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CONTINUOUS DIVISION OF LINEAR DIFFERENTIAL OPERATORS AND FAITHFUL FLATNESS OF $\mathscr{D}_X^\infty$ OVER $\mathscr{D}_X$

by

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Abstract. — In these notes we prove the faithful flatness of the sheaf of infinite order linear differential operators over the sheaf of finite order linear differential operators on a complex analytic manifold. We give the Mebkhout-Narváez's proof based on the continuity of the division of finite order differential operators with respect to a natural topology. We reproduce the proof of the continuity theorem given by Hauser-Narváez, which is simpler than the original proof.

***Résumé* (Continuité de la division des opérateurs différentiels et fidèle platitude de $\mathscr{D}_X^\infty$ sur $\mathscr{D}_X$)**

Dans ce cours on démontre la fidèle platitude du faisceau d'opérateurs différentiels linéaires d'ordre infini sur le faisceau d'opérateurs différentiels linéaires d'ordre fini d'une variéte analytique complexe lisse. La preuve que nous donnons est celle de Mebkhout-Narváez, qui utilise la continuité de la division d'opérateurs différentiels d'ordre fini par rapport à une topologie naturelle. Nous réproduisons la preuve de Hauser-Narváez du théorème de continuité, qui est plus simple que la preuve originale.

Introduction

The sheaf $\mathcal{O}_X$ of holomorphic functions on a complex analytic manifold X is the first natural example of left module over the sheaf of linear differential operators $\mathcal{D}_X$ on X. Here, as usual, differential operators have (locally) finite order. In fact, there is another natural sheaf of noncommutative rings extending $\mathcal{D}_X$, called the *sheaf of linear differential operators of infinite order*, $\mathcal{D}_X^\infty$, introduced by Sato. The left $\mathcal{D}_X$-module structure on $\mathcal{O}_X$ extends to a left $\mathcal{D}_X^\infty$-module structure in such a way that $\mathcal{D}_X^\infty \otimes_{\mathcal{D}_X} \mathcal{O}_X = \mathcal{O}_X$.

For any holonomic left $\mathcal{D}_X$-module $\mathcal{M}$, we know by the constructibility theorem of Kashiwara [**9**] (see also [**12**], [**13**]) that the complex of holomorphic solutions of $\mathcal{M}$, $R\,\mathit{Hom}_{\mathcal{D}_X}(\mathcal{M},\mathcal{O}_X)$, is constructible. The canonical $\mathcal{D}_X$-linear biduality morphism

$$\mathcal{M} \to R\,\mathit{Hom}_{\mathbb{C}_X}(R\,\mathit{Hom}_{\mathcal{D}_X}(\mathcal{M},\mathcal{O}_X),\mathcal{O}_X)$$

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induces a $\mathcal{D}_X^\infty$-linear morphism

$$(*)\qquad \mathcal{D}_X^\infty \otimes_{\mathcal{D}_X} \mathcal{M} \to R\,\mathcal{H}om_{\mathbb{C}_X}(R\,\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X), \mathcal{O}_X).$$

The *local biduality theorem* of Mebkhout asserts that (*) is an isomorphism for any holonomic module $\mathcal{M}$ (see [**11**, 11.3] in this volume). This theorem is an essential ingredient for the "full" Riemann-Hilbert correspondence, which establishes an equivalence between three categories: the bounded derived category of regular holonomic complexes of $\mathcal{D}_X$-modules, the bounded derived category of holonomic complexes of $\mathcal{D}_X^\infty$-modules and the bounded derived category of analytic constructible complexes (see 11.4 in loc. cit.). The sheaf $\mathcal{D}_X^\infty$ does not have any known finiteness properties like $\mathcal{D}_X$, but to prove the full Riemann-Hilbert correspondence one needs to know that the extension $\mathcal{D}_X \subset \mathcal{D}_X^\infty$ is faithfully flat. This result has been stated and proved for the first time in [**17**] (see also [**1**]), and its proof depended on the microlocal machinery.

The aim of these notes is to give an elementary self-contained proof of the faithful flatness of the sheaf of differential operators of infinite order over the sheaf of differential operators of finite order. The method we follow is that of [**14**], whose first step consists in considering the ring of differential operators of infinite order as the completion of the corresponding ring of finite order for a natural topology, and then mimic Serre's proof of the faithful flatness of the completion of a noetherian local ring over the ring itself [**18**]. The essential technical tool is the continuity of the Weierstrass-Grauert-Hironaka division of differential operators [**2, 3**]. We reproduce with detail the proof given in [**8**], which simplifies the original proof in [**14**]. As a complement we sketch the results of [**15**] for the case of differential operators with polynomial coefficients (Weyl algebra).

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1. Topological structure on rings of linear differential operators with analytic coefficients

Let X be a complex analytic manifold of pure dimension n, countable at infinity. Let us denote by $\mathcal{O}_X$ the sheaf of holomorphic functions and by $\mathcal{D}_X$ the sheaf of linear differential operators (*cf.* [**6**]). For each open set $U \subset X$, the space $\mathcal{O}_X(U)$ endowed with the topology of uniform convergence on compact sets is a *Fréchet space*, *i.e.* a complete metrizable locally convex space (it is also a *nuclear* space, *cf.* [**5**] for details). The Banach open mapping theorem shows that the property of being continuous for a $\mathbb{C}$-linear endomorphism $P : \mathcal{O}_X \to \mathcal{O}_X$ is a local property. For that, let $\{U_i\}$ be an open covering of X, that we can take as countable, such that each restriction $P|_{U_i} : \mathcal{O}_X|_{U_i} \to \mathcal{O}_X|_{U_i}$ is continuous. For any open set $U \subset X$, the canonical injection $\mathcal{O}_X(U) \hookrightarrow \prod \mathcal{O}_X(U \cap U_i)$ is a closed inmersion by the open mapping theorem (its image is the kernel of the Čech map $\prod \mathcal{O}_X(U \cap U_i) \to \prod \mathcal{O}_X(U \cap U_i \cap U_j)$ by the sheaf condition). Hence, the continuity of $P(U) : \mathcal{O}_X(U) \to \mathcal{O}_X(U)$ comes from the continuity of $\prod P(U \cap U_i) : \prod \mathcal{O}_X(U \cap U_i) \to \prod \mathcal{O}_X(U \cap U_i)$. As a consequence, the pre-sheaf of $\mathbb{C}$-linear continuous endomorphisms of $\mathcal{O}_X$, $\mathit{Homtop}(\mathcal{O}_X, \mathcal{O}_X)$, is actually a sheaf.

The following proposition is well-known (*cf.* [**14**], prop. 2.1.4):

Proposition 1.1. — *For any continuous $\mathbb{C}$-linear endomorphism $P : \mathcal{O}_X \to \mathcal{O}_X$ and for any system $(U; x_1, \dots, x_n)$ of local coordinates of X, there are unique holomorphic functions $a_\alpha \in \mathcal{O}_X(U)$, $\alpha \in \mathbb{N}^n$, such that*

$$P|_U = \sum_{\alpha \in \mathbb{N}^n} a_\alpha \frac{1}{\alpha!} \partial^\alpha,$$

with $\partial = (\partial/\partial x_1, \dots, \partial/\partial x_n)$ and $\lim_{|\alpha| \to \infty} |a_\alpha|^{1/|\alpha|} = 0$ uniformly on any compact set of U. Equivalently, the function

$$(p, \xi) \in U \times \mathbb{C}^n \longmapsto \sum_{\alpha \in \mathbb{N}^n} a_\alpha(p) \xi^\alpha \in \mathbb{C}$$

is holomorphic.

From now on, we will denote $\mathcal{D}_X^\infty = \mathit{Homtop}(\mathcal{O}_X, \mathcal{O}_X)$ and call it *sheaf of infinite order linear differential operators*. From the above proposition we deduce that it coincides with the sheaf of infinite order linear differential operators defined in [**16, 17**].

The following proposition is proved in [**14**], prop. 2.1.3.

Proposition 1.2. — *Let $P : \mathcal{O}_X \to \mathcal{O}_X$ be a $\mathbb{C}$-linear endomorphism. The following properties are equivalent:*

a) *P is continuous.*

b) *For any pair $K, K' \subseteq X$ of compact sets with $K \subset \mathring{K'}$, there is a constant $C_{K,K'} > 0$ such that $|P(f)|_K \leqslant C_{K,K'} |f|_{K'}$ for any holomorphic function f defined on a neighborhood of K'.*

Corollary 1.3. — *The sheaf $\mathcal{D}_X$ of (finite order) linear differential operators is a subsheaf (of rings) of $\mathit{Homtop}(\mathcal{O}_X, \mathcal{O}_X)$.*

Proof. — Let P be a section of $\mathcal{D}_X$ over an open set $U \subset X$. Since continuity is a local property, we can suppose that U is a connected open set of $\mathbb{C}^n$. Then P admits a unique expression

$$P = \sum_{\alpha \in \mathbb{N}^n, |\alpha| \leqslant d} a_\alpha \frac{1}{\alpha!} \partial^\alpha,$$

where d is the order of P and the a_α are holomorphic functions on U. Let $K, K' \subseteq U$ be a pair of compact sets as in proposition 1.2, b) and let f be a holomorphic function on a neighborhood of K'. From Cauchy inequalities we deduce that

$$|P(f)|_K = \Big|\sum_{|\alpha|\leqslant d} a_\alpha \frac{1}{\alpha!}\partial^\alpha(f)\Big|_K \leqslant \sum_{|\alpha|\leqslant d} |a_\alpha|_K r^{-|\alpha|}|f|_{K'}$$

where r is the distance between K and $U - \overset{\circ}{K'}$. By proposition 1.2, we conclude that P is continuous. □

***Definition 1.4* ([14], déf. 2.1.6).** — For any open set $U \subseteq X$, the *canonical topology* of $\mathcal{D}_X^\infty(U)$ or $\mathcal{D}_X(U)$ is defined as the locally convex topology given by the semi-norms

$$p_{(K,K')} : P \in \mathcal{D}_X^\infty(U) \longmapsto p_{(K,K')}(P) := \sup\{|P(f)|_K/|f|_{K'} \mid f \in \mathcal{O}_X(K'), f \neq 0\},$$

indexed by pairs (K, K') of compact sets in U with $K \subset \overset{\circ}{K'}$.

