Séminaires & Congrès
8, 2004, p. 59–87

THE LOCAL DUALITY THEOREM IN $\mathscr{D}$-MODULE THEORY

by

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Abstract. — These notes are devoted to the Local Duality Theorem for $\mathscr{D}$-modules, which asserts that the topological Grothendieck-Verdier duality exchanges the de Rham complex and the solution complex of holonomic modules over a complex analytic manifold. We give Mebkhout's original proof and the relationship with Kashiwara-Kawai's proof. In that way we are able to precise the commutativity of some diagrams appearing in the last one.

***Résumé* (Le théorème de dualité locale dans la théorie des $\mathscr{D}$-modules).** — Ce cours est consacré au théorème de du alité locale pour les $\mathscr{D}$-modules, qui affirme que la dualité topologique de Grothendieck-Verdier échange le complexe de de Rham et le complexe des solutions des modules holonomes sur une variété analytique complexe. On donne la preuve originale de Mebkhout en faisant le rapport avec la preuve de Kashiwara-Kawai. Ceci nous permet de préciser la commutativité de certains diagrammes dans cette dernière.

Contents

2000 ***Mathematics Subject Classification.*** — 32C38, 32S60.
Key words and phrases. — de Rham complex, Grothendieck-Verdier duality.

Supported by DGES PR97, BFM2001-3207 and FEDER.

Introduction

These notes are issued from a course taught in the C.I.M.P.A. School on Differential Systems, held at Seville (Spain) from September 2 through September 13, 1996. They are an improved version of the handwritten notes distributed during the School.

The aim of these notes is to introduce the reader to the Local Duality Theorem in $\mathcal{D}$-module Theory —LDT for short— and to explain in a detailed way the proofs of it in [**Me3**], [**K-K**]. This theorem asserts that the Verdier duality for analytic constructible complexes interchanges the "De Rham" and the "Solutions" of every bounded holonomic complex of $\mathcal{D}$-modules on a complex manifold. Besides the importance and the beauty of such a result, it is a good representative of the relationship between discrete and continuous coefficients, an important idea in contemporary Algebraic Geometry.

The first published duality type result is a punctual one due to Kashiwara [**Ka**], § 5. The LDT in the way we currently use was first stated by Mebkhout in [**Me1**], 4.1, [**Me2**], 5.2, but its proof depended on a still conjectural theory of Topological Homological Algebra. A complete proof was given in [**Me3**], III.1.1 (see also [**Me4**], 1.1, [**Me5**], ch. I, 4.3). Kashiwara and Kawai proposed another proof in [**K-K**], 1.4.6 based on the punctual result above.

The proof of the punctual result of Kashiwara uses the Local Duality in Analytic Geometry (residues). Mebkhout's proof of the LDT uses Serre and Poincaré-Verdier dualities to construct the duality morphism and to prove it is an isomorphism. Kashiwara and Kawai define the duality morphism as the formal one and reduce the proof of the LDT to the former result of Kashiwara by means of the Biduality Theorem for analytic constructible complexes. However, this reduction demands the commutativity of some diagram involving the global formal duality morphism and the punctual one, which is not obvious. Both proofs are evidently based on the Kashiwara's Constructibility Theorem.

In these notes we prove that the duality morphism defined by Mebkhout coincides with the formal one and, as a consequence, that the diagram needed in Kashiwara-Kawai's proof is commutative. This fact is explained by the relationship between the Global Serre Duality and the Local Duality in Analytic Geometry (*cf.* [**Li**]).

As we could expect, to do the task we need to be especially attentive to the definition and the properties of the different formal objects involved. In particular, we have to manage some signs. A complete reference for these questions is [**De2**], 1.1. For the sake of completeness and for the ease of the reader, we have collected (a big portion of) them in the Appendix.

Other somewhat different proofs of the LDT are available in [**Bo2**], § 19, [**Sa**], 2.7, [**Bj**], III, 3.3.10. We have chosen to present the first proof of the LDT, due to Mebkhout, and the proof of Kashiwara-Kawai because they are conceptually simple and they fit in this collective work as a continuation of [**M-S1, M-S2**].

This work has been done during a sabbatical year at the Institute for Advanced Study, Princeton. I would like to thank this institution for its hospitality. Discussions with Pierre Deligne have been of great value to me. I am grateful to him. I am also grateful to Leo Alonso and Ana Jeremías for good suggestions.

Notation

Given a sheaf of rings $\mathcal{R}_X$ on a topological space X, we shall denote by $C^*(\mathcal{R}_X), K^*(\mathcal{R}_X)$ and $D^*(\mathcal{R}_X)$ the category of complexes, the homotopy category of complexes and the derived category of the abelian category of left $\mathcal{R}_X$-modules respectively. We shall use ${}_r\mathcal{R}_X$ for referying to the category of right $\mathcal{R}_X$-modules.

The symbols $\mathcal{A}^\bullet, \mathcal{B}^\bullet, \mathcal{C}^\bullet$, etc. will be used for complexes of sheaves on a topological space: the objects of $\mathcal{A}^\bullet$ are the $\mathcal{A}^n$ and the differentials are $d^n_{\mathcal{A}} : \mathcal{A}^n \to \mathcal{A}^{n+1}$, for every $n \in \mathbb{Z}$.

Given a complex $\mathcal{A}^\bullet$ and an integer d, we shall denote by $h^d(\mathcal{A}^\bullet)$ its dth cohomology object.

Given a complex $A^\bullet$ (of objects in some additive category), the complex $A^\bullet[1]$ is defined by $A^\bullet[1]^n = A^{n+1}$, $d_{A^\bullet[1]} = -d_A$.

The total derived functors of $\mathrm{Hom}^\bullet_{\mathcal{R}_X}(-,-), \mathcal{H}om^\bullet_{\mathcal{R}_X}(-,-)$ and $- \overset{\bullet}{\otimes}_{\mathcal{R}_X} -$ will be denoted by $\mathbf{R}\,\mathrm{Hom}^\bullet_{\mathcal{R}_X}(-,-), \mathbf{R}\,\mathcal{H}om^\bullet_{\mathcal{R}_X}(-,-)$ and of $- \overset{\bullet}{\otimes}_{\mathcal{R}_X} -$ respectively, and $\mathcal{E}xt^d_{\mathcal{R}_X}(-,-) = h^d\,\mathbf{R}\,\mathcal{H}om^\bullet_{\mathcal{R}_X}(-,-)$.

If $\mathcal{R}_X$ is the constant sheaf associated to a fixed ring K and no confusion is possible, we shall abreviate $\mathrm{Hom}^\bullet_{K_X}(-,-)$, $\mathcal{H}om^\bullet_{K_X}(-,-)$, $\mathbf{R}\,\mathcal{H}om^\bullet_{K_X}(-,-)$ and $\mathcal{E}xt^d_{K_X}(-,-)$ by $\mathrm{Hom}^\bullet_X(-,-)$, $\mathcal{H}om^\bullet_X(-,-)$, $\mathbf{R}\,\mathcal{H}om^\bullet_X(-,-)$ and $\mathcal{E}xt^d_X(-,-)$ respectively.

1. Duality for Analytic Constructible Sheaves

Throughout this section X denotes a connected complex analytic manifold countable at infinity of dimension d, and $D^b_c(\mathbb{C}_X)$ the derived category of bounded complexes of sheaves of $\mathbb{C}$-vector spaces with analytic constructible cohomology (*cf.* [**Ve**], [**Ka**], [**M-N3**]). We denote $\mathbb{T}_X = \mathbb{C}_X[2d]$.

1.1. The Topological Biduality Morphism.

— The abelian category of sheaves of complex vector spaces over X has finite injective dimension (*cf.* [**DP**], exp. 2, 4.3). The functor $\mathbf{R}\,\mathcal{H}om^\bullet_X(-,-)$ induces a functor

$$\mathbf{R}\,\mathcal{H}om^\bullet_X(-,-) : D^b(\mathbb{C}_X) \times D^b(\mathbb{C}_X) \longrightarrow D^b(\mathbb{C}_X)$$

which can be computed by taking injective resolutions of the second argument, or locally free resolutions of the first argument if they exist.

1.1.1. Proposition. — *If $\mathcal{F}_1^\bullet, \mathcal{F}_2^\bullet$ are two complexes in $D_c^b(\mathbb{C}_X)$, then $\mathbf{R}\, \mathcal{H}om_X^\bullet(\mathcal{F}_1^\bullet, \mathcal{F}_2^\bullet)$ is also in $D_c^b(\mathbb{C}_X)$. Furthermore, if the $\mathcal{F}_i^\bullet$ are constructible with respect to a Whitney stratification Σ of X, then $\mathbf{R}\, \mathcal{H}om_X^\bullet(\mathcal{F}_1^\bullet, \mathcal{F}_2^\bullet)$ is also constructible with respect to Σ.*

Proof[(1)]. — We can suppose that the $\mathcal{F}_i^\bullet$ are single constructible sheaves $\mathcal{F}_i$ (*cf.* [**M-N3**], II.5). The question being local (*cf. loc. cit.*, I.4.21) we can suppose that $\mathcal{F}_1 = \sigma_! \mathcal{L}$, for $\sigma : S \hookrightarrow X$ the inclusion of a stratum of Σ and $\mathcal{L}$ a local system (of finite rank) on S (*cf. loc. cit.*, I.4.14). In this case we have $\mathbf{R}\, \mathcal{H}om_X^\bullet(\sigma_! \mathcal{L}, \mathcal{F}_2) \simeq \mathbf{R}\, \sigma_* \, \mathbf{R}\, \mathcal{H}om_S^\bullet(\mathcal{L}, \sigma^! \mathcal{F}_2)$, and we can conclude by induction on the dimension of X and Thom-Whitney's isotopy theorem (*cf. loc. cit.*, I.4.15). $\square$

1.1.2. Definition. — For every bounded complex $\mathcal{F}^\bullet$ in $D_c^b(\mathbb{C}_X)$ we define its *dual* by

$$\mathcal{F}^{\bullet\vee} := \mathbf{R}\, \mathcal{H}om_X^\bullet(\mathcal{F}^\bullet, \mathbb{C}_X)$$

and the *topological biduality morphism* $\boldsymbol{\beta}_{\mathcal{F}^\bullet} : \mathcal{F}^\bullet \to (\mathcal{F}^{\bullet\vee})^\vee$ as in A.2.

