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GEOMETRIC IRREGULARITY AND $\mathscr{D}$-MODULES

by

Yves Laurent

Abstract. — In the one dimensional case, J.-P. Ramis associated a Newton polygon to an analytic differential operator. On this polygon may be read the irregularity of the operator as well as its indices in various functional spaces. This result is here generalized in the higher dimensional case. We define a Newton polygon and positive microcharacteristic cycles. We get so a purely algebraic definition of the characteristic cycle of the irregularity of a holonomic $\mathscr{D}$-module.

***Résumé* (Irrégularité géométrique et $\mathscr{D}$-modules).** — En une variable, J.-P. Ramis a associé à un opérateur différentiel analytique un polygone de Newton sur lequel on peut lire l'irrégularité de cet opérateur ainsi que ses indices dans divers espaces fonctionnels. On montre ici que ce résultat se généralise en dimension quelconque, en définissant un polygone de Newton et des cycles microcaractéristiques positifs. En particulier, on obtient une définition purement algébrique du cycle caractéristique de l'irrégularité d'un $\mathscr{D}$-module holonome.

Introduction

Let X be a complex manifold and $\mathscr{D}_X$ the sheaf of differential operators with holomorphic coefficients on X. Regular holonomic $\mathscr{D}_X$-modules are completely determined by the Riemann-Hilbert correspondence which is an equivalence of categories between these modules and the perverse sheaves on X. In the non regular case, things are much more complicated.

When the dimension of X is 1, the irregularity of an ordinary differential equation is just a positive number. In higher dimensions, it may be understood as a perverse sheaf as explained by Mebkhout in this Summer School or as a positive cycle as we will see here. The relation between these two points of view is simply the fact that the positive cycle is the characteristic cycle associated to the perverse sheaf. But in fact, the two methods are completely different and give complementary results.

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As shown in dimension 1 by the results of Ramis, the irregularity itself is not sufficient and we have to define a finite family of positive cycles, not only one. This will be done by a method which is very similar to the definition of the characteristic cycle. We will define a family of filtrations on the sheaf $\mathscr{D}_X$ and from it we will get the family of microcharacteristic cycles.

More precisely, if Y is a submanifold of X, the microcharacteristic cycles form a finite family of lagrangian cycles on the space $T^*T^*_YX$ (cotangent to the conormal space to Y). They give a formula to compute the index of solutions of the $\mathscr{D}_X$-module. In particular they compute the index of the sheaf of irregularity introduced by Mebkhout.

But these cycles are not the good ones. Let us assume that Y is a hypersurface. Then the sheaf of irregularity is a sheaf on Y and we need cycles on T^*Y not on $T^*T^*_YX$. We show that to each lagrangian cycle on $T^*T^*_YX$ with a suitable action of $\mathbb{C}^*$ is associated a cycle on T^*Y, called its irregularity, which has good properties of homogeneity and positivity.

Applying this to $\mathscr{D}_X$-modules, we get positive cycles on T^*Y which compute the index of the sheaf of irregularity and which vanishes if and only if $\mathscr{M}$ is regular along Y. Moreover we show that these cycles are positive (positivity of the irregularity) and that they are divisible by an integer (the denominator of the slope). These properties generalize the properties of the irregularity in dimension one. In particular it generalizes the positivity while the last property is the generalization of the fact that the vertices of Newton Polygon have integral coordinates.

The detailed proofs are not given here but may be found in [**5**] and [**6**].

1. Ordinary differential equations

1.1. Newton Polygon (*cf.* Ramis [9]). — Let X be an open neighborhood of 0 in $\mathbb{C}$ and and P a differential operator on X:

$$P(t, D_t) = \sum_{0\leqslant j\leqslant m} p_j(t) D_t^j$$

(with $D_t^j = d^j/dt^j$). Developing the p_j functions in Taylor series near 0 we get:

$$P(t, D_t) = \sum_{\substack{0\leqslant j\leqslant m\\ i\geqslant 0}} p_{ij} t^i D_t^j$$

For $0 \leqslant j \leqslant m$, we denote by k_j the valuation of the function p_j at 0 (i.e. the highest power of t dividing p_j) and we define:

$$S_j = \{ (\lambda, \mu) \in \mathbb{R}^2 \mid \lambda \leqslant j,\ \mu \geqslant k_j - j \}$$

Then $S_0(P)$ is the union of the sets S_j and the Newton Polygon $\mathbb{N}_0(P)$ is the convex hull of $S_0(P)$. It is a convex subset of $\mathbb{R}^2$ (Figure 1).

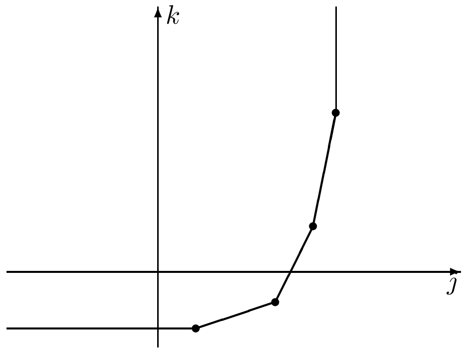

FIGURE 1. Newton Polygon

The operator P is said to be regular at 0, or to have "regular singularities" if the Newton Polygon has only one vertex.

In the general case, this polygon is made of two half-lines (one vertical, one horizontal) and of a finite number of segments. We denote by $0 < s_N < \cdots < s_1 < +\infty$ the slopes of these segments and by $1 < r_1 < \cdots < r_N < +\infty$ the rational numbers given by $(r_i - 1)s_i = 1$. The numbers r_i are, by definition, the slopes of P or the "algebraic slopes" of P (sometimes also called the critical indexes of P).

The sum $\sum p_{ij} t^i \tau^j$ over $(j, i-j)$ on the vertical half-line of the Newton Polygon is nothing else than the function $p_m(t)\tau^m$ where m is the order of P that is the principal symbol of P. In a similar way, we define the the determining equation of P relative to the index r as the restriction to $t = 1$ of the sum $\sum p_{ij} t^i \tau^j$ over $(j, i-j)$ on the segment of slope $1/(r-1)$.

If r is not a slope of P, the corresponding determining equation is monomial, otherwise it is a polynomial function of τ. The Newton Polygon is determined up to a translation by the list of the degrees and valuations of the determining equations.

1.2. The algebraic case. — If all the coefficients of P are polynomial in t, we may define a "negative part" of the Newton Polygon. Keeping the previous notations, we denote by d_j the degree of p_j and replace the sets S_j by the two families:

$$S'_j = \{ (\lambda, \mu) \in \mathbb{R}^2 \mid \lambda \leqslant j,\ \mu = k_j - j \} \tag{1.2.1}$$

$$S''_j = \{ (\lambda, \mu) \in \mathbb{R}^2 \mid \lambda \leqslant j,\ \mu = d_j - j \} \tag{1.2.2}$$

We get a Newton Polygon with positive and negative slopes (Figure 2).

1.3. Formal power series. — When P is regular at 0, Fuchs theorem says that all formal power series which are solutions of the equation $Pu = 0$ are convergent.

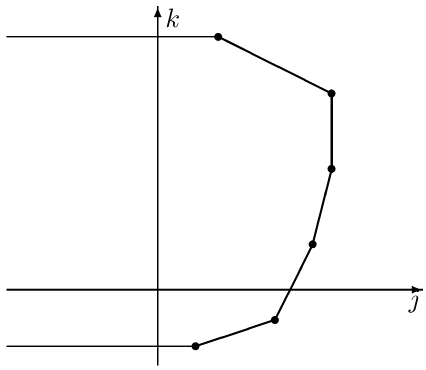

Figure 2. Second Newton Polygon

In [**7**], Malgrange has defined the irregularity of P as

$$\mathrm{Irr}(P) = \chi(P, \mathbb{C}[[t]]) - \chi(P, \mathbb{C}\{t\})$$

where $\mathbb{C}[[t]]$ is the ring of formal power series and $\mathbb{C}\{t\}$ the ring of convergent series.

Let us recall that if P is an operator on a $\mathbb{C}$-vector space F, P has finite index if the kernel and the cokernel of P are finite dimensional $\mathbb{C}$-vector spaces and the index $\chi(P, F)$ of P is:

$$\chi(P, F) = \dim_{\mathbb{C}} \mathrm{Ker}(P) - \dim_{\mathbb{C}} \mathrm{Coker}(P)$$

Malgrange proved that the irregularity is equal to the height between the higher and the lower vertex of the Newton Polygon of P (with the definition of section 1.1). It is thus a positive number which vanishes if and only if P is regular.