For any coordinate system $(U; x_1, \ldots, x_n)$ in X, we can use Cauchy inequalities as in corollary 1.3 and proposition 1.1 to prove that the map

$$\sum_{\alpha\in\mathbb{N}^n} a_\alpha(\underline{x})\frac{1}{\alpha!}\partial^\alpha \longmapsto \sum_{\alpha\in\mathbb{N}^n} a_\alpha(\underline{x})\underline{y}^\alpha$$

is an isomorphism of locally convex vector spaces between $\mathcal{D}_X^\infty(U)$ endowed with the canonical topology and the space of holomorphic functions on $U \times \mathbb{C}^n$ endowed with the topology of uniform convergence on compact sets. This isomorphism depends on local coordinates and carries the space $\mathcal{D}_X(U)$ into the space of holomorphic functions on $U \times \mathbb{C}^n$ which are polynomials with respect to the second factor. Consequently, $\mathcal{D}_X^\infty(U)$ is a Fréchet (and nuclear) space and $\mathcal{D}_X(U)$ is dense in $\mathcal{D}_X^\infty(U)$. We can write then $\mathcal{D}_X^\infty(U) = \widehat{\mathcal{D}_X(U)}$.

In fact, in [**14**, §2] it is proved that $\mathcal{D}_X^\infty$ endowed with the canonical topology is a sheaf with values in the category of Fréchet $\mathbb{C}$-algebras.

Let us denote by $\mathcal{O}_n, \mathcal{D}_n, \mathcal{D}_n^\infty$ the stalk at the origin of the sheaves $\mathcal{O}_{\mathbb{C}^n}, \mathcal{D}_{\mathbb{C}^n}, \mathcal{D}_{\mathbb{C}^n}^\infty$ respectively. For $\rho = (\rho_1, \ldots, \rho_n)$, $L = (L_1, \ldots, L_n)$ in $(\mathbb{R}_+^*)^n$ let us consider the pseudo-norm $|-|_\rho^L : \mathcal{D}_n^\infty \to \mathbb{R}_+ \cup \{+\infty\}$ whose value at $P = \sum a_\beta \partial^\beta = \sum_{\alpha\beta} a_{\alpha\beta}x^\alpha\partial^\beta$ is

$$|P|_\rho^L = \sum_\beta |a_\beta|_\rho |\beta|! L^\beta = \sum_{\alpha\beta} |a_{\alpha\beta}| \cdot |\beta|! \rho^\alpha L^\beta \in \mathbb{R}_+ \cup \{+\infty\}. \tag{1}$$

Since $\beta! \leqslant |\beta|! \leqslant n^{|\beta|}\beta!$, we could also use $\beta!$ instead $|\beta|!$ in (1) to obtain an equivalent system of pseudo-norms. Nevertheless, the choice of $|\beta|!$ is forced by the proofs of the majorations needed to obtain the norm estimates of theorem 2.11 (see [**14**, 2.2.4] and [**8**]).

Let us denote by $\mathcal{D}_n^\infty(\rho)$ the subspace of $\mathcal{D}_n^\infty$ where $|-|_\rho^L$ takes finite values for any $L \in (\mathbb{R}_+^*)^n$ and let us write $\mathcal{D}_n(\rho) := \mathcal{D}_n \cap \mathcal{D}_n^\infty(\rho)$. The semi-norms $|-|_\rho^L$, $L \in (\mathbb{R}_+^*)^n$, define a Fréchet topology on $\mathcal{D}_n^\infty(\rho)$.

Following [**8**], we consider weights $\lambda, \mu \in (\mathbb{N}^*)^n$ and, for real numbers $s, t > 0$, $\rho = s^\lambda = (s^{\lambda_1}, \ldots, s^{\lambda_n})$, $L = t^{-\mu} = (t^{-\mu_1}, \ldots, t^{-\mu_n})$. When λ is fixed, we denote $|-|_s^{\mu,t} := |-|_\rho^L$, $\mathcal{D}_n^\infty(s) := \mathcal{D}_n^\infty(\rho)$ and $\mathcal{D}_n(s) := \mathcal{D}_n(\rho)$.

In the case where U is an open polycylinder of $\mathbb{C}^n$ centered at 0 of polyradius $\sigma = s_0^\lambda$, $0 < s_0 \leqslant +\infty$, we have

$$\mathcal{D}_{\mathbb{C}^n}^\infty(U) = \bigcap_{0<s<s_0} \mathcal{D}_n^\infty(s), \quad \mathcal{D}_{\mathbb{C}^n}(U) = \bigcap_{0<s<s_0} \mathcal{D}_n(s),$$

and the canonical topology of $\mathcal{D}_{\mathbb{C}^n}^\infty(U)$ (resp. $\mathcal{D}_{\mathbb{C}^n}(U)$) is the (topological) inverse limit of the $\mathcal{D}_n^\infty(s)$ (resp. $\mathcal{D}_n(s)$), for $0 < s < s_0$. In other words, the canonical topologies of $\mathcal{D}_{\mathbb{C}^n}^\infty(U)$ and $\mathcal{D}_{\mathbb{C}^n}(U)$ are given by the semi-norms $|-|_s^{\mu,t}$, $0 < s < s_0$, $t^{-\mu} \gg 0$ [**14**], 2.2.3. The last condition can be obtained with μ fixed and $t \to 0$, or taking $t = t(s) < 1$ and $\mu \gg 0$.

For vectors $P = (P_1, \ldots, P_q) \in (\mathcal{D}_n^\infty)^q$, following [**7**] we also define

$$|P|_s^{\mu,t} := \sum_{i=1}^q |P_i|_s^{\mu,t} s^{-(i-1)},$$

where $\lambda \in (\mathbb{N}^*)^n$ is fixed.

In the above situation, the product topology on $\mathcal{D}_{\mathbb{C}^n}^\infty(U)^q$ and $\mathcal{D}_{\mathbb{C}^n}(U)^q$ is also given by the semi-norms $|-|_s^{\mu,t}$, $0 < s < s_0$, $t^{-\mu} \gg 0$.

2. The continuity theorem

In this section, we fix $M_1, \ldots, M_r \in \mathcal{D}_n^q$ and a total well ordering $<$ in $\mathbb{N}^{2n}$ compatible with sums (*cf.* [**4**, 1.3]). Whenever we speak about the ordering $<$ in $\mathbb{N}^{2n} \times \{1, \ldots, q\}$ we mean the ordering induced by $<$ in the following way:

$$(\alpha, \beta, i) < (\alpha', \beta', j) \iff \begin{cases} (\alpha, \beta) < (\alpha', \beta') \\ \text{or} \\ (\alpha, \beta) = (\alpha', \beta') \text{ and } i > j \end{cases}$$

Given

$$N = (N_1, \ldots, N_q) = \sum_{i=1}^q N_i e_i = \sum_{i=1}^q \sum_{\alpha,\beta} a_{\alpha\beta i} x^\alpha \partial^\beta e_i \in \mathcal{D}_n^q, \quad a_{\alpha\beta i} \in \mathbb{C},$$

where $\{e_i\}_{i=1\ldots q}$ stands for the canonical basis of $\mathcal{D}_n^q$ as a free $\mathcal{D}_n$-module, we denote by $\mathscr{N}(N)$, the *Newton diagram* of N, the set of (α, β, i) in $\mathbb{N}^{2n} \times \{1, \ldots, q\}$ such that $a_{\alpha\beta i} \neq 0$ and by $\sigma(N)$ its *symbol*, *i.e.* the homogeneous component of N of maximal degree with respect to the grading given by the total degree in ∂:

$$\sigma(N) = \sum_{i=1}^q \sum_{|\beta|=d} \sum_\alpha a_{\alpha\beta i} x^\alpha \partial^\beta e_i, \quad d = \deg_{\mathrm{T}}(N) = \max \deg(N_i).$$

The *exponent* of N is $\exp(N) := \min\{(\alpha,\beta,i) | a_{\alpha\beta i} \neq 0, |\beta| = d\}$, and the corresponding monomial of $\sigma(N)$ (and therefore of N) is, by definition, the *initial monomial* of N.

Let (α_j, β_j, i_j) be the exponent of M_j with respect to the given ordering (we can assume without loss of generality that its coefficient is 1). Also let $M'_j = M_j - x^{\alpha_j}\partial^{\beta_j} e_{i_j}$. We will denote by $\mathscr{F}$ the r-tuple $(M_1, \ldots, M_r)$.

The following notion is needed in the continuity theorem 2.6:

Definition 2.1. — We say that a given weight $\lambda \in (\mathbb{N}^*)^n$ is *adapted* to $\mathscr{F}$ if for every positive constant K there exists $\mu \in \mathbb{N}^n$ with $\mu_i > K, \lambda_i$ for every $i = 1, \ldots, n$ such that

$$\lambda\alpha_j - \mu\beta_j - i_j < \lambda\alpha - \mu\beta - i$$

for every $j = 1, \ldots, r$ and every $(\alpha, \beta, i) \in \mathscr{N}(M'_j)$. We say that such a μ is K-*admissible*, or simply *admissible*, for $(\mathscr{F}, \lambda)$.

Lemma 2.2. — *For any $\mathscr{F} = (M_1, \ldots, M_r)$ as above, there exists a weight λ adapted to $\mathscr{F}$.*

Proof. — Consider first the case $q = 1$. Let $\pi_1, \pi_2 : \mathbb{N}^{2n} = \mathbb{N}^n \times \mathbb{N}^n \to \mathbb{N}^n$ be the canonical projections. For every $j = 1, \ldots, r$ and every $\beta \in \pi_2(\mathscr{N}(M_j))$ let $\mathscr{M}^j_\beta$ be the set of (α, β) in $\mathscr{N}(M_j)$ such that α is minimal in $\mathscr{A} = \{\alpha : (\alpha, \beta) \in \mathscr{N}(M_j)\}$ with respect to the componentwise order. The set $\mathscr{M}^j_\beta$ is finite, in fact it consists of the elements $(\alpha_1, \beta), \ldots, (\alpha_s, \beta)$, where $\{\alpha_1, \ldots, \alpha_s\}$ is the minimal set of generators of the ideal $\mathscr{A} + \mathbb{N}^n$ of $\mathbb{N}^n$. Therefore, the set $\mathscr{M} = \bigcup_j \bigcup_\beta (\mathscr{M}^j_\beta \cup \{(0, \beta)\})$ is also finite.