1.1.3. Proposition. — *If $\mathcal{F}^\bullet$ is a bounded constructible complex on X, then for each point $x \in X$ and for every small ball B centered in x with respect to some local coordinates, the complex $\mathbf{R}\, \Gamma_c(B, \mathcal{F}^\bullet)$ has finite dimensional cohomology.*

Proof. — According to proposition 1.1.1, the complex $\mathcal{F}^{\bullet\vee}$ is bounded and constructible. Then, for every small ball B centered in x, the canonical morphism $\mathbf{R}\, \Gamma(B, \mathcal{F}^{\bullet\vee}) \to (\mathcal{F}^{\bullet\vee})_x$ is an isomorphism (*cf.* [**M-N3**], I.4.16) and we conclude by the Poincaré-Verdier duality

$$\mathbf{R}\, \Gamma(B, \mathcal{F}^{\bullet\vee}) = \mathbf{R}\, \mathrm{Hom}_B^\bullet(\mathcal{F}^\bullet|_B, \mathbb{C}_B) \xrightarrow{\simeq} \mathbf{R}\, \mathrm{Hom}_{\mathbb{C}}^\bullet(\mathbf{R}\, \Gamma_c(B, \mathcal{F}^\bullet), \mathbb{C})[-2d]$$

(*cf.* [**DP**], exp. 5). $\square$

1.2. The Biduality Theorem. — The Biduality Theorem for analytic constructible sheaves has been first stated and proved by Verdier in [**Ve**], 6.2 using Resolution of Singularities. Other proofs in the setting of *cohomologically constructible sheaves* are available in [**DP**], exp. 10, § 2, [**Bo1**], V, 8.10, [**K-S**], 3.4. We sketch here a proof following the lines in [**SGA 4$\frac{1}{2}$**], Th. finitude, 4.3 and [**M-N3**], III.2.1,III.2.6 and based on the Poincaré-Verdier duality *cf.* [**DP**], exp. 4,5, [**Bo1**], V, 7.17, [**Iv**], VII.5.2, [**K-S**], 3.1.10.

1.2.1. Theorem. — *For each bounded constructible complex $\mathcal{F}^\bullet$ on X, the biduality morphism $\boldsymbol{\beta}_{\mathcal{F}^\bullet} : \mathcal{F}^\bullet \to (\mathcal{F}^{\bullet\vee})^\vee$ is an isomorphism.*

Proof. — We can suppose that $\mathcal{F}^\bullet$ is a single constructible sheaf $\mathcal{F}$ (*cf.* [**M-N3**], II.5). The result is clear if $\mathcal{F}$ is a local system (of finite rank).

(1) This proof is also valid in the case of an arbitrary complex analytic space.

As the question is local, we can also suppose that $X = D_1^{d-1} \times D_2$, where the D_i are open disks in $\mathbb{C}$, $\mathcal{F}$ is a local system on the complement of an hypersurface $Z \subset X$ and the first projection $p\colon X \to D_1^{d-1}$ is finite over Z (*cf. loc. cit.*, I.4.20).

We can extend our data, first to a constructible sheaf $\widetilde{\mathcal{F}}$ on $\widetilde{X} = D_1^{d-1} \times \mathbb{C}$ and second to $\overline{\mathcal{F}} = \sigma_! \widetilde{\mathcal{F}}$, where $\sigma : \widetilde{X} \hookrightarrow \overline{X} = D_1^{d-1} \times \mathbb{P}_1$ is the (open) inclusion. Call $\overline{p} : \overline{X} \to Y = D_1^{d-1}$ the first projection, which is proper.

Let us consider the triangle

$$\overline{\mathcal{F}} \xrightarrow{\boldsymbol{\beta}_{\overline{\mathcal{F}}}} (\overline{\mathcal{F}}^\vee)^\vee \longrightarrow \mathcal{Q}^\bullet \longrightarrow \overline{\mathcal{F}}[1] \tag{1}$$

where the support of the (bounded) complex $\mathcal{Q}^\bullet$ is contained in $Z \cup (Y \times \{\infty\})$ and then it is finite over Y.

By taking direct images by $\overline{p}$ we obtain a new triangle in $D_c^b(\mathbb{C}_Y)$

$$\mathbf{R}\,\overline{p}_*\overline{\mathcal{F}} \xrightarrow{\mathbf{R}\,\overline{p}_*\boldsymbol{\beta}_{\overline{\mathcal{F}}}} \mathbf{R}\,\overline{p}_*(\overline{\mathcal{F}}^\vee)^\vee \longrightarrow \mathbf{R}\,\overline{p}_*\mathcal{Q}^\bullet \longrightarrow \mathbf{R}\,\overline{p}_*\overline{\mathcal{F}}[1]$$

(*cf.* [**M-N3**], I.4.23).

In order to prove that $\boldsymbol{\beta}_{\overline{\mathcal{F}}}$ is an isomorphism we need to prove that $\mathcal{Q}^\bullet = 0$, but that is equivalent to $\mathbf{R}\,\overline{p}_*\mathcal{Q}^\bullet = 0$ because $\overline{p}$ is finite over the support of $\mathcal{Q}^\bullet$.

Let $\mathrm{Tr}_{X/Y} : \mathbf{R}\,\overline{p}_*\mathbb{T}_{\overline{X}} \to \mathbb{T}_Y$ be the *topological trace morphism* for the proper map $\overline{p}$. According to the local form of the Poincaré-Verdier duality (*cf.* [**Iv**], VII.5, [**K-S**], 3.1.10) the morphism $\boldsymbol{\rho}_{\mathcal{K}^\bullet}$ composition of

$$\begin{aligned}\mathbf{R}\,\overline{p}_*\,\mathbf{R}\,\mathcal{H}om^\bullet_{\overline{X}}(\mathcal{K}^\bullet, \mathbb{T}_{\overline{X}}) \xrightarrow{\text{nat.}} \mathbf{R}\,\mathcal{H}om^\bullet_Y(\mathbf{R}\,\overline{p}_*\mathcal{K}^\bullet, \mathbf{R}\,\overline{p}_*\mathbb{T}_{\overline{X}})&\\ \xrightarrow{(\mathrm{Tr}_{X/Y})_*} \mathbf{R}\,\mathcal{H}om^\bullet_Y(\mathbf{R}\,\overline{p}_*\mathcal{K}^\bullet, \mathbb{T}_Y)&\end{aligned}$$

is an isomorphism for every bounded complex of sheaves of $\mathbb{C}$-vector spaces $\mathcal{K}^\bullet$.

Call $\boldsymbol{\rho}^*_{\overline{\mathcal{F}}} := \mathbf{R}\,\mathcal{H}om^\bullet_Y(\boldsymbol{\rho}_{\overline{\mathcal{F}}}, \mathbb{T}_Y)$ the isomorphism induced by $\boldsymbol{\rho}_{\overline{\mathcal{F}}}$. According to A.5, we can "redefine"

$$(\overline{\mathcal{F}}^\vee)^\vee = \mathbf{R}\,\mathcal{H}om^\bullet_{\overline{X}}(\mathbf{R}\,\mathcal{H}om^\bullet_{\overline{X}}(\overline{\mathcal{F}}, \mathbb{T}_{\overline{X}}), \mathbb{T}_{\overline{X}})$$

and using A.2 and lemma A.15 we deduce the relation

$$\Big(\boldsymbol{\rho}_{\mathbf{R}\,\mathcal{H}om^\bullet_{\overline{X}}(\overline{\mathcal{F}},\mathbb{T}_{\overline{X}})}\Big) \circ \mathbf{R}\,\overline{p}_*\boldsymbol{\beta}_{\overline{\mathcal{F}}} = \boldsymbol{\rho}^*_{\overline{\mathcal{F}}} \circ \boldsymbol{\beta}_{\mathbf{R}\,\overline{p}_*\overline{\mathcal{F}}}.$$

By induction hypothesis, the morphism $\boldsymbol{\beta}_{\mathbf{R}\,\overline{p}_*\overline{\mathcal{F}}}$ is an isomorphism, then $\mathbf{R}\,\overline{p}_*\boldsymbol{\beta}_{\overline{\mathcal{F}}}$ too and we obtain the desired $\mathbf{R}\,\overline{p}_*\mathcal{Q}^\bullet = 0$. □

1.2.2. As X is a connected oriented manifold of (topological) dimension $2d$, the *topological trace morphism* $\mathrm{tr}_X : \mathrm{H}^{2d}_c(X, \mathbb{C}_X) \to \mathbb{C}$ given by integration of top C^∞-forms with compact support is an isomorphism. Then, for each point $x \in X$, denoting by $i : \{x\} \hookrightarrow X$ the inclusion, the canonical morphism $i^!\mathbb{C}_X \to \mathbf{R}\,\Gamma_c(X, \mathbb{C}_X)$ gives rise to a *punctual topological trace* isomorphism

$$\mathrm{tr}_x : \mathrm{H}^{2d}_x(\mathbb{C}_X) \xrightarrow[\simeq]{\text{nat.}} \mathrm{H}^{2d}_c(X, \mathbb{C}_X) \xrightarrow[\simeq]{\mathrm{tr}_X} \mathbb{C}.$$

1.2.3. Proposition. — *Let $\mathcal{F}^\bullet$ be a complex in $D^b_c(\mathbb{C}_X)$ and $x \in X$. Denote $i : \{x\} \hookrightarrow X$ the (closed) inclusion. Then, the natural morphism*

$$\boldsymbol{n} : (\mathcal{F}^\bullet)^\vee_x = i^{-1}\,\mathbf{R}\,\mathcal{H}om^\bullet_X(\mathcal{F}^\bullet, \mathbb{C}_X) \longrightarrow \mathbf{R}\,\mathrm{Hom}^\bullet_{\mathbb{C}}(i^!\mathcal{F}^\bullet, i^!\mathbb{C}_X)$$

is an isomorphism. In particular, using 1.2.2, we obtain an isomorphism

$$((\mathcal{F}^\bullet)^\vee)_x \simeq \mathbf{R}\,\mathrm{Hom}^\bullet_{\mathbb{C}}(i^!\mathcal{F}^\bullet, \mathbb{C})[-2d].$$

Proof. — As $(\mathcal{F}^\bullet)^\vee_x$ is a bounded complex of $\mathbb{C}$-vector spaces with finite dimensional cohomology and $i^!\mathbb{C}_X \simeq \mathbb{C}[-2d]$, the natural morphism A.2

$$\boldsymbol{\beta}_0 : (\mathcal{F}^\bullet)^\vee_x \longrightarrow \mathbf{R}\,\mathrm{Hom}^\bullet_{\mathbb{C}}(\mathbf{R}\,\mathrm{Hom}^\bullet_{\mathbb{C}}((\mathcal{F}^\bullet)^\vee_x, i^!\mathbb{C}_X), i^!\mathbb{C}_X)$$

is an isomorphism. We also have a canonical isomorphism (*cf.* A.11)

$$\boldsymbol{g} : i^!((\mathcal{F}^\bullet)^\vee)^\vee = i^!\,\mathbf{R}\,\mathcal{H}om^\bullet_X((\mathcal{F}^\bullet)^\vee, \mathbb{C}_X) \xrightarrow{\ \simeq\ } \mathbf{R}\,\mathrm{Hom}^\bullet_{\mathbb{C}}((\mathcal{F}^\bullet)^\vee_x, i^!\mathbb{C}_X).$$

Call $\boldsymbol{g}^* := \mathbf{R}\,\mathrm{Hom}^\bullet_{\mathbb{C}}(\mathbf{R}\,\mathrm{Hom}^\bullet_{\mathbb{C}}(\boldsymbol{g}, i^!\mathbb{C}_X), i^!\mathbb{C}_X)$ the isomorphism induced by $\boldsymbol{g}$, and $(i^!\boldsymbol{\beta}_{\mathcal{F}^\bullet})^* := \mathbf{R}\,\mathrm{Hom}^\bullet_{\mathbb{C}}(i^!\boldsymbol{\beta}_{\mathcal{F}^\bullet}, i^!\mathbb{C}_X)$ the morphism induced by $i^!\boldsymbol{\beta}_{\mathcal{F}^\bullet}$, which is an isomorphism according to theorem 1.2.1. To conclude, we observe that $\boldsymbol{n} = (i^!\boldsymbol{\beta}_{\mathcal{F}^\bullet})^* \circ \boldsymbol{g}^* \circ \boldsymbol{\beta}_0$ according to A.12. ☐

2. The Local Duality Morphism in $\mathcal{D}$-module Theory

Throughout this section X denotes a complex analytic manifold countable at infinity of dimension d, $\mathcal{D}_X$ the sheaf of linear differential operators with coefficients in $\mathcal{O}_X$ (*cf.* [**G-M**], I) and $D^b_c(\mathcal{D}_X)$ the derived category of bounded complexes of left $\mathcal{D}_X$-modules with coherent cohomology.

