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LECTURES ON HEIGHT ZETA FUNCTIONS OF TORIC VARIETIES

by

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Abstract. — We explain the main ideas and techniques involved in recent proofs of asymptotics of rational points of bounded height on toric varieties.

1. Introduction

Toric varieties are an ideal testing ground for conjectures: their theory is sufficiently rich to reflect general phenomena and sufficiently rigid to allow explicit combinatorial computations. In these notes I explain a conjecture in arithmetic geometry and describe its proof for toric varieties.

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1.1. Counting problems

Example 1.1.1. — Let $X \subset \mathbb{P}^n$ be a smooth hypersurface given as the zero set of a homogeneous form f of degree d (with coefficients in $\mathbb{Z}$). Let

$$N(X,B) = \#\{\boldsymbol{x} \mid f(\boldsymbol{x}) = 0,\ \max(|x_j|) \leqslant B\}$$

(where $\boldsymbol{x} = (x_0, \ldots, x_n) \in \mathbb{Z}^{n+1}/(\pm 1)$ with $\gcd(x_j) = 1$) be the number of $\mathbb{Q}$-rational points on X of "height" $\leqslant B$. Heuristically, the probability that f represents 0 is about B^{-d} and the number of "events" about B^{n+1}. Thus we expect that

$$\lim_{B\to\infty} N(X,B) \sim B^{n+1-d}.$$

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This can be proved by the circle method, at least when $n \gg 2^d$. The above heuristic leads to a natural trichotomy, corresponding to the possibilities when $n+1-d$ positive, zero or negative. In the first case we expect many rational points on X, in the third case very few and in the intermediate case we don't form an opinion.

Example 1.1.2. — Let $X \subset \mathbb{P}^n \times \mathbb{P}^n$ be a hypersurface given as the zero set of a bihomogeneous diagonal form of bidegree (d_1, d_2):

$$f(\boldsymbol{x}, \boldsymbol{y}) = \sum_{k=0}^{n} a_k x_k^{d_1} \cdot y_k^{d_2},$$

with $a_k \in \mathbb{Z}$. Each pair of positive integers $L = (l_1, l_2)$ defines a counting function on rational points $X(\mathbb{Q})$ by

$$N(X, L, B) = \#\{(\boldsymbol{x}, \boldsymbol{y}) \mid f(\boldsymbol{x}, \boldsymbol{y}) = 0,\, \max(|x_i|)^{l_1} \cdot \max(|y_j|)^{l_2} \leqslant B\}$$

(where $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{Z}^{(n+1)}/(\pm 1)$ with $\gcd(x_i) = \gcd(y_j) = 1$). Heuristics as above predict that the asymptotic should depend on the vector

$$-K = (n + 1 - d_1, n + 1 - d_2)$$

and on the location of L with respect to $-K$.

An interesting open problem is, for example, the case when $(d_1, d_2) = (1, 2), n = 3$ and $L = (3, 2)$. Notice that this variety is a compactification of the affine space. For appropriate a_k one expects $\sim B \log(B)$ rational points of height bounded by B.

Trying to systematize such examples one is naturally lead to the following problems:

Problem 1.1.3. — Let $X \subset \mathbb{P}^n$ be an algebraic variety over a number field. Relate the asymptotics of rational points of bounded height to geometric invariants of X.

Problem 1.1.4. — Develop analytic techniques proving such asymptotics.

1.2. Zariski density. — Obviously, not every variety is a hypersurface in a projective space or product of projective spaces. To get some systematic understanding of the distribution of rational points we need to use ideas from classification theories of algebraic varieties. On a very basic level (smooth projective) algebraic varieties are distinguished according to the ampleness of the canonical class: Fano varieties (big anticanonical class), varieties of general type (big canonical class) and varieties of intermediate type (neither). The conjectures of Bombieri-Lang-Vojta predict that on varieties of general type the set of rational points is not Zariski dense (see [**46**]). Faltings proved this for subvarieties of abelian varieties ([**16**]). It is natural to ask for a converse. As the examples of Colliot-Thélène, Swinnerton-Dyer and Skorobogatov suggest (see [**11**]), the most optimistic possibility would be: if X does not have finite étale covers which dominate a variety of general type then there exists a finite extension E/F such that $X(E)$ is Zariski dense in X. In particular, this should hold for Fano varieties. I have no idea how to prove this for a general smooth quintic hypersurface in $\mathbb{P}^5$. Quartic hypersurfaces in $\mathbb{P}^4$ are treated in [**22**] (see also [**23**]).

Clearly, we need Zariski density of rational points on X before attempting to establish a connection between the global geometry of X and $X(F)$. Therefore, we will focus on varieties birational to the projective space or possessing a large group of automorphisms so that rational points are a priori dense, at least after a finite extension. In addition to allowing finite field extensions we will need to restrict to some appropriate Zariski open subsets.

Example 1.2.1. — Let X be the cubic surface $x_0^3 + x_1^3 + x_2^3 + x_3^3 = 0$ over $\mathbb{Q}$. We expect $\sim B(\log(B))^3$ rational points of height $\max(|x_j|) \leqslant B$. However, on the lines like $x_0 = -x_1$ and $x_2 = -x_3$ we already have $\sim B^2$ rational points. Numerical experiments in [**39**] confirm the expected growth rate on the complement to the lines; and Heath-Brown proved the upper bound $O(B^{4/3+\epsilon})$ [**24**]. Thus the asymptotic of points on the whole X will be dominated by the contribution from lines, and it is futile to try to read off geometric invariants of X from what is happening on the lines.

Such Zariski closed subvarieties will be called *accumulating*. Notice that this notion may depend on the projective embedding.

1.3. Results. — Let X be a smooth projective algebraic variety over a number field F and L a very ample line bundle on X. It defines an embedding $X \hookrightarrow \mathbb{P}^n$. Fix a "height" on the ambient projective space. For example, we may take

$$H(\boldsymbol{x}) := \prod_v \max_j(|x_j|_v),$$

where $\boldsymbol{x} = (x_0, \dots, x_n) \in \mathbb{P}^n(F)$ and the product is over all (normalized) valuations of F. To highlight the choice of the height we will write $\mathcal{L}$ for the pair (L-embedding, height). We get an induced (exponential) *height function*

$$H_{\mathcal{L}} : X(F) \longrightarrow \mathbb{R}_{>0}$$

on the set of F-rational points $X(F)$ (see 4.1 for more details). The set of F-rational points of height bounded by $B > 0$ is finite and we can define the *counting function*

$$N(U, \mathcal{L}, B) := \#\{x \in U(F) \mid H_{\mathcal{L}}(x) \leqslant B\},$$

where $U \subset X$ is a Zariski open subset.

Theorem 1.3.1. — *Let X/F be one of the following varieties:*

- *toric variety* [**5**]*;*
- *equivariant compactification of* $\mathbf{G}_a^n$ [**9**]*;*
- *flag variety* [**18**]*;*
- *equivariant compactification of* $\mathbf{G}/\mathbf{U}$ *- horospherical variety (where* $\mathbf{G}$ *is a semi-simple group and* $\mathbf{U} \subset \mathbf{G}$ *a maximal unipotent subgroup)* [**41**]*;*
- *smooth complete intersection of small degree (for example,* [**6**]*).*

Let $\mathcal{L}$ be an appropriate height on X such that the class $L \in \mathrm{Pic}(X)$ is contained in the interior of the cone of effective divisors.