Let $(\sigma, \rho) \in \mathbb{N}^n \times \mathbb{N}^n = \mathbb{N}^{2n}$ be a vector defining the given ordering restricted to the finite set $\mathscr{M}$. We claim that $\lambda = \sigma$ is adapted to $\mathscr{F}$. Fix a positive constant K, and let p be an integer such that $p > \max\{K + |\rho|, \sigma\alpha + \rho\beta : (\alpha, \beta) \in \mathscr{M}\}$. Set $\mu = (p, \ldots, p) - \rho$. We have then

$$\lambda\alpha - \mu\beta = \sigma\alpha + \rho\beta - p|\beta|.$$

We will show that the minimum of $\lambda\alpha - \mu\beta$ for $(\alpha, \beta) \in \mathscr{N}(M_j)$ is attained in the exponent of M_j. First, we see that if $\lambda\alpha - \mu\beta$ is minimal, then $(\alpha, \beta) \in \mathscr{M}$. Otherwise, there would be $(\alpha', \beta) \in \mathscr{N}(M_j), \gamma \in \mathbb{N}^n \backslash \{0\}$ such that $\alpha = \alpha' + \gamma$, so $\lambda\alpha - \mu\beta = \sigma\gamma + (\lambda\alpha' - \mu\beta) > \lambda\alpha' - \mu\beta$.

Furthermore, (α, β) must be in $\mathscr{N}(\sigma(M_j))$. Otherwise, there would be $(\alpha', \beta') \in \mathscr{N}(M_j) \cap \mathscr{M}$ with $|\beta'| > |\beta|$. Then $\lambda\alpha - \mu\beta = \sigma\alpha + \rho\beta - p|\beta| \geqslant \sigma\alpha + \rho\beta - p(|\beta'| - 1) > p - p|\beta'| > \sigma\alpha' + \rho\beta' - p|\beta'| = \lambda\alpha' - \mu\beta'$. Therefore, $\min\{\lambda\alpha - \mu\beta : (\alpha, \beta) \in \mathscr{N}(M_j)\} = \min\{\sigma\alpha + \rho\beta - p|\beta| : (\alpha, \beta) \in \mathscr{N}(\sigma(M_j)) \cap \mathscr{M}\}$. In this set, $|\beta|$ is constant and the ordering is defined by (σ, ρ), so the minimum is attained in the smallest element of $\mathscr{N}(\sigma(M_j))$ with respect to the ordering, i.e the initial monomial of M_j.

Now assume $q \neq 1$. Let $M_j = \sum_{i\alpha\beta} a^j_{i\alpha\beta} x^\alpha \partial^\beta e_i$, and define $\overline{M}_j = \sum_{i\alpha\beta} |a^j_{i\alpha\beta}| x^\alpha \partial^\beta \in \mathcal{D}_n$. Let $(\overline{\alpha}_j, \overline{\beta}_j)$ be the exponent of $\overline{M}_j$ with respect to the given ordering in $\mathbb{N}^{2n}$. Let (α_j, β_j, i_j) be the exponent of M_j. We have $\alpha_j = \overline{\alpha}_j$ and $\beta_j = \overline{\beta}_j$. Otherwise, we would have $(\alpha_j, \beta_j) > (\overline{\alpha}_j, \overline{\beta}_j)$. Let i be such that $(\overline{\alpha}_j, \overline{\beta}_j, i) \in \mathscr{N}(M_j)$. Then we have by definition of the exponent that $(\overline{\alpha}_j, \overline{\beta}_j, i) \geqslant (\alpha_j, \beta_j, i_j)$, and therefore $(\overline{\alpha}_j, \overline{\beta}_j) \geqslant (\alpha_j, \beta_j)$, which is in contradiction with the last inequality.

By the first part of the proof, there exists $\overline{\lambda}$ adapted to $(\overline{M}_1, \dots, \overline{M}_r)$. Now given a positive constant K there is $\overline{\mu} \in \mathbb{N}^n$ with $\overline{\mu}_j > K/q$ and $\overline{\lambda}_j < \overline{\mu}_j$ such that $\overline{\lambda}\alpha_j - \overline{\mu}\beta_j < \overline{\lambda}\alpha - \overline{\mu}\beta$ for every $(\alpha, \beta) \in \mathscr{N}(\overline{M}'_j)$. Let us see that $\lambda = q\overline{\lambda}$ is adapted to $\mathscr{F}$ and $\mu = q\overline{\mu}$ is K-admissible for $(\mathscr{F}, \lambda)$. Let $(\alpha, \beta, i) \in \mathscr{N}(M'_j)$, then $(\alpha, \beta) \in \mathscr{N}(\overline{M}_j)$, hence $\overline{\lambda}\alpha_j - \overline{\mu}\beta_j \leqslant \overline{\lambda}\alpha - \overline{\mu}\beta$ by construction. Now we distinguish two cases:

If $(\alpha, \beta) > (\alpha_j, \beta_j)$, then $\lambda\alpha_j - \mu\beta_j < \lambda\alpha - \mu\beta$. But λ as well as μ are multiples of q, and therefore $\lambda\alpha_j - \mu\beta_j \leqslant \lambda\alpha - \mu\beta - q$, and $\lambda\alpha_j - \mu\beta_j - i_j < \lambda\alpha_j - \mu\beta_j \leqslant \lambda\alpha - \mu\beta - q \leqslant \lambda\alpha - \mu\beta - i$.

If $(\alpha, \beta) = (\alpha_j, \beta_j)$, then we must have $i < i_j$, hence $\lambda\alpha_j - \mu\beta_j - i_j < \lambda\alpha_j - \mu\beta_j - i = \lambda\alpha - \mu\beta - i$. In either case, we get the desired inequality.

This completes the proof of the lemma. □

Lemma 2.3. — *Let $\mathscr{F}_1, \dots, \mathscr{F}_m$ be a finite number of vectors whose coordinates are in $\mathcal{D}^q_n$ as above (they may have distinct lengths). Then there exists $\lambda \in \mathbb{N}^n$ which is adapted to all of them.*

Proof. — This is a direct consequence of the following lemma applied to the vector constructed by concatenation of $\mathscr{F}_1, \dots, \mathscr{F}_m$. □

The following lemma is clear:

Lemma 2.4. — *Let $\mathscr{F} = (M_1, \dots, M_r)$ be a vector in $(\mathcal{D}^q_n)^r$ and let $\mathscr{G} = (M_{i_1}, \dots, M_{i_k})$, with $1 \leqslant i_1 < \dots < i_k \leqslant r$. Then every $\lambda \in \mathbb{N}^n$ adapted to $\mathscr{F}$ is also adapted to $\mathscr{G}$.*

Before stating the main theorem of this section we make one further definition:

Definition 2.5. — Let $\lambda \in (\mathbb{N}^*)^n$ be a weight. A basis $\mathscr{B}$ of open neighborhoods of $0 \in \mathbb{C}^n$ is said to be a λ-*basis* if it consists of open polycylinders of polyradius s^λ for $0 < s < s_0$, for some $s_0 > 0$. We will say that $\mathscr{B}$ is adapted to $\mathscr{F}$ if it is a λ-basis for some λ adapted to $\mathscr{F}$.

From the lemmas above it follows that we can always find a basis of neighborhoods of 0 adapted to $\mathscr{F}$, and even a basis adapted to a finite number of vectors $\mathscr{F}_1, \dots, \mathscr{F}_m$.

After these preliminaries we are ready to state the continuity theorem of the division of linear differential operators:

Theorem 2.6. — *Let $\mathscr{F} = (M_1, \dots, M_r)$, with $M_i \in \mathcal{D}^q_n$ and let $Q_i(\mathscr{F}; E)$, $i = 1, \dots, r$ (resp. $R(\mathscr{F}; E)$) be the quotients (resp. the remainder) of the division of $E \in \mathcal{D}^q_n$ by $\mathscr{F}$*

*(see [**4**]). Then, for any weight $\lambda \in (\mathbb{N}^*)^n$ adapted to $\mathscr{F}$, there exists a λ-basis $\mathscr{B}$ of open neighborhoods of $0 \in \mathbb{C}^n$, such that for every $U \in \mathscr{B}$ the $\mathbb{C}$-linear morphisms $Q_i(\mathscr{F}; -)$ (resp. $R(\mathscr{F}; -)$) map $\mathcal{D}_{\mathbb{C}^n}(U)^q$ into $\mathcal{D}_{\mathbb{C}^n}(U)$ (resp. into $\mathcal{D}_{\mathbb{C}^n}(U)^q$). Furthermore,*

$$Q_i(\mathscr{F}; -) \colon \mathcal{D}_{\mathbb{C}^n}(U)^q \longrightarrow \mathcal{D}_{\mathbb{C}^n}(U),$$
$$R(\mathscr{F}; -) \colon \mathcal{D}_{\mathbb{C}^n}(U)^q \longrightarrow \mathcal{D}_{\mathbb{C}^n}(U)^q$$

are continuous with respect to the canonical topology.

The proof of theorem 2.6 will be obtained after some majorations, as in [**14**], [**8**], and it will not be finished until the end of 2.13. Our task consists of adapting the proof in [**8**] to the vector case. Roughly speaking, as explained in loc. cit., the key point is to approximate the $\mathcal{D}_n$-linear map $\mathcal{D}_n^r \to \mathcal{D}_n^q$ defined by the finite system of vectors $\mathscr{F} = (M_1, \ldots, M_r) \in (\mathcal{D}_n^q)^r$ instead of approximating the system itself by their initial monomials. This idea has been introduced in [**7**] in the commutative case of vectors of convergent power series.

Let (α_j, β_j, i_j) be the exponent of M_j. For every $i = 1, \ldots, n$, let $T_j : \mathcal{D}_n^q \to \mathcal{D}_n^q$ be the $\mathbb{C}$-linear map defined by $T_j x^\alpha \partial^\beta e_i = x^\alpha \partial^{\beta + e_j} e_i$. Given $A = \sum c_{\gamma\delta} x^\gamma \partial^\delta \in \mathcal{D}_n$, we will denote by A^o the map $\sum c_{\gamma\delta} x^\gamma T^\delta : \mathcal{D}_n \to \mathcal{D}_n$, and $A' = A - A^o$ (A is considered here to be acting by multiplication on the left). Let also $\{\Delta_1, \ldots, \Delta_r, \overline{\Delta}\}$ be the partition of $\mathbb{N}^{2n} \times \{1, \ldots, q\}$ defined by $M_1, \ldots, M_r$ (see [**3**] and [**4**] in this volume).