2.1. The Solution and the De Rham Functors. — Here, our basic functor is $\mathbf{R}\,\mathcal{H}om^\bullet_{\mathcal{D}_X}(-,-)$ which can be computed by taking injective resolutions of the second argument, or locally free resolutions of the first argument if they exist.

Since $\mathcal{D}_X$ is a coherent sheaf of rings and every single $\mathcal{D}_X$-module admits locally a finite free resolution (*cf.* [**Me5**], ch. I, 2.1.16), we have an induced functor

$$\mathbf{R}\,\mathcal{H}om^\bullet_{\mathcal{D}_X}(-,-) : D^b_c(\mathcal{D}_X) \times D^b(\mathcal{D}_X) \longrightarrow D^b(\mathbb{C}_X).$$

The *De Rham* functor is

$$\mathbf{DR} = \mathbf{R}\,\mathcal{H}om^\bullet_{\mathcal{D}_X}(\mathcal{O}_X, -) : D^b(\mathcal{D}_X) \longrightarrow D^b(\mathbb{C}_X)$$

and the *Solutions* functors are

$$\mathsf{S} = \mathcal{H}om^\bullet_{\mathcal{D}_X}(-, \mathcal{O}_X) : K^b(\mathcal{D}_X) \longrightarrow K^b(\mathbb{C}_X),$$
$$\mathbf{S} = \mathbf{R}\,\mathcal{H}om^\bullet_{\mathcal{D}_X}(-, \mathcal{O}_X) : D^b_c(\mathcal{D}_X) \longrightarrow D^b(\mathbb{C}_X).$$

We will also consider the *"external" duality functors*

$$\begin{aligned}\mathsf{D} &= \mathit{Hom}^\bullet_{\mathcal{D}_X}(-,\mathcal{D}_X) : K^b_c(\mathcal{D}_X) \longrightarrow K^b_c({}_r\mathcal{D}_X),\\ \mathbf{D} &= \mathbf{R}\,\mathit{Hom}^\bullet_{\mathcal{D}_X}(-,\mathcal{D}_X) : D^b_c(\mathcal{D}_X) \longrightarrow D^b_c({}_r\mathcal{D}_X).\end{aligned}$$

We have $\mathsf{S}(\mathcal{D}_X) = \mathbf{S}(\mathcal{D}_X) = \mathcal{O}_X$.

The De Rham functor can be computed by means of the *Spencer resolution* $Sp^\bullet_X$ (*cf.* [**Me5**], ch. I, 2.1.17), whose objects are defined by $Sp^{-p}_X = \mathcal{D}_X \otimes_{\mathcal{O}_X} \overset{p}{\wedge} \mathit{Der}_{\mathbb{C}}(\mathcal{O}_X), p = 0,\dots,d$ and the differential $\epsilon^{-p} : Sp^{-p}_X \to Sp^{-(p-1)}_X$ is given by:

$$\begin{aligned}\epsilon^{-p}(P\otimes(\delta_1\wedge\cdots\wedge\delta_p)) &= \sum_{i=1}^{p}(-1)^{i-1}(P\delta_i)\otimes(\delta_1\wedge\cdots\wedge\widehat{\delta_i}\wedge\cdots\wedge\delta_p)\\ &\quad+\sum_{1\leqslant i<j\leqslant p}(-1)^{i+j}P\otimes([\delta_i,\delta_j]\wedge\delta_1\wedge\cdots\wedge\widehat{\delta_i}\wedge\cdots\wedge\widehat{\delta_j}\wedge\cdots\wedge\delta_p)\end{aligned}$$

for $p = 2,\dots,d$ and $\epsilon^{-1}(P\otimes\delta) = P\delta$ for $p = 1$.

There is an obvious augmentation $\epsilon^0 : Sp^0_X = \mathcal{D}_X \to \mathcal{O}_X$, $\epsilon^0(P) = P(1)$, that makes $Sp^\bullet_X$ into a (canonical) locally free resolution of $\mathcal{O}_X$ as left $\mathcal{D}_X$-module. We will always consider this augmentation to identify the functors $\mathbf{DR}(-) = \mathit{Hom}^\bullet_{\mathcal{D}_X}(Sp^\bullet_X,-)$.

Every left $\mathcal{D}_X$-module $\mathcal{E}$ carries an integrable connection $\nabla : \mathcal{E} \to \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{E}$ and we can then consider its *classical De Rham complex* $\Omega^\bullet_X(\mathcal{E})$ (*cf.* [**De1**], I.2). It is defined by $\Omega^p_X(\mathcal{E}) = \Omega^p_X \otimes_{\mathcal{O}_X} \mathcal{E}$ for $p = 0,\dots,d$, and the differential $\nabla^p : \Omega^p_X(\mathcal{E}) \to \Omega^{p+1}_X(\mathcal{E})$ is given by $\nabla^p(\omega\otimes e) = (d\omega)\otimes e + (-1)^p\omega\wedge\nabla(e)$.

2.1.1. Lemma. — *For each left $\mathcal{D}_X$-module $\mathcal{E}$, the morphisms*

$$\alpha^p_{\mathcal{E}} : \Omega^p_X \otimes_{\mathcal{O}_X} \mathcal{E} \longrightarrow \mathit{Hom}^p_{\mathcal{D}_X}(Sp^\bullet_X,\mathcal{E}) = \mathit{Hom}_{\mathcal{D}_X}(Sp^{-p}_X,\mathcal{E}),\quad p = 0,\dots,d$$

defined by $\alpha^p_{\mathcal{E}}(\theta\otimes e)(P\otimes\delta) = (-1)^{p(p+1)/2}P\cdot\langle\delta,\theta\rangle\cdot e$, commute with the differentials and gives rise to a natural isomorphism of complexes

$$\alpha^\bullet_{\mathcal{E}} : \Omega^\bullet_X(\mathcal{E}) \longrightarrow \mathit{Hom}^\bullet_{\mathcal{D}_X}(Sp^\bullet_X,\mathcal{E}).$$

The proof of the lemma is straightforward. It should be noticed that the sign $(-1)^{p(p+1)/2}$ is imposed by the definition of the functor $\mathit{Hom}^\bullet_{\mathcal{D}_X}(-,-)$ (*cf.* A.1).

We will denote

$$\begin{aligned}\alpha^\bullet_0 &:= \alpha^\bullet_{\mathcal{D}_X} : \Omega^\bullet_X(\mathcal{D}_X) \xrightarrow{\ \simeq\ } \mathit{Hom}^\bullet_{\mathcal{D}_X}(Sp^\bullet_X,\mathcal{D}_X) = \mathsf{D}(Sp^\bullet_X) = \mathbf{D}(\mathcal{O}_X),\\ \alpha^\bullet_1 &:= \alpha^\bullet_{\mathcal{O}_X} : \Omega^\bullet_X = \Omega^\bullet_X(\mathcal{O}_X) \xrightarrow{\ \simeq\ } \mathit{Hom}^\bullet_{\mathcal{D}_X}(Sp^\bullet_X,\mathcal{O}_X) = \mathsf{S}(Sp^\bullet_X) = \mathbf{S}(\mathcal{O}_X).\end{aligned}$$

Obviously $\alpha^\bullet_0$ is right $\mathcal{D}_X$-linear.

2.1.2. Denote by ω_X the sheaf of top differential forms Ω_X^d on X. It carries a canonical right $\mathcal{D}_X$-module structure (*cf.* [**G-M**], prop. 15, [**M-N2**], 1.1.5). Call $\sigma^\bullet : \Omega_X^\bullet(\mathcal{D}_X) \to \omega_X[-d]$ the right $\mathcal{D}_X$-linear morphism given by $\sigma^d(\theta \otimes P) = \theta \cdot P$. It is a quasi-isomorphism (*cf.* [**Me5**], ch. I, 2.6.6) admitting a $\mathbb{C}_X$-linear section $\tau^\bullet$ given by $\tau^d(\theta) = \theta \otimes 1$. Let us consider the following morphisms:

$$\alpha^\bullet := \alpha_0^\bullet \circ \tau^\bullet : \omega_X[-d] \longrightarrow \mathit{Hom}^\bullet_{\mathcal{D}_X}(\mathit{Sp}_X^\bullet, \mathcal{D}_X) = \mathsf{D}(\mathit{Sp}_X^\bullet),$$

$$\boldsymbol{\alpha} : \omega_X[-d] \xrightarrow{\alpha_0^\bullet \circ (\sigma^\bullet)^{-1}} \mathit{Hom}^\bullet_{\mathcal{D}_X}(\mathit{Sp}_X^\bullet, \mathcal{D}_X) = \mathsf{D}(\mathit{Sp}_X^\bullet) \xrightarrow[\simeq]{\text{can}} \mathbf{D}(\mathit{Sp}_X^\bullet) = \mathbf{D}(\mathcal{O}_X).$$

The first one is a $\mathbb{C}_X$-linear quasi-isomorphism, and the second one is an isomorphism in the derived category of right $\mathcal{D}_X$-modules. Both morphisms coincide in $D^b(\mathbb{C}_X)$.

In particular, the cohomology of the complex $\mathbf{DR}(\mathcal{D}_X)$ vanishes in degree different from d and then $\mathbf{DR}(\mathcal{D}_X) = \mathit{Ext}^d_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{D}_X)[-d]$.

According to the Poincaré lemma, the inclusion $\mathbb{C}_X \subset \Omega_X^0$ gives rise to a quasi-isomorphism $\kappa_0^\bullet : \mathbb{C}_X \to \Omega_X^\bullet = \Omega_X^\bullet(\mathcal{O}_X)$. Using the isomorphism of complexes $\alpha_1^\bullet$ we obtain an isomorphism in the derived category

$$\boldsymbol{\kappa} : \mathbb{C}_X \xrightarrow{\simeq} \mathit{Hom}^\bullet_{\mathcal{D}_X}(\mathit{Sp}_X^\bullet, \mathcal{O}_X) = \mathbf{S}(\mathcal{O}_X) = \mathbf{DR}(\mathcal{O}_X). \tag{2}$$

2.2. The Duality Morphism

2.2.1. Definition. — For every bounded complex of left $\mathcal{D}_X$-modules $\mathcal{M}^\bullet$ with coherent cohomology we define the duality morphism

$$\boldsymbol{\xi}_{\mathcal{M}^\bullet} : \mathbf{DR}(\mathcal{M}^\bullet) \longrightarrow \mathbf{S}(\mathcal{M}^\bullet)^\vee = \mathbf{R}\,\mathit{Hom}^\bullet_{\mathbb{C}_X}(\mathbf{S}(\mathcal{M}^\bullet), \mathbb{C}_X)$$

by composing the natural morphism (*cf.* A.2)

$$\boldsymbol{\xi} : \mathbf{R}\,\mathit{Hom}^\bullet_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{M}^\bullet) \longrightarrow \mathbf{R}\,\mathit{Hom}^\bullet_{\mathbb{C}_X}(\mathbf{S}(\mathcal{M}^\bullet), \mathbf{S}(\mathcal{O}_X))$$

with the isomorphism induced by $\boldsymbol{\kappa}$ (2).