Ramis [**9**] improved this results using the rings $\mathbb{C}[[t]]_r$ of Gevrey formal powers. A formal power series $u(t) = \sum_{k\geqslant 0} u_k t^k$ is an element of $\mathbb{C}[[t]]_r$ if and only if:

$$F_r[u](t) := \sum_{k\geqslant 0} u_k \frac{t^k}{(k!)^{r-1}}$$

is convergent.

***Theorem 1.3.1* (Ramis [9])**

(1) *The operator P has a finite index on $\mathbb{C}[[t]]_r$ for any $r \geqslant 1$.*

(2) *If u is a formal power series solution of the equation $Pu = 0$, it belongs to one of the spaces $\mathbb{C}[[t]]_r$ where r is a slope of P and the convergence radius of $F_r[u]$ is equal to the inverse of the modulus of one of the roots of the corresponding determining equation.*

(3) *The index $\chi(P, \mathbb{C}[[t]]_r)$, as a function of r, is constant outside of the points r which are slopes of P. Its jump at one of these points is equal to the height of the segment of slope $1/(r-1)$ of the Newton Polygon of P.*

1.4. Holomorphic microfunctions. — The previous result may be stated and proved in other families of functions. We will consider in particular the family of holomorphic microfunctions with support in $\{0\}$ which is easier to generalize in higher dimensional case as we shall see later.

If U is a neighborhood of 0 in $\mathbb{C}$ the quotient $\mathscr{O}(U-\{0\})\,/\mathscr{O}(U)$ does not depend on U, it is denoted by $\mathscr{B}^{\infty}_{\{0\}|\mathbb{C}}$. The subspace generated by meromorphic functions at 0 is denoted by $\mathscr{B}_{\{0\}|\mathbb{C}}$.

The space $\mathscr{B}^{\infty}_{\{0\}|\mathbb{C}}$ operates on germs of holomorphic functions at 0 by the Cauchy formula. If f is a holomorphic function on a neighborhood U of 0 and if $u\in\mathscr{B}^{\infty}_{\{0\}|\mathbb{C}}$ is represented by a function $\varphi(t)$ on $U-\{0\}$, we choose a path γ in $U-\{0\}$ such that the index of 0 is 1, e.g. a small circle centered at 0 and we set:

$$\langle u,f\rangle=\int_{\gamma}\varphi(t)f(t)dt$$

In this way, the class of the function $\frac{1}{2i\pi}\frac{1}{t}$ is identified to the Dirac operator δ : $f\mapsto f(0)$ and the function

$$\Phi_k(t)=\frac{(-1)^{k+1}}{2i\pi}\frac{k!}{t^{k+1}}$$

to the k-th derivative $\delta^{(k)}(t)$. This shows that an element of $\mathscr{B}^{\infty}_{\{0\}|\mathbb{C}}$ is written in a unique way:

$$u(t)=\sum_{k\in\mathbb{N}}a_k\delta^{(k)}(t)$$

where a_k is a sequence of complex numbers satisfying:

$$\forall\,\varepsilon>0,\,\exists\,C_\varepsilon>0,\,\forall\,k\geqslant 0,\quad |a_k|\leqslant C_\varepsilon\varepsilon^k\frac{1}{k!}$$

and such an u is an element of $\mathscr{B}_{\{0\}|\mathbb{C}}$ if and only if the sum is finite. $\mathscr{B}_{\{0\}|\mathbb{C}}$ is thus the space of distributions with support $\{0\}$ and $\mathscr{B}^{\infty}_{\{0\}|\mathbb{C}}$ the space of hyperfunctions with support $\{0\}$.

For $r\geqslant 1$, we define the spaces of ultradistributions $\mathscr{B}_{\{0\}|\mathbb{C}\{r\}}$ and $\mathscr{B}_{\{0\}|\mathbb{C}(r)}$. An element $u(t)=\sum a_k\delta^{(k)}(t)$ of $\mathscr{B}^{\infty}_{\{0\}|\mathbb{C}}$ is in $\mathscr{B}_{\{0\}|\mathbb{C}\{r\}}$ if the sequence a_k satisfies:

$$\forall\,\varepsilon>0,\,\exists\,C_\varepsilon>0,\,\forall\,k\geqslant 0,\quad |a_k|\leqslant C_\varepsilon\varepsilon^k\frac{1}{(k!)^r}$$

and it is in $\mathscr{B}_{\{0\}|\mathbb{C}(r)}$ if

$$\exists\,C>0,\,\forall\,k\geqslant 0,\quad |a_k|\leqslant C^{k+1}\frac{1}{(k!)^r}$$

The spaces $\mathbb{C}[[t]]_r$ et $\mathscr{B}_{\{0\}|\mathbb{C}\{r\}}$ carry natural topologies for which they are topologically duals and the theorem 1.3.1 may be translated to $\mathscr{B}_{\{0\}|\mathbb{C}\{r\}}$.

Theorem 1.4.1 **(Ramis [9])**

(1) *The operator P has a finite index on $\mathscr{B}_{\{0\}|\mathbb{C}\{r\}}$ and $\mathscr{B}_{\{0\}|\mathbb{C}(r)}$ for any $r\geqslant 1$.*

(2) *If r_i and r_{i+1} are two consecutive slopes of P, then* $\mathrm{Ker}(P, \mathscr{B}_{\{0\}|\mathbb{C}\{r\}})$ *is independent of* $r \in [r_i, r_{i+1}[$, $\mathrm{Ker}(P, \mathscr{B}_{\{0\}|\mathbb{C}^{(r)}})$ *is independent of* $r \in]r_i, r_{i+1}]$, *and they are equal. The same is true for the cokernel.*

(3) *The indices* $\chi(P, \mathscr{B}_{\{0\}|\mathbb{C}\{r\}})$ *and* $\chi(P, \mathscr{B}_{\{0\}|\mathbb{C}^{(r)}})$, *are constant in r except at the slopes of P where their jump (which is also equal to* $\chi(P, \mathscr{B}_{\{0\}|\mathbb{C}^{(r)}}) - \chi(P, \mathscr{B}_{\{0\}|\mathbb{C}\{r\}})$*) is equal to the height of the segment of slope* $1/(r-1)$ *of the Newton Polygon of P.*

2. Microcharacteristic Varieties

We quit now the one dimensional case and we consider a complex analytic manifold X of dimension n and a submanifold Y of X. We will denote by $\Lambda = T^*_Y X$ the conormal bundle to Y in X. In fact, what we are going to do here may be extended to the case where Λ is any homogeneous lagrangian submanifold of T^*X (see [**4**]).

2.1. Newton Polygon of an operator.— Let $(x_1, \ldots, x_p, t_1, \ldots, t_q)$ be local coordinates of X such that $Y = \{\, (x,t) \in X \mid t = 0 \,\}$, then Λ is given by

$$\Lambda = T^*_Y X = \{\, (x,t,\xi,\tau) \in T^*X \mid t = 0, \xi = 0 \,\}$$

We have thus coordinates (x, τ) on Λ and we will denote by (x, τ, x^*, τ^*) the corresponding coordinates on the conormal bundle $T^*\Lambda$.

Let P be a differential operator defined in a neighborhood of Y (or a microdifferential operator defined in a neighborhood of Λ). Its symbol is a formal series:

$$P(x,t,\xi,\tau) = \sum_{j \leqslant m} p_j(x,t,\xi,\tau)$$

where p_j is a homogeneous function of degree j in (ξ, τ) (if P is a differential operator, p_j is polynomial in (ξ, τ) and $p_j \equiv 0$ if $j < 0$).

In both cases, p_j is defined near Λ and thus has a Taylor expansion in (t, ξ):

$$p_j(x,t,\xi,\tau) = \sum_{(\alpha,\beta) \in \mathbb{N}^q \times \mathbb{N}^p} p^{(j)}_{\alpha,\beta}(x,\tau) t^\alpha \xi^\beta$$

and we may define the following functions:

$$p_{ij}(x,\tau,x^*,\tau^*) = \sum_{|\alpha|+|\beta|=i} p^{(j)}_{\alpha,\beta}(x,\tau)(-\tau^*)^\alpha (x^*)^\beta$$

The Newton Polygon of the operator P is defined as the convex hull $\mathbb{N}_\Lambda(P)$ of the union of the sets

$$S_j = \{\, (\lambda, \mu) \in \mathbb{R}^2 \mid \lambda \leqslant j,\ \mu \geqslant i - j \,\}$$

over all (i,j) such that $p_{ij} \not\equiv 0$.