We define now the following sets L and J:

$$L = \{A \in \mathcal{D}_n^r : \exp(M_j) + \mathscr{N}(A_j) \subset \Delta_j,\ \forall j = 1, \ldots, r\}$$
$$J = \{B \in \mathcal{D}_n^q : \mathscr{N}(B) \subset \overline{\Delta}\}$$

and the linear map $u : L \oplus J \to \mathcal{D}_n^q$ given by $u(A, B) = \sum_{j=1}^r A_j M_j + B$.

From the division theorem ([**4**], th. 2.4.1) we see that L and J are the sets where quotients and the remainder of the division by $M_1, \ldots, M_r$ are “allowed” to lie, the A_j and B are just the quotients and the remainder of the division of $u(A, B)$ by $\mathscr{F}$ and the map u is bijective.

We start by splitting u as a sum $v + w_1 + w_2$, with

$$v(A, B) = \sum A_j^o x^{\alpha_j} \partial^{\beta_j} e_{i_j} + B$$
$$w_1(A, B) = \sum A_j' x^{\alpha_j} \partial^{\beta_j} e_{i_j}$$
$$w_2(A, B) = \sum A_j M_j'.$$

The $\mathbb{C}$-linear map v is easily seen to be an isomorphism of $\mathbb{C}$-vector spaces, by definition of L and J.

We follow the notation in the previous section regarding the seminorms $| - |_s^{\mu,t}$. Let $E \in \mathcal{D}_n^q$, and $(A, B) = v^{-1}(E) \in L \oplus J$, with $A_j = \sum_{\gamma\delta} a_{\gamma\delta}^j x^\gamma \partial^\delta$. Then, $E = \sum_{j\gamma\delta} a_{\gamma\delta}^j x^{\alpha_j + \gamma} \partial^{\beta_j + \delta} e_{i_j} + B$. If we take the $| - |_s^{\mu,t}$ norm on both sides, and keep in mind that $(\alpha_j,\beta_j,i_j)+\mathscr{N}(A_j)\subset\Delta_j$, $\mathscr{N}(B)\subset\overline{\Delta}$ and that the sets $\Delta_j,\overline{\Delta}$ are pairwise disjoint, we get:

$$(2)\quad |E|_s^{\mu,t}=\sum\nolimits_j\Big|\sum\nolimits_{\gamma\delta}a^j_{\gamma\delta}x^{\alpha_j+\gamma}\partial^{\beta_j+\delta}e_{i_j}\Big|_s^{\mu,t}+\big|B\big|_s^{\mu,t}$$
$$\geqslant\sum\nolimits_{j\gamma\delta}\big|a^j_{\gamma\delta}\big|\big|\beta_j+\delta\big|!s^{\lambda(\alpha_j+\gamma)-(i_j-1)}t^{-\mu(\beta_j+\delta)}.$$

Proposition 2.7. — *There is a constant $C_1>0$ such that $|(w_1v^{-1})E|_s^{\mu,s}\leqslant C_1 s|E|_s^{\mu,s}$ for every $E\in\mathcal{D}_n^q$ and for every μ admissible for $(\mathscr{F},\lambda)$.*

Proof. — Let $(A,B)=v^{-1}(E)$, with $A_j=\sum_{\gamma\delta}a^j_{\gamma\delta}x^\gamma\partial^\delta$. First, we have

$$|(w_1v^{-1})E|_s^{\mu,s}=|w_1(A,B)|_s^{\mu,s}=\Big|\sum_j A'_jx^{\alpha_j}\partial^{\beta_j}e_{i_j}\Big|_s^{\mu,s}$$
$$=\Big|\sum_{j\gamma\delta}a^j_{\gamma\delta}x^\gamma(\partial^\delta-T^\delta)x^{\alpha_j}\partial^{\beta_j}e_{i_j}\Big|_s^{\mu,s}.$$

Expanding the inner product, we get

$$|(w_1v^{-1})E|_s^{\mu,s}=\Big|\sum_{j\gamma\delta}a^j_{\gamma\delta}x^\gamma\sum_{0<\varepsilon\leqslant\alpha_j,\delta}\binom{\alpha_j}{\varepsilon}\frac{\delta!}{(\delta-\varepsilon)!}x^{\alpha_j-\varepsilon}\partial^{\beta_j+\delta-\varepsilon}e_{i_j}\Big|_s^{\mu,s}$$
$$\leqslant\sum_{j\gamma\delta}\sum_{0<\varepsilon\leqslant\alpha_j,\delta}|a^j_{\gamma\delta}|\binom{\alpha_j}{\varepsilon}\frac{\delta!}{(\delta-\varepsilon)!}|\beta_j+\delta-\varepsilon|!s^{\lambda(\alpha_j+\gamma-\varepsilon)-(i_j-1)-\mu(\beta_j+\delta-\varepsilon)}$$
$$\leqslant\sum_j\sum_{0<\varepsilon\leqslant\alpha_j}\sum_{\delta\geqslant\varepsilon}\sum_\gamma|a^j_{\gamma\delta}|2^{|\alpha_j|}\frac{\delta!}{(\delta-\varepsilon)!}|\beta_j+\delta-\varepsilon|!s^{\lambda(\alpha_j+\gamma-\varepsilon)-(i_j-1)-\mu(\beta_j+\delta-\varepsilon)}.$$

Therefore

$$\frac{|(w_1v^{-1})E|_s^{\mu,s}}{|E|_s^{\mu,s}}$$
$$\leqslant\frac{\sum_j\sum_{0<\varepsilon\leqslant\alpha_j}\sum_{\delta\geqslant\varepsilon}\sum_\gamma|a^j_{\gamma\delta}|2^{|\alpha_j|}\dfrac{\delta!}{(\delta-\varepsilon)!}|\beta_j+\delta-\varepsilon|!s^{\lambda(\alpha_j+\gamma-\varepsilon)-(i_j-1)-\mu(\beta_j+\delta-\varepsilon)}}{\sum_{j\gamma\delta}|a^j_{\gamma\delta}||\beta_j+\delta|!s^{\lambda(\alpha_j+\gamma)-(i_j-1)-\mu(\beta_j+\delta)}}$$
$$\leqslant\sum_{j\varepsilon}\frac{\sum_{\delta\geqslant\varepsilon}\sum_\gamma|a^j_{\gamma\delta}|2^{|\alpha_j|}\dfrac{\delta!}{(\delta-\varepsilon)!}|\beta_j+\delta-\varepsilon|!s^{\lambda(\alpha_j+\gamma-\varepsilon)-(i_j-1)-\mu(\beta_j+\delta-\varepsilon)}}{\sum_{\gamma\delta}|a^j_{\gamma\delta}||\beta_j+\delta|!s^{\lambda(\alpha_j+\gamma)-(i_j-1)-\mu(\beta_j+\delta)}}$$
$$=\sum_{j\varepsilon}\frac{\sum_{\delta\geqslant\varepsilon}\sum_\gamma|a^j_{\gamma\delta}|2^{|\alpha_j|}\dfrac{\delta!}{(\delta-\varepsilon)!}|\beta_j+\delta-\varepsilon|!s^{\lambda(\alpha_j+\gamma-\varepsilon)-\mu(\beta_j+\delta-\varepsilon)}}{\sum_{\gamma\delta}|a^j_{\gamma\delta}||\beta_j+\delta|!s^{\lambda(\alpha_j+\gamma)-\mu(\beta_j+\delta)}}$$
$$\leqslant\sum_{j\varepsilon}\frac{\sum_{\gamma\delta}|a^j_{\gamma\delta}|2^{|\alpha_j|}\dfrac{\delta!}{(\delta-\varepsilon)!}|\beta_j+\delta-\varepsilon|!s^{\lambda(\alpha_j+\gamma-\varepsilon)-\mu(\beta_j+\delta-\varepsilon)}}{\sum_{\gamma\delta}|a^j_{\gamma\delta}||\beta_j+\delta|!s^{\lambda(\alpha_j+\gamma)-\mu(\beta_j+\delta)}}.$$

Now, using that

$$\frac{\delta!}{(\delta-\varepsilon)!} \leqslant \frac{|\beta_i+\delta|!}{|\beta_i+\delta-\varepsilon|!} \qquad \text{for} \quad \beta_i \geqslant 0,$$

we get

$$\begin{aligned}\frac{|(w_1v^{-1})E|_s^{\mu,s}}{|E|_s^{\mu,s}} &\leqslant \sum_{j\varepsilon} \frac{\sum_{\gamma\delta}|a^j_{\gamma\delta}|2^{|\alpha_j|}|\beta_j+\delta|!s^{\lambda(\alpha_j+\gamma-\varepsilon)-\mu(\beta_j+\delta-\varepsilon)}}{\sum_{\gamma\delta}|a^j_{\gamma\delta}||\beta_j+\delta|!s^{\lambda(\alpha_j+\gamma)-\mu(\beta_j+\delta)}}\\ &= \sum_{j\varepsilon} 2^{|\alpha_j|}s^{(\mu-\lambda)\varepsilon}.\end{aligned}$$

For each j, there is only a finite number of $\varepsilon \leqslant \alpha_j$, so this last sum is finite. Moreover, since $\varepsilon > 0$ and $\mu-\lambda$ has positive components, the exponent of s in every term of the sum is greater that 1. Assuming $s<1$, we can therefore bound this sum by $C_1 s$, with $C_1 = \sum_i 2^{|\alpha_i|}\#\{\varepsilon\in\mathbb{N}^n : 0<\varepsilon\leqslant\alpha_i\}$. We end up with the following bound

$$|(w_1v^{-1})E|_s^{\mu,s} \leqslant C_1 s|E|_s^{\mu,s} \tag{3}$$

for $s<1$. □

Proposition 2.8. — *There is a constant $C_2 > 0$, and an $s_0 > 0$ such that $|(w_2v^{-1})E|_s^{\mu,s} \leqslant C_2 s|E|_s^{\mu,s}$ for $0<s<s_0$ and for every $E\in\mathcal{D}_n^q$ and every μ admissible for $(\mathscr{F},\lambda)$.*