Then there exists a dense Zariski open subset $U \subset X$ and constants

$$a(L), b(L), \Theta(U, \mathcal{L}) > 0$$

such that

$$N(U, \mathcal{L}, B) = \frac{\Theta(U, \mathcal{L})}{a(L)(b(L)-1)!} B^{a(L)} (\log(B))^{b(L)-1}(1 + o(1)),$$

as $B \to \infty$.

Remark 1.3.2. — The constants $a(L)$ and $b(L)$ depend only on the class of L in $\mathrm{Pic}(X)$. The constant $\Theta(U, \mathcal{L})$ depends, of course, not only on the geometric data (U, L) but also on the choice of the height. It is interpreted, in a general context, in [**5**].

Remark 1.3.3. — Notice that with the exception of complete intersections the varieties from Theorem 1.3.1 have a rather simple “cellular” structure. In particular, we can parametrize all rational points in some dense Zariski open subset. The theorem is to be understood as a statement about *heights*: even the torus $\mathbf{G}_m^2$ has very nontrivial embeddings into projective spaces and in each of these embeddings we have a different counting problem.

1.4. Techniques. — Let $\mathbf{G}$ be an algebraic torus or the group $\mathbf{G}_a^n$. The study of height asymptotics in these cases uses harmonic analysis on the adelic points $\mathbf{G}(\mathbb{A})$:

(1) Define a height pairing

$$H = \prod_v H_v \; : \; \mathrm{Pic}^{\mathbf{G}}(X)_{\mathbb{C}} \times \mathbf{G}(\mathbb{A}) \longrightarrow \mathbb{C},$$

(where $\mathrm{Pic}^{\mathbf{G}}(X)$ is the group of isomorphism classes of $\mathbf{G}$-linearized line bundles on X) such that its restriction to $L \in \mathrm{Pic}(X) \times \mathbf{G}(F)$ is the usual height $\mathcal{L}$ as before and such that H is invariant under a standard compact subgroup $\mathbf{K} \subset \mathbf{G}(\mathbb{A})$.

(2) Define the height zeta function

$$Z(\mathbf{G}, \boldsymbol{s}) = \sum_{x \in \mathbf{G}(F)} H(\boldsymbol{s}; x)^{-1}.$$

The projectivity of X implies that $Z(\mathbf{G}, \boldsymbol{s})$ converges for $\Re(\boldsymbol{s})$ in some (shifted) open cone in $\mathrm{Pic}^{\mathbf{G}}(X)_{\mathbb{R}}$.

(3) Apply the Poisson formula to obtain a representation

$$Z(\mathbf{G}, \boldsymbol{s}) = \int_{(\mathbf{G}(\mathbb{A})/\mathbf{G}(F)\mathbf{K})^*} \widehat{H}(\boldsymbol{s}; \chi) d\chi,$$

where the integral is over the group of unitary characters χ of $\mathbf{G}(\mathbb{A})$ which are trivial on $\mathbf{G}(F)\mathbf{K}$ and $d\chi$ is an appropriate Haar measure.

(4) Compute the Fourier transforms $\widehat{H}_v$ at almost all nonarchimedean places and find estimates at the remaining places.

(5) Prove a meromorphic continuation of $Z(\mathbf{G}, \boldsymbol{s})$ and identify the poles.
(6) Apply a Tauberian theorem.

2. Algebraic tori

For simplicity, we will always assume that $\mathbf{T}$ is a *split* algebraic torus over a number field F, that is, a connected reductive group isomorphic to $\mathbf{G}_{m,F}^d$, where $\mathbf{G}_{m,F} := \operatorname{Spec}(F[x, x^{-1}])$.

2.1. Adelization

***Notations 2.1.1* (Fields)**. — Let F be a number field and $\operatorname{disc}(F)$ the discriminant of F (over $\mathbb{Q}$). The set of places of F will be denoted by $\operatorname{Val}(F)$. We shall write $v|\infty$ if v is archimedean and $v \nmid \infty$ if v is nonarchimedean. For any place v of F we denote by F_v the completion of F at v and by $\mathcal{O}_v$ the ring of v-adic integers (for $v \nmid \infty$). Let q_v be the cardinality of the residue field $\mathbb{F}_v$ of F_v for nonarchimedean valuations and put $q_v = e$ for archimedean valuations. The local absolute value $|\cdot|_v$ on F_v is the multiplier of the Haar measure, i.e., $d(ax_v) = |a|_v dx_v$ for some Haar measure dx_v on F_v. We denote by $\mathbb{A} = \mathbb{A}_F = \prod_v' F_v$ the adele ring of F.

***Notations 2.1.2* (Groups)**. — Let $\mathbf{G}$ be a connected reductive algebraic group defined over a number field F. Denote by $\mathbf{G}(\mathbb{A})$ the adelic points of $\mathbf{G}$ and by

$$\mathbf{G}^1(\mathbb{A}) := \Big\{ g \in \mathbf{G}(\mathbb{A}) \mid \prod_{v \in \operatorname{Val}(F)} |m(g_v)|_v = 1 \ \ \forall m \in \widehat{\mathbf{G}}_F \Big\}$$

the kernel of F-rational characters $\widehat{\mathbf{G}}_F$ of $\mathbf{G}$.

***Notations 2.1.3* (Tori)**. — Denote by $M = \widehat{\mathbf{T}}_F = \mathbb{Z}^d$ the group of F-rational characters of $\mathbf{T}$ and by $N = \operatorname{Hom}(M, \mathbb{Z})$ the dual group (as customary in toric geometry). Put $M_v := M$ (resp. $N_v := N$) for nonarchimedean valuations and $M_v := M \otimes \mathbb{R}$ for archimedean valuations.

Write $K_v \subset \mathbf{T}(F_v)$ for the maximal compact subgroup of $\mathbf{T}(F_v)$ (after fixing an integral model for $\mathbf{T}$ we have $K_v = \mathbf{T}(\mathcal{O}_v)$ for almost all v).

Choose a Haar measure $d\mu = \prod_v d\mu_v$ on $\mathbf{T}(\mathbb{A})$ normalized by $\operatorname{vol}(K_v) = 1$ (for nonarchimedean v the induced measure on $\mathbf{T}(F_v)/K_v$ is the discrete measure).

The adelic picture of a split torus $\mathbf{T}$ is as follows:
- $\mathbf{T}(\mathbb{A})/\mathbf{T}^1(\mathbb{A}) \simeq (\mathbf{G}_m(\mathbb{A})/\mathbf{G}_m^1(\mathbb{A}))^d \simeq \mathbb{R}^d$;
- $\mathbf{T}^1(\mathbb{A})/\mathbf{T}(F) = (\mathbf{G}_m^1(\mathbb{A})/\mathbf{G}_m(F))^d$ is compact;
- $\mathbf{K} = \prod_{v \in \operatorname{Val}(F)} K_v$;
- $\mathbf{T}^1(\mathbb{A})/\mathbf{T}(F)\mathbf{K}$ is a product of a finite group and a connected compact abelian group;
- $\mathbf{K} \cap \mathbf{T}(F)$ is a finite group of torsion elements.

- For all v the map

$$\begin{aligned}\log_v\ :\ \mathbf{T}(F_v)/K_v &\hookrightarrow N_v\\ t_v &\longmapsto \bar{t}_v \in N_v\end{aligned}$$

is an isomorphism.

For more details the reader could consult Tate's thesis ([**42**]).

