]/I \cong \mathbb{C}[G]\},$$

where $|G| = n$. This is a union of components of fixed points of G-action on $\mathrm{Hilb}^n(\mathbb{C}^2)$ and in fact it is just the minimal resolution of the quotient singularity $\mathbb{C}^2/G$. It was proved for $G \subset \mathrm{SL}(2, \mathbb{C})$ in [**11**] first by the properties of $\mathrm{Hilb}^n(\mathbb{C}^2)$ and finite group action of G and a McKay correspondence in terms of ideals of G-Hilbert schemes was stated.

Later Kidoh ([**14**]) proved that the G-Hilbert scheme for any small cyclic subgroup of $\mathrm{GL}(2,\mathbb{C})$ is also the minimal resolution of the corresponding cyclic quotient singularities and Riemenschneider conjectured that the irreducible representations which are given from the ideals of G-Hilbert scheme, so-called Ito-Nakamura type McKay correspondence, are just same as the special representations which were defined by himself ([**19**]). Recently A. Ishii ([**9**]) proved more generally that the G-Hilbert scheme for any small $G \subset \mathrm{GL}(2,\mathbb{C})$ is always isomorphic to the minimal resolution of the singularity $\mathbb{C}^2/G$ and the conjecture is true:

Theorem 3.1 **([9])**. — *Let G be a finite small subgroup of* $\mathrm{GL}(2,\mathbb{C})$.

(i) G-Hilbert scheme $\mathrm{Hilb}^G(\mathbb{C}^2)$ *is the minimal resolution of* $\mathbb{C}^2/G$.

(ii) For $y \in \mathrm{Hilb}^G(\mathbb{C}^2)$, denote by I_y the ideal corresponding to y and let m be the maximal ideal of $\mathcal{O}_{\mathbb{C}^2}$ corresponding to the origin 0. If y is in the exceptional locus, then, as representations of G, we have

$$I_y/mI_y \cong \begin{cases} \rho_i \oplus \rho_0 & \text{if } y \in E_i \text{ and } y \notin E_j \text{ for } j \neq i, \\ \rho_i \oplus \rho_j \oplus \rho_0 & \text{if } y \in E_i \cap E_j, \end{cases}$$

where ρ_i is the special representation associated with the ireducible exceptional curve E_i.

Remark 3.2. — In dimension two, we can say that the G-Hilbert scheme is the same as a 2-dimensional irreducible component of the G-fixed set of $\mathrm{Hilb}^n(\mathbb{C}^2)$. A similar statement holds in dimension three, for $G \subset \mathrm{SL}(3,\mathbb{C})$, that is, the G-Hilbert scheme is a 3-dimensional irreducible component of the G-fixed set of $\mathrm{Hilb}^n(\mathbb{C}^3)$, and a crepant resolution of the quotient singularity $\mathbb{C}^3/G$ (cf. [**15**], [**3**]). In this case, note that $\mathrm{Hilb}^n(\mathbb{C}^3)$ is not smooth.

Moreover, Haiman proved that the $\mathcal{S}_n$-Hilbert scheme $\mathrm{Hilb}^{\mathcal{S}_n}(\mathbb{C}^{2n})$ is a crepant resolution of $\mathbb{C}^{2n}/\mathcal{S}_n = n$-th symmetric product of $\mathbb{C}^2$, i.e.,

$$\mathrm{Hilb}^{\mathcal{S}_n}(\mathbb{C}^{2n}) \cong \mathrm{Hilb}^n(\mathbb{C}^2)$$

in the process of the proof of $n!$ conjecture. (cf. [**13**])

From now on, we restrict our considerations to $G \subset \mathrm{GL}(2,\mathbb{C})$ cyclic. Wunram constructed the generalized McKay correspondence for cyclic surface singularities in the paper [**20**] and we have to consider the corresponding geometrical informations (the minimal resolution, reflexive sheaves and so on) to obtain the special representations. Here we would like to give a new characterization of the special representations in terms of combinatorics. It is much easier to find the special representation because we don't need any geometrical objects, but based on the result of G-Hilbert schemes.