Proof. — We have

$$\begin{aligned}|(w_2v^{-1})E|_s^{\mu,s} &= |w_2(A,B)|_s^{\mu,s} = \Big|\sum_j A_jM'_j\Big|_s^{\mu,s}\\ &= \Big|\sum_j(\sum_{\gamma\delta} a^j_{\gamma\delta}x^\gamma\partial^\delta)(\sum_{\alpha\beta i} c^j_{\alpha\beta i}x^\alpha\partial^\beta e_i)\Big|_s^{\mu,s} = \Big|\sum_{j\alpha\beta\gamma\delta i} a^j_{\gamma\delta}c^j_{\alpha\beta i}x^\gamma\partial^\delta x^\alpha\partial^\beta e_i\Big|_s^{\mu,s}\\ &= \Big|\sum_{j\alpha\beta\gamma\delta i} a^j_{\gamma\delta}c^j_{\alpha\beta i}x^\gamma(\sum_{\varepsilon\leqslant\alpha,\delta}\binom{\alpha}{\varepsilon}\frac{\delta!}{(\delta-\varepsilon)!}x^{\alpha-\varepsilon}\partial^{\delta-\varepsilon})\partial^\beta e_i\Big|_s^{\mu,s}\\ &= \Big|\sum_{j\alpha\beta\gamma\delta\varepsilon i} a^j_{\gamma\delta}c^j_{\alpha\beta i}\binom{\alpha}{\varepsilon}\frac{\delta!}{(\delta-\varepsilon)!}x^{\gamma+\alpha-\varepsilon}\partial^{\beta+\delta-\varepsilon}e_i\Big|_s^{\mu,s}\\ &\leqslant \sum_{j\alpha\beta\gamma\delta\varepsilon i}|a^j_{\gamma\delta}||c^j_{\alpha\beta i}|\binom{\alpha}{\varepsilon}\frac{\delta!}{(\delta-\varepsilon)!}|\beta+\delta-\varepsilon|!s^{\lambda(\gamma+\alpha-\varepsilon)-(i-1)-\mu(\beta+\delta-\varepsilon)}\\ &= \sum_{j\alpha\beta\varepsilon i}|c^j_{\alpha\beta i}|\binom{\alpha}{\varepsilon}\sum_{\gamma\delta}|a^j_{\gamma\delta}|\frac{\delta!}{(\delta-\varepsilon)!}|\beta+\delta-\varepsilon|!s^{\lambda(\alpha+\gamma-\varepsilon)-(i-1)-\mu(\beta+\delta-\varepsilon)}\end{aligned}$$

with ε and δ satisfying $0 \leqslant \varepsilon \leqslant \alpha$ and $\varepsilon \leqslant \delta$. Now, using (2), we get

$$\begin{aligned}
&\frac{|(w_2v^{-1})E|_s^{\mu,s}}{|E|_s^{\mu,s}}\\
&\quad\leqslant \frac{\sum_{j\alpha\beta\varepsilon i}|c^j_{\alpha\beta i}|\binom{\alpha}{\varepsilon}\sum_{\gamma\delta\geqslant\varepsilon}|a^j_{\gamma\delta}|\dfrac{\delta!}{(\delta-\varepsilon)!}|\beta+\delta-\varepsilon|!s^{\lambda(\alpha+\gamma-\varepsilon)-(i-1)-\mu(\beta+\delta-\varepsilon)}}{\sum_{j\gamma\delta}|a^j_{\gamma\delta}||\beta_j+\delta|!s^{\lambda(\alpha_j+\gamma)-(i_j-1)-\mu(\beta_j+\delta)}}\\
&\quad\leqslant \sum_{\alpha\beta\varepsilon i}\frac{\sum_j|c^j_{\alpha\beta i}|\binom{\alpha}{\varepsilon}\sum_{\gamma\delta}|a^j_{\gamma\delta}|\dfrac{\delta!}{(\delta-\varepsilon)!}|\beta+\delta-\varepsilon|!s^{\lambda(\alpha+\gamma-\varepsilon)-(i-1)-\mu(\beta+\delta-\varepsilon)}}{\sum_{j\gamma\delta}|a^j_{\gamma\delta}||\beta_j+\delta|!s^{\lambda(\alpha_j+\gamma)-(i_j-1)-\mu(\beta_j+\delta)}}\\
&\quad\leqslant \sum_{j\alpha\beta\varepsilon i}|c^j_{\alpha\beta i}|\binom{\alpha}{\varepsilon}\frac{\sum_{\gamma\delta}|a^j_{\gamma\delta}|\dfrac{\delta!}{(\delta-\varepsilon)!}|\beta+\delta-\varepsilon|!s^{\lambda(\alpha+\gamma-\varepsilon)-(i-1)-\mu(\beta+\delta-\varepsilon)}}{\sum_{\gamma\delta}|a^j_{\gamma\delta}||\beta_j+\delta|!s^{\lambda(\alpha_j+\gamma)-(i_j-1)-\mu(\beta_j+\delta)}}.
\end{aligned}$$

We use the fact that $\frac{\delta!}{(\delta-\varepsilon)!} \leqslant \frac{|\delta|!}{|\delta-\varepsilon|!} \leqslant \frac{|\beta_j+\delta|!}{|\beta+\delta-\varepsilon|!}$ and therefore

$$\begin{aligned}
\frac{|(w_2v^{-1})E|_s^{\mu,s}}{|E|_s^{\mu,s}} &\leqslant \sum_{j\alpha\beta\varepsilon i}|c^j_{\alpha\beta i}|\binom{\alpha}{\varepsilon}\frac{\sum_{\gamma\delta}|a^j_{\gamma\delta}||\beta_j+\delta|!s^{\lambda(\alpha+\gamma-\varepsilon)-(i-1)-\mu(\beta+\delta-\varepsilon)}}{\sum_{\gamma\delta}|a^j_{\gamma\delta}||\beta_j+\delta|!s^{\lambda(\alpha_j+\gamma)-(i_j-1)-\mu(\beta_j+\delta)}}\\
&= \sum_{j\alpha\beta\varepsilon i}|c^j_{\alpha\beta i}|\binom{\alpha}{\varepsilon}s^{\lambda(\alpha-\alpha_j-\varepsilon)-(i-i_j)-\mu(\beta-\beta_j-\varepsilon)}\\
&\leqslant \sum_{j\alpha\beta\varepsilon i}|c^j_{\alpha\beta i}|2^{|\alpha|}s^{\lambda(\alpha-\alpha_j-\varepsilon)-(i-i_j)-\mu(\beta-\beta_j-\varepsilon)}.
\end{aligned}$$

For every β appearing in this sum as part of the exponent of a monomial in M'_j, let $(\alpha_{j\beta},\beta,i_{j\beta}) \in \mathscr{N}(M'_j)$ be such that $\lambda\alpha_{j\beta} = \min\{\lambda\alpha : (\alpha,\beta,i)\in\mathscr{N}(M'_j)\}$. Then

$$\begin{aligned}
&\sum_{j\alpha\beta\varepsilon i}|c^j_{\alpha\beta i}|2^{|\alpha|}s^{\lambda(\alpha-\alpha_j-\varepsilon)-(i-i_j)-\mu(\beta-\beta_j-\varepsilon)}\\
&\qquad\qquad= \sum_{j\alpha\beta\varepsilon i}|c^j_{\alpha\beta i}|2^{|\alpha|}s^{(\mu-\lambda)\varepsilon}s^{\lambda(\alpha-\alpha_{j\beta})}s^{(\lambda\alpha_{j\beta}-\mu\beta-i)-(\lambda\alpha_j-\mu\beta_j-i_j)}
\end{aligned}$$

and the exponents $(\lambda\alpha_{j\beta}-\mu\beta-i)-(\lambda\alpha_j-\mu\beta_j-i_j)$ are integers greater or equal than one. Therefore for $s<1$ the last sum is bounded by

$$s\sum_{j\alpha\beta\varepsilon i}|c^j_{\alpha\beta i}|2^{|\alpha|}s^{(\mu-\lambda)\varepsilon}s^{\lambda(\alpha-\alpha_{j\beta})} \leqslant s\Big(\sum_{\varepsilon}s^{(\mu-\lambda)\varepsilon}\Big)\sum_{j\beta}\Big(\sum_{i\alpha}|c^j_{\alpha\beta i}|2^{|\alpha|}s^{\lambda(\alpha-\alpha_{j\beta})}\Big)$$

and this series converges for s small enough.

Lemma 2.9. — *Given j,β, there are $s_{j\beta}>0$ and $C_{j\beta}>0$ such that the series $\sum_{i\alpha}|c^j_{\alpha\beta i}|2^{|\alpha|}s^{\lambda(\alpha-\alpha_{j\beta})}$ converges with sum bounded by $C_{j\beta}$ for $0<s<s_{j\beta}$.*

Proof. — The power series $\sum_{i\alpha}|c^j_{\alpha\beta i}|x^\alpha$ defines an analytic function $F(x)$ in a neighborhood of $0\in\mathbb{C}^n$. Therefore, the function $f(s)=F(2s^{\lambda_1},\dots,2s^{\lambda_n})$ is analytic in a neighborhood of $0\in\mathbb{C}$. Its expansion as a power series near 0 is $\sum_{i\alpha}|c^j_{\alpha\beta i}|2^{|\alpha|}s^{\lambda\alpha}$.

Let r be its radius of convergence, and let $s_{j\beta} < \min\{1, r\}$. The function f is analytic in the disc $|s| \leqslant s_{j\beta}$.