2.2. Hecke characters. — Let

$$\mathcal{A}_{\mathbf{T}} := (\mathbf{T}(\mathbb{A})/\mathbf{T}(F)\mathbf{K})^*$$

be the group of (unitary) Hecke characters which are invariant under the closed subgroup $\mathbf{T}(F)\mathbf{K}$. The local components of a character $\chi \in \mathcal{A}_{\mathbf{T}}$ are given by

$$\chi_v(t_v) = \chi_v(\bar{t}_v) = q_v^{i\langle m_v, \bar{t}_v\rangle}$$

for some $m_v = m_v(\chi) \in M_v$ (a character χ_v trivial on K_v is called unramified). We have a homomorphism

$$\begin{aligned}\mathcal{A}_{\mathbf{T}} &\longrightarrow M_{\mathbb{R},\infty}\\ \chi &\longmapsto m_\infty(\chi) := (m_v(\chi))_{v|\infty},\end{aligned}$$

where $M_{\mathbb{R},\infty} := \oplus_{v|\infty} M_v$. We also have an embedding

$$\begin{aligned}M_{\mathbb{R}} &\hookrightarrow \mathcal{A}_{\mathbf{T}},\\ m &\longmapsto \Big(t \mapsto \prod_{v\in \mathrm{Val}(F)} e^{i\log(|m(t)|_v)}\Big).\end{aligned}$$

We can choose a splitting

$$\mathcal{A}_{\mathbf{T}} = M_{\mathbb{R}} \oplus \mathcal{U}_{\mathbf{T}}$$

where

$$\mathcal{U}_{\mathbf{T}} := (\mathbf{T}^1(\mathbb{A})/\mathbf{T}(F)\mathbf{K})^*.$$

We have a decomposition

$$M_{\mathbb{R},\infty} = M_{\mathbb{R}} \oplus M^1_{\mathbb{R},\infty},$$

where $M^1_{\mathbb{R},\infty}$ contains the image of $\mathcal{U}_{\mathbf{T}}$ (under the map $\mathcal{A}_{\mathbf{T}} \to M_{\mathbb{R},\infty}$) as a lattice of maximal rank. The kernel of $\mathcal{U}_{\mathbf{T}} \to M^1_{\mathbb{R},\infty}$ is a finite group.

2.3. Tamagawa numbers. — Let $\mathbf{G}$ be a connected linear algebraic group of dimension d over F and Ω a $\mathbf{G}$-invariant F-rational algebraic differential d-form. One can use this form to define a v-adic measure ω_v on $\mathbf{G}(F_v)$ for all $v \in \mathrm{Val}(F)$ (see [**35**], [**47**], Chapter 2, [**37**]). For almost all v we have

$$\tau_v(\mathbf{G}) := \int_{\mathbf{G}(\mathcal{O}_v)} \omega_v = \frac{\#\mathbf{G}(\mathbb{F}_v)}{q_v^d}$$

(to make sense of $\mathbf{G}(\mathcal{O}_v)$ one fixes a model of $\mathbf{G}$ over $\mathrm{Spec}(\mathcal{O}_{S'})$ for some finite set of valuations S'). One introduces a set of convergence factors to obtain a measure on the adelic space $\mathbf{G}(\mathbb{A})$ as follows: Choose a finite set S of valuations, including the archimedean valuations, such that for $v \notin S$,

$$\lambda_v := L_v(1, \widehat{\mathbf{G}}) \neq 0,$$

where L_v is the local factor of the Artin L-function associated to the Galois-module $\widehat{\mathbf{G}}$ of characters of $\mathbf{G}$ (see Section 6.2). For $v \in S$ put $\lambda_v = 1$. The measure on $\mathbf{G}(\mathbb{A})$ associated with the set $\{\lambda_v\}$ is

$$\omega := L_S^*(1, \widehat{\mathbf{G}})^{-1} \cdot |\mathrm{disc}(F)|^{-d/2} \prod_{v \in \mathrm{Val}(F)} \lambda_v \omega_v,$$

where $L_S^*(1, \widehat{\mathbf{G}})$ is the coefficient at the leading pole at $s = 1$ of the (partial) Artin L-function attached to $\widehat{\mathbf{G}}$ (see Section 6.2). On the space $\mathbf{G}(\mathbb{A})/\mathbf{G}^1(\mathbb{A}) = \mathbb{R}^r$ (where $r = \mathrm{rk}\, \widehat{\mathbf{G}}_F$) we have the standard Lebesgue measure dx normalized in such a way that the covolume of the lattice $\widehat{\mathbf{G}}_F \subset \widehat{\mathbf{G}}_F \otimes \mathbb{R}$ is equal to 1. There exists a unique measure ω^1 on $\mathbf{G}^1(\mathbb{A})$ such that $\omega = dx \cdot \omega^1$. Use this measure to define

$$\tau(\mathbf{G}) := \int_{\mathbf{G}^1(\mathbb{A})/\mathbf{G}(F)} \omega^1.$$

Remark 2.3.1. — The adelic integral defining $\tau(\mathbf{G})$ converges (see [**47**],[**33**]). The definition does not depend on the choices made (splitting field, finite set S, F-rational differential d-form).

3. Toric varieties

3.1. Geometry. — When we say X is a (split), smooth, proper, d-dimensional toric variety over F we mean the following collection of data:

- $\mathbf{T} = \mathbf{G}_{m,F}^d$, $M = \mathrm{Hom}(\mathbf{T}, \mathbf{G}_m) = \mathbb{Z}^d$ and the dual N;
- Σ - a complete regular d-dimensional fan: a collection of strictly convex polyhedral cones generated by vectors $e_1, \ldots, e_n \in N$ such that the set of generators of every cone σ can be extended to a basis of N.

We denote by $\Sigma(j)$ the set of j-dimensional cones and by d_σ the dimension of the cone σ ($\Sigma(1) = \{e_1, \ldots, e_n\}$). Denote by

$$\check{\sigma} = \{m \in M \mid \langle m, n \rangle \geqslant 0 \;\; \forall n \in \sigma\}$$

the dual cone to σ. Then

$$X = X_\Sigma = \cup_{\sigma \in \Sigma} \mathrm{Spec}(F[M \cap \check{\sigma}])$$

is the associated smooth complete toric variety over F. A toric structure on a variety X is unique, up to automorphisms of X (this follows from the fact that maximal tori in linear algebraic groups are conjugated; see [**23**], Section 2.1 for more details). The variety X has a stratification as a disjoint union of tori $\mathbf{T}_\sigma = \mathbf{G}_m^{d-d_\sigma}$; in particular, $\mathbf{T}_0 = \mathbf{T}$. Denote by $\mathrm{Pic}^{\mathbf{T}}(X)$ the group of isomorphism classes of $\mathbf{T}$-linearized line bundles. It is identified with the group PL of (continuous) $\mathbb{Z}$-valued functions on N

which are additive on each $\sigma \in \Sigma$. For $\varphi \in \mathrm{PL}$ we denote by L_φ the corresponding $\mathbf{T}$-linearized line bundle on X. Since we will work with $\mathrm{PL}_{\mathbb{C}}$ it will be convenient to introduce coordinates identifying the vector $\boldsymbol{s} = (s_1, \ldots, s_n)$ with the function $\varphi_{\boldsymbol{s}} \in \mathrm{PL}_{\mathbb{C}}$ determined by $\varphi_{\boldsymbol{s}}(e_j) = s_j$ for $j = 1, \ldots, n$.