2.2.2. Proposition. — *For $\mathcal{M}^\bullet \in D^b_c(\mathcal{D}_X)$ there exist (local) natural isomorphisms*

$$\boldsymbol{\lambda}_{\mathcal{M}^\bullet} : \mathbf{DR}(\mathcal{D}_X) \overset{\bullet}{\otimes}_{\mathcal{D}_X} \mathcal{M}^\bullet \longrightarrow \mathbf{DR}(\mathcal{M}^\bullet),$$

$$\boldsymbol{\mu}_{\mathcal{M}^\bullet} : \mathbf{S}(\mathcal{D}_X)^\vee \overset{\bullet}{\otimes}_{\mathcal{D}_X} \mathcal{M}^\bullet \longrightarrow \mathbf{S}(\mathcal{M}^\bullet)^\vee$$

such that the following diagram commutes

$$\begin{tikzcd}
\mathbf{DR}(\mathcal{D}_X) \overset{\bullet}{\otimes}_{\mathcal{D}_X} \mathcal{M}^\bullet \arrow[r, "\boldsymbol{\xi}_{\mathcal{D}_X} \otimes \mathit{Id}_{\mathcal{M}^\bullet}"] \arrow[d, "\boldsymbol{\lambda}_{\mathcal{M}^\bullet}"', "\simeq"] & \mathbf{S}(\mathcal{D}_X)^\vee \overset{\bullet}{\otimes}_{\mathcal{D}_X} \mathcal{M}^\bullet \arrow[d, "\boldsymbol{\mu}_{\mathcal{M}^\bullet}", "\simeq"'] \\
\mathbf{DR}(\mathcal{M}^\bullet) \arrow[r, "\boldsymbol{\xi}_{\mathcal{M}^\bullet}"] & \mathbf{S}(\mathcal{M}^\bullet)^\vee .
\end{tikzcd}$$

Proof. — Take a flat resolution $\mathcal{P}^\bullet \to \mathcal{M}^\bullet$ and an injective Godement resolution $\mathcal{O}_X \to \mathcal{J}^\bullet$. We have

$$\begin{aligned}\mathbf{DR}(\mathcal{D}_X)\overset{\bullet}{\otimes}_{\mathcal{D}_X}\mathcal{M}^\bullet &= \mathcal{H}om^\bullet_{\mathcal{D}_X}(\mathcal{S}p^\bullet_X,\mathcal{D}_X)\overset{\bullet}{\otimes}_{\mathcal{D}_X}\mathcal{P}^\bullet,\\ \mathbf{DR}(\mathcal{M}^\bullet) &= \mathcal{H}om^\bullet_{\mathcal{D}_X}(\mathcal{S}p^\bullet_X,\mathcal{M}^\bullet) = \mathcal{H}om^\bullet_{\mathcal{D}_X}(\mathcal{S}p^\bullet_X,\mathcal{P}^\bullet),\\ \mathbf{S}(\mathcal{D}_X)^\vee\overset{\bullet}{\otimes}_{\mathcal{D}_X}\mathcal{M}^\bullet &= \mathcal{O}_X^\vee\overset{\bullet}{\otimes}_{\mathcal{D}_X}\mathcal{M}^\bullet = \mathcal{H}om^\bullet_{\mathbb{C}_X}(\mathcal{J}^\bullet,\mathcal{H}om^\bullet_{\mathcal{D}_X}(\mathcal{S}p^\bullet_X,\mathcal{J}^\bullet))\overset{\bullet}{\otimes}_{\mathcal{D}_X}\mathcal{P}^\bullet,\\ \mathbf{S}(\mathcal{M}^\bullet)^\vee &= \cdots = \mathcal{H}om^\bullet_{\mathbb{C}_X}(\mathcal{H}om^\bullet_{\mathcal{D}_X}(\mathcal{P}^\bullet,\mathcal{J}^\bullet),\mathcal{H}om^\bullet_{\mathcal{D}_X}(\mathcal{S}p^\bullet_X,\mathcal{J}^\bullet))\end{aligned}$$

and we are reduced to lemma A.10.

The fact that $\boldsymbol{\lambda}_{\mathcal{M}^\bullet}$ and $\boldsymbol{\mu}_{\mathcal{M}^\bullet}$ are isomorphisms is a local question. So, we can suppose that $\mathcal{M}^\bullet$ has a finite free resolution and we are reduced to the obvious fact that $\boldsymbol{\lambda}_{\mathcal{D}_X}$ and $\boldsymbol{\mu}_{\mathcal{D}_X}$ are isomorphisms. □

2.2.3. Corollary. — *For every bounded complex of left $\mathcal{D}_X$-modules $\mathcal{M}^\bullet$ with coherent cohomology, the duality morphism $\boldsymbol{\xi}_{\mathcal{M}^\bullet}$ is an isomorphism if and only if $\boldsymbol{\xi}_{\mathcal{D}_X}\otimes \mathrm{Id}_{\mathcal{M}^\bullet}$ is an isomorphism.*

3. Proof of the Local Duality Theorem

Throughout this section X denotes a complex analytic manifold countable at infinity of dimension d.

3.1. Statement of the Local Duality Theorem

3.1.1. Theorem. — *For every bounded complex of left $\mathcal{D}_X$-modules $\mathcal{M}^\bullet$ with holonomic cohomology, the duality morphism*

$$\boldsymbol{\xi}_{\mathcal{M}^\bullet} : \mathbf{DR}(\mathcal{M}^\bullet) \longrightarrow \mathbf{S}(\mathcal{M}^\bullet)^\vee$$

is an isomorphism (in the derived category).

3.2. The Basic Commutative Diagram

3.2.1. Proposition **([Me3], [Me4], [Me5])**. — *The complex $\mathbf{S}(\mathcal{D}_X)^\vee = \mathcal{O}_X^\vee$ is concentrated in degree $d = \dim X$.*

Proof. — For every integer $i \geqslant 0$, the sheaf $\mathcal{E}xt^i_{\mathbb{C}_X}(\mathcal{O}_X,\mathbb{C}_X)$ is the sheaf associated to the presheaf $U \mapsto \mathrm{Ext}^i_{\mathbb{C}_U}(\mathcal{O}_U,\mathbb{C}_U)$. It is enough to prove that $\mathrm{Ext}^i_{\mathbb{C}_U}(\mathcal{O}_U,\mathbb{C}_U)=0$ for all $i \neq d$ and for every Stein open set $U \subset X$.

Now, by the Poincaré-Verdier duality (*cf.* [**DP**], exp. 5, [**Iv**], VI) the space $\mathrm{Ext}^i_{\mathbb{C}_U}(\mathcal{O}_U,\mathbb{C}_U)$ is isomorphic to the algebraic dual of $\mathrm{H}^{2d-i}_c(U,\mathcal{O}_U)$, and by the Serre duality [**Se**], if U is Stein, the space $\mathrm{H}^{2d-i}_c(U,\mathcal{O}_U)$ is isomorphic to the topological dual of $\mathrm{H}^{i-d}(U,\omega_U)$, but for such open sets $\mathrm{H}^{i-d}(U,\omega_U)=0$ and then $\mathrm{Ext}^i_{\mathbb{C}_U}(\mathcal{O}_U,\mathbb{C}_U)=0$, for all $i \neq d$. □

3.2.2. Call

$$\xi := h^d(\boldsymbol{\xi}_{\mathcal{D}_X}) : \mathcal{E}xt^d_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{D}_X) \longrightarrow \mathcal{E}xt^d_{\mathbb{C}_X}(\mathcal{O}_X, \mathbb{C}_X),$$
$$\alpha := h^d(\boldsymbol{\alpha}) : \omega_X \longrightarrow \mathcal{E}xt^d_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{D}_X)$$

where $\boldsymbol{\alpha}$ is the isomorphism in 2.1.2. Both morphisms are right $\mathcal{D}_X$-linear.

As $\mathcal{O}_X^\vee$ is concentrated in degree d, for every open set $U \subset X$ we have

$$\begin{aligned}\Gamma(U, \mathcal{E}xt^d_{\mathbb{C}_X}(\mathcal{O}_X, \mathbb{C}_X)) &= \mathbf{R}^d\,\Gamma(U, \mathcal{E}xt^d_{\mathbb{C}_X}(\mathcal{O}_X, \mathbb{C}_X)[-d]) \\ &= \mathbf{R}^d\,\Gamma(U, \mathbf{R}\,\mathcal{H}om^\bullet_{\mathbb{C}_X}(\mathcal{O}_X, \mathbb{C}_X)) = h^d\,\mathbf{R}\,\mathrm{Hom}^\bullet_{\mathbb{C}_U}(\mathcal{O}_U, \mathbb{C}_U),\end{aligned}$$

and using the natural isomorphism $\boldsymbol{\nu}^d$ (*cf.* A.5) we obtain an isomorphism

$$\varepsilon_U : \Gamma(U, \mathcal{E}xt^d_{\mathbb{C}_X}(\mathcal{O}_X, \mathbb{C}_X)) \xrightarrow{\simeq} \mathrm{Hom}_{D(\mathbb{C}_U)}(\mathcal{O}_U, \mathbb{C}_U[d]).$$

The ε_U commute with restrictions and each ε_U is right $\mathcal{D}_X(U)$-linear, where the right $\mathcal{D}_X(U)$-module structure on $\mathrm{Hom}_{D(\mathbb{C}_U)}(\mathcal{O}_U, \mathbb{C}_U[d])$ comes from the left action of $\mathcal{D}_X(U)$ on $\mathcal{O}_U$.

The Poincaré quasi-isomorphism $\kappa_0^\bullet : \mathbb{C}_X \to \Omega_X^\bullet$ and the inclusion map $\kappa_1^\bullet : \omega_X[-d] \to \Omega_X^\bullet$ gives rise to a *Poincaré-De Rham morphism* in the derived category

$$\boldsymbol{\kappa}' := (\kappa_0^\bullet[d])^{-1} \circ \kappa_1^\bullet[d] : \omega_X \longrightarrow \mathbb{C}_X[d].$$

We will denote by $\beta_U : \Gamma(U, \omega_X) \to \mathrm{Hom}_{D(\mathbb{C}_U)}(\mathcal{O}_U, \mathbb{C}_U[d])$ the composition of $(\boldsymbol{\kappa}')_*$ with the map

$$\Gamma(U, \omega_X) = \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{O}_U, \omega_U) \xrightarrow{\text{forget}} \mathrm{Hom}_{\mathbb{C}_U}(\mathcal{O}_U, \omega_U) = \mathrm{Hom}_{D(\mathbb{C}_U)}(\mathcal{O}_U, \omega_U).$$

In corollary 3.2.5 we will see that β_U is right $\mathcal{D}_X(U)$-linear.