Let us discuss the new characterization of the special representations in terms of combinatorics. Let G be the cyclic group $C_{r,a}$, generated by the matrix $\begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^a \end{pmatrix}$ where $\varepsilon^r = 1$ and $\gcd(r,a) = 1$ and consider the character map $\mathbb{C}[x,y] \longrightarrow \mathbb{C}[t]/t^r$ given by

$x \mapsto t$ and $y \mapsto t^a$. Then we have a corresponding character for each monomial in $\mathbb{C}[x, y]$.

Let I_p be the ideal of the a point p in the G-Hilbert scheme, then we can define the following sets.

Consider a G-invariant subscheme $Z_p \subset \mathbb{C}^2$ for which $H^0(Z_p, \mathcal{O}_{Z_p}) = \mathcal{O}_{\mathbb{C}^2}/I_p$ is the regular representation of G. Then the G-Hilbert scheme can be regarded as a moduli space of such Z_p.

Definition 3.3. — A set $Y(Z_p)$ of monomials in $\mathbb{C}[x, y]$ is called *G-cluster* if all monomials in $Y(Z_p)$ are not in I_p, and $Y(Z_p)$ can be drawn as a Young diagram of $|G|$ boxes.

Definition 3.4. — For any small cyclic group G, let $B(G)$ be the set of monomials which are not divisible by any G-invariant monomial. We call $B(G)$ G-basis.

Definition 3.5. — If $|G| = r$, then let $L(G)$ be $\{1, x, \dots, x^{r-1}, y, \dots, y^{r-1}\}$, i.e., the set of monomials which cannot be divided by x^r, y^r or xy. We call it L-space for G because the shape of this diagram looks like the capital letter “L”.

Definition 3.6. — The monomial $x^m y^n$ is of *weight* k if $m + an = k$.

Let us describe the method to find the special representations of G with these diagrams:

Theorem 3.7. — *For a small finite cyclic subgroup of* $\mathrm{GL}(2, \mathbb{C})$*, the irreducible representation ρ_i is special if and only if the corresponding monomials in $B(G)$ are not contained in the set of monomials $B(G) \smallsetminus L(G)$.*

Proof. — In Theorem 2.4 (3), we have the definition of the special representation, and it is not easy to compute all special representations. However look at the behavior of the monomials in $\mathbb{C}[x, y]$ under the map $\Phi_i : (\Omega^2_{\mathbb{C}^2})^G \otimes (\mathcal{O}_{\mathbb{C}^2} \otimes V_i)^G \to (\Omega^2_{\mathbb{C}^2} \otimes V_i)^G$ for each representation ρ_i:

First, let us consider the monomial bases of each set. Let $V_i = \mathbb{C}e_i$ and $\rho(g)e_i = \varepsilon^{-i}$. An element $f(x, y)dx \wedge dy \otimes \rho_i$ is in $(\Omega^2_{\mathbb{C}^2} \otimes V_i)^G$ if and only if

$$g^* f(x, y)dx \wedge dy \cdot \varepsilon^{1+a} \otimes \varepsilon^{-i} = f(x, y)dx \wedge dy,$$

that is,

$$g^*(f(x, y)dx \wedge dy) = \varepsilon^{i-(a+1)}(f(x, y)dx \wedge dy).$$

Therefore the monomial basis for $(\Omega^2_{\mathbb{C}^2} \otimes V_i)^G$ is the set of monomials $f(x, y)$ such that

$$g : f(x, y) \longmapsto \varepsilon^{i-(a+1)} f(x, y)$$

under the action of G, that is, monomials of weight $i - (a + 1)$.

Similarly, we have the monomial basis for $(\Omega^2_{\mathbb{C}^2})^G$ as the set of monomials $f(x.y)$ of weight $r - (a + 1)$.

The monomial basis for $(\mathcal{O}_{\mathbb{C}^2}\otimes V_i)^G$ is given as the set of monomials $f(x,y)$ of weight i.

Let us check the surjectivity of the map Φ_i. If Φ_i is surjective, then the monomial basis in $(\Omega^2_{\mathbb{C}^2}\otimes V_i)^G$ can be obtained as the product of the monomial bases of two other sets. Therefore the degree of the monomials in $(\Omega^2_{\mathbb{C}^2}\otimes V_i)^G$ must be higher than the degree of the monomials in $(\mathcal{O}_{\mathbb{C}^2}\otimes V_i)^G$.