Proposition 3.1.1

$$0 \longrightarrow M \longrightarrow \mathrm{PL} \xrightarrow{\psi} \mathrm{Pic}(X) \longrightarrow 0 \tag{3.1}$$

$$-K_X = \psi((1, \ldots, 1)).$$

Let $\varphi \in \mathrm{PL}$ be a piecewise linear function on N and L_φ the associated $\mathbf{T}$-linearized line bundle. The space of global sections $H^0(X, L_\varphi)$ is identified with the set of lattice points in a polytope $\square_\varphi \subset M$:

$$m \in \square_\varphi \iff \varphi(e_j) \geqslant \langle m, e_j \rangle \quad \forall j \in [1, \ldots, n]$$

(these characters m are the weights of the representation of $\mathbf{T}$ on $H^0(X, L_\varphi)$).

3.2. Digression: Characters. — Dualizing the sequence (3.1) we get a map of tori $\widetilde{\mathbf{T}} \to \mathbf{T}$ (where $\widetilde{\mathbf{T}}$ is dual to PL). Every character χ of $\mathbf{T}(\mathbb{A})$ gives rise to a character $\widetilde{\chi}$ of $\widetilde{\mathbf{T}}(\mathbb{A})$. We have

$$\widetilde{\mathbf{T}} = \mathbf{G}_m^n$$

and every character $\widetilde{\chi}$ determines characters χ_j $(j = 1, \ldots, n)$ of $\mathbf{G}_m(\mathbb{A})$. This gives an *injective* homomorphism

$$\begin{array}{ccc} (\mathbf{T}(\mathbb{A})/\mathbf{T}(F))^* & \longrightarrow & \prod_{j=1}^n (\mathbf{G}_m(\mathbb{A})/\mathbf{G}_m(F))^* \\ \chi & \longmapsto & (\chi_j)_{j \in [1, \ldots, n]}. \end{array}$$

4. Heights

4.1. Metrizations of line bundles

Definition 4.1.1. — *Let X be an algebraic variety over F and L a line bundle on X. A v-adic metric on L is a family $(\|\cdot\|_x)_{x \in X(F_v)}$ of v-adic Banach norms on L_x such that for every Zariski open $U \subset X$ and every section $g \in H^0(U, L)$ the map*

$$U(F_v) \longrightarrow \mathbb{R}, \quad x \longmapsto \|g\|_x,$$

is continuous in the v-adic topology on $U(F_v)$.

Example 4.1.2. — Assume that L is generated by global sections. Choose a basis $(g_j)_{j \in [0, \ldots, n]}$ of $H^0(X, L)$ (over F). If g is a section such that $g(x) \neq 0$ then

$$\|g\|_x := \max_{0 \leqslant j \leqslant n} \left(\left| \frac{g_j}{g}(x) \right|_v \right)^{-1},$$

otherwise $\|g\|_x := 0$. This defines a v-adic metric on L. Of course, this metric depends on the choice of $(g_j)_{j \in [0, \ldots, n]}$.

Definition 4.1.3. — *Assume that L is generated by global sections. An adelic metric on L is a collection of v-adic metrics (for every $v \in \mathrm{Val}(F)$) such that for* all but finitely many $v \in \mathrm{Val}(F)$ *the v-adic metric on L is defined by means of some* fixed *basis $(g_j)_{j\in[0,\dots,n]}$ of $H^0(X, L)$.*

We shall write $(\|\cdot\|_v)$ for an adelic metric on L and call a pair $\mathcal{L} = (L, (\|\cdot\|_v))$ an adelically metrized line bundle. Metrizations extend naturally to tensor products and duals of metrized line bundles. Take an arbitrary line bundle L and represent it as $L = L_1 \otimes L_2^{-1}$ with very ample L_1 and L_2. Assume that L_1, L_2 are adelically metrized. An adelic metrization of L is any metrization which for all but finitely many v is induced from the metrizations on L_1, L_2.

Definition 4.1.4. — *Let $\mathcal{L} = (L, \|\cdot\|_v)$ be an adelically metrized line bundle on X and g an F-rational local section of L. Let $U \subset X$ be the maximal Zariski open subset of X where g is defined and is $\neq 0$. For all $x = (x_v)_v \in U(\mathbb{A})$ we define the local*

$$H_{\mathcal{L},g,v}(x_v) := \|g\|_{x_v}^{-1}$$

and the global height function

$$H_{\mathcal{L},g}(x) := \prod_{v\in\mathrm{Val}(F)} H_{\mathcal{L},g,v}(x_v).$$

By the product formula, the restriction of the global height to $U(F)$ does not depend on the choice of g.

4.2. Heights on toric varieties. — We need explicit formulas for heights on toric varieties.

Definition 4.2.1. — *For $\varphi \in \mathrm{PL}$ the local height pairing is given by:*

$$H_v(\varphi; t_v) := e^{\varphi(\overline{t}_v)\log(q_v)}.$$

Globally, for $\varphi \in \mathrm{PL}$,

$$H(\varphi; t) := \prod_{v\in\mathrm{Val}(F)} H_v(\varphi; t_v).$$

Proposition 4.2.2. — *The pairing*

- *is invariant under K_v for all v;*
- *for $t \in \mathbf{T}(F)$ descends to the complexified Picard group $\mathrm{Pic}(X)_{\mathbb{C}}$ (the value of $H(\varphi; t)$ depends only on φ mod $M_{\mathbb{C}}$);*
- *for $\varphi \in \mathrm{PL}$ gives a classical height (with respect to some metrization on L_φ.*

Proof. — The first part is clear. The second claim follows from the product formula. The third claim is verified on very ample L_φ: recall that the global sections $H^0(X, L_\varphi)$

are identified with monomials whose exponents are lattice points in the polytope $\square_\varphi$. For every $t_v \in K_v$ and every $m \in M^{\Gamma_v}$ we have $|m(t_v)| = 1$. Finally,

$$\varphi(\bar{t}_v) = \max_{m \in \square_\varphi} (|m(\bar{t}_v)|_v).$$

For more details the reader could consult [**30**]. □

Example 4.2.3. — Let $X = \mathbb{P}^1 = (x_0 : x_1)$ and $\mathrm{Pic}^{\mathbf{T}}(X) = \mathbb{Z}^2$, spanned by the classes of $0, \infty$ and $\varphi_{\boldsymbol{s}}(e_1) = s_1, \varphi_{\boldsymbol{s}}(e_2) = s_2$. Then

$$H_v(\varphi_{\boldsymbol{s}}, x_v) = \begin{cases} \left|\dfrac{x_0}{x_1}\right|_v^{s_1} & \text{if } \left|\dfrac{x_0}{x_1}\right|_v \geqslant 1, \\ \left|\dfrac{x_0}{x_1}\right|_v^{-s_2} & \text{otherwise.} \end{cases}$$

The following sections are devoted to the computation of the Fourier transforms of H with respect to characters $\chi \in \mathcal{A}_{\mathbf{T}}$. By definition,

$$\widehat{H}(\varphi; \chi) := \int_{\mathbf{T}(\mathbb{A})} H(\varphi; t)\chi(t) d\mu = \prod_{v \in \mathrm{Val}(F)} \int_{\mathbf{T}(F_v)} H_v(\varphi; t_v)\chi_v(t_v) d\mu_v,$$

where $d\mu$ is the normalized Haar measure and χ_v are trivial on K_v (unramified) for all v (see Section 2.1).