Since $\lambda\alpha \geqslant \lambda\alpha_{j\beta}$ for every α, this series does not have any term in which the exponent of s is less than $\lambda\alpha_{j\beta}$. Hence $f(s)$ has a zero of order at least $\lambda\alpha_{j\beta}$ in $s = 0$, and therefore the function $g(s) = f(s)/s^{\lambda\alpha_{j\beta}}$ in analytic in the disc $|s| \leqslant s_{j\beta}$. In particular, it is bounded inside the interval $[0, s_{j\beta}]$ of the real line. And this is what we want, since the power series expansion of g in a neighborhood of 0 is $\sum_{i\alpha} |c^j_{\alpha\beta i}| 2^{|\alpha|} s^{\lambda(\alpha-\alpha_{j\beta})}$. □

Lemma 2.10. — *The series $\sum_\varepsilon s^{(\mu-\lambda)\varepsilon}$ converges and its sum is uniformly bounded for $s < 1/2$.*

Proof. — Since $\mu_j > \lambda_j$ for every $j = 1, \dots, n$, we have:

$$\sum_\varepsilon s^{(\mu-\lambda)\varepsilon} = \sum_\varepsilon \prod_j s^{(\mu_j-\lambda_j)\varepsilon_j} = \prod_j \sum_{\varepsilon_j} s^{(\mu_j-\lambda_j)\varepsilon_j} \leqslant \prod_j \sum_{\varepsilon_j} s^{\varepsilon_j} = \left(\frac{1}{1-s}\right)^n \leqslant 2^n$$

□

Now take $s_0 = \min\{\frac{1}{2}, \min_{j\beta} s_{j\beta}\}$ and $C_2 = 2^n \sum_{j\beta} C_{j\beta}$. From the last two lemmas it follows that

$$\frac{|(w_2 v^{-1})E|^{\mu,s}_s}{|E|^{\mu,s}_s} \leqslant C_2 s \tag{4}$$

for $0 < s < s_0$. □

From these two propositions we can deduce the following

Theorem 2.11. — *There are constants $s_0 > 0$ and $C > 0$ such that for every $E = u(A, B) \in \mathcal{D}^q_n$ we have*

$$\sum_j |A_j|^{\mu,s}_s |M_j|^{\mu,s}_s + |B|^{\mu,s}_s \leqslant C|E|^{\mu,s}_s$$

for $0 < s < s_0$ and for every μ admissible for $(\mathscr{F}, \lambda)$.

Proof. — Let $\mathcal{D}_n(s)_d$ denote the subspace of $\mathcal{D}_n(s)$ whose elements are germs of differential operators of degree at most d. So far we know that there are $s_0 > 0$ and $C_1, C_2 > 0$ such that

$$|((w_1 + w_2)v^{-1})E|^{\mu,s}_s \leqslant |(w_1 v^{-1})E|^{\mu,s}_s + |(w_2 v^{-1})E|^{\mu,s}_s \leqslant (C_1 + C_2)s|E|^{\mu,s}_s$$

for $0 < s < s_0$. Choose $\varepsilon > 0$. Taking $s < \min\{s_0, \frac{1-\varepsilon}{C_1+C_2}\}$, the map $(w_1 + w_2)v^{-1}$ has norm $| - |^{\mu,s}_s$ less than $1 - \varepsilon$. Hence the series $\sum(-(w_1 + w_2)v^{-1})^n$ converges to a continuous endomorphism on every $\mathcal{D}_n(s)^q_d$ with norm at most $1/\varepsilon$, since the map does not increase the degree on ∂ and $\mathcal{D}_n(s)^q_d$ is a Banach space for the $| - |^{\mu,s}_s$ norm. Therefore, this series defines a continuous endomorphism of the whole $\mathcal{D}_n(s)^q$

which is nothing but the inverse of $(\mathrm{Id} + (w_1 + w_2)v^{-1})$. In particular we see that $u = (\mathrm{Id} + (w_1 + w_2)v^{-1})v$ is an isomorphism. Furthermore, we have the inequality

$$\begin{aligned}\sum_j |A^o_j x^{\alpha_j}\partial^{\beta_j} e_{i_j}|_s^{\mu,s} + |B|_s^{\mu,s} &= |v(A,B)|_s^{\mu,s}\\ &= |(\mathrm{Id} + (w_1+w_2)v^{-1})^{-1}u(A,B)|_s^{\mu,s} \leqslant \varepsilon^{-1}|E|_s^{\mu,s}\end{aligned}$$

for every $E = u(A,B) \in \mathcal{D}_n(s)^q$. We conclude with two lemmas.

Lemma 2.12. — *For every $j = 1,\dots,r$ we have $|A^o x^{\alpha_j}\partial^{\beta_j} e_{i_j}|_s^{\mu,t} \geqslant |x^{\alpha_j}\partial^{\beta_j} e_{i_j}|_s^{\mu,t}|A|_s^{\mu,t}$ for every $A \in \mathcal{D}_n$ and every $s,t > 0$.*

Proof. — Let $A = \sum_{\alpha\beta} a_{\alpha\beta}x^\alpha\partial^\beta$. Then $A^o x^{\alpha_j}\partial^{\beta_j} e_{i_j} = \sum_{\alpha\beta} a_{\alpha\beta}x^{\alpha+\alpha_j}\partial^{\beta+\beta_j} e_{i_j}$ and therefore

$$\begin{aligned}|A^o x^{\alpha_j}\partial^{\beta_j} e_{i_j}|_s^{\mu,t} &= \sum_{\alpha\beta} |a_{\alpha\beta}|\cdot|\beta+\beta_j|!\cdot s^{\lambda(\alpha+\alpha_j)-(i_j-1)}t^{-\mu(\beta+\beta_j)}\\ &= s^{\lambda\alpha_j-(i_j-1)}t^{-\mu\beta_j}\sum_{\alpha\beta}|a_{\alpha\beta}|\cdot|\beta+\beta_j|!\cdot s^{\lambda\alpha}t^{-\mu\beta}\\ &\geqslant s^{\lambda\alpha_j-(i_j-1)}t^{-\mu\beta_j}|\beta_j|!\sum_{\alpha\beta}|a_{\alpha\beta}|\cdot|\beta|!\cdot s^{\lambda\alpha}t^{-\mu\beta}\\ &= |x^{\alpha_j}\partial^{\beta_j} e_{i_j}|_s^{\mu,t}|A|_s^{\mu,t}.\end{aligned}$$ □

Lemma 2.13. — *For every $j = 1,\dots,r$, there is a constant $C_j > 0$ such that $|M_j|_s^{\mu,s} \leqslant C_j|x^{\alpha_j}\partial^{\beta_j} e_{i_j}|_s^{\mu,s}$ for $0 < s < s_0$.*

Proof. — We have

$$\begin{aligned}\frac{|M_j|_s^{\mu,s}}{|x^{\alpha_j}\partial^{\beta_j} e_{i_j}|_s^{\mu,s}} &= \sum_{\alpha\beta i}|c^j_{\alpha\beta i}|\frac{|\beta|!}{|\beta_j|!}s^{\lambda\alpha-\lambda\alpha_j+(i_j-i)+\mu\beta_j-\mu\beta}\\ &\leqslant \sum_{\alpha\beta i}|c^j_{\alpha\beta i}|s^{(\lambda\alpha-\mu\beta-i)-(\lambda\alpha_j-\mu\beta_j-i_j)}\end{aligned}$$

Now with an argument analogous to that in the proof of 2.9, we see that the series $\sum_\alpha |c^j_{\alpha\beta i}|s^{\lambda\alpha}$ defines an analytic function in a neighborhood of 0, with a zero of order at least $\mu\beta + i + (\lambda\alpha_j - \mu\beta_j - i_j)$ at the origin, hence our series defines a continuous function in a neighborhood of the origin, from where the existence of the constants C_j is deduced. □

This concludes the proof of the theorem 2.11, by taking $C = \varepsilon^{-1}\cdot\max_j\{C_j\}$. □

We see that the fact that u is bijective implies the division theorem, more precisely, it implies the existence and uniqueness of the quotients and the remainder of the division of E by $M_1,\dots,M_r$, assuming that their Newton diagrams are contained in L and J respectively.

Proof of Theorem 2.6. — Using the same notation as in the previous theorem, we have $A_j = Q_j(\mathscr{F}; E)$ and $B = R(\mathscr{F}; E)$. Let $\mathscr{B}$ be the basis of open neighborhoods of 0 consisting of open polycylinders of polyradius s^λ with $0 < s < s_0$. The canonical topology on $\mathcal{D}_{\mathbb{C}^n}(U)^q$ for any open polycylinder U centered at 0 of polyradius s^λ, $s > 0$, is given by the seminorms $|-|_{s'}^{\mu,s'}$, $0 < s' < s$ and μ admissible for $(\mathscr{F}, \lambda)$ (see section 1). Hence the bound in Theorem 2.11 implies the continuity of Q_j and R. $\square$

Let us suppose now the following additional hypothesis on the chosen ordering:

$(\star)$ For every $(\alpha, \beta) \in \mathbb{N}^{2n}, (\alpha, \beta) < (\alpha', \beta') \Longrightarrow (0, \beta) < (0, \beta') \Longrightarrow |\beta| \leqslant |\beta'|$.

There are many monomial orderings satisfying this condition. For instance, if $<'$ is a well ordering in $\mathbb{N}^n$ compatible with sums, we can set

$$(\alpha, \beta) < (\alpha', \beta') \quad \text{if} \quad \begin{cases} |\beta| < |\beta'| & \text{or} \\ |\beta| = |\beta'| \text{ and } \beta <' \beta' & \text{or} \\ \beta = \beta' \text{ and } \alpha <' \alpha'. \end{cases}$$

We will see that under hypothesis $(\star)$ we can obtain a sharper result, namely (see [**8**]):

Theorem 2.14. — *There are constants $s_0 > 0, C > 0$ such that for every $E = u(A, B) \in \mathcal{D}_n^q$ we have*

$$\sum_j |A_j|_s^{\mu,t} |M_j|_s^{\mu,t} + |B|_s^{\mu,t} \leqslant C |E|_s^{\mu,t}$$

for every $0 < t < s < s_0$ and for every μ admissible for $(\mathscr{F}, \lambda)$.