Recall that (*cf.* 2.1.2)

$$\alpha^\bullet = \alpha_0^\bullet \circ \tau^\bullet : \omega_X[-d] \longrightarrow \mathcal{H}om^\bullet_{\mathcal{D}_X}(\mathcal{S}p_X^\bullet, \mathcal{D}_X) = \mathsf{D}(\mathcal{S}p_X^\bullet),$$

and denote

$$\beta^\bullet := (\kappa_1^\bullet)_* \circ (\text{forget}) : \omega_X[-d] \longrightarrow \mathcal{H}om^\bullet_{\mathbb{C}_X}(\mathcal{O}_X, \Omega_X^\bullet),$$

where "(forget)" is the morphism

$$\omega_X[-d] = \mathcal{H}om^\bullet_{\mathcal{O}_X}(\mathcal{O}_X, \omega_X[-d]) \xrightarrow{\text{forget}} \mathcal{H}om^\bullet_{\mathbb{C}_X}(\mathcal{O}_X, \omega_X[-d]),$$

and

$$\gamma^\bullet := (\alpha_1^\bullet)_* : \mathcal{H}om^\bullet_{\mathbb{C}_X}(\mathcal{O}_X, \Omega_X^\bullet) \xrightarrow{\simeq} \mathcal{H}om^\bullet_{\mathbb{C}_X}(\mathcal{O}_X, \mathsf{S}(\mathcal{S}p_X^\bullet)).$$

3.2.3. Proposition. — *The following diagram of complexes of sheaves of vector spaces*

$$
\begin{tikzcd}
\mathsf{D}(Sp^\bullet_X) \arrow[r, "\xi^\bullet"] & \mathcal{H}om^\bullet_{\mathbb{C}_X}(\mathcal{O}_X, \mathsf{S}(Sp^\bullet_X)) \\
\omega_X[-d] \arrow[u, "\alpha^\bullet"] \arrow[r, "\beta^\bullet"] & \mathcal{H}om^\bullet_{\mathbb{C}_X}(\mathcal{O}_X, \Omega^\bullet_X) \arrow[u, "\simeq"', "\gamma^\bullet"]
\end{tikzcd}
\tag{3}
$$

is commutative.

Proof. — As $\alpha^i = \beta^i = 0$ for all $i \neq d$, we need only to prove that $\xi^d \circ \alpha^d = \gamma^d \circ \beta^d$, but $\mathsf{S}(\mathcal{D}_X) = \mathcal{O}_X$ is a complex vanishing in degrees different from 0 and then there is no signs in the expression for ξ^d (*cf.* A.2). We deduce that the degree d part of the diagram (3) can be identified with the diagramm

$$
\begin{tikzcd}
\mathcal{H}om_{\mathcal{D}_X}(\mathrm{Sp}^{-d}_X, \mathcal{D}_X) \arrow[r, "\text{nat.}"] & \mathcal{H}om_{\mathbb{C}_X}(\mathcal{O}_X, \mathcal{H}om_{\mathcal{D}_X}(\mathrm{Sp}^{-d}_X, \mathcal{O}_X)) \\
\omega_X \arrow[u, "\alpha^d"] \arrow[r, "\text{forget}"] & \mathcal{H}om_{\mathbb{C}_X}(\mathcal{O}_X, \omega_X). \arrow[u, "\simeq"', "(\alpha^d_1)_*"]
\end{tikzcd}
\tag{4}
$$

For a top differential form θ on an open set $U \subset X$, the section

$$\varphi = \alpha^d(\theta) \in \Gamma(U, \mathcal{H}om_{\mathcal{D}_X}(\mathrm{Sp}^{-d}_X, \mathcal{D}_X)) = \mathrm{Hom}_{\mathcal{D}_U}(\mathrm{Sp}^{-d}_U, \mathcal{D}_U)$$

is given by

$$\varphi(P \otimes \delta) = \alpha^d_0(\theta \otimes 1)(P \otimes \delta) = (-1)^{d(d+1)/2} P \cdot \langle \delta, \theta \rangle.$$

Call $\psi \in \Gamma(U, \mathcal{H}om_{\mathbb{C}_X}(\mathcal{O}_X, \mathcal{H}om_{\mathcal{D}_X}(\mathrm{Sp}^{-d}_X, \mathcal{O}_X))) = \mathrm{Hom}_{\mathbb{C}_U}(\mathcal{O}_U, \mathcal{H}om_{\mathcal{D}_U}(\mathrm{Sp}^{-d}_U, \mathcal{O}_U))$ the morphism determined by φ. For each local section f of $\mathcal{O}_U$ we have

$$\psi(f)(P \otimes \delta) = \varphi(P \otimes \delta)(f) = (-1)^{d(d+1)/2}(P \cdot \langle \delta, \theta \rangle)(f) = (-1)^{d(d+1)/2} P(\langle \delta, \theta \rangle f).$$

On the other hand, the section

$$\varphi' = \mathrm{forget}(\theta) \in \Gamma(U, \mathcal{H}om_{\mathbb{C}_X}(\mathcal{O}_X, \omega_X)) = \mathrm{Hom}_{\mathbb{C}_U}(\mathcal{O}_U, \omega_U)$$

is given by $\varphi'(f) = f\theta$. Call

$$\psi' = (\alpha^d_1)_*(\varphi') = \alpha^d_1 \circ \varphi' \in \mathrm{Hom}_{\mathbb{C}_U}(\mathcal{O}_U, \mathcal{H}om_{\mathcal{D}_U}(\mathrm{Sp}^{-d}_U, \mathcal{O}_U)).$$

We have

$$\psi'(f)(P \otimes \delta) = \alpha^d_1(\varphi'(f))(P \otimes \delta) = \alpha^d_1(f\theta)(P \otimes \delta) = (-1)^{d(d+1)/2} P(\langle \delta, f\theta \rangle)$$

and we conclude that the diagram (4) is commutative, and then (3) too. □

3.2.4. Proposition. — *For every open set $U \subset X$, the following diagram of vector spaces*

$$\begin{tikzcd}
\Gamma(U, \mathit{Ext}^d_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{D}_X)) \arrow[r, "\Gamma(U,\xi)"] & \Gamma(U, \mathit{Ext}^d_{\mathbb{C}_X}(\mathcal{O}_X, \mathbb{C}_X)) \arrow[d, "\simeq"', "\varepsilon_U"] \\
\Gamma(U, \omega_X) \arrow[u, "\Gamma(U,\alpha)", "\simeq"'] \arrow[r, "\beta_U"] & \mathrm{Hom}_{D(\mathbb{C}_U)}(\mathcal{O}_U, \mathbb{C}_U[d])
\end{tikzcd}$$

is commutative.

Proof. — Let us call

- $a_U : h^d\Gamma(U, \omega_X[-d]) \xrightarrow{=} \Gamma(U, \omega_X)$ the identity map,
- $b_U =$ the composition of

$$\begin{aligned}
h^d\Gamma(U, \mathsf{D}(Sp^\bullet_X)) &\xrightarrow{\text{can}} \mathbf{R}^d\,\Gamma(U, \mathsf{D}(Sp^\bullet_X)) \\
&\xrightarrow{(\text{can})} \mathbf{R}^d\,\Gamma(U, \mathbf{D}(Sp^\bullet_X)) = \mathbf{R}^d\,\Gamma(U, \mathbf{D}(\mathcal{O}_X)) \\
&= \mathbf{R}^d\,\Gamma(U, \mathit{Ext}^d_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{D}_X)[-d]) = \Gamma(U, \mathit{Ext}^d_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{D}_X)),
\end{aligned}$$

- $c_U =$ the composition of

$$\begin{aligned}
&h^d\Gamma(U, \mathit{Hom}^\bullet_{\mathbb{C}_X}(\mathcal{O}_X, \mathsf{S}(Sp^\bullet_X))) \xrightarrow{\text{can}} \mathbf{R}^d\,\Gamma(U, \mathit{Hom}^\bullet_{\mathbb{C}_X}(\mathcal{O}_X, \mathbf{S}(Sp^\bullet_X))) \\
&\xrightarrow{(\text{can})} \mathbf{R}^d\,\Gamma(U, \mathbf{R}\,\mathit{Hom}^\bullet_{\mathbb{C}_X}(\mathcal{O}_X, \mathbf{S}(Sp^\bullet_X))) = \mathbf{R}^d\,\Gamma(U, \mathbf{R}\,\mathit{Hom}^\bullet_{\mathbb{C}_X}(\mathcal{O}_X, \mathbf{S}(\mathcal{O}_X))) \\
&\xrightarrow[\simeq]{(\boldsymbol{\kappa}^{-1})_*} \mathbf{R}^d\,\Gamma(U, \mathcal{O}^\vee_X) = \mathbf{R}^d\,\Gamma(U, \mathit{Ext}^d_{\mathbb{C}_X}(\mathcal{O}_X, \mathbb{C}_X)[-d]) = \Gamma(U, \mathit{Ext}^d_{\mathbb{C}_X}(\mathcal{O}_X, \mathbb{C}_X)),
\end{aligned}$$

- and $d_U =$ the composition of

$$\begin{aligned}
h^d\Gamma(U, \mathit{Hom}^\bullet_{\mathbb{C}_X}(\mathcal{O}_X, \Omega^\bullet_X)) &\xrightarrow{\text{can}} \mathbf{R}^d\,\Gamma(U, \mathit{Hom}^\bullet_{\mathbb{C}_X}(\mathcal{O}_X, \Omega^\bullet_X)) \\
&\xrightarrow{(\text{can})} \mathbf{R}^d\,\Gamma(U, \mathbf{R}\,\mathit{Hom}^\bullet_{\mathbb{C}_X}(\mathcal{O}_X, \Omega^\bullet_X)) = h^d\,\mathbf{R}\,\mathrm{Hom}^\bullet_{\mathbb{C}_U}(\mathcal{O}_U, \Omega^\bullet_U) \\
&\xrightarrow[\simeq]{\boldsymbol{\nu}^d} \mathrm{Hom}_{D(\mathbb{C}_U)}(\mathcal{O}_U, \Omega^\bullet_U[d]) \xrightarrow[\simeq]{(\kappa^\bullet_0)^{-1}_*} \mathrm{Hom}_{D(\mathbb{C}_U)}(\mathcal{O}_U, \mathbb{C}_U[d]).
\end{aligned}$$

We are going to prove that the following relations:

$$\Gamma(U,\alpha) \circ a_U = b_U \circ h^d\Gamma(U, \alpha^\bullet), \quad \Gamma(U,\xi) \circ b_U = c_U \circ h^d\Gamma(U, \xi^\bullet),$$
$$\varepsilon_U \circ c_U \circ h^d\Gamma(U, (\gamma^\bullet)) = d_U, \quad \beta_U \circ a_U = d_U \circ h^d\Gamma(U, \beta^\bullet)$$

hold, and then we can conclude by using proposition 3.2.3.