4.3. Height integrals - nonarchimedean valuations. — Let X be a smooth d-dimensional equivariant compactification of a linear algebraic group $\mathbf{G}$ over F such that the boundary is a strict normal crossing divisor consisting of (geometrically) irreducible divisors

$$X \smallsetminus \mathbf{G} = \cup_{j \in [1,\ldots,n]} D_j.$$

We put $D_\varnothing = \mathbf{G}$ and define for every subset $J \subset [1, \ldots, n]$

$$\begin{aligned} D_J &= \quad \cap_{j \in J} D_j \\ D_J^0 &= D_J \smallsetminus \cup_{J' \supsetneq J} D_{J'}. \end{aligned}$$

Choose for each v a Haar measure dg_v on $\mathbf{G}(F_v)$ such that for almost all v

$$\int_{\mathbf{G}(\mathcal{O}_v)} dg_v = 1.$$

As in Section 4.1, one can define a pairing between

$$\mathrm{Div}_{\mathbb{C}} := \mathbb{C}D_1 \oplus \cdots \oplus \mathbb{C}D_n$$

and $\mathbf{G}(\mathbb{A})$. In the above basis, we have coordinates $\boldsymbol{s} = (s_1, \ldots, s_n)$ on $\mathrm{Div}_{\mathbb{C}}$. Choose an F-rational (bi-)invariant differential form d-form on $\mathbf{G}$. Then it has poles along each boundary component, and we denote by κ_j the corresponding multiplicities. For all but finitely many nonarchimedean valuations v, one has (see [**9**] and [**13**])

$$(4.1) \qquad \int_{\mathbf{G}(F_v)} H_v(\boldsymbol{s}; g_v)^{-1} dg_v = \tau_v(\mathbf{G})^{-1} \Big(\sum_{J \subseteq [1,\ldots,n]} \frac{\# D_J^0(\mathbb{F}_v)}{q_v^d} \prod_{j \in J} \frac{q_v - 1}{q_v^{(s_j - \kappa_j + 1)} - 1} \Big).$$

Remark 4.3.1. — Notice that for almost all v

$$\int_{\mathbf{G}(F_v)} H_{-K_X}(g_v)^{-1}dg_v = \frac{\#X(\mathbb{F}_v)}{\#\mathbf{G}(\mathbb{F}_v)}. \tag{4.2}$$

In particular, for some $\Re(s) > 1-\delta$ (and some $\delta > 0$)

$$\prod_{v\in \mathrm{Val}(F)} \int_{\mathbf{G}(F_v)} \zeta_F(s)^{-n} H_{-K_X}(g_v)^{-s}dg_v \tag{4.3}$$

is an absolutely convergent Euler product (see [**9**], Section 7).

For toric varieties, we can compute the integral (4.1) combinatorially.

Example 4.3.2. — Let $X = \mathbb{P}^1$, $H_v(\varphi_{\boldsymbol{s}}; x_v)$ the local height as in Example 4.2.3 and $d\mu_v$ the normalized Haar measure on $\mathbf{G}_m(F_v)$ as in 2.1. Then $N_v = \mathbb{Z}$ and

$$\int_{\mathbf{G}_m(F_v)} H_v(\boldsymbol{s}; x_v)^{-1} d\mu_v = \sum_{n_v\in\mathbb{Z}} q_v^{-\varphi_{\boldsymbol{s}}(n_v)} = \frac{1}{1-q_v^{-s_1}} + \frac{1}{1-q_v^{-s_2}} - 1. \tag{4.4}$$

If X is a *split* smooth (!) toric variety of dimension d then

$$\int_{\mathbf{G}_m^d(F_v)} H_v(\boldsymbol{s}; x_v)^{-1} d\mu_v = \sum_{\sigma\in\Sigma} (-1)^{d-d_\sigma} \prod_{e_j\in\sigma} \frac{1}{1-q_v^{-s_j}}. \tag{4.5}$$

Remark 4.3.3. — As the formula (4.5) and the Example 4.3.2 suggest, the height integral is an alternating sum of (sums of) geometric progressions, labeled by cones $\sigma \in \Sigma$ (which are, of course, in bijection with tori forming the boundary stratification by disjoint locally closed subvarieties). The smoothness of the toric variety is crucial — we need to know that the set of generators of each cone can be extended to a *basis* of N_v.

Proposition 4.3.4. — *There exists an* $\varepsilon > 0$ *such that for all* $\boldsymbol{s} \in \mathrm{PL}$ *with* $\Re(s_j) \geqslant 1-\varepsilon$ *(for all* j*)*

$$\int_{\mathbf{T}(F_v)} H_v(\boldsymbol{s}; t_v)^{-1}\chi(t_v) d\mu_v = Q_v(\boldsymbol{s};\chi) \cdot \prod_{j=1}^n \zeta_{F,v}(s_j, \chi_{j,v}),$$

where χ_j *is as in Section 3.2,* $\zeta_v(s_j, \chi_j)$ *is the local factor of the Hecke L-function of* F *with character* χ_j *and* $Q_v(\boldsymbol{s}, \chi)$ *is a holomorphic function on* $\mathrm{PL}_{\mathbb{C}}$*. Moreover, for* $\boldsymbol{s}$ *in this domain the Euler product*

$$Q(\boldsymbol{s};\chi) := \prod_{v\nmid\infty} Q_v(\boldsymbol{s};\chi_v)$$

is absolutely and uniformly convergent and there exist positive constants C_1, C_2 *such that for all* χ *one has*

$$C_1 < |Q(\boldsymbol{s};\chi)| \leqslant C_2.$$

Proof. — This is Theorem 3.1.3. in [**2**]. □

4.4. Height integrals - archimedean valuations. — Similarly to the combinatorics in Example 4.3.2 one obtains

$$\begin{aligned}(4.6)\qquad \int_{\mathbf{T}(F_v)/K_v} H_v(\varphi;t_v)^{-1}\chi_v(t_v)d\mu_v &= \int_{\mathbb{R}^d} e^{-\varphi(n)-i\langle m_v,\bar{t}_v\rangle}dn \\ &= \sum_{\sigma\in\Sigma(d)}\int_\sigma e^{-\varphi(n)-i\langle m_v,\bar{t}_v\rangle}dn,\end{aligned}$$

where $m_v = m_v(\chi)$ as in Section 6.1 and dn is the Lebesgue measure on $N_{\mathbb{R}}$ normalized by N. Using the regularity of the fan Σ we have

$$\widehat{H}_v(-\varphi_{\boldsymbol{s}};\chi_v) = \sum_{\sigma\in\Sigma(d)}\prod_{e_j\in\sigma}\frac{1}{s_j+i\langle m_v,e_j\rangle}. \tag{4.7}$$

Example 4.4.1. — For $\mathbb{P}^1$ we get (keeping the notations of Example 4.2.3)

$$\widehat{H}_v(-\varphi_{\boldsymbol{s}};\chi) = \frac{1}{s_1+im}+\frac{1}{s_2-im}. \tag{4.8}$$

In the next section we will need to integrate $\prod_{v|\infty}\widehat{H}_v$ over $M_{\mathbb{R},\infty}$. Notice that each term in Equation (4.7) decreases as $\|m_v\|^{-d}$ and is not integrable. However, some cancelations help.

Lemma 4.4.2. — *For every $\varepsilon > 0$ and every compact K in the domain $\Re(s_j) > \varepsilon$ (for all j) there exists a constant $C(K)$ such that*

$$|\widehat{H}_v(-\varphi_{\boldsymbol{s}};\chi_v)| \leqslant C(K)\sum_{\sigma\in\Sigma(d)}\prod_{e_j\in\sigma}\frac{1}{(1+|\langle m_v,e_j\rangle|)^{1+1/d}}.$$

This is Proposition 2.3.2 in [**2**]. One uses integration by parts.