The proof of the theorem in this case follows exactly the same steps as in the former case, the required hypothesis allows us to find admissible μ's with arbitrarily large components satisfying the following additional condition:

$$\mu\beta_j \geqslant \mu\beta \text{ for every } (\alpha, \beta) \in \mathscr{N}(M_j') \text{ and for every } j = 1, \ldots, r.$$

Proof. — We keep the notation used in Lemma 1. We will first check that $\beta \in \pi_2(\mathscr{N}(\sigma(M_j)))$ when $\mu\beta$ is maximal. Otherwise there would be $(\alpha', \beta') \in \mathscr{N}(M_j) \cap \mathscr{M}$ with $|\beta'| > |\beta|$, hence

$$\mu\beta = p|\beta| - \rho\beta < p|\beta| \leqslant p|\beta'| - p < p|\beta'| - \sigma\alpha' - \rho\beta' < p|\beta'| - \rho\beta' = \mu\beta'.$$

Let $(\alpha, \beta) \in \mathscr{N}(\sigma(M_j))$. Then, $(\alpha_j, \beta_j) \leqslant (\alpha, \beta)$ implies $(0, \beta_j) \leqslant (0, \beta)$ because of the hypothesis $(\star)$. Since $(0, \beta_j)$ and $(0, \beta)$ are in $\mathscr{M}$ and the ordering restricted to this set is defined by the weights (σ, ρ), we have $\rho\beta_j \leqslant \rho\beta$, hence $\mu\beta_j \geqslant \mu\beta$ as we wanted to show.

This permits to verify the inequalities in the proof of the theorem for every $t < s$, and not only for $t = s$. $\square$

It is possible to obtain a similar result if we restrict ourselves to the Weyl algebra $A_n(\mathbb{C})$ (*cf.* [**6**], I.6) and consider the division operator inside this ring (*cf.* [**4**], Thm. 2.4.1, with $L_2(\beta) = |\beta|$). The proof is basically the same one (now we do not have to take care of the convergence problems, since every element of $A_n(\mathbb{C})$ consists only of a finite number of monomials). We will just state the corresponding results without giving the proofs, which can be found in [**15**].

In this case, the canonical topology on $A_n(\mathbb{C})$ that we consider is the one induced by the canonical topology on $\mathcal{D}_{\mathbb{C}^n}(\mathbb{C}^n)$ (or on $\mathcal{D}^{\infty}_{\mathbb{C}^n}(\mathbb{C}^n)$). It is defined by the family of seminorms (which in fact are now norms) parameterized by $s^{-\lambda}$ and $t^{-\mu}$ for suitable λ and μ, such that s tends to zero and $t^{-\mu} \gg 0$. Here we say that a weight $\lambda \in (\mathbb{N}^*)^n$ is *adapted* to an r-tuple $(M_1, \dots, M_r)$ of elements of $A_n(\mathbb{C})$ if for every positive constant K there exists $\mu \in \mathbb{N}^n$ with components greater than K such that

$$\lambda\alpha_j + \mu\beta_j - i_j > \lambda\alpha + \mu\beta - i$$

for every $(\alpha, \beta, i) \in \mathscr{N}(M'_j)$ and every $j = 1, \dots, r$, where (α_j, β_j, i_j) is the *exponent* of M_j, defined in this case as follows:

Let $\sigma(M) = \sum_i \sum_{\alpha\beta} a_{\alpha\beta i} x^\alpha \partial^\beta e_i$ be the *symbol* of M (defined as above). The *exponent* of M is $\max\{(\alpha, \beta, i) | a_{\alpha\beta i} \neq 0\}$ (this is a finite set in this case, so its maximum with respect to the ordering considered is well defined). The *initial monomial* of M is the monomial corresponding to its exponent.

In the same way as above, one can show that there is always a weight λ adapted to any r-tuple $\mathscr{F}$, and even one adapted to several r-tuples (for distinct r's) simultaneously.

Theorem 2.15. — *Let $\mathscr{F} = (M_1, \dots, M_r)$, with $M_i \in A_n(\mathbb{C})^q$ and let $Q_i(\mathscr{F}; E)$, $i = 1, \dots, r$ (resp. $R(\mathscr{F}; E)$) be the quotients (resp. the remainder) of the division of $E \in A_n(\mathbb{C})^q$ by $\mathscr{F}$. Then the $\mathbb{C}$-linear maps*

$$Q_i(\mathscr{F}; -) \colon A_n(\mathbb{C})^q \longrightarrow A_n(\mathbb{C}),$$
$$R(\mathscr{F}; -) \colon A_n(\mathbb{C})^q \longrightarrow A_n(\mathbb{C})^q$$

are continuous with respect to the canonical topology.

Proposition 2.16. — *There is a constant $C_1 > 0$ such that $|(w_1 v^{-1})E|^{s^{-\mu}}_{s^{-\lambda}} \leqslant C_1 s |E|^{s^{-\mu}}_{s^{-\lambda}}$ for every $E \in A_n(\mathbb{C})^q$ and $0 < s < 1$.*

Proposition 2.17. — *There are constants $C_2 > 0$ and $s_0 > 0$ such that $|(w_2 v^{-1})E|^{s^{-\mu}}_{s^{-\lambda}} \leqslant C_2 s |E|^{s^{-\mu}}_{s^{-\lambda}}$ for $0 < s < s_0$ and for every $E \in A_n(\mathbb{C})^q$.*

Theorem 2.18. — *The map $u : L \oplus J \to A_n(\mathbb{C})^q$ is a bi-continuous isomorphism. There are constants $s_0 > 0, C > 0$ such that for every $E = u(A, B) \in A_n(\mathbb{C})^q$ we have*

$$\sum_j |A_j|^{s^{-\mu}}_{s^{-\lambda}} |M_j|^{s^{-\mu}}_{s^{-\lambda}} + |B|^{s^{-\mu}}_{s^{-\lambda}} \leqslant C|E|^{s^{-\mu}}_{s^{-\lambda}}$$

for $0 < s < s_0$.

As in the previous case, it is also possible to obtain a sharper result if we assume the additional hypothesis $(\star)$ given earlier on the ordering. The new stronger statement is the following:

Theorem 2.19. — *If the chosen ordering satisfies hypothesis $(\star)$, the map $u : L \oplus J \to A_n(\mathbb{C})^q$ is a bi-continuous isomorphism. There are constants $s_0 > 0$, $C > 0$ such that for every $E = u(A, B) \in A_n(\mathbb{C})^q$ we have, for $0 < t < s < s_0$,*

$$\sum_j |A_j|_{s^{-\lambda}}^{t^{-\mu}} |M_j|_{s^{-\lambda}}^{t^{-\mu}} + |B|_{s^{-\lambda}}^{t^{-\mu}} \leqslant C|E|_{s^{-\lambda}}^{t^{-\mu}}.$$

3. Continuous scissions

Proposition 3.1. — *Let $V \subset \mathbb{C}^n$ be an open neighborhood of 0, $r_i, q_i \geqslant 1$, $1 \leqslant i \leqslant m$ integers and $F_i : \mathcal{D}_V^{r_i} \to \mathcal{D}_V^{q_i}, 1 \leqslant i \leqslant m$ a finite family of $\mathcal{D}_V$-linear maps. There exists a weight $\lambda \in (\mathbb{N}^*)^n$, a λ-basis $\mathscr{B}$ of open neighborhoods of $0 \in \mathbb{C}^n$ and a family of continuous scissions $\{\sigma_U^i : \mathcal{D}_V(U)^{q_i} \to \mathcal{D}_V(U)^{r_i}\}_{U \in \mathscr{B}}$ of F_i, i.e. :*

$$F_i(U) \circ \sigma_U^i \circ F_i(U) = F_i(U), \quad U \in \mathscr{B},\ 1 \leqslant i \leqslant m$$

compatible with restrictions, i.e. $\sigma_U^i|_W = \sigma_W^i$ for $W \subset U$.

Proof. — First, let us write $f_i = (F_i)_0$ and let us take a Gröbner (or standard) basis $\mathscr{G}_i = \{N_1^i, \dots, N_{s_i}^i\}$ of $\operatorname{im}(f_i) \subset \mathcal{D}_n^{q_i}$ for each $i = 1, \dots, m$ (see [**4**] in this volume) , and consider the corresponding linear map $g_i : \mathcal{D}_n^{s_i} \to \mathcal{D}_n^{q_i}$. By shrinking V if necessary, we can suppose that g_i is the stalk at 0 of a linear map $G_i : \mathcal{D}_V^{s_i} \to \mathcal{D}_V^{q_i}$. Let us consider

$$\tau^i = (Q_1(\mathscr{G}_i; -), \dots, Q_{s_i}(\mathscr{G}_i; -)) : \mathcal{D}_n^{q_i} \longrightarrow \mathcal{D}_n^{s_i}.$$

Let $\lambda \in (\mathbb{N}^*)^n$ be a weight adapted to $\mathscr{G}_i$, for every $i = 1, \dots, m$ (see lemma 2.3). By theorem 2.6, there exists a λ-basis $\mathscr{B}$ of open neighborhoods of $0 \in \mathbb{C}^n$, such that for every $U \in \mathscr{B}$ and every i the $\mathbb{C}$-linear map $\tau_U^i := \tau^i|_{\mathcal{D}_V(U)^{q_i}} : \mathcal{D}_V(U)^{q_i} \to \mathcal{D}_V(U)^{s_i}$ is continuous. Furthermore, the fact that $\mathscr{G}_i$ is a Gröbner basis implies that $g_i \circ \tau^i \circ g_i = g_i$. By analytic continuation we obtain

$$G_i(U) \circ \tau_U^i \circ G_i(U) = G_i(U), \quad \forall\, U \in \mathscr{B},\ \forall\, i = 1, \dots, m.$$

Let $h_i : \mathcal{D}_n^{s_i} \to \mathcal{D}_n^{r_i}$ be the linear map such that $f_i \circ h_i = g_i$. By shrinking V again if necessary, we can suppose that h_i is the stalk at 0 of a linear map $H_i : \mathcal{D}_V^{s_i} \to \mathcal{D}_V^{r_i}$. Let $\sigma^i := h_i \circ \tau^i : \mathcal{D}_n^{q_i} \to \mathcal{D}_n^{r_i}$ and $\sigma_U^i := h_i(U) \circ \tau_U^i : \mathcal{D}_V(U)^{q_i} \to \mathcal{D}_V(U)^{r_i}$, $U \in \mathscr{B}$, which are continuous.