As in dimension 1, the polygon is made of two half-lines and of a finite number of segments. We will be interested in the functions p_{ij} corresponding to (i,j) such that $(j, i-j)$ belongs to one of these segments.

We fix a rational number r between 1 and infinity, denote by D_r the segment of slope $1/(r-1)$ of the boundary of the Newton polygon of P (of course, it is reduced to one point except for a finite number of r). The function $\sigma[r](P)$ is the sum of all functions p_{ij} such that $(j, i-j) \in D_r$.

We define also the functions $\sigma^{(r)}(P)$ and $\sigma^{\{r\}}(P)$ in the following way:

- If r is not a slope of P, that is if D_r is a point, then $\sigma^{(r)}(P) = \sigma^{\{r\}}(P) = \sigma[r](P)$.
- If r is a slope, then $\sigma^{(r)}(P) = \sigma[r-\varepsilon](P)$ and $\sigma^{\{r\}}(P) = \sigma[r+\varepsilon](P)$ for small $\varepsilon > 0$.
- If $r = \infty$, we still define $\sigma^{(\infty)}(P)$ as the function $\sigma^{(r)}(P)$ with r greater than the last slope of P.

The functions $\sigma^{(r)}(P)$ and $\sigma^{\{r\}}(P)$ are always bihomogeneous, that is homogeneous separately in (x^*, τ^*) and (x^*, τ). A function $\sigma[r](P)$ is bihomogeneous if and only if r is not a slope. Moreover $\sigma^{(r)}(P)$ (resp. $\sigma^{\{r\}}(P)$) is the homogeneous part of higher (resp. lower) degree of the Taylor expansion of $\sigma[r](P)$ relative to the variables (x^*, τ^*).

If $\mathscr{M}$ is a coherent $\mathscr{D}_X$-module of the form $\mathscr{D}_X/\mathscr{I}$ with $\mathscr{I}$ ideal of $\mathscr{D}_X$, we may define its microcharacteristic variety of type r as

$$\Sigma_\Lambda^{(r)}(\mathscr{M}) = \{\, (x,\tau,x^*,\tau^*) \in T^*\Lambda \mid \forall\, P \in \mathscr{I},\ \sigma[r](P)(x,\tau,x^*,\tau^*) = 0 \,\}$$

and the same definition applied to $\sigma^{(r)}(P)$ and $\sigma^{\{r\}}(P)$ defines the varieties $\mathrm{Ch}_{\Lambda\,(r)}(\mathscr{M})$ and $\mathrm{Ch}_{\Lambda\,\{r\}}(\mathscr{M})$.

However, these definitions are not very convenient and it is not easy to show that they do not depend on local coordinates or local presentation of $\mathscr{M}$. As for characteristic varieties, we will use filtrations to redefine them.

2.2. The V-filtration. — The sheaf $\mathscr{D}_X|_Y$ of differential operators defined near Y is provided with two canonical filtrations. First, the filtration by the usual order of operators denoted by $(\mathscr{D}_{X,m})_{m\geqslant 0}$ and second the V-filtration of Kashiwara [**3**]:

$$V_k\mathscr{D}_X = \{P \in \mathscr{D}_X|_Y \mid \forall\, j \in \mathbb{Z},\ \ P\mathscr{I}_Y^j \subset \mathscr{I}_Y^{j-k}\}$$

where $\mathscr{I}_Y$ is the definition ideal of Y and $\mathscr{I}_Y^j = \mathscr{O}_X$ if $j \leqslant 0$. These definitions may be extended to microdifferential operators [**4**].

In the local coordinates such that $Y = \{t = 0\}$, we can see that $\mathscr{I}^k$, $k \geqslant 0$, is the sheaf of holomorphic functions

$$\sum_{|\alpha|=k} f_\alpha(x,t)t^\alpha$$

hence the operators x_i and D_{x_i} have order 0 for the V-filtration while the operators t_i have order -1 and D_{t_i} order $+1$.

This proves that $V_k\mathscr{D}_X$ is the sheaf of operators whose Newton Polygon is above the horizontal line with ordinate $-k$, while $\mathscr{D}_{X,k}$ is the sheaf of operators whose Newton Polygon is to the left of the vertical line with abscissa k.

The associated graduate is defined by:

$$\operatorname{gr}_V \mathscr{D}_X := \oplus \operatorname{gr}_V^k \mathscr{D}_X, \qquad \operatorname{gr}_V^k \mathscr{D}_X := V_k\mathscr{D}_X \,/V_{k-1}\mathscr{D}_X$$

By definition of the V-filtration, $\operatorname{gr}_V \mathscr{D}_X$ operates on the direct sum $\bigoplus \left(\mathscr{I}^k \,/\mathscr{I}^{(k+1)}\right)$. But this sheaf is canonically isomorphic to the direct image by the projection $p : T_YX \to Y$ of the sheaf $\mathscr{O}_{[T_YX]}$ of holomorphic functions on the normal bundle T_YX which are polynomial in the fibers of p. In this way $\operatorname{gr}_V \mathscr{D}_X$ is a subsheaf of $p_* \mathscr{H}om_{\mathbb{C}}(\mathscr{O}_{[T_YX]}, \mathscr{O}_{[T_YX]})$ and it is easily verified in local coordinates that this subsheaf is exactly the sheaf of differential operators on T_YX with coefficients in $\mathscr{O}_{[T_YX]}$.

The principal symbol of such an operator is a function on the cotangent bundle $T^*(T_YX)$ but it is more convenient to work on the cotangent bundle T^*_YX where we can use homogeneous symplectic transformations.

As the fiber bundles T_YX et T^*_YX are dual each other, there is a canonical isomorphism between their cotangent bundles $T^*(T_YX)$ and $T^*(T^*_YX)$. In local coordinates this isomorphism is given by $(x, \widetilde{t}, x^*, \widetilde{t^*}) \mapsto (x, -\widetilde{t^*}, x^*, \widetilde{t})$.

To a differential operator P on X, we associate thus its image in $\operatorname{gr}_V \mathscr{D}_X$ which is a differential operator on T_YX, then the principal symbol of this operator which is a function on $T^*(T_YX)$ and finally a function on $T^*(T^*_YX)$. In coordinates this map associates to the operators x_i, D_{x_i}, t_j, D_{t_j} respectively the functions x_i, x^*_i, $-\tau^*_j$ and τ_j.

Moreover, it is clear from the definition that the function associated to P is the p_{ij}-function associated to the lowest vertex of the Newton Polygon of P, that is $\sigma^{(\infty)}(P)$ and also that, for any operators P and Q, we have $\sigma^{(\infty)}(PQ) = \sigma^{(\infty)}(P)\sigma^{(\infty)}(Q)$.

2.3. The F_r-filtrations. — Let r be a rational number written in its irreducible form as $r = p/q$ with $p \geqslant q \geqslant 1$, the F_r-filtration is defined as:

$$F^k_r\mathscr{D}_X = \sum_{(p-q)m+qn=k} \mathscr{D}_{X,n} \cap V_m\mathscr{D}_X$$

In this way, we get a family of filtrations for $1 < r < +\infty$.

By the definition, $F^k_r\mathscr{D}_X$ is the sheaf of operators whose Newton Polygon is above the line which equation is $qj + (q-p)(i-j) = k$. This shows in particular that the Newton Polygon is independent of local coordinates.

It is easily shown [**4**, Proposition 2.2.4] that if $(j, i-j)$ is a point of the distinguished boundary of the Newton Polygon of P (that is a vertex of the polygon or a point on a segment joining two vertices), there exists an operator Q in the intersection of $\mathscr{D}_{X,j}$ and of $V_{j-i}\mathscr{D}_X$ such that $P - Q$ belongs to $\mathscr{D}_{X,j-1} + V_{j-i-1}\mathscr{D}_X$ and $\sigma^{(\infty)}(Q) \not\equiv 0$. This function $\sigma^{(\infty)}(Q)$ is equal to p_{ij} and this proves that the functions p_{ij} associated to points of the distinguished boundary of the Newton Polygon of P are independent of local coordinates.