Now look at the map Φ_{a+1}. The vector space $(\mathcal{O}_{\mathbb{C}^2}\otimes V_{a+1})^G$ is generated by the monomials of weight $a+1$, i.e., $x^{a+1}, xy, \dots, y^b$ where $ab = a+1 \mod r$. On the other hand, $(\Omega^2_{\mathbb{C}^2}\otimes V_{a+1})^G$ is generated by the degree 0 monomial 1. Then the map Φ_{a+1} is not surjective.

By this, if a monomial of type $x^m y^n$, where $mn \neq 0$, is a generator of $(\mathcal{O}_{\mathbb{C}^2}\otimes V_i)^G$, then there exists a monomial $x^{m-1}y^{n-1}$ in $(\Omega^2_{\mathbb{C}^2}\otimes V_i)^G$ and the degree become smaller under the map Φ_i. This means Φ_i is not surjective.

Moreover, if the bases of $(\mathcal{O}_{\mathbb{C}^2}\otimes V_i)^G$ is generated only by x^i and y^j where $aj \equiv i \mod r$, then the degrees of the monomials in $(\Omega^2_{\mathbb{C}^2}\otimes V_i)^G$ are larger and Φ_i is surjective. Thus we have the assertion. □

Remark 3.8. — From this theorem, we can also say that a representation ρ_i is special if and only if the number of the generators of the space $(\mathcal{O}_{\mathbb{C}^2}\otimes V_i)^G$ is 2. However, as a module over the invariant ring $\mathcal{O}^G_{\mathbb{C}^2}$ it is minimally generated by 2 elements. In this form, the remark is not new. It follows easily in one direction from the remark after Theorem 2.1 in Wunram's paper [**21**], and in the other direction from Theorem 2.1 in combination with the fact proven in the first appendix of that paper that in the case of cyclic quotient surface singularities a reflexive module is determined by the "Chern numbers" of its torsionfree preimage on the minimal resolution.

Theorem 3.9. — *Let p be a fixed point by the G-action, then we can define an ideal I_p by the G-cluster and the configuration of the exceptional divisors can be described by these data.*

Proof. — The defining equation of the ideal I_p is given by

$$\begin{cases} x^a = \alpha y^c, \\ y^b = \beta x^d, \\ x^{a-d}y^{b-c} = \alpha\beta, \end{cases}$$

where α and β are complex numbers and both x^a and y^c (resp. y^b and x^d) correspond to the same representation (or character).

The pair (α,β) is a local affine coordinate near the fixed point p and it is also obtained from the calculation with toric geometry. Moreover each axis of the affine chart is just a exceptional curve or the original axis of $\mathbb{C}^2$. The exceptional curve is isomorphic to a $\mathbb{P}^1$ and the points on it are written by the ratio like $(x^a : y^c)$ (resp.

$(x^d : y^b)$) which is corresponding to a special representation ρ_a (resp. ρ_d). The fixed point p is the intersection point of 2 exceptional curves E_a and E_d.

Thus we can get the whole space of exceptional locus by deforming the point p and patching the affine pieces. □

We will see a concrete example in the following section. Here we would like to make one remark as a corollary:

Corollary 3.10. — *For A_n-type simple singularities, all $n+1$ affine charts can be described by $n+1$ Young diagrams of type $(1,\ldots,1,k)$.*

Proof. — In A_n case, xy is always G-invariant, hence $B(G) = L(G)$. Therefore we have $n+1$ G-clusters and each of them corresponds to the monomial ideal (x^k, y^{n-k+2}, xy). □

4. Example and related topics

First, we recall the toric resolution of cyclic quotient singularities because the quotient space $\mathbb{C}^2/G$ is a toric variety.