The first relation $\Gamma(U,\alpha) \circ a_U = b_U \circ h^d\Gamma(U, \alpha^\bullet)$ is an straightforward consequence of the facts that $\alpha^\bullet$ and $\boldsymbol{\alpha}$ induce the same isomorphism in $D^b(\mathbb{C}_X)$ (*cf.* 2.1.2), and that the isomorphisms $\boldsymbol{\alpha}$ and $\alpha[-d]$ coincide after the canonical identification $\mathbf{D}(\mathcal{O}_X) = \mathit{Ext}^d_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{D}_X)[-d]$.

The second relation $\Gamma(U,\xi)\circ b_U = c_U \circ h^d\Gamma(U,\xi^\bullet)$ comes from the standard properties of the total derived (bi)functors $\mathbf{R}\,\mathcal{H}om^\bullet(-,-)$ and of the natural morphism

$$\boldsymbol{\xi} : \mathbf{R}\,\mathcal{H}om^\bullet_{\mathcal{D}_X}(-,?) \longrightarrow \mathbf{R}\,\mathcal{H}om^\bullet_{\mathbb{C}_X}(\mathbf{S}(?),\mathbf{S}(-))$$

(*cf.* A.4), and from the fact that the morphisms $\boldsymbol{\xi}_{\mathcal{D}_X}$ and $\xi[-d]$ coincide after the canonical identifications $\mathbf{D}(\mathcal{O}_X) = \mathcal{E}xt^d_{\mathcal{D}_X}(\mathcal{O}_X,\mathcal{D}_X)[-d]$, $\mathcal{O}_X^\vee = \mathcal{E}xt^d_{\mathbb{C}_X}(\mathcal{O}_X,\mathbb{C}_X)[-d]$.

The third relation $\varepsilon_U\circ c_U\circ h^d\Gamma(U,(\gamma^\bullet)) = d_U$ follows from the commutativity of the following diagram

$$\begin{tikzcd}
\mathbf{R}^d\,\Gamma(U,\mathbf{R}\,\mathcal{H}om^\bullet_{\mathbb{C}_X}(\mathcal{O}_X,\mathbb{C}_X) \arrow[r, "(\kappa_0^\bullet)_*"] \arrow[d, "="'] & \mathbf{R}^d\,\Gamma(U,\mathbf{R}\,\mathcal{H}om^\bullet_{\mathbb{C}_X}(\mathcal{O}_X,\Omega^\bullet_X)) \arrow[d, "="] \\
h^d\,\mathbf{R}\,\mathrm{Hom}^\bullet_{\mathbb{C}_U}(\mathcal{O}_U,\mathbb{C}_U) \arrow[r, "(\kappa_0^\bullet)_*"] \arrow[d, "\boldsymbol{\nu}^d"'] & h^d\,\mathbf{R}\,\mathrm{Hom}^\bullet_{\mathbb{C}_U}(\mathcal{O}_U,\Omega^\bullet_U) \arrow[d, "\boldsymbol{\nu}^d"] \\
\mathrm{Hom}_{D(\mathbb{C}_U)}(\mathcal{O}_U,\mathbb{C}_U[d]) \arrow[r, "(\kappa_0^\bullet[d])_*"] & \mathrm{Hom}_{D(\mathbb{C}_U)}(\mathcal{O}_U,\Omega^\bullet_U[d])
\end{tikzcd}$$

and from standard naturality properties.

The last relation $\beta_U\circ a_U = d_U\circ h^d\Gamma(U,\beta^\bullet)$ is a consequence of the commutativity of the following diagramms (see A.7)

$$\begin{tikzcd}
h^d\,\mathrm{Hom}^\bullet_{\mathbb{C}_U}(\mathcal{O}_U,\Omega^\bullet_U) \arrow[r, "\mathrm{can}"] \arrow[d, "\nu_2^d"'] & h^d\,\mathbf{R}\,\mathrm{Hom}^\bullet_{\mathbb{C}_U}(\mathcal{O}_U,\Omega^\bullet_U) \arrow[d, "\boldsymbol{\nu}^d"] \\
\mathrm{Hom}_{K(\mathbb{C}_U)}(\mathcal{O}_U,\Omega^\bullet_U[d]) \arrow[r, "\mathrm{can}"] & \mathrm{Hom}_{D(\mathbb{C}_U)}(\mathcal{O}_U,\Omega^\bullet_U[d])
\end{tikzcd}$$

and

$$\begin{tikzcd}
h^d\Gamma(U,\omega_X[-d]) \arrow[r, "h^d\Gamma(U,\beta^\bullet)"] \arrow[d, "="'] & h^d\Gamma(U,\mathcal{H}om^\bullet_{\mathbb{C}_X}(\mathcal{O}_X,\Omega^\bullet_X)) \arrow[d, "="] \\
\Gamma(U,\omega_X) \arrow[r, "\beta'"] \arrow[d, "\mathrm{forget}"'] & h^d\,\mathrm{Hom}^\bullet_{\mathbb{C}_U}(\mathcal{O}_U,\Omega^\bullet_U) \arrow[d, "\nu_2^d"] \\
\mathrm{Hom}_{K(\mathbb{C}_U)}(\mathcal{O}_U,\omega_U) \arrow[r, "(\kappa_1^\bullet[d])_*"] & \mathrm{Hom}_{K(\mathbb{C}_U)}(\mathcal{O}_U,\Omega^\bullet_U[d])
\end{tikzcd}$$

where $\beta'(\theta)$ is the cohomology class of $\Gamma(U,\beta^d)(\theta)$ for every top differential form θ on U. $\square$

3.2.5. Corollary. — *For every open set $U\subset X$, the morphism*

$$\beta_U : \Gamma(U,\omega_X) \longrightarrow \mathrm{Hom}_{D(\mathbb{C}_U)}(\mathcal{O}_U,\mathbb{C}_U[d])$$

is right $\mathcal{D}_X(U)$-linear.

3.3. Compatibility of the Duality Morphism with the Serre and the Poincaré-Verdier Dualities: Mebkhout's Proof

3.3.1. For each open set $U \subset X$ we consider the *analytic trace morphism* $\mathrm{Tr}_U : \mathrm{H}^d_c(U,\omega_U) \to \mathbb{C}$ and the *topological trace morphism* $\mathrm{tr}_U : \mathrm{H}^{2d}_c(U,\mathbb{C}_U) \to \mathbb{C}$ given by integration of top differential forms (of type (d,d)) with compact support:

The smooth De Rham complex

$$0 \longrightarrow \mathbb{C}_X \longrightarrow \mathcal{E}^0_X \longrightarrow \cdots \longrightarrow \mathcal{E}^{2d}_X \longrightarrow 0$$

gives rise to a morphism in the derived category $\theta_1 : \mathcal{E}^{2d}_X[-2d] \to \mathbb{C}_X$ which induces another one

$$\overline{\theta_1} : \Gamma_c(U,\mathcal{E}^{2d}_X) = \mathbf{R}^{2d}\,\Gamma_c(U,\mathcal{E}^{2d}_X[-2d]) \xrightarrow{\mathbf{R}^{2d}\,\Gamma_c(U,\theta_1)} \mathrm{H}^{2d}_c(U,\mathbb{C}_U).$$

The topological trace morphism is defined by the relation: $\mathrm{tr}_U \circ \overline{\theta_1} = \int_U$.

In a similar way, the Dolbeault resolution

$$0 \longrightarrow \omega_X \longrightarrow \mathcal{E}^{d,0}_X \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} \mathcal{E}^{d,d}_X = \mathcal{E}^{2d}_X \longrightarrow 0$$

gives rise to a morphism in the derived category $\theta_2 : \mathcal{E}^{2d}_X[-d] \to \omega_X$ inducing another one

$$\overline{\theta_2} : \Gamma_c(U,\mathcal{E}^{2d}_X) = \mathbf{R}^{d}\,\Gamma_c(U,\mathcal{E}^{2d}_X[-d]) \xrightarrow{\mathbf{R}^{d}\,\Gamma_c(U,\theta_2)} \mathrm{H}^{d}_c(U,\omega_U).$$

The analytic trace morphism is defined by the relation: $\mathrm{Tr}_U \circ \overline{\theta_2} = \int_U$.

The Poincaré-De Rham morphism $\boldsymbol{\kappa}' : \omega_X \to \mathbb{C}_X[d]$ induces a map

$$\beta'_U : \mathrm{H}^d_c(U,\omega_U) \longrightarrow \mathrm{H}^d_c(U,\mathbb{C}_U[d]) = \mathrm{H}^{2d}_c(U,\mathbb{C}_U).$$

A straightforward computation shows that $\boldsymbol{\kappa}' \circ \theta_2 = (-1)^d \theta_1[d]$, and then we obtain

$$\mathrm{tr}_U \circ \beta'_U = (-1)^d \mathrm{Tr}_U. \tag{5}$$

The *analytic Serre pairing* [**Se**]

$$\langle -,-\rangle_S : \Gamma(U,\omega_X) \times \mathrm{H}^d_c(U,\mathcal{O}_U) \longrightarrow \mathbb{C}$$

is given by the composition of the analytic trace morphism Tr_U with the Yoneda pairing

$$\Gamma(U,\omega_X) \times \mathrm{H}^d_c(U,\mathcal{O}_U) \xrightarrow{\text{forget} \times \mathrm{Id}} \mathrm{Hom}_{\mathbb{C}_U}(\mathcal{O}_U,\omega_U) \times \mathrm{H}^d_c(U,\mathcal{O}_U) \xrightarrow{\text{Yoneda}} \mathrm{H}^d_c(U,\omega_U).$$

The vector space $\Gamma(U,\omega_X)$ has a natural Fréchet-Schwartz structure and, if U is Stein, the pairing $\langle -,-\rangle_S$ identifies $\mathrm{H}^d_c(U,\mathcal{O}_U)$ with the topological dual $\Gamma(U,\omega_X)'$. Then, $\mathrm{H}^d_c(U,\mathcal{O}_U)$ carries a natural DFS structure and $\Gamma(U,\omega_X) \simeq \mathrm{H}^d_c(U,\mathcal{O}_U)'$ (*cf.* [**Se**], [**B-S**], ch. 1, § 1 (c), 2.1).

The *Poincaré-Verdier pairing* (*cf.* [**DP**], exp. 5)

$$\langle -,-\rangle_{PV} : \mathrm{Hom}_{D(\mathbb{C}_U)}(\mathcal{O}_U,\mathbb{C}_U[d]) \times \mathrm{H}^d_c(U,\mathcal{O}_U) \longrightarrow \mathbb{C}$$

is given by the composition of the topological trace morphism tr_U with the Yoneda map

$$\mathrm{Hom}_{D(\mathbb{C}_U)}(\mathcal{O}_U, \mathbb{C}_U[d]) \times \mathrm{H}^d_c(U, \mathcal{O}_U) \xrightarrow{\text{Yoneda}} \mathrm{H}^{2d}_c(U, \mathbb{C}_U).$$

According to the Poincaré-Verdier duality, the pairing $\langle -, - \rangle_{PV}$ indentifies $\mathrm{Hom}_{D(\mathbb{C}_U)}(\mathcal{O}_U, \mathbb{C}_U[d])$ with the algebraic dual $\mathrm{H}^d_c(U, \mathcal{O}_U)^*$ (*cf. loc. cit.*).