Remark 4.4.3. — In particular, Lemma 4.4.2 implies that for all $m\in M_{\mathbb{R}}$ one has

$$\sum_{\sigma\in\Sigma(d)}\prod_{e_j\in\sigma}\frac{1}{\langle m,e_j\rangle} = 0.$$

5. Height zeta functions

5.1. $\mathcal{X}$-functions. — Let (A,Λ) be a pair consisting of a lattice and a strictly convex (closed) cone in $A_{\mathbb{R}}$ and $(\check{A},\check{\Lambda})$ the pair consisting of the dual lattice and the dual cone. The lattice $\check{A}$ determines the normalization of the Lebesgue measure $d\check{a}$ on $\check{A}_{\mathbb{R}}$ (covolume =1). For $a\in A_{\mathbb{C}}$ define

$$\mathcal{X}_\Lambda(a) := \int_{\check{\Lambda}} e^{-\langle a,\check{a}\rangle}d\check{a}.$$

Remark 5.1.1. — The integral converges absolutely and uniformly for $\Re(a)$ in compacts contained in the interior Λ° of Λ.

Example 5.1.2. — Consider $(\mathbb{Z}^n, \mathbb{R}^n_{\geqslant 0})$. Then

$$\mathcal{X}_\Lambda(a_1, \ldots, a_n) = \frac{1}{a_1 \cdots a_n},$$

where (a_j) are the standard coordinates on $\mathbb{R}^n$.

Remark 5.1.3. — The $\mathcal{X}$-functions of cones appeared in the work of Köcher [**28**], Vinberg [**43**], and others (see [**40**], [**1**] pp. 57-78, [**17**]).

5.2. Iterated residues. — Let (A, Λ) be a pair as above with $\Lambda \subset \mathbb{A}_\mathbb{R}$ generated by finitely many vectors in A. Such Λ are called (rational) polyhedral cones. It will be convenient to fix a basis in A.

Remark 5.2.1. — To compute $\mathcal{X}_\Lambda(a)$ explicitly one could decompose the dual cone $\check{\Lambda}$ into simplicial subcones and then apply Example 5.1.2. Thus there is a finite set $\mathcal{A}$ such that

$$\mathcal{X}_\Lambda(a) = \sum_{\alpha \in \mathcal{A}} \mathcal{X}_\alpha \cdot \frac{1}{\prod_{\beta=1}^n \ell^\alpha_\beta(a)}, \tag{5.1}$$

where $n = \dim \mathbb{A}_\mathbb{R}$ and $\mathcal{X}_\alpha = \det(\ell^\alpha_\beta)$ ((ℓ^α_β) are n-tuples of linearly independent linear forms on $A_\mathbb{R}$ with coefficients in $\mathbb{R}$).

Remark 5.2.2. — Using this decomposition one can show that $\mathcal{X}_\Lambda$ has simple poles along the hyperplanes defining Λ. The terms in the sum (5.1) may have poles in the domain $\Re(a) \in \Lambda^\circ$, but these must cancel (by Remark 5.1.1).

Proposition 5.2.3. — *Let (A, Λ) be a pair as above and $\psi : A \to \widetilde{A}$ a surjective homomorphism of lattices with kernel M. Let $\widetilde{\Lambda} = \psi(\Lambda) \subset \mathbb{A}_\mathbb{R}$ be the image of Λ - it is obtained by projecting Λ along the linear subspace $M_\mathbb{R} \subset A_\mathbb{R}$ ($M_\mathbb{R} \cap \Lambda = 0$). Let dm be the Lebesgue measure on $M_\mathbb{R}$ normalized by the lattice M. Then for all a with $\Re(a) \in \Lambda^\circ$ one has*

$$\mathcal{X}_{\widetilde{\Lambda}}(\psi(a)) = \frac{1}{(2\pi)^d} \int_{M_\mathbb{R}} \mathcal{X}(a + im) dm,$$

where $d = \dim M_\mathbb{R}$.

Proof. — First one verifies that $\mathcal{X}_\Lambda(a)$ is integrable over $iM_\mathbb{R}$ (and the integral descends to $\widetilde{A}_\mathbb{C}$, by the Cauchy-Riemann equations). The formula is a consequence of Theorem 6.3.1. □

Example 5.2.4. — The cone $\mathbb{R}_{\geqslant 0} \subset \mathbb{R}$ is the image of the cone $\mathbb{R}^2_{\geqslant 0} \subset \mathbb{R}^2$ under the projection $(a_1, a_2) \mapsto a_1 + a_2$ (with kernel $\{(m, -m)\} \subset \mathbb{R}^2$). According to Proposition 5.2.3 we have

$$\frac{1}{2\pi} \int_\mathbb{R} \frac{1}{(s_1 + im)(s_2 - im)} dm = \frac{1}{s_1 + s_2}.$$

Example 5.2.5. — Similarly, consider

$$\mathcal{X}(\boldsymbol{s}) := \frac{1}{2\pi}\int_{\mathbb{R}} \frac{1}{\prod_{j=1}^{k}(s_j+im)\prod_{j'=1}^{k'}(s_{j'}-im)}dm.$$

We can deform the contour of integration to the left or to the right. In the first case, we get

$$\mathcal{X}(\boldsymbol{s}) = \sum_{j'=1}^{k'} \frac{1}{\prod_j (s_j+s_{j'})\prod_{j''\neq j'}(s_{j''}-s_{j'})}.$$

In the second expansion,

$$\mathcal{X}(\boldsymbol{s}) = \sum_{j=1}^{k} \frac{1}{\prod_{j''\neq j}(s_{j''}-s_j)\prod_{j'}(s_{j'}+s_j)}.$$

Of course, both formulas define the same function. The two expansions correspond to two different subdivisions of the image cone into simplicial subcones.

Example 5.2.6. — The fan in $N=\mathbb{Z}^2$ spanned by the vectors

$$e_1=(1,0),\ e_2=(1,1),\ e_3=(0,1),\ e_4=(-1,0),\ e_5=(-1,-1),\ e_6=(0,-1)$$

defines a Del Pezzo surface X of degree 6 - a blowup of 3 (non-collinear) points in $\mathbb{P}^2$. Let $\Lambda=\Lambda_{\mathrm{eff}}(X)\subset\mathbb{R}^4$ be the cone of effective divisors of X. In the proof of our main theorem for X we encounter an integral similar to

$$\mathcal{X}(s_1,\ldots,s_6) = \frac{1}{(2\pi)^2}\int_{M_{\mathbb{R}}}\prod_{j=1}^{6}\frac{1}{s_j+i\langle m,e_j\rangle}dm.$$

(where $M_{\mathbb{R}}=\mathbb{R}^2$). Choosing a generic path in the space $M_{\mathbb{R}}$ and shifting the contour of integration we can reduce this integral to a sum of 1-dimensional integrals of type 5.2.5. Then we use the previous example and, finally, collect the terms. The result is

$$\mathcal{X}(s_1,\ldots,s_6) = \frac{s_1+s_2+s_3+s_4+s_5+s_6}{(s_1+s_4)(s_2+s_5)(s_3+s_6)(s_1+s_3+s_5)(s_2+s_4+s_6)}.$$

Definition 5.2.7. — *Let (A,Λ) and $(\widetilde{A},\widetilde{\Lambda})$ be as above. We say that a function f on $A_{\mathbb{C}}$ has Λ-poles if:*

- *f is holomorphic for $\Re(a)\in\Lambda^\circ$;*
- *there exist an $\varepsilon>0$ and a finite set $\mathcal{A}$ of n-tuples of linearly independent linear forms $(\ell_\beta^\alpha)_{\alpha\in\mathcal{A}}$, functions f_α and a constant $c\neq 0$ such that*

$$f(a) = \sum_{\alpha\in\mathcal{A}} \mathcal{X}_\alpha\cdot f_\alpha(a)\cdot\prod_{\beta=1}^{n}\frac{1}{\ell_\beta^\alpha(a)},$$

where

$$\sum_{\alpha\in\mathcal{A}} \mathcal{X}_\alpha\cdot\prod_{\beta=1}^{n}\frac{1}{\ell_\beta^\alpha(a)} = \mathcal{X}_\Lambda(a)$$

(as in 5.2.1) and for every $\alpha \in \mathcal{A}$ the function f_α is holomorphic in the domain $\|\Re(a)\| < \varepsilon$ with $f_\alpha(\mathbf{0}) = c$ (compare with Remark 5.2.1).