Let $\mathbb{R}^2$ be the 2-dimensional real vector space, $\{e^i \mid i = 1,2\}$ its standard base, L the lattice generated by e^1 and e^2, $N := L + \sum \mathbb{Z}v$, where the summation runs over all the elements $v = 1/r(1,a) \in G = C_{r,a}$, and

$$\sigma := \left\{ \sum_{i=1}^{2} x_i e^i \in \mathbb{R}^2, \quad x_i \geqslant 0, \forall i, 1 \leqslant i \leqslant 2 \right\}$$

the naturally defined rational convex polyhedral cone in $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$. The corresponding affine torus embedding X_σ is defined as $\operatorname{Spec}(\mathbb{C}[\check{\sigma} \cap M])$, where M is the dual lattice of N and $\check{\sigma}$ the dual cone of σ in $M_{\mathbb{R}}$ defined as $\check{\sigma} := \{\xi \in M_{\mathbb{R}} | \xi(x) \geqslant 0, \forall x \in \sigma\}$.

Then $X = \mathbb{C}^2/G$ corresponds to the toric variety X_σ which is induced by the cone σ within the lattice N.

Fact 1. — We can construct a simplicial decomposition S with the vertices on the Newton Boundary, that is, the convex hull of the lattice points in σ except the origin.

Fact 2. — If $\widetilde{X} := X_S$ is the corresponding torus embedding, then X_S is non-singular. Thus, we obtain the minimal resolution $\pi = \pi_S : \widetilde{X} = X_S \longrightarrow \mathbb{C}^2/G = Y$. Moreover, each lattice point of the Newton boundary corresponds to an exceptional divisor.

Example. — Let us look at the example of the cyclic quotient singularity of type $C_{7,3}$ which is generated by the matrix $\begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^3 \end{pmatrix}$ where $\varepsilon^7 = 1$. The toric resolution of this quotient singularity is given by the triangulation of the lattice $N\colon = \mathbb{Z}^2 + \frac{1}{7}(1,3)\mathbb{Z}$ with the lattice points: See Figure 4.1.

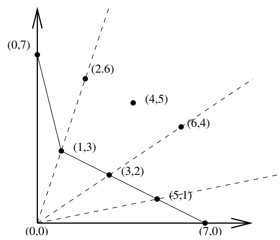

FIGURE 4.1. Toric resolution of $\mathbb{C}^2/G$

From this Newton polytope, we can see that there are 3 exceptional divisors and the dual graph gives the configuration of the exceptional components with a deformed coordinate from the original coordinate (x, y) on $\mathbb{C}^2$ as in Figure 4.2.

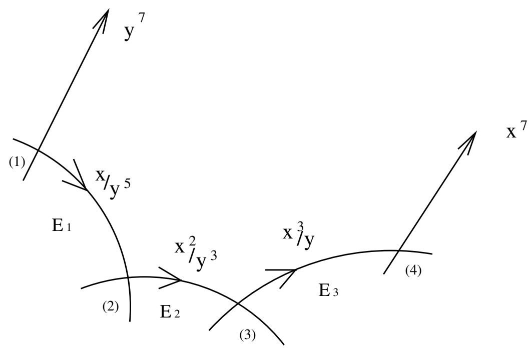

FIGURE 4.2. Configuration of $\tilde{X}$

Therefore we have 4 affine pieces in this example and we have 4 coordinate systems corresponding to each affine piece. In this picture, we will see the corresponding special irreducible representations, but we would like to use our method in the previous section to find the representations.

Let us draw the diagram which corresponds to the G-basis and L-space. First we have the following G-basis $B(G)$ and the corresponding characters in a same diagram. In Figure 4.3 we draw the L-space as shaded part in $B(G)$.

Now we have three monomials xy, x^2y and x^3y in $B(G) \smallsetminus L(G)$ and they correspond to the characters (resp. representations) 4, 5 and 6 (resp. ρ_4, ρ_5 and ρ_6). Therefore we can find a set of special representations, that is, $\{\rho_1, \rho_2, \rho_3\}$, and find the corresponding G-clusters, representing the origin of the affine charts of the resolution, can be drawn as 4 Young diagrams and get the corresponding special representations in this case. See Figure 4.4.