We saw that $F_r^k\mathscr{D}_X$ is the sheaf of operators whose Newton Polygon is above the line with equation $qj + (q-p)(i-j) = k$. The map which associates to an operator of $F_r^k\mathscr{D}_X$ the function $\sum_{qj+(q-p)(i-j)=k} p_{ij}$ induces a bijection

$$\operatorname{gr}^k_{F_r}\mathscr{D}_X \xrightarrow{\ \sim\ } \bigoplus_{qj+(q-p)(i-j)=k} \pi_*\mathscr{O}_{[T^*\Lambda]}[i,j]$$

where $\mathscr{O}_{[T^*\Lambda]}[i,j]$ is the sheaf of holomorphic functions on $T^*\Lambda$ which are polynomial in the fibers of $T^*\Lambda \to Y$, and homogeneous of degree i in (x^*,τ^*) and of degree j in (x^*,τ) while π is the projection of $T^*\Lambda$ on Y. We have thus an isomorphism

$$\operatorname{gr}_{F_r}\mathscr{D}_X \xrightarrow{\ \sim\ } \pi_*\mathscr{O}_{[T^*\Lambda]}$$

compatible with the ring structures.

The definition of the microcharacteristic variety of a $\mathscr{D}_X$-module is now similar to the definition of the characteristic variety. If $\mathscr{M}$ is a coherent $\mathscr{D}_X$-module, a filtration $(F_r^k\mathscr{M})_{k\in\mathbb{Z}}$ of $\mathscr{M}$ is said to be a good F_r-filtration if locally there exists generators $u_1,\dots,u_N$ of $\mathscr{M}$ and integers $k_1,\dots,k_N$ such that

$$\mathscr{M}_k = \sum_{i=1,\dots,N} F_r^{(k-k_i)}\mathscr{D}_X u_i$$

It may be shown that if $F_r^{\bullet}\mathscr{M}$ is a good filtration of $\mathscr{M}$, the associated graduate $\pi^{-1}\operatorname{gr}_{F_r}\mathscr{M}$ is a coherent $\mathscr{O}_{[T^*\Lambda]}$-module. To this module, we associate the support of $\mathscr{O}_{T^*\Lambda}\otimes\pi^{-1}\operatorname{gr}_{F_r}\mathscr{M}$ which is an analytic subset of $T^*\Lambda$ independent of the choice of the good filtration. This variety is the microcharacteristic variety of type r of $\mathscr{M}$, denoted by $\Sigma^{(r)}_\Lambda(\mathscr{M})$.

We may also consider the positive analytic cycle associated to $\mathscr{O}_{T^*\Lambda}\otimes\pi^{-1}\operatorname{gr}_{F_r}\mathscr{M}$, this means that we may associate a positive multiplicity to each irreducible component of $\Sigma^{(r)}_\Lambda(\mathscr{M})$. The microcharacteristic cycle of type r of $\mathscr{M}$ is denoted by $\widetilde{\Sigma}^{(r)}_\Lambda(\mathscr{M})$.

We can show that if P is a differential operator we have:

$$\Sigma^{(r)}_\Lambda(\mathscr{D}_X/\mathscr{D}_X P) = \{\,\omega\in T^*\Lambda \mid \sigma[r](P)(\omega)=0\,\}$$

and if $\mathscr{I}$ is a coherent ideal of $\mathscr{D}_X$:

$$\Sigma^{(r)}_\Lambda(\mathscr{D}_X/\mathscr{I}) = \{\,\omega\in T^*\Lambda \mid \forall\, P\in\mathscr{I},\ \sigma[r](P)(\omega)=0\,\}$$

2.4. The case of a vector bundle. — In this section we assume that X is a vector bundle on Y and we identify Y with the zero section of X. We denote by $\mathscr{D}_{[X]}$ the sheaf of differential operators on X whose coefficients are polynomial in the fibers of $p: X\to Y$.

To a fiber bundle is associated the Euler vector field θ_X characterized by the fact that $\theta_X f = kf$ for any homogeneous function of degree k.

In local coordinates $(x_1,\dots,x_p,t_1,\dots,t_q)$ where x is a system of local coordinates on Y and t linear coordinates of the fibers, $\theta_X = \langle t, D_t\rangle$.

Let us denote by $\mathscr{D}_{[X]}[k]$ the subsheaf of $\mathscr{D}_{[X]}$ of differential operators P such that $[P, \theta_X] = kP$. This equivalent to the fact that for any function f homogeneous of degree ℓ, then Pf is homogeneous of degree $\ell - k$.

In local coordinates, t_i is in $\mathscr{D}_{[X]}[-1]$, D_{t_i} in $\mathscr{D}_{[X]}[1]$,x_i and D_{x_i} in $\mathscr{D}_{[X]}[0]$.

The sheaf $\mathscr{D}_{[X]}$ is equal to $\bigoplus_k \mathscr{D}_{[X]}[k]$ and the filtration is given by

$$V_k\mathscr{D}_{[X]} = \bigoplus_{j\leqslant k} \mathscr{D}_{[X]}[j]$$

We may define another canonical filtration by

$$V_k^-\mathscr{D}_{[X]} = \bigoplus_{j\geqslant k} \mathscr{D}_{[X]}[j]$$

and then define the F_r-filtration for $r = p/q \leqslant 1$ by the formula:

$$F_r^k\mathscr{D}_X = \sum_{(q-p)m+qn=k} \mathscr{D}_{X,n} \cap V_m^-\mathscr{D}_X$$

Then we may adapt all definition of section 2.3 to any rational number r. In dimension 1, we recover the results of section 1.2.

2.5. Properties of microcharacteristic varieties

Theorem 2.5.1

(1) *The microcharacteristic varieties $\Sigma_\Lambda^{(r)}(\mathscr{M})$ are involutive subsets of $T^*\Lambda$.*

(2) *The dimension of $\Sigma_\Lambda^{(r)}(\mathscr{M})$ is less or equal to the dimension of the characteristic variety of $\mathscr{M}$.*

(3) *It exists a finite number of algebraic slopes (or critical indexes) $1 = r_0 < r_1 < \cdots < r_N = +\infty$ (or $-\infty = r_0 < r_1 < \cdots < r_N = +\infty$ in the case of section 2.4) such that $\Sigma_\Lambda^{(r)}(\mathscr{M})$ is independent of r on each open interval $]r_i, r_{i+1}[$.*

This result have been proved in [**4**].

By construction, the variety $\Sigma_\Lambda^{(r)}(\mathscr{M})$ is homogeneous under the action of $\mathbb{C}$ given in coordinates by $(x, \tau, x^*, \tau^*) \mapsto (x, \lambda^p\tau, \lambda^q x^*, \lambda^{q-p}\tau^*)$ (with $r = p/q$).

If this variety is independent of r for r in an open interval, it is homogeneous for any r hence bihomogeneous, that is separately homogeneous relatively to (x^*, τ^*) and (x^*, τ).

So, varieties $\Sigma_\Lambda^{(r)}(\mathscr{M})$ are of two different kinds, a non bihomogeneous variety for each critical index and a bihomogeneous variety for each interval between critical indexes. In particular, there is only a finite number of distinct microcharacteristic varieties.

That is why we introduce new notations:

$$\mathrm{Ch}_{\Lambda\,(r)}(\mathscr{M}) = \mathrm{Ch}_{\Lambda\,\{r\}}(\mathscr{M}) = \Sigma_\Lambda^{(r)}(\mathscr{M})$$

if r is not a critical index and

$$\mathrm{Ch}_{\Lambda\,(r)}(\mathscr{M}) = \Sigma_\Lambda^{(r-\varepsilon)}(\mathscr{M}), \quad \mathrm{Ch}_{\Lambda\,\{r\}}(\mathscr{M}) = \Sigma_\Lambda^{(r+\varepsilon)}(\mathscr{M})$$

for small $\varepsilon > 0$ if r is a critical index.

The variety $\mathrm{Ch}_{\Lambda\,(r)}(\mathscr{M})$ corresponds to the function $\sigma^{(r)}(P)$ while $\mathrm{Ch}_{\Lambda\,\{r\}}(\mathscr{M})$ corresponds to $\sigma^{\{r\}}(P)$. They are involutive bihomogeneous varieties.

All these definitions extend to the microcharacteristic cycles and we will denote by $\widetilde{\mathrm{Ch}}_{\Lambda(r)}(\mathscr{M})$ and $\widetilde{\mathrm{Ch}}_{\Lambda\{r\}}(\mathscr{M})$ the cycles corresponding to the varieties $\mathrm{Ch}_{\Lambda\,(r)}(\mathscr{M})$ and $\mathrm{Ch}_{\Lambda\,\{r\}}(\mathscr{M})$ and which are defined in the same way.