3.3.2. Lemma. — *The Serre pairing is $\mathcal{D}_X(U)$-balanced.*

Proof. — Let c be a class in $\mathrm{H}^d_c(U, \mathcal{O}_U)$, $\theta \in \Gamma(U, \omega_X)$ and $P \in \mathcal{D}_X(U)$. For each $i, j = 0, \ldots, d$ let $\mathcal{E}^{i,j}_U$ be the sheaf of smooth differential forms of type (i, j). The Dolbeault resolution

$$0 \longrightarrow \mathcal{O}_U \longrightarrow \mathcal{E}^{0,0}_U \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} \mathcal{E}^{0,d}_U \longrightarrow 0$$

$$(\text{resp. } 0 \longrightarrow \omega_U \longrightarrow \mathcal{E}^{d,0}_U \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} \mathcal{E}^{d,d}_U \longrightarrow 0)$$

is a complex of left (resp. right) $\mathcal{D}_U$-modules, and the morphism

$$\theta \wedge : \mathcal{E}^{0,\bullet}_U \longrightarrow \mathcal{E}^{d,\bullet}_U$$

is a lifting of $\theta : \mathcal{O}_U \to \omega_U$. Let $\alpha \in \Gamma_c(U, \mathcal{E}^{0,d}_U)$ a section representing the class c. We have

$$\langle \theta, Pc \rangle_S = \cdots = \int_U \theta \wedge (P\alpha), \quad \langle \theta P, c \rangle_S = \cdots = \int_U (\theta P) \wedge \alpha.$$

Both integrals coincide when P is a holomorphic function. For the general case we can work in local coordinates $z = (z_1, \ldots, z_d), \overline{z} = (\overline{z}_1, \ldots, \overline{z}_d)$, and it is enough to consider $P = \partial_{z_i}$. Let $\alpha = f d\overline{z}$, $\theta = g dz$ be the local expressions. We have $P\alpha = f_{z_i} d\overline{z}, \theta P = P^t(g) dz = -g_{z_i} dz$, where P^t is the transposed operator. The difference $\langle Pc, \theta \rangle_S - \langle c, \theta P \rangle_S$ is the integral of the closed form $(fg)_{z_i} dz d\overline{z}$, and then it vanishes. $\square$

3.3.3. Lemma. — *The Poincaré-Verdier pairing is $\mathcal{D}_X(U)$-balanced.*

Proof. — This is a consequence of the easy fact that the Yoneda map

$$\mathrm{Hom}_{D(\mathbb{C}_U)}(\mathcal{O}_U, \mathbb{C}_U[d]) \times \mathrm{H}^d_c(U, \mathcal{O}_U) \xrightarrow{(-,-)} \mathrm{H}^{2d}_c(U, \mathbb{C}_U)$$

is $\mathcal{D}_X(U)$-balanced. To see that, take $c \in \mathrm{H}^d_c(U, \mathcal{O}_U)$, $\varphi \in \mathrm{Hom}_{D(\mathbb{C}_U)}(\mathcal{O}_U, \mathbb{C}_U[d])$ and $P \in \mathcal{D}_X(U) \subset \mathrm{Hom}_{\mathbb{C}_U}(\mathcal{O}_U, \mathcal{O}_U)$. Then we have

$$(\varphi, Pc) = (\varphi, P_*(c)) = \varphi_*(P_*(c)) = (\varphi P)_*(c) = (\varphi P, c). \qquad \square$$

3.3.4. Proposition. — *The following relation*

$$\langle -, - \rangle_{PV} \circ (\beta_U \times Id) = (-1)^d \langle -, - \rangle_S$$

holds.

Proof. — According to the definition of β_U, the following diagram

$$\begin{tikzcd}
\Gamma(U,\omega_X)\times \mathrm{H}^d_c(U,\mathcal{O}_U) \arrow[r, "\text{Yoneda}"] \arrow[d, "\beta_U\times \mathrm{Id}"'] & \mathrm{H}^d_c(U,\omega_U) \arrow[d, "\beta'_U"] \\
\mathrm{Hom}_{D(\mathbb{C}_U)}(\mathcal{O}_U,\mathbb{C}_U[d])\times \mathrm{H}^d_c(U,\mathcal{O}_U) \arrow[r, "\text{Yoneda}"] & \mathrm{H}^{2d}_c(U,\mathbb{C}_U)
\end{tikzcd}$$

is commutative. The propostion then follows from (5). □

3.3.5. Proposition. — *For each Stein open set $U\subset X$, there exist natural right $\mathcal{D}_X(U)$-linear isomorphisms*

$$\mathrm{H}^d_c(U,\mathcal{O}_U)' \xrightarrow{\simeq} \Gamma(U,\mathcal{E}xt^d_{\mathcal{D}_X}(\mathcal{O}_X,\mathcal{D}_X))$$
$$\mathrm{H}^d_c(U,\mathcal{O}_U)^* \xrightarrow{\simeq} \Gamma(U,\mathcal{E}xt^d_{\mathbb{C}_X}(\mathcal{O}_X,\mathbb{C}_X))$$

such that the following diagram

$$\begin{tikzcd}
\Gamma(U,\mathcal{E}xt^d_{\mathcal{D}_X}(\mathcal{O}_X,\mathcal{D}_X)) \arrow[r, "\Gamma(U,\xi)"] & \Gamma(U,\mathcal{E}xt^d_{\mathbb{C}_X}(\mathcal{O}_X,\mathbb{C}_X)) \\
\mathrm{H}^d_c(U,\mathcal{O}_U)' \arrow[u, "\simeq"] \arrow[r, "inclusion"] & \mathrm{H}^d_c(U,\mathcal{O}_U)^* \arrow[u, "\simeq"']
\end{tikzcd}$$

commutes.

Proof. — It is a consequence of propositions 3.2.4, 3.3.4, of lemmas 3.3.2, 3.3.3 and of Serre and Poincaré-Verdier dualities. □

3.3.6. According to 2.1.2, corollary 2.2.3 and proposition 3.2.1, the question in the theorem 3.1.1 is equivalent to prove that

$$\xi\otimes \mathrm{Id}_{\mathcal{M}^\bullet} : \mathcal{E}xt^d_{\mathcal{D}_X}(\mathcal{O}_X,\mathcal{D}_X)\overset{\bullet}{\otimes}_{\mathcal{D}_X}\mathcal{M}^\bullet \longrightarrow \mathcal{E}xt^d_{\mathbb{C}_X}(\mathcal{O}_X,\mathbb{C}_X)\overset{\bullet}{\otimes}_{\mathcal{D}_X}\mathcal{M}^\bullet$$

is an isomorphism.

We can suppose (*cf.* [**M-N3**], II.5) that $\mathcal{M}^\bullet$ is a single holonomic module $\mathcal{M}$. The problem being local, we can also suppose that there exists a finite free resolution $\mathcal{P}^\bullet$

$$0\longrightarrow \mathcal{D}_X^{r_m}\longrightarrow\cdots\longrightarrow \mathcal{D}_X^{r_0}\longrightarrow\mathcal{M}\longrightarrow 0.$$

We have to prove that

$$\xi\otimes \mathrm{Id}_{\mathcal{P}^\bullet} : \mathcal{E}xt^d_{\mathcal{D}_X}(\mathcal{O}_X,\mathcal{D}_X)\otimes_{\mathcal{D}_X}\mathcal{P}^\bullet \longrightarrow \mathcal{E}xt^d_{\mathbb{C}_X}(\mathcal{O}_X,\mathbb{C}_X)\otimes_{\mathcal{D}_X}\mathcal{P}^\bullet$$

is a quasi-isomorphism.

According to proposition 3.3.5, for each Stein open set $U \subset X$ the morphism $\Gamma(U, \xi \otimes \mathrm{Id}_{\mathcal{P}^\bullet})$ can be identified with

$$\begin{array}{ccc} \left[\mathrm{H}^d_c(U, \mathcal{O}_U)^{r_m}\right]' \longrightarrow \cdots \longrightarrow & \left[\mathrm{H}^d_c(U, \mathcal{O}_U)^{r_0}\right]' \\ \Big\downarrow \text{inc.} & \Big\downarrow \text{inc.} \\ \left[\mathrm{H}^d_c(U, \mathcal{O}_U)^{r_m}\right]^* \longrightarrow \cdots \longrightarrow & \left[\mathrm{H}^d_c(U, \mathcal{O}_U)^{r_0}\right]^* . \end{array}$$

But the complex

$$\mathrm{H}^d_c(U, \mathcal{O}_U)^{r_m} \longleftarrow \cdots \longleftarrow \mathrm{H}^d_c(U, \mathcal{O}_U)^{r_0}$$

is quasi-isomorphic to $\mathbf{R}\Gamma_c(U, \mathbf{S}(\mathcal{M}))$ (up to some shift), and so, by Kashiwara's constructibility theorem [**Ka**] (see also [**M-N3**]) and by proposition 1.1.3, it has finite dimensional cohomology for all small balls U with respect to some local coordinates. According to Serre's lemma for DFS spaces [**Se**], 10.1, [**B-S**], ch. 1, § 1 (c), we deduce that $\Gamma(U, \xi \otimes \mathrm{Id}_{\mathcal{P}^\bullet})$ is a quasi-isomorphism for many open sets U, and then $\xi \otimes \mathrm{Id}_{\mathcal{P}^\bullet}$ is a quasi-isomorphism too.

3.3.7. Remark. — The duality morphism $\mathbf{DR}(\mathcal{M}^\bullet) \to \mathbf{S}(\mathcal{M}^\bullet)^\vee$ considered by Mebkhout in [**Me3**], III.1.1 comes from the isomorphisms $\boldsymbol{\lambda}_{\mathcal{M}^\bullet}$ and $\boldsymbol{\mu}_{\mathcal{M}^\bullet}$ of proposition 2.2.2 and from the morphism $\mathcal{E}xt^d_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{D}_X) \simeq \omega_X \to \mathcal{E}xt^d_{\mathbb{C}_X}(\mathcal{O}_X, \mathbb{C}_X)$ induced by Serre and Poincaré-Verdier dualities. According to proposition 3.3.5, Mebkhout's duality morphism coincides with the formal one.

3.4. Compatibility of the Duality Morphism with the Local Analytic Duality: Kashiwara-Kawai's proof. — Let $\mathcal{M}^\bullet$ be a bounded complex of left $\mathcal{D}_X$-modules with holonomic cohomology. In order to proof the Local Duality Theorem 3.1.1 it is enough to proof that the stalk

$$(\boldsymbol{\xi}_{\mathcal{M}^\bullet})_x : \mathbf{DR}(\mathcal{M}^\bullet)_x \longrightarrow \mathbf{S}(\mathcal{M}^\bullet)^\vee_x$$

is an isomorphism for every point $x \in X$.