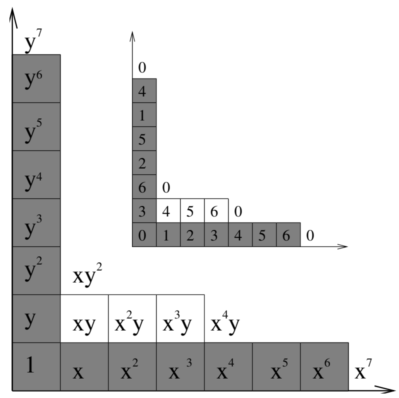

FIGURE 4.3. G-basis $B(G)$ and the characters

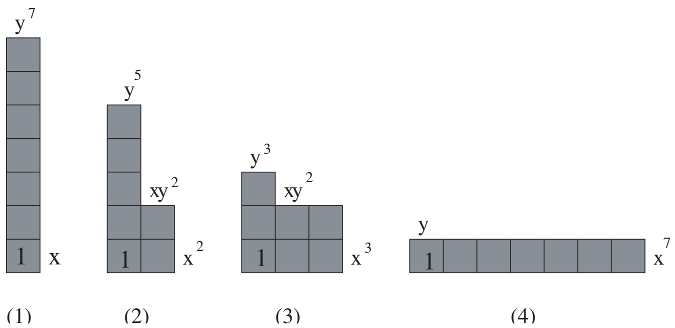

FIGURE 4.4. G-cluster $Y(Z_p)$

Let us see the meaning of the corresponding G-clusters in this case. From $Y(Z_p)$ for (2), we obtain an ideal $I_2 = (y^5, x^2, xy^2)$ for the origin of the affine chart (2) in Figure 4.2, and the corresponding representations are ρ_1, ρ_2 and ρ_0. If we take the maximal ideal m of $\mathcal{O}_{\mathbb{C}^2}$ corresponding to the origin 0, then we have

$$I_2/mI_2 \cong \rho_1 \oplus \rho_2 \oplus \rho_0.$$

Similarly we have the ideal $I_3 = (y^3, x^3, xy^2)$ and

$$I_3/mI_3 \cong \rho_2 \oplus \rho_3 \oplus \rho_0.$$

These descriptions coincide with the results of Theorem 3.1 for a point at the intersection $E_1 \cap E_2$.

For any other points p on the exceptional component E_i, we must have

$$(*) \qquad I_p/mI_p \cong \rho_i \oplus \rho_0.$$

In fact, we can see that a point on the exceptional divisor E_2 in this example was determined by the ratio $x^2 : y^3$, that is, the corresponding ideal of a point on E_2 can be described as $I_p = (\alpha x^2 - \beta y^3, xy^2 - \gamma)$. Therefore the ratio $(\alpha : \beta)$ gives the coordinate of the exceptional curve ($\cong \mathbb{P}^1$) and we also have $(*)$.

We discussed special McKay correspondence in 2-dimensional case in this paper. In dimension three, it is convenient to consider crepant resolutions as minimal resolutions and we have a much more complicated situation. Even in the case $G \subset \mathrm{SL}(3, \mathbb{C})$, we have $H^4(\tilde{X}, \mathbb{Q}) \neq 0$ in general. Of course we can use the same definition for the special representations in the higher dimensional case, but all non-trivial irreducible representations of $G \subset \mathrm{SL}(3, \mathbb{C})$ are special. On the other hand, the number of the exceptional divisors is less than that of the non-trivial irreducible representations. Therefore, it looks very difficult to generalize this special McKay correspondence. That is, we should make a difference, say a kind of the grading, in the set of the special (or non-trivial) representations like "age" of the conjugacy classes.

However, there are good news: In 2000, Craw [**4**] constructed a cohomological McKay correspondence for the G-Hilbert schemes where G is an abelian group, and in this correspondence we can see the 2-dimensional special McKay correspondence. And recently, the author found a way to obtain a polytope which corresponds to the 3-dimensional G-Hilbert schemes for abelian subgroups in $\mathrm{SL}(3, \mathbb{C})$ by combinatorics. There are many crepant resolutions in general in higher dimension, but the G-Hilbert scheme for $G \subset \mathrm{SL}(3, \mathbb{C})$ is a unique crepant resolution, and the configuration of the exceptional locus of the special crepant resolution, G-Hilbert scheme, can be determined in terms of a Gröbner basis. (Let us call this the Gröbner method.) Moreover, we can get another characterization of special representations for cyclic quotient surface singularities by this Gröbner method. So the author is dreaming of having a more simple and beautiful formulation of the McKay correspondence in the future.