The most interesting case is the case of holonomic $\mathscr{D}_X$-modules. Then the characteristic variety is lagrangian hence has the minimal dimension, that is the dimension n of X. But the previous theorem shows that the dimension of the microcharacteristic varieties is less than n and that they are involutive, they are thus lagrangian too. We will see in section 4.1 that bihomogeneous lagrangian varieties have very particular properties.

3. Sheaves of solutions

As stated in the first section, we will state the index formulas with the sheaves of holomorphic hyperfunctions, but similar results could be stated with formal completions of holomorphic functions. Holomorphic hyperfunctions may be microlocalized and this is very interesting for example to change the codimension of the submanifold Y.

3.1. Holomorphic Hyperfunctions. — We assume that Y has codimension 1 and we denote by $\mathscr{B}^\infty_{Y|X}$ the sheaf of holomorphic hyperfunctions, which is defined as the cohomology sheaf:

$$\mathscr{B}^\infty_{Y|X} = \mathscr{H}^1_Y(\mathscr{O}_X)$$

that is the sheaf $j_*j^*\mathscr{O}_X/\mathscr{O}_X$, if $j : X - Y \hookrightarrow X$ is the canonical immersion. This means that it is the quotient of holomorphic functions on X with singularities on Y modulo holomorphic functions.

The sheaf $\mathscr{B}_{Y|X}$ is the subsheaf of $\mathscr{B}^\infty_{Y|X}$ generated by meromorphic functions on Y, it is the algebraic cohomology sheaf:

$$\mathscr{B}_{Y|X} = \mathscr{H}^1_{[Y]}(\mathscr{O}_X)$$

Let us fix local coordinates (x, t) of X such that $Y = \{\, t = 0 \,\}$. Let V be an open subset of Y and U a neighborhood of Y in X. A holomorphic function on $U - V$ is given by a Laurent expansion:

$$f(x, t) = \sum_{k\in\mathbb{Z}} f_k(x)t^k$$

The sum $\sum_{k\geqslant 0} f_k(x)t^k$ is holomorphic near Y hence its class in $\mathscr{B}^\infty_{Y|X}$ is 0. Using the functions $\Phi_k(t)$ of section 1.4, the negative part of the expansion may be written

$$f(x,t) = \sum_{k\geqslant 0} f_k(x)\Phi_k(t)$$

and thus the class of f is a section of $\mathscr{B}^\infty_{Y|X}$ on V given by:

$$u(x,t) = \sum_{k\geqslant 0} f_k(x)\delta^{(k)}(t) \tag{3.1.1}$$

where the functions f_k are holomorphic on V.

The fact that f has essential singularity on Y is equivalent to the following conditions on the functions f_k:

$$\forall\, \varepsilon > 0,\ \forall\, K \subset\subset Y,\ \exists\, C_\varepsilon > 0,\ \forall\, k \geqslant 0, \quad \sup_K |f_k(x)| \leqslant C_\varepsilon \varepsilon^k \frac{1}{k!} \tag{3.1.2}$$

The set $\Gamma(V, \mathscr{B}^\infty_{Y|X})$ of holomorphic hyperfunctions on Y is thus equal to the set of series 3.1.1 satisfying the condition 3.1.2. The subsheaf $\mathscr{B}_{Y|X}$ correspond to meromorphic functions hence $\Gamma(V, \mathscr{B}_{Y|X})$ is equal to the set of series 3.1.1 where the sum is finite.

As in dimension 1, we define now the sheaves of holomorphic hyperfunctions with Gevrey growth. For any $r \geqslant 1$, $\Gamma(V, \mathscr{B}_{Y|X\{r\}})$ is the set of series 3.1.1 satisfying:

$$\forall\, \varepsilon > 0,\ \forall\, K \subset\subset V,\ \exists\, C_\varepsilon > 0,\ \forall\, k \geqslant 0, \quad \sup_K |f_k(x)| \leqslant C_\varepsilon \varepsilon^k \frac{1}{(k!)^r} \tag{3.1.3}$$

while $\Gamma(V, \mathscr{B}_{Y|X(r)})$ is the set of series 3.1.1 satisfying:

$$\forall\, K \subset\subset V,\ \exists\, C_\varepsilon > 0,\ \forall\, k \geqslant 0, \quad \sup_K |f_k(x)| \leqslant C^{k+1} \frac{1}{(k!)^r} \tag{3.1.4}$$

It is easy to verify that these conditions are independent of local coordinates and define subsheaves $\mathscr{B}_{Y|X\{r\}}$ and $\mathscr{B}_{Y|X(r)}$ of $\mathscr{B}^\infty_{Y|X}$.

If Y is a submanifold of codimension d of X, the sheaf $\mathscr{B}^\infty_{Y|X}$ is defined as $\mathscr{H}^d_Y(\mathscr{O}_X)$, the sheaf $\mathscr{B}_{Y|X}$ as $\mathscr{H}^d_{[Y]}(\mathscr{O}_X)$ and we can still define sheaves $\mathscr{B}_{Y|X\{r\}}$ and $\mathscr{B}_{Y|X(r)}$, see [**4**] for the details.

3.2. Index theorems. — The index of a coherent $\mathscr{D}_X$-module $\mathscr{M}$ with value in a sheaf $\mathscr{F}$ is the function:

$$\chi(\mathscr{M}, \mathscr{F})_x = \sum_j (-1)^j \dim_{\mathbb{C}} \mathscr{E}xt^j_{\mathscr{D}_X}(\mathscr{M}, \mathscr{F})_x$$

that is the Euler characteristic of the complex $\mathbb{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{F})$. This index is well defined if all $\mathscr{E}xt^j$ are finite dimensional $\mathbb{C}$-vector spaces, especially if the sheaves $\mathscr{E}xt^j(\mathscr{M}, \mathscr{F})$ are constructible.

Kashiwara proved in [**2**] that for any holonomic $\mathscr{D}_X$-module $\mathscr{M}$, the sheaves of holomorphic solutions $\mathscr{E}xt^j_{\mathscr{D}_X}(\mathscr{M},\mathscr{O}_X)$ are constructible and the index $\chi(\mathscr{M},\mathscr{O}_X)$ is equal to the Euler obstruction of the characteristic cycle of $\mathscr{M}$.

Let us recall that an analytic cycle is a variety with integers, called multiplicities, attached to each irreducible component. Such a cycle is written:

$$\widetilde{\Sigma} = \sum n_i[S_i]$$

where n_i are integers and S_i irreducible varieties.

The support of $\widetilde{\Sigma}$ is the union of all S_i such that $n_i \neq 0$. If all n_i are $\geqslant 0$, the cycle is said to be positive. If all S_i are lagrangian the cycle is lagrangian.

In particular, the characteristic variety of a holonomic $\mathscr{D}_X$-module is a lagrangian subvariety of the cotangent bundle and thus each irreducible component is of pure dimension $n = \dim_{\mathbb{C}} X$. The analytic cycle of $\mathscr{M}$ is, by definition a positive cycle with support the characteristic variety.

To each lagrangian analytic cycle $\widetilde{\Sigma}$ of T^*X is associated a constructible function on X, the local Euler obstruction $\mathrm{Eu}\big(\widetilde{\Sigma}\big)$. (A map $X \to \mathbb{Z}$ is constructible if it is constant on the strata of some stratification of X). We do not give the precise definition of this function (see [**1**]) but we recall its main properties:

- The Euler obstruction is a one-to-one map from the lagrangian analytic cycles of T^*X to the constructible functions on X.
- It is additive, in particular

$$\mathrm{Eu}\Big(\sum n_i[\Lambda_i]\Big) = \sum n_i \,\mathrm{Eu}\,(\Lambda_i)$$

- If Z is a submanifold of X with codimension d then $\mathrm{Eu}\,(T^*_Z X) = (-1)^d \mathbb{C}_Z$.

So, Kashiwara's theorem relates the Euler characteristic of the complex

$$\mathbb{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\mathscr{O}_X)$$

and the Euler obstruction of the characteristic cycle of $\mathscr{M}$. In fact this complex is a perverse sheaf and this correspond to the fact that the characteristic cycle is positive.