The main technical result is

Theorem 5.2.8. — *Let (A, Λ) be as above and f a function on $A_{\mathbb{C}}$ with Λ-poles. Assume that there exists an $\varepsilon > 0$ such that for every compact K in the domain $\|\Re(a)\| < \varepsilon$ there exist positive constants ε' and $C(K)$ such that*

- *for all $b \in A_{\mathbb{R}}$, $\alpha \in \mathcal{A}$ and $a \in K$ one has*

$$|f_\alpha(a + ib)| \leqslant C(K)(1 + \|b\|)^{\varepsilon'};$$

- *for $a \in K$ and every subspace $M'_{\mathbb{R}} \subset M_{\mathbb{R}}$ of dimension d'*

$$\Big| f(a + im') \prod_{\alpha,\beta} \frac{\ell^\alpha_\beta(a)}{\ell^\alpha_\beta(a) + 1} \Big| \leqslant C(K)(1 + \|m'\|)^{-(d'+\delta)}$$

for all $m' \in M'_{\mathbb{R}}$ and some $\delta > 0$.

Then

$$\widetilde{f}(\psi(a)) := \frac{1}{(2\pi)^d} \int_{M_{\mathbb{R}}} f(a + im) dm$$

is a function on $\widetilde{A}_{\mathbb{C}}$ with $\psi(\Lambda)$-poles.

Proof. — Decompose the projection with respect to $M_{\mathbb{R}}$ into a sequence of (appropriate) 1-dimensional projections and apply the residue theorem. A refined statement with a detailed proof is in [**8**], Section 3. □

Corollary 5.2.9. — *For f as in Theorem 5.2.8 and $a \in \Lambda^\circ \subset A_{\mathbb{R}}$ we have*

$$\lim_{s \to 0^+} \frac{\widetilde{f}(sa)}{\mathcal{X}_{\widetilde{\Lambda}}(\psi(sa))} = \lim_{s \to 0^+} \frac{f(sa)}{\mathcal{X}_\Lambda(sa)}.$$

5.3. Meromorphic continuation

Proposition 5.3.1. — *For $\Re(s_j) > 1$ (for all j) one has*

$$Z(\boldsymbol{s}) = \sum_{t \in \mathbf{T}(F)} H(\boldsymbol{s}; t)^{-1} = \int_{\mathcal{A}_{\boldsymbol{T}}} \widehat{H}(-\boldsymbol{s}; \chi) d\chi = (*) \int_{M_{\mathbb{R}}} f(\boldsymbol{s} + im) dm,$$

where

$$f(\boldsymbol{s}) = \sum_{\chi \in \mathcal{U}_{\mathbf{T}}} \widehat{H}(-\boldsymbol{s}; \chi)$$

and $(*)$ is an appropriate constant (comparison between the measures).

Proof. — Application of the general Poisson formula 6.3.1. The integrability of both sides of the equation follows from estimates similar to 4.4.2 (see Theorem 3.2.5 in [**2**]). Then we use the decomposition of characters as in Section 2. □

Now we are in the situation of Theorem 5.2.8. From the computations in Sections 4.3 and 4.4 we know that

$$\widehat{H}(-\boldsymbol{s};\chi) = \prod_{v|\infty} \widehat{H}_v(-\boldsymbol{s};\chi_v) \cdot \prod_{v\nmid\infty} Q_v(\boldsymbol{s};\chi_v) \cdot \prod_{j=1}^{n} L(s_j,\chi_j),$$

where $Q(\boldsymbol{s};\chi) = \prod_v Q_v(\boldsymbol{s};\chi)$ is a holomorphic bounded function in the domain $\Re(s_j) > 1-\delta$ (for some $\delta > 0$). The poles of $\widehat{H}(-\boldsymbol{s};\chi)$ come from the poles of the Hecke L-functions $L(s_j,\chi)$ (that is from trivial characters χ_j and at $s_j = 1$). Using uniform estimates from Theorem 6.1.1 and bounds on $\widehat{H}_v$ for $v \in S$ we see that the function

$$f(\boldsymbol{s}) \prod_{j=1}^{n} (s_j - 1)$$

is holomorphic for $\Re(s_j) > 1-\delta$ (for some $\delta > 0$) and satisfies the growth conditions of Theorem 5.2.8. Once we know that

$$\Theta = \lim_{\boldsymbol{s}\to\boldsymbol{1}} \prod_{j=1}^{n} (s_j - 1) \cdot f(\boldsymbol{s}) \neq 0$$

we can apply that theorem.

Theorem 5.3.2. — *The function $Z(\boldsymbol{s}+K_X)$ has $\Lambda_{\mathrm{eff}}(X)$-poles. The 1-parameter function $Z(s(-K_X))$ has a representation*

$$Z(s(-K_X)) = \frac{\Theta(\mathbf{T},-\mathcal{K}_X)}{(s-1)^{n-d}} + \frac{h(s)}{(s-1)^{n-d-1}},$$

where $h(s)$ is a holomorphic function for $\Re(s) > 1-\delta$ (for some $\delta > 0$) and $\Theta(\mathbf{T},-\mathcal{K}_X) > 0$ (interpreted in [**5**]*).*

Proof (Sketch). — We need to identify Θ. First of all,

$$\Theta = \lim_{s\to 1} (s-1)^n \cdot \sum_{\chi} \widehat{H}(-s\mathbf{1};\chi),$$

where the summation is over all $\chi \in \mathcal{U}_{\mathbf{T}}$ such that the corresponding components χ_j are trivial for all $j = 1,\ldots,n$. There is only one such character — the trivial character. We obtain

$$\Theta = \lim_{s\to 1} (s-1)^n \int_{\mathbf{T}(\mathbb{A})} H(-s\mathbf{1};t)d\mu.$$

The nonvanishing follows from (4.3.1). □

5.4. Digression on cones. — Let $(A, \Lambda, -K)$ be a triple consisting of a (torsion free) lattice $A = \mathbb{Z}^n$, a (closed) strictly convex polyhedral cone in $A_{\mathbb{R}}$ generated by finitely many vectors in $\mathbb{A}$ and a vector $-K \subset \Lambda^\circ$ (the interior of Λ). For $L \in A$ we define

$$a(\Lambda, L) = \inf\{a \mid aL + K \in \Lambda\}$$

and $b(\Lambda, L)$ as the codimension of the minimal face $\Lambda(L)$ of Λ containing $a(\Lambda, L)L+K$. Obviously, for $L = -K$ we get $a(\Lambda, -K) = 1$ and $b(\Lambda, -K) = n$.