References

[1] M. Artin and J.-L. Verdier, *Reflexive modules over rational double points*, Math. Ann. **270** (1985), 79–82.

[2] V. Batyrev, *Non-Archimedean integrals and stringy Euler numbers of log-terminal pairs*, J. Eur. Math. Soc. **1** (1999), 5–33.

[3] T. Bridgeland, A. King and M. Reid, *The McKay correspondence as an equivalence of derived categories*, J. AMS, **14**, (2001), 535–554.

[4] A. Craw, *An explicit construction of the McKay correspondence for A-Hilb* $\mathbb{C}^3$, preprint, math.AG/0010053.

[5] H. Esnault, *Reflexive modules on quotient surface singularities*, J. Reine Angew. Math. **362** (1985), 63–71.

[6] H. Esnault and H. Knörrer, *Reflexive modules over rational double points*, Math. Ann. **272** (1985), 545–548.

[7] G. Gonzalez-Sprinberg, *Calcul de l'invariant local d'Euler pour les singularités quotient de surfaces*, C. R. Acad. Sci Paris, t. **288**, (1979), 989-992.

[8] G. Gonzalez-Sprinberg and J.-L. Verdier, *Construction géometrique de la correspondance de McKay*, Ann. Sci. École Norm. Sup. **16** (1983), 409–449.

[9] A. Ishii, *On McKay correspondence for a finite small subgroup of* $\mathrm{GL}(2,\mathbb{C})$, to appear in J. Reine Angew. Math.

[10] Y. Ito and H. Nakajima, *McKay correspondence and Hilbert schemes in dimension three*, Topology **39** (2000), 1155–1191.

[11] Y. Ito and I. Nakamura, *Hilbert schemes and simple singularities*, in: New trends in Algebraic Geometry (Warwick, June 1996), K. Hulek and others Eds., CUP (1999), 151–233.

[12] Y. Ito and M. Reid, *The McKay correspondence for finite subgroups of* $\mathrm{SL}(3,\mathbb{C})$, in: Higher dimensional complex varieties (Trento, Jun 1994), M. Andreatta and others Eds., de Gruyter, 1996, 221–240.

[13] M. Haiman, *Hilbert schemes, polygraphs, and the Macdonald positivity conjecture* J. AMS, **14**, (2001), 941–1006.

[14] R. Kidoh, *Hilbert schemes and cyclic quotient singularities*, Hokkaido Mathematical Journal, **30** (2001), 91–103.

[15] I. Nakamura, *Hilbert schemes of abelian group orbits*, J. Algebraic Geom., **10**, (2001), 757–779.

[16] J. McKay, *Graphs, singularities and finite groups*, Proc. Symp. Pure Math., **37** (1980), Amer. Math. Soc. 183–186.

[17] M. Reid, *La correspondance de McKay*, Séminaire Bourbaki, 52ème année, novembre 1999, no. 867, Astérisque (2000), 1-21.

[18] O. Riemenschneider, *Characterization and application of special reflexive modules on rational surface singularities*, Institut Mittag-Leffler Report No.3 (1987).

[19] O. Riemenschneider, *On the two dimensional McKay correspondence*, Hamburger Beiträge zur Mathematik aus dem Mathematischen Seminar, Heft 94. Hamburg (2000).

[20] J. Wunram, *Reflexive modules on cyclic quotient surface singularities*, in Singularities, Representations of Algebras, and Vector Bundles. Greuel, Trautmann (eds.) Lect. Notes math. 1273. Springer (1987).

[21] J. Wunram, *Reflexive modules on quotient surface singularities*, Math Ann. **279** (1988), 583–598.

Y. Ito, Department of Mathematics, Tokyo Metropolitan University, Hachioji, Tokyo 192-0397, Japan • *E-mail :* `yukari@comp.metro-u.ac.jp` • *Url :* `http://www.comp.metro-u.ac.jp/~yukari`