A similar theorem has been proved in [**5**] relating the holomorphic hyperfunction solutions with the microcharacteristic cycles:

Theorem 3.2.1. — *Let $\mathscr{M}$ be a holonomic $\mathscr{D}_X$-module defined near Y, then for each $r \in\,]1,+\infty[$, the sheaves of solutions $\mathscr{E}xt^j_{\mathscr{D}_X}(\mathscr{M},\mathscr{B}_{Y|X^{(r)}})$ and $\mathscr{E}xt^j_{\mathscr{D}_X}(\mathscr{M},\mathscr{B}_{Y|X\{r\}})$ are constructible and*

$$\chi\left(\mathscr{M},\mathscr{B}_{Y|X^{(r)}}\right) = \mathrm{Eu}\left(\widetilde{\mathrm{Ch}}_{\Lambda^{(r)}}(\mathscr{M})\right) \tag{3.2.1}$$

$$\chi\left(\mathscr{M},\mathscr{B}_{Y|X\{r\}}\right) = \mathrm{Eu}\left(\widetilde{\mathrm{Ch}}_{\Lambda\{r\}}(\mathscr{M})\right) \tag{3.2.2}$$

It also been proved in [**5**] that the sheaves $\mathscr{E}xt^j_{\mathscr{D}_X}(\mathscr{M},\mathscr{B}_{Y|X\{r\}})$ are independent of r outside the critical indexes that is:

Theorem 3.2.2. — *Let $\mathscr{M}$ be a holonomic $\mathscr{D}_X$-module defined near Y, r_k and r_{k+1} two* successive *critical indexes, then for any r in $]r_k, r_{k+1}[$, there are isomorphisms:*

$$\begin{aligned}
&\mathbb{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{B}_{Y|X(r_k)}) = \mathbb{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{B}_{Y|X(r)}) \qquad (3.2.3)\\
&\qquad = \mathbb{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{B}_{Y|X\{r\}})\\
&\qquad = \mathbb{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{B}_{Y|X\{r_{k+1}\}}) \qquad (3.2.4)
\end{aligned}$$

The problem with theorem 3.2.1 is that it makes a connection between a complex of sheaves on Y, $\mathbb{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{B}_{Y|X(r)})$ and an analytic cycle on $T^*T^*_YX$, $\widetilde{\mathrm{Ch}}_{\Lambda(r)}(\mathscr{M})$. This means that to compute the index, we take a cycle on $T^*T^*_YX$, compute its Euler obstruction which is a function on T^*_YX and then restrict this function to Y. But in this way we have no connection between perverse sheaves and positivity of cycles.

Indeed $\widetilde{\mathrm{Ch}}_{\Lambda(r)}(\mathscr{M})$ is positive and the complex $\mathbb{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{B}_{Y|X(r)})$ is not perverse while the complex

$$\mathbb{R}\mathscr{H}om_{\mathscr{D}_X}\left(\mathscr{M}, \mathscr{B}_{Y|X(r)}\big/\mathscr{B}_{Y|X\{r\}}\right)$$

is perverse but its index is computed by the cycle $\widetilde{\mathrm{Ch}}_{\Lambda(r)}(\mathscr{M}) - \widetilde{\mathrm{Ch}}_{\Lambda\{r\}}(\mathscr{M})$ which is not positive. To solve this problem, we have to study more closely lagrangian cycles on $T^*T^*_YX$ and connect them to cycles on T^*Y, this the object of section 4.

4. Geometric irregularity

4.1. Bihomogeneous lagrangian varieties. — Let $\Lambda \to Y$ be a fiber bundle $\Lambda \to Y$. It is provided with an Euler vector field θ_Λ which is characterized by the fact that $\theta_\Lambda f = kf$ for any function f homogeneous of degree k in the fibers of Λ. This vector field may be considered as a function on the cotangent bundle $T^*\Lambda$. This function defines a canonical hypersurface of $T^*\Lambda$ which will be denoted by S_Λ.

This hypersurface may also be defined in another way. The action of $\mathbb{C}^*$ on the fibers of Λ induces an action H_∞ of $\mathbb{C}^*$ on $T^*\Lambda$. The manifold $T^*\Lambda$ is thus provided with two actions of $\mathbb{C}^*$, H_∞ and the usual action H_0 on the fibers of $T^*\Lambda \to \Lambda$.

To each action is associated an Euler vector field. The manifold $T^*\Lambda$ is thus provided with two vector fields u_∞ et u_0. As any cotangent bundle, $T^*\Lambda$ has also a symplectic structure hence a canonical 2-form Ω. The function $\Omega(u_\infty, u_0)$ is thus well defined on $T^*\Lambda$ and equal to the function defined by θ_Λ as we can see in coordinates.

Let us fix local coordinates $(x_1, \dots, x_p, \tau_1, \dots, \tau_q)$ of Λ where Λ is a system of local coordinates on Y and τ linear coordinates of the fibers as in section 2.4. The action of $\mathbb{C}^*$ is given by $(x, \tau) \mapsto (x, \lambda\tau)$, the Euler vector field is $\theta_\Lambda = \langle \tau, \partial/\partial\tau\rangle$ and define the function $\langle \tau, \tau^*\rangle$ in the coordinates (x, τ, x^*, τ^*) of $T^*\Lambda$.

The two actions of $\mathbb{C}^*$ on $T^*\Lambda$ are $H_\infty : (x, \tau, x^*, \tau^*) \mapsto (x, \lambda\tau, x^*, \lambda^{-1}\tau^*)$ associated to the vector field $u_\infty = \langle \tau, \partial/\partial\tau\rangle - \langle \tau^*, \partial/\partial\tau^*\rangle$ and $H_0 : (x, \tau, x^*, \tau^*) \mapsto (x, \tau, \lambda x^*, \lambda\tau^*)$ associated to $u_0 = \langle \tau^*, \partial/\partial\tau^*\rangle + \langle x^*, \partial/\partial x^*\rangle$. The value of the canonical 2-form $\Omega = dx^* \wedge dx + d\tau^* \wedge d\tau$ on these vector field is $\langle \tau, \tau^*\rangle$ i.e. θ_Λ.

Let us consider a lagrangian bihomogeneous submanifold Σ of $T^*\Lambda$. The 2-form Ω vanishes on the vector fields tangent to Σ because it is lagrangian. If it is bihomogeneous, the two vector fields u_∞ and u_0 are tangent to Σ and thus the function $\Omega(u_\infty, u_0)$ vanishes on Σ. If Σ is not smooth the same result is still true for the smooth part of Σ.

Finally, we proved that the bihomogeneous lagrangian varieties are all included in the canonical hypersurface.

We will use later the following family of actions of $\mathbb{C}^*$ on $T^*\Lambda$: Let $r = p/q$ be a rational number, p/q being irreducible, we define the action $H_r = H_\infty^p H_0^q$ associated to the vector field $pu_\infty + qu_o$. In coordinates it is $H_r : (x, t, x^*, \tau^*) \mapsto (x, \lambda^p t, \lambda^q x^*, \lambda^{q-p}\tau^*)$.

The canonical immersion $Y \hookrightarrow \Lambda$ and the projection $\Lambda \to Y$ define maps:

$$T^*Y \xleftarrow{p_1} (T^*Y) \times_Y \Lambda \xrightarrow{j_1} T^*\Lambda$$

$$T^*Y \xleftarrow{p_2} (T^*\Lambda) \times_\Lambda Y \xrightarrow{j_2} T^*\Lambda$$

The maps p_1 et p_2 are submersions while j_1 and j_2 are immersions. In local coordinates we have:

$$p_1(x, x^*, \tau) = (x, x^*) \qquad j_1(x, x^*, \tau) = (x, \tau, x^*, 0)$$
$$p_2(x, x^*, \tau^*) = (x, x^*) \qquad j_2(x, x^*, \tau^*) = (x, 0, x^*, \tau^*)$$

We assume now that Λ is a line bundle over Y, i.e. that the fibers of $p : \Lambda \to Y$ are of dimension 1. In local coordinate we have $S_\Lambda = \{ (x, \tau, x^*, \tau^*) \in T^*\Lambda \mid \tau\tau^* = 0 \}$ and therefore the union of $j_1((T^*Y) \times_Y \Lambda)$ and of $j_2((T^*\Lambda) \times_\Lambda Y)$ is exactly S_Λ.

Lemma 4.1.1. — *Let Λ be a line bundle over Y and Σ be a bihomogeneous lagrangian subvariety of $T^*\Lambda$. There exists two homogeneous lagrangian subvarieties S_1 and S_2 of T^*Y such that:*

$$\Sigma = j_1 p_1^{-1} S_1 \cup j_2 p_2^{-1} S_2$$

Proof. — Let us consider an irreducible component of Σ. It is included in the set $S_\Lambda = \{ \tau\tau^* = 0 \}$ but, because it is irreducible, it is included in $\{ \tau = 0 \}$ or in $\{ \tau^* = 0 \}$. As it is lagrangian, if it is included in $\{ \tau^* = 0 \}$, it is invariant by the Hamiltonian vector field of τ^* hence of the form:

$$\{ (x, \tau, x^*, \tau^*) \in T^*\Lambda \mid \tau = 0, (x, x^*) \in S_1 \}$$

and it is clear that S_1 is a homogeneous lagrangian subvariety of T^*Y.