From $\operatorname{im}(f_i) = \operatorname{im}(g_i)$ ($\mathscr{G}_i$ is a Gröbner basis of $\operatorname{im}(f_i)$) and $g_i \circ \tau^i \circ g_i = g_i$ we deduce $f_i \circ \sigma^i \circ f_i = f_i$. Analytic continuation gives again $F_i(U) \circ \sigma_U^i \circ F_i(U) = F_i(U)$ for all $U \in \mathscr{B}$, $1 \leqslant i \leqslant m$. $\square$

Corollary 3.2. — *Let $V \subset \mathbb{C}^n$ be an open neighborhood of* 0, $r_i, s_i, q_i \geqslant 1$, $i = 1, \dots, m$ *integers and*

$$\mathcal{D}_V^{r_i} \xrightarrow{F_i} \mathcal{D}_V^{s_i} \xrightarrow{G_i} \mathcal{D}_V^{t_i}, \quad i = 1, \dots, m$$

a finite family of exact sequences of $\mathcal{D}_V$-linear maps. There exists a weight $\lambda \in (\mathbb{N}^*)^n$ and a λ-basis $\mathscr{B}$ of open neighborhoods of $0 \in \mathbb{C}^n$ such that for any $U \in \mathscr{B}$ and any $i = 1, \dots, m$ the sequence

$$\mathcal{D}_V(U)^{r_i} \xrightarrow{F_i(U)} \mathcal{D}_V(U)^{s_i} \xrightarrow{G_i(U)} \mathcal{D}_V(U)^{t_i}$$

is exact and topologically split.

Proof. — By proposition 3.1, there exists a weight $\lambda \in (\mathbb{N}^*)^n$, a λ-basis $\mathscr{B}$ of open neighborhoods of $0 \in \mathbb{C}^n$ and a family of continuous scissions $\{\sigma_U^i : \mathcal{D}_V(U)^{q_i} \to \mathcal{D}_V(U)^{r_i}\}_{U \in \mathscr{B}}$ of F_i, $i = 1, \dots, m$ compatible with restrictions. Let us write $f_i = (F_i)_0, g_i = (G_i)_0$ and $\sigma^i = \lim\limits_{U \in \mathscr{B}} \sigma_U^i$ for each $i = 1, \dots, m$. We have $f_i = f_i \circ \sigma^i \circ f_i$ and

$$\ker g_i = \operatorname{im} f_i = \ker(1 - f_i \circ \sigma^i).$$

Then, for any $U \in \mathscr{B}$ and any $M \in \ker G_i(U)$ we have

$$0 = M_0 - f_i(\sigma^i(M_0)) = \big(M - F_i(U)(\sigma_U^i(M))\big)_0,$$

and by analytic continuation we deduce $M \in \ker(1 - F_i(U) \circ \sigma_U^i) \subset \operatorname{im} F_i(U)$. $\square$

Proposition 3.3. — *Let $V \subset \mathbb{C}^n$ be an open neighborhood of* 0, *$\mathcal{M}_i$ a coherent $\mathcal{D}_V$-module, $i = 1, \dots, m$, and*

$$\mathcal{D}_V^{r_i} \xrightarrow{F_i} \mathcal{D}_V^{s_i} \xrightarrow{\pi_i} \mathcal{M}_i \longrightarrow 0$$

a finite presentation. Then, for any λ-basis $\mathscr{B}$ of open neighborhoods of $0 \in \mathbb{C}^n$ such that the morphisms $F_i(U)$ split for $U \in \mathscr{B}$ and $i = 1, \dots, m$, the sequence

$$\mathcal{D}_V(U)^{r_i} \xrightarrow{F_i(U)} \mathcal{D}_V(U)^{s_i} \xrightarrow{\pi_i(U)} \mathcal{M}_i(U) \longrightarrow 0$$

is exact for $U \in \mathscr{B}$ and $i = 1, \dots, m$.

Proof. — By shrinking V if needed, we can suppose that the kernel of F_i has a good filtration on V (*cf.* [**6**], prop. 10). Then, for any compact polycylinder $K \subset V$ and any $i = 1, \dots, m$ the sequence

$$\mathcal{D}_V(K)^{r_i} \xrightarrow{F_i(K)} \mathcal{D}_V(K)^{s_i} \xrightarrow{\pi_i(K)} \mathcal{M}_i(K) \longrightarrow 0 \tag{5}$$

is exact by the Cartan-Oka theorem (*cf.* loc. cit., prop. 11). By proposition 3.1, there exist a weight $\lambda \in (\mathbb{N}^*)^n$ and a λ-basis $\mathscr{B}$ of open neighborhoods of $0 \in \mathbb{C}^n$ such that for any $U \in \mathscr{B}$ and any $i = 1, \dots, m$ the maps $F_i(U) : \mathcal{D}_V(U)^{r_i} \to \mathcal{D}_V(U)^{s_i}$ split, with scissions compatible with restrictions.

If K is the closure of a $U' \in \mathscr{B}$, the sequence (5) is the inductive limit of the sequences

$$\mathcal{D}_V(U'')^{r_i} \xrightarrow{F_i(U'')} \mathcal{D}_V(U'')^{s_i} \longrightarrow \operatorname{coker} F_i(U'') \longrightarrow 0$$

with $U'' \in \mathscr{B}$, $K \subset U''$, and hence it splits. Now, for any $U \in \mathscr{B}$ the sequence

$$\mathcal{D}_V(U)^{r_i} \xrightarrow{F_i(U)} \mathcal{D}_V(U)^{s_i} \xrightarrow{\pi_i(U)} \mathcal{M}_i(U) \longrightarrow 0$$

is the projective limit of sequences (5), with $K = \overline{U'} \subset U$, $U' \in \mathscr{B}$, and consequently it is exact. □

4. Faithful flatness of $\mathcal{D}_X^\infty$ over $\mathcal{D}_X$

Faithful flatness of the ring of differential operators of infinite order over the ring of differential operators of finite order has been stated for the first time by Sato, Kashiwara and Kawai in [**17**]. Their proof used microlocal methods. In this section we reproduce the proof given in [**14**], based on the continuity of division of differential operators studied in the precedent sections.

Theorem 4.1. — *For any complex analytic manifold X, the extension $\mathcal{D}_X \to \mathcal{D}_X^\infty$ is faithfully flat.*

Proof. — It is enough to prove that the ring extension $\mathcal{D}_n \to \mathcal{D}_n^\infty$ is faithfully flat.

Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be an exact sequence of $\mathcal{D}_n$-modules. It is the stalk at 0 of an exact sequence $0 \to \mathcal{M}_1 \to \mathcal{M}_2 \to \mathcal{M}_3 \to 0$ of $\mathcal{D}_V$-modules, where V is an open neighborhood of 0. We can find a commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{M}_1 & \longrightarrow & \mathcal{M}_2 & \longrightarrow & \mathcal{M}_3 & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{D}_V^{r_1} & \longrightarrow & \mathcal{D}_V^{r_2} & \longrightarrow & \mathcal{D}_V^{r_3} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{D}_V^{s_1} & \longrightarrow & \mathcal{D}_V^{s_2} & \longrightarrow & \mathcal{D}_V^{s_3} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{D}_V^{t_1} & \longrightarrow & \mathcal{D}_V^{t_2} & \longrightarrow & \mathcal{D}_V^{t_3} & \longrightarrow & 0
\end{array}$$

with exact rows and columns. By propositions 3.1, 3.3 and corollary 3.2, there exist a weight λ and a λ-basis of neighborhoods of 0 such that for any $U \in \mathscr{B}$ the diagram

$$\begin{array}{ccccccccc}
& & 0 & & 0 & & 0 & & \\
& & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{M}_1(U) & \longrightarrow & \mathcal{M}_2(U) & \longrightarrow & \mathcal{M}_3(U) & \longrightarrow & 0 \\
& & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{D}_V(U)^{r_1} & \longrightarrow & \mathcal{D}_V(U)^{r_2} & \longrightarrow & \mathcal{D}_V(U)^{r_3} & \longrightarrow & 0 \\
& & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{D}_V(U)^{s_1} & \longrightarrow & \mathcal{D}_V(U)^{s_2} & \longrightarrow & \mathcal{D}_V(U)^{s_3} & \longrightarrow & 0 \\
& & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{D}_V(U)^{t_1} & \longrightarrow & \mathcal{D}_V(U)^{t_2} & \longrightarrow & \mathcal{D}_V(U)^{t_3} & \longrightarrow & 0
\end{array}$$

has exact and topologically splitting rows and columns.

Then, the corresponding diagram of topological completions has also exact rows and columns. As for any open polycylinder W we have $\widehat{\mathcal{D}_V(W)} = \mathcal{D}_V^\infty(W)$, we deduce $\widehat{\mathcal{M}_i(U)} = \mathcal{D}_V^\infty(U) \otimes_{\mathcal{D}_V(U)} \mathcal{M}_i(U)$ and the exactness of the sequence

$$0 \to \mathcal{D}_V^\infty(U) \otimes_{\mathcal{D}_V(U)} \mathcal{M}_1(U) \to \mathcal{D}_V^\infty(U) \otimes_{\mathcal{D}_V(U)} \mathcal{M}_2(U) \to \mathcal{D}_V^\infty(U) \otimes_{\mathcal{D}_V(U)} \mathcal{M}_3(U) \to 0$$

for every $U \in \mathscr{B}$. Taking direct limits we obtain the exactness of

$$0 \to \mathcal{D}_n^\infty \otimes_{\mathcal{D}_n} M_1 \to \mathcal{D}_n^\infty \otimes_{\mathcal{D}_n} M_2 \to \mathcal{D}_n^\infty \otimes_{\mathcal{D}_n} M_3 \to 0,$$

and so the extension $\mathcal{D}_n \to \mathcal{D}_n^\infty$ is flat.

To conclude, we observe that the quotient topology on $\mathcal{M}_i(U)$, $U \in \mathscr{B}$, is separated. Then $\mathcal{M}_i(U) \hookrightarrow \widehat{\mathcal{M}_i(U)} = \mathcal{D}_V^\infty(U) \otimes_{\mathcal{D}_V(U)} \mathcal{M}_i(U)$ and $M_i \hookrightarrow \mathcal{D}_n^\infty \otimes_{\mathcal{D}_n} M_i$. In particular $M_i \neq 0$ implies $\mathcal{D}_n^\infty \otimes_{\mathcal{D}_n} M_i \neq 0$ and the extension $\mathcal{D}_n \to \mathcal{D}_n^\infty$ is faithfully flat. $\square$

Using the corresponding results for the Weyl algebra (Theorems 2.15 and 2.18) it is possible to obtain the faithful flatness for the extension $A_n(\mathbb{C}) \to \mathcal{D}_{\mathbb{C}^n}^\infty(\mathbb{C}^n)$, as done in [**15**].

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