5.5. General L. — Let $\mathcal{L}$ be an adelically metrized line bundle of X such that L is contained in $\Lambda^\circ_{\mathrm{eff}}(X)$. The 1-parameter height zeta function

$$Z(sL) = \sum_{t \in \mathbf{T}(F)} H(sL; t)^{-1}$$

is absolutely convergent for $\Re(s) > a(\Lambda_{\mathrm{eff}}(X), L)$ and, by Theorem 5.2.8, has an isolated pole at $s = a(\Lambda_{\mathrm{eff}}(X), L)$ of order *at most* $b(\Lambda_{\mathrm{eff}}(X), L)$. Denote by $\Sigma(L) \subset$ PL the set of generators projecting onto the face $\Lambda(L)$ (under ψ). Let

$$M'_{\mathbb{R}} := \{m \in M_{\mathbb{R}} \mid \langle m, e_j \rangle = 0 \ \forall e_j \notin \Sigma(L)\}$$

and $M' := M'_{\mathbb{R}} \cap M$. Then $M'' = M/M'$ is torsion free. Again, we are in the situation of Theorem 5.2.8, this time with $\mathrm{PL}_{\mathbb{R}}/M'_{\mathbb{R}}$ projecting with kernel M''. We need to compute

$$\lim_{\boldsymbol{s} \to \mathbf{1}} \prod_{e_j \notin \Sigma(L)} (s_j - 1) \cdot f(\boldsymbol{s}),$$

where

$$f(\boldsymbol{s}) = (*) \int_{M'_{\mathbb{R}}} \{\sum_{\mathcal{U}'_{\mathbf{T}}} \widehat{H}(\boldsymbol{s} + im'; \chi)\} dm',$$

the summation is over all characters in $\mathcal{U}_{\mathbf{T}}$ such that $\chi_j = 1$ if $e_j \notin \Sigma(L)$ and $(*)$ is an appropriate constant. We apply the Poisson formula 6.3.1 and convert $f(\boldsymbol{s})$ into a sum of adelic integrals of $H(\boldsymbol{s}, t)$ (up to rational factors) over the set of certain fibers of a natural fibration induced by the exact sequence of tori

$$1 \to \mathbf{T}'' \to \mathbf{T} \to \mathbf{T}' \to 1,$$

where $\mathbf{T}' := \mathrm{Spec}(F[M'])$. The regularized adelic integrals over the fibers are Tamagawa type numbers similar to those encountered in Theorem 5.3.2. However, even if X is smooth - the compactifications of these fibers need not be! This explains the technical setup in [**5**].

6. Appendix: Facts from algebra and analysis

6.1. Hecke L-functions. — Let $\chi : \mathbf{G}_m(\mathbb{A})/\mathbf{G}_m(F) \to S^1$ be an unramified (unitary) character and χ_v its components on $\mathbf{G}_m(F_v)$. For all $v \in \mathrm{Val}(F)$ there exists an $m_v \in \mathbb{R}$ such that

$$\chi_v(x_v) = q_v^{im_v \log(|x_v|_v)}.$$

Put

$$\chi_\infty = (m_v)_{v|\infty} \in \mathbb{R}^{\mathrm{Val}_\infty(F)} \text{ and } \|\chi_\infty\| = \max_{v|\infty}(|m_v|).$$

Theorem 6.1.1. — *For every $\varepsilon > 0$ there exist a $\delta > 0$ and a constant $c(\varepsilon) > 0$ such that for all s with $\Re(s) > 1 - \delta$ and all unramified Hecke characters χ which are nontrivial on $\mathbf{G}_m^1(\mathbb{A})$ one has*

$$|L(s,\chi)| \leqslant c(\varepsilon)(1 + |\Im(s)| + \|\chi_\infty\|)^\varepsilon. \tag{6.1}$$

For the trivial character $\chi = 1$ one has

$$|L(s,1)| \leqslant c(\varepsilon)\left|\frac{1+s}{1-s}\right|(1 + |\Im(s)|)^\varepsilon \tag{6.2}$$

6.2. Artin L-functions. — Let E/F be a Galois extension of number fields with Galois group Γ, M a torsion free finitely generated Γ-module and M^Γ its submodule of Γ-invariants. We have an integral representation of Γ on $\mathrm{Aut}(M)$. Let $S \subset \mathrm{Val}(F)$ be a finite set including all v which ramify in E and all archimedean valuations. For $v \notin S$ define

$$L_v(s,M) := \frac{1}{\det(\mathrm{Id} - q_v^{-s}\Phi_v)},$$

where Φ_v is the image in $\mathrm{Aut}(M)$ of a local Frobenius element (this is well defined since the characteristic polynomial of the matrix Φ_v only depends on its conjugacy class). The partial Artin L-function is

$$L_S(s,M) := \prod_{v \notin S} L_v(s,M).$$

The Euler product converges for $\Re(s) > 1$. The function $L_S(s,M)$ has a meromorphic continuation with an isolated pole at $s = 1$ of order $r = \mathrm{rk}\, M^\Gamma$. Denote by

$$L_S^*(1,M) = \lim_{s\to 1}(s-1)^r L_S(s,M)$$

the leading coefficient at this pole.

6.3. Poisson formula

Theorem 6.3.1. — *Let G be a locally compact abelian group with Haar measure dg. For $f \in L^1(G)$ and $\chi : G \to S^1$ a unitary character of G define the Fourier transform*

$$\widehat{f}(\chi) = \int_G f(g)\chi(g)dg.$$

Let $H \subset G$ be a closed subgroup with Haar measure dh and

$$H^{\perp} = \{\chi : G \to S^1 \mid \chi(h) = 1\ \forall h \in H\}.$$

Then there exists a unique Haar measure $d\chi$ on $H^{\perp}$ such that for all $f \in L^1(G)$ with $\widehat{f} \in L^1(H^{\perp})$ one has

$$\int_H f(h)dh = \int_{H^{\perp}} \widehat{f}(\chi)d\chi.$$

6.4. Convexity. — Let $U \subset \mathbb{R}^d$ be any subset. A tube domain $\mathcal{T}(U) \subset \mathbb{C}^d$ is

$$\mathcal{T}(U) := \{z \in \mathbb{C}^d \mid \Re(z) \subset U\}.$$

Theorem 6.4.1. — *Let $U \subset \mathbb{R}^d$ be a connected open subset and $d \geqslant 2$. Any holomophic function in $\mathcal{T}(U)$ extends to a holomophic function in $\mathcal{T}(\overline{U})$ where $\overline{U}$ is the convex hull of U.*

Proof. — See Proposition 6, p. 122 in [**31**]. □

6.5. Tauberian theorem

Theorem 6.5.1. — *Let $(h_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$ be two sequences of positive real numbers. Assume that the first sequence is strictly increasing and consider*

$$f(s) = \sum_{n=0}^{\infty} \frac{c_n}{h_n^s}.$$

Assume further that

(1) *the series defining $f(s)$ converges for $\Re(s) > a > 0$;*

(2) *it admits meromorphic continuation to $\Re(s) > a - \delta > 0$ (for some $\delta > 0$) with a unique pole at $s = a$ of order $b \in \mathbb{N}$;*

(3) *there exist a real number $\kappa > 0$ and a constant k such that*

$$\left|\frac{f(s)(s-a)^b}{s^b}\right| \leqslant k(1 + \Im(s))^{\kappa}$$

for $\Re(s) > a - \delta$.

Then there exist a polynomial P of degree $b-1$ with leading coefficient 1 and a constant $\delta' > 0$ such that

$$N(B) = \sum_{h_n \leqslant B} c_n = \frac{\Theta}{a(b-1)!} B^a P(\log(B)) + O(B^{a-\delta'}),$$

for $B \to \infty$, where

$$\Theta = \lim_{s \to a} (s-a)^b f(s) > 0.$$

This is a standard Tauberian theorem, see [**12**] or the Appendix to [**8**].

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