The proof is the same for a component included in $\{ \tau^* = 0 \}$. □

An analytic cycle $\widetilde{\Sigma}$ whose support satisfy the hypothesis of lemma 4.1.1 decomposes in the sum of two cycles:

$$\widetilde{\Sigma} = j_1 p_1^{-1} \widetilde{S}_1 + j_2 p_2^{-1} \widetilde{S}_2$$

where $\widetilde{S}_1$ and $\widetilde{S}_2$ are two analytic cycles of T^*Y.

4.2. Filtrations. — We keep the notations of the previous section, Λ being a fiber bundle but perhaps not a line bundle. Let $\mathscr{O}_{[T^*\Lambda]}$ be the sheaf of holomorphic functions on $T^*\Lambda$ which are polynomial in the fibers of $T^*\Lambda \to Y$ and $\mathscr{O}_{(T^*\Lambda)}(i,j)$ be the subsheaf of functions which are homogeneous of degree i for H_0 and of degree $j-i$ for H_∞. If f is a section of $\mathscr{O}_{(T^*\Lambda)}(i,j)$ we have thus $u_0 f = if$ and $u_\infty f = (j-i)f$.

We consider two filtrations on $\mathscr{O}_{[T^*\Lambda]}$ relative to H_∞, that is

$$F_k^+\mathscr{O}_{[T^*\Lambda]} = \bigoplus_{i-j\geqslant k} \mathscr{O}_{[T^*\Lambda]}(i,j)$$
$$F_k^-\mathscr{O}_{[T^*\Lambda]} = \bigoplus_{i-j\leqslant k} \mathscr{O}_{[T^*\Lambda]}(i,j)$$

It is easy to verify that these filtrations are "noetherian" [**4**, proposition 3.2.3.] and this means in particular that if $\mathscr{M}$ is a coherent $\mathscr{O}_{[T^*\Lambda]}$-module provided with a good F^+ or F^- filtration, the associated graduate is a coherent $\mathscr{O}_{[T^*\Lambda]}$-module.

Let us now consider an irreducible subvariety Σ of $T^*\Lambda$ with equations in $\mathscr{O}_{[T^*\Lambda]}$ and denote by $[\Sigma]$ the associated cycle with multiplicity 1. The definition ideal of Σ is:

$$\mathscr{I}_\Sigma = \{\, f \in \mathscr{O}_{[T^*\Lambda]} \mid f|_\Sigma = 0 \,\}$$

It is provided with the two filtrations F^+ and F^- induced by those of $\mathscr{O}_{[T^*\Lambda]}$ and we denote by $\mathscr{I}_\Sigma^\pm$ the corresponding graduate.

More explicitly, a function f of $\mathscr{O}_{[T^*\Lambda]}$ is written as a finite sum $\sum f_{ij}$ with f_{ij} in $\mathscr{O}_{[T^*\Lambda]}(i,j)$. Let k_+ and k_- be respectively the highest and the lowest values of $i-j$ such that $f_{ij} \not\equiv 0$, we set:

$$\sigma^\pm(f) = \sum_{i-j=k_\pm} f_{ij}$$

and we have:

$$\mathscr{I}_\Sigma^\pm = \{\, \sigma^\pm(f) \mid f \in \mathscr{I}_\Sigma \,\}$$

The coherent ideals $\mathscr{I}_\Sigma^\pm$ define two positive analytic cycles denoted by $C^\pm([\widetilde{\Sigma}])$ with supports $C^\pm(\Sigma)$.

If $\widetilde{\Sigma}$ is an analytic cycle of $T^*\Lambda$ whose support is defined by equations in $\mathscr{O}_{[T^*\Lambda]}$ we define $C^\pm(\widetilde{\Sigma})$ by linearity, that is if $\widetilde{\Sigma} = \sum n_i[\Sigma_i]$ with Σ_i irreducible, we set:

$$C^\pm(\widetilde{\Sigma}) = \sum_i n_i C^\pm([\Sigma_i])$$

The support of $C^\pm(\widetilde{\Sigma})$ is $C^\pm(\Sigma)$, it is homogeneous for H_∞. We can prove [**6**] that if the support Σ of $\widetilde{\Sigma}$ is lagrangian than the same is true for the supports of $C^\pm(\widetilde{\Sigma})$.

Assume that Σ is homogeneous under H_r (see sec. 4.1), then the same is true for $C^\pm(\Sigma)$ hence they are bihomogeneous.

We assume now that Λ is a line bundle and that $\widetilde{\Sigma}$ is an analytic cycle of $T^*\Lambda$ whose support is a lagrangian subvariety of $T^*\Lambda$ which is homogeneous for H_r. Than we have two bihomogeneous lagrangian cycles $C^{\pm}(\widetilde{\Sigma})$ to which we may apply the results of the previous section to get the following decomposition:

$$C^{\pm}(\widetilde{\Sigma}) = j_1 p_1^{-1} \widetilde{S}_1^{\pm} + j_2 p_2^{-1} \widetilde{S}_2^{\pm}$$

and we define the irregularity of $\widetilde{\Sigma}$ as:

$$\operatorname{Irr}(\widetilde{\Sigma}) = C_2^{-}(\widetilde{\Sigma}) - C_2^{+}(\widetilde{\Sigma}) - C_1^{-}(\widetilde{\Sigma}) + C_1^{+}(\widetilde{\Sigma})$$

The main result of [**6**] is:

Theorem 4.2.1. — *If $\widetilde{\Sigma}$ is a* positive *lagrangian analytic cycle, homogeneous for H_r, then* $\operatorname{Irr}(\widetilde{\Sigma})$ *is a* positive *lagrangian analytic cycle of T^*Y. Moreover $r \operatorname{Irr}(\widetilde{\Sigma})$ has integral coefficients and* $\operatorname{Irr}(\widetilde{\Sigma})$ *is zero if and only if $\widetilde{\Sigma}$ is bihomogeneous.*

This means that the multiplicities of $\operatorname{Irr}(\widetilde{\Sigma})$ are positive integers, multiple of the denominator q of r.

4.3. Monoidal transform. — If Λ is a fiber bundle with fibers of dimension greater than 1, we may no more apply lemma 4.1.1, but we can still make a similar decomposition. To do that we consider the monoidal transform of Λ.

Let us denote by $\dot{\Lambda} = \Lambda - Y$, the fiber bundle less its zero section and by $\mathbb{P}\Lambda$ the associated projective bundle that is $\dot{\Lambda}/\mathbb{C}^*$. The monoidal transform of Λ is by definition the set:

$$\widetilde{\Lambda} = \{\, (x, \omega) \in \Lambda \times \mathbb{P}\Lambda \mid x \in \omega \,\}$$

By definition, a point ω of $\mathbb{P}\Lambda$ is a class of $\dot{\Lambda}$ modulo the action of $\mathbb{C}^*$ hence a line of a fiber of Λ. So, $x \in \omega$ means either that $x \in Y$ and then x is the projection of ω under $\mathbb{P}\Lambda \to Y$ or that $x \in \dot{\Lambda}$ and in in this case ω is the class of x. This shows that $\widetilde{\Lambda}$ is the disjoint union of $\dot{\Lambda}$ and of $\mathbb{P}\Lambda$.

The set $\widetilde{\Lambda}$ has a structure of line bundle over $\mathbb{P}\Lambda$ and there is a canonical projection $\pi : \widetilde{\Lambda} \to \Lambda$ (which is the identity on $\dot{\Lambda}$).

If Λ is a line bundle, then $\mathbb{P}\Lambda$ is isomorphic to Y and $\pi : \widetilde{\Lambda} \to \Lambda$ is an isomorphism of line bundles.

Now let Σ be a lagrangian irreducible subvariety of $T^*\Lambda$ which is homogeneous for H_r. We assume that Σ is not contained in $T^*\Lambda \times_\Lambda Y$. We intersect Σ with $T^*\dot{\Lambda} = T^*\Lambda \times_\Lambda (\Lambda - Y)$, this set may be considered as a subset of $T^*\widetilde{\Lambda}$ by π and we take its closure in $T^*\widetilde{\Lambda}$ which we denote by $\overline{\Sigma}$. It is easy to see that $\overline{\Sigma}$ is lagrangian and homogeneous for H_r. So, as $\widetilde{\Lambda}$ is a line bundle over $\mathbb{P}\Lambda$ we may apply the definition of the previous section to get the irregularity of $\widetilde{\Sigma} = [\Sigma]$ which is a positive cycle on $\mathbb{P}\Lambda$.

If Σ is contained in $T^*\Lambda \times_\Lambda Y$, it is bihomogeneous and we set $\operatorname{Irr}(\widetilde{\Sigma}) = 0$. By linearity, we get a definition of $\operatorname{Irr}(\widetilde{\Sigma})$ for any lagrangian H_r-homogeneous cycle $\widetilde{\Sigma}$ and by the definition the theorem 4.2.1 is still true.

5. Application to $\mathscr{D}_X$-modules

5.1. The irregularity of a $\mathscr{D}_X$-module. — If $\mathscr{M}$ is a holonomic $\mathscr{D}_X$-module, we have defined its microcharacteristic cycles $\widetilde{\Sigma}^{(r)}_\Lambda(\mathscr{M})$ and by theorem 2.5.1 they satisfy the conditions of lemma 4.1.1 hence we may consider its irregularity:

Definition 5.1.1. — If $\mathscr{M}$ is a holonomic $\mathscr{D}_X$-module, the irregularity of type r of $\mathscr{M}$ is the cycle:

$$\operatorname{Irr}_{(r)}\mathscr{M} = \operatorname{Irr}\left(\widetilde{\Sigma}^{(r)}_\Lambda(\mathscr{M})\right)$$

If r is not a critical index for $\mathscr{M}$ then $\Sigma^{(r)}_\Lambda(\mathscr{M})$ is bihomogeneous hence $\operatorname{Irr}_{(r)}\mathscr{M} = 0$. On the other hand, if r is a critical index we have

$$C^-(\widetilde{\Sigma}^{(r)}_\Lambda(\mathscr{M})) = \widetilde{\operatorname{Ch}}_{\Lambda(r)}(\mathscr{M}) \quad \text{and} \quad C^+(\widetilde{\Sigma}^{(r)}_\Lambda(\mathscr{M})) = \widetilde{\operatorname{Ch}}_{\Lambda\{r\}}(\mathscr{M})$$

The irregularity of type r is thus equal to

$$\operatorname{Irr}_{(r)}\mathscr{M} = C_2(\widetilde{\operatorname{Ch}}_{\Lambda\{r\}}(\mathscr{M})) - C_1(\widetilde{\operatorname{Ch}}_{\Lambda\{r\}}(\mathscr{M})) - C_2(\widetilde{\operatorname{Ch}}_{\Lambda(r)}(\mathscr{M})) + C_1(\widetilde{\operatorname{Ch}}_{\Lambda(r)}(\mathscr{M}))$$

The irregularity of $\mathscr{M}$ is the finite sum:

$$\operatorname{Irr}\mathscr{M} = \sum_{r>1} \operatorname{Irr}_{(r)}\mathscr{M}$$

Applying theorem 4.2.1, we get:

Corollary 5.1.2. — *$\operatorname{Irr}_{(r)}\mathscr{M}$ is a positive lagrangian analytic cycle of T^*Y such that $r\operatorname{Irr}(\widetilde{\Sigma})$ has integral multiplicities. It is the zero if and only if r is not a critical index of $\mathscr{M}$.*

$\operatorname{Irr}\mathscr{M}$ is a positive lagrangian analytic cycle of T^*Y which is the zero if and only if $\mathscr{M}$ has no critical index.

As an exercise, we may verify that the previous definition are compatible with the definitions of section 1.1. If the dimension of X is 1 then T^*Y is a point and the cycles on T^*Y are ordinary numbers.

Let P is a differential operator and $t^a\tau^b + \cdots + t^\ell\tau^m$ its determining equation associated to the critical index r. We assume $b > m$ and as the slope is $1/r - 1$ we have $(r-1)a - rb = (r-1)\ell - rm$.

Let $\mathscr{M} = \mathscr{D}_X/\mathscr{D}_X P$. The cycle $\widetilde{\operatorname{Ch}}_{\Lambda\{r\}}(\mathscr{M})$ is given by the function $(\tau^*)^\ell\tau^m$ hence is $\ell[\tau^* = 0] + m[\tau^* = 0]$ while $\widetilde{\operatorname{Ch}}_{\Lambda(r)}(\mathscr{M})$ is given by $(\tau^*)^a\tau^b$ hence is $a[\tau^* = 0] + b[\tau^* = 0]$. We get:

$$C_2(\widetilde{\operatorname{Ch}}_{\Lambda\{r\}}(\mathscr{M})) = m,\ C_1(\widetilde{\operatorname{Ch}}_{\Lambda\{r\}}(\mathscr{M})) = \ell,\ C_2(\widetilde{\operatorname{Ch}}_{\Lambda(r)}(\mathscr{M})) = b,\ C_1(\widetilde{\operatorname{Ch}}_{\Lambda(r)}(\mathscr{M})) = a$$

and the irregularity of type r is $(a - b) - (\ell - m)$ which is the height of the segment of slope $1/(r-1)$ of the Newton Polygon of P.

We may remark that the fact that the coefficients of $\mathrm{Irr}(\widetilde{\Sigma})$ are multiple of the denominator of r for each r is equivalent to the fact that the vertices of the Newton Polygon of P have integral coordinates.

5.2. Positivity and perversity. — We can now connect these results with the sheaf of Irregularity defined by Mebkhout [**8**]. From the definition of Euler obstruction and lemma 4.1.1 , it is easy to see that if x is a point of Y and $\widetilde{\Sigma}$ a lagrangian bihomogeneous cycle of $T^*\Lambda$ (with $\Lambda = T^*_Y X$), then the value of $\mathrm{Eu}(\widetilde{\Sigma})$ is $\mathrm{Eu}(S_1) - \mathrm{Eu}(S_2)$.

Applying theorem 3.2.1 we get:

$$\begin{aligned}\chi\left(\mathscr{M}, \mathscr{B}_{Y|X(r)} \middle/ \mathscr{B}_{Y|X\{r\}}\right) &= \chi\left(\mathscr{M}, \mathscr{B}_{Y|X(r)}\right) - \chi\left(\mathscr{M}, \mathscr{B}_{Y|X\{r\}}\right) \\ &= \mathrm{Eu}\left(\widetilde{\mathrm{Ch}}_{\Lambda(r)}(\mathscr{M})\right) - \mathrm{Eu}\left(\widetilde{\mathrm{Ch}}_{\Lambda\{r\}}(\mathscr{M})\right) = \mathrm{Eu}\left(\mathrm{Irr}_{(r)}\mathscr{M}\right)\end{aligned}$$

Theorem 5.2.1. — *If $\mathscr{M}$ is a holonomic module defined near Y, then we have:*

$$\chi\left(\mathscr{M}, \mathscr{B}_{Y|X(r)} \middle/ \mathscr{B}_{Y|X\{r\}}\right) = \mathrm{Eu}\left(\mathrm{Irr}_{(r)}\mathscr{M}\right)$$

and

$$\chi\left(\mathscr{M}, \mathscr{B}^{\infty}_{Y|X} \middle/ \mathscr{B}_{Y|X}\right) = \mathrm{Eu}\left(\mathrm{Irr}\,\mathscr{M}\right)$$

Now, we have connected the index of sheaves on Y with the analytic cycles on T^*Y and the perverse sheaf $\mathbb{R}\mathscr{H}om_{\mathscr{D}_X}\left(\mathscr{M}, \mathscr{B}_{Y|X(r)} \middle/ \mathscr{B}_{Y|X\{r\}}\right)$ correspond to a positive analytic cycle.

This index is zero if and only if r is not a critical index of the module. This shows in particular that $\mathscr{M}$ has no critical index if and only if $\mathbb{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{B}_{Y|X})$ is equal to $\mathbb{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{B}^{\infty}_{Y|X})$.

We may also use holomorphic functions and consider the formal completion of $\mathscr{O}_X$ along Y denoted by $\mathscr{O}_{\widehat{X|Y}}$. We have then analogous results and $\mathscr{M}$ has no critical index if and only if $\mathbb{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X)$ is equal to $\mathbb{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_{\widehat{X|Y}})$.

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Y. Laurent, Institut Fourier, Unité Mixte de Recherche du CNRS 5582, Université Joseph Fourier, BP 74, Saint Martin d'Hères Cedex, France • *E-mail :* Yves.Laurent@ujf-grenoble.fr