Let $i : \{x\} \hookrightarrow X$ be the inclusion. Denote

$$\mathbf{S}_{!x}(\mathcal{M}^\bullet) := \mathbf{R}\,\mathrm{Hom}^\bullet_{\mathcal{D}_{X,x}}(\mathcal{M}^\bullet_x, i^!\mathcal{O}_X), \quad \mathbf{DR}_x(\mathcal{M}^\bullet) := \mathbf{R}\,\mathrm{Hom}^\bullet_{\mathcal{D}_{X,x}}(\mathcal{O}_{X,x}, \mathcal{M}^\bullet_x).$$

We have a natural isomorphism (*cf.* A.11)

$$i^! \circ \mathbf{S} \xrightarrow{\ \simeq\ } \mathbf{S}_{!x}, \tag{6}$$

which induces, joint with (2), another one

$$\mathbf{S}_{!x}(\mathcal{O}_X) \simeq i^!\mathbb{C}_X. \tag{7}$$

Call

$$\boldsymbol{\xi}_{\mathcal{M}^\bullet}(x) : \mathbf{DR}_x(\mathcal{M}^\bullet) \longrightarrow \mathbf{R}\,\mathrm{Hom}^\bullet_{\mathbb{C}}(\mathbf{S}_{!x}(\mathcal{M}^\bullet), i^!\mathbb{C}_X)$$

the natural morphism defined as in definition 2.2.1, now using (7) instead of (2).

Call

$$\boldsymbol{\zeta} : \mathbf{S}(\mathcal{M}^\bullet)^\vee_x \longrightarrow \mathbf{R}\,\mathrm{Hom}^\bullet_{\mathbb{C}}(\mathbf{S}_{!x}(\mathcal{M}^\bullet), i^!\mathbb{C}_X)$$

the composition of the natural morphism (*cf.* A.11)

$$\mathbf{R}\,\mathcal{H}om^{\bullet}_{\mathbb{C}_X}(\mathbf{S}(\mathcal{M}^{\bullet}),\mathbb{C}_X)_x \longrightarrow \mathbf{R}\,\mathrm{Hom}^{\bullet}_{\mathbb{C}}(i^!\,\mathbf{S}(\mathcal{M}^{\bullet}), i^!\mathbb{C}_X))$$

with the isomorphism induced by (6).

3.4.1. Lemma. — *Let $\mathcal{M}^{\bullet}$ be a bounded complex of left $\mathcal{D}_X$-modules. The following diagram*

$$\begin{array}{ccc}
\mathbf{DR}(\mathcal{M}^{\bullet})_x & \xrightarrow{(\boldsymbol{\xi}_{\mathcal{M}^{\bullet}})_x} & (\mathcal{M}^{\bullet})^{\vee}_x \\
{\scriptstyle nat.}\downarrow & & \downarrow{\scriptstyle \boldsymbol{\zeta}} \\
\mathbf{DR}_x(\mathcal{M}^{\bullet}) & \xrightarrow{\boldsymbol{\xi}_{\mathcal{M}^{\bullet}}(x)} & \mathbf{R}\,\mathrm{Hom}^{\bullet}_{\mathbb{C}}(\mathbf{S}_{!x}(\mathcal{M}^{\bullet}), i^!\mathbb{C}_X)
\end{array}$$

is commutative.

Proof. — It is a consequence of lemma A.12. □

3.4.2. Corollary. — *Let $\mathcal{M}^{\bullet}$ be a bounded complex of left $\mathcal{D}_X$-modules with holonomic cohomology. Then, $(\boldsymbol{\xi}_{\mathcal{M}^{\bullet}})_x$ is an isomorphism if and only if $(\boldsymbol{\xi}_{M^{\bullet}})(x)$ is an isomorphism.*

Proof. — As $\mathcal{M}^{\bullet}$ has coherent cohomology, the natural morphism

$$\mathbf{DR}(\mathcal{M}^{\bullet})_x \longrightarrow \mathbf{DR}_x(\mathcal{M}^{\bullet})$$

is an isomorphism. Also, as $\mathcal{M}^{\bullet}$ has holonomic cohomology, by the constructibility theorem of Kashiwara [**Ka**] (see also [**M-N3**]) and by proposition 1.2.3, the morphism $\boldsymbol{\zeta}$ is an isomorphism. We conclude by applying the preceeding lemma. □

We can repeat the arguments in proposition 2.2.2 and corollary 2.2.3 to obtain the following.

3.4.3. Proposition. — *For every bounded complex of left $\mathcal{D}_X$-modules $\mathcal{M}^{\bullet}$ with coherent cohomology objects, the duality morphism $\boldsymbol{\xi}_{\mathcal{M}^{\bullet}}(x)$ is an isomorphism if and only if $\boldsymbol{\xi}_{\mathcal{D}_X}(x)\otimes Id_{\mathcal{M}^{\bullet}_x}$ is an isomorphism.*

Now, we are going to give the punctual analogous of results in section 3.2.

First, the complexes $\mathbf{DR}_x(\mathcal{D}_X)$ and $i^!\mathcal{O}_X$ are concentrated in degree d and we have an isomorphism of right $\mathcal{D}_{X,x}$-modules

$$\alpha(x) := h^d(\mathrm{nat.})\circ\boldsymbol{\alpha}_x : \omega_{X,x} \xrightarrow{\simeq} h^d\,\mathbf{R}\,\mathrm{Hom}^{\bullet}_{\mathcal{D}_{X,x}}(\mathcal{O}_{X,x},\mathcal{D}_{X,x}) = \mathrm{Ext}^d_{\mathcal{D}_{X,x}}(\mathcal{O}_{X,x},\mathcal{D}_{X,x}).$$

The complex $i^!\mathbb{C}_X$ is concentrated in degree $2d$ (*cf.* 1.2.2). We then obtain that the complex $\mathbf{R}\,\mathrm{Hom}^{\bullet}_{\mathbb{C}}(\mathbf{S}_{!x}(\mathcal{D}_X), i^!\mathbb{C}_X) = \mathbf{R}\,\mathrm{Hom}^{\bullet}_{\mathbb{C}}(i^!\mathcal{O}_X, i^!\mathbb{C}_X)$ is also concentrated in degree d, and we have a canonical identification

$$\mathrm{Hom}_{D(\mathbb{C})}(i^!\mathcal{O}_X, i^!\mathbb{C}_X[d]) = \mathrm{Hom}_{\mathbb{C}}(\mathrm{H}^d_x(\mathcal{O}_X), \mathrm{H}^{2d}_x(\mathbb{C}_X)).$$

Call

$$\xi(x) := h^d(\boldsymbol{\xi}_{\mathcal{D}_X}(x)) : \operatorname{Ext}^d_{\mathcal{D}_{X,x}}(\mathcal{O}_{X,x}, \mathcal{D}_{X,x}) \longrightarrow \operatorname{Ext}^d_{\mathbb{C}}(i^!\mathcal{O}_X, i^!\mathbb{C}_X),$$

which is right $\mathcal{D}_{X,x}$-linear.

As in 3.2, we find an isomorphism

$$\varepsilon(x) : \operatorname{Ext}^d_{\mathbb{C}}(i^!\mathcal{O}_X, i^!\mathbb{C}_X) \xrightarrow{\simeq} \operatorname{Hom}_{D(\mathbb{C})}(i^!\mathcal{O}_X, i^!\mathbb{C}_X[d]) = \operatorname{Hom}_{\mathbb{C}}(\mathrm{H}^d_x(\mathcal{O}_X), \mathrm{H}^{2d}_x(\mathbb{C}_X))$$

and a map

$$\beta(x) : \omega_{X,x} \longrightarrow \operatorname{Hom}_{D(\mathbb{C})}(i^!\mathcal{O}_X, i^!\mathbb{C}_X[d]) = \operatorname{Hom}_{\mathbb{C}}(\mathrm{H}^d_x(\mathcal{O}_X), \mathrm{H}^{2d}_x(\mathbb{C}_X)),$$

which are compatibles in the obvious way with the ε_U and the β_U defined in 3.2.2, for $x \in U$.

The following proposition is then a direct consequence of proposition 3.2.4.

3.4.4. Proposition. — *For every point $x \in X$, the following diagramm*

$$\begin{tikzcd}
\operatorname{Ext}^d_{\mathcal{D}_{X,x}}(\mathcal{O}_{X,x}, \mathcal{D}_{X,x}) \arrow[r, "\xi(x)"] & \operatorname{Ext}^d_{\mathbb{C}}(i^!\mathcal{O}_X, i^!\mathbb{C}_X) \arrow[d, "\simeq"', "\varepsilon(x)"] \\
\omega_{X,x} \arrow[u, "\alpha(x)", "\simeq"'] \arrow[r, "\beta(x)"] & \operatorname{Hom}_{\mathbb{C}}(\mathrm{H}^d_x(\mathcal{O}_X), \mathrm{H}^{2d}_x(\mathbb{C}_X))
\end{tikzcd}$$

is commutative.

As in 3.3.1, call

a) $\operatorname{Tr}_x : \mathrm{H}^d_x(\omega_X) \to \mathbb{C}$ the *local analytic trace morphism*, which is induced by the global analytic trace morphism Tr_X,

b) $\beta'(x) : \mathrm{H}^d_x(\omega_X) \to \mathrm{H}^{2d}_x(\mathbb{C}_X)$ the morphism induced by the Poincaré-De Rham morphism $\omega_X \to \mathbb{C}_X[d]$,

c) $\langle -, -\rangle^{\mathrm{an}}_x : \omega_{X,x} \times \mathrm{H}^d_x(\mathcal{O}_X) \to \mathbb{C}$ the *local duality pairing* obtained by composing the local analytic trace morphism Tr_x with the Yoneda pairing,

d) $\langle -, -\rangle^{top}_x : \operatorname{Hom}_{\mathbb{C}}(\mathrm{H}^d_x(\mathcal{O}_X), \mathrm{H}^{2d}_x(\mathbb{C}_X)) \times \mathrm{H}^d_x(\mathcal{O}_X) \to \mathbb{C}$ the composition of the punctual topological trace tr_x (see 1.2.2) with the evaluation map.

As in 3.3.1, we have the following assertions:

(1) The pairings $\langle -, -\rangle^{\mathrm{an}}_x$ and $\langle -, -\rangle^{top}_x$ are $\mathcal{D}_{X,x}$-balanced.

(2) $\langle -, -\rangle^{top}_x \circ (\beta(x) \times \mathrm{Id}) = (-1)^d \langle -, -\rangle^{\mathrm{an}}_x$.

Using the Local Analytic Duality Theorem we obtain the following.

3.4.5. Proposition. — *There exist natural right $\mathcal{D}_{X,x}$-linear isomorphisms*

$$\mathrm{H}^d_x(\mathcal{O}_X)' \xrightarrow{\simeq} \operatorname{Ext}^d_{\mathcal{D}_{X,x}}(\mathcal{O}_{X,x}, \mathcal{D}_{X,x})$$
$$\mathrm{H}^d_x(\mathcal{O}_X)^* \xrightarrow{\simeq} \operatorname{Ext}^d_{\mathbb{C}}(i^!\mathcal{O}_X, i^!\mathbb{C}_X)$$

such that the following diagram

$$\begin{tikzcd}
\mathrm{Ext}^d_{\mathcal{D}_{X,x}}(\mathcal{O}_{X,x},\mathcal{D}_{X,x}) \arrow[r, "\xi(x)"] & \mathrm{Ext}^d_{\mathbb{C}}(i^!\mathcal{O}_X, i^!\mathbb{C}_X) \\
\mathrm{H}^d_x(\mathcal{O}_X)' \arrow[u, "\simeq"] \arrow[r, "inclusion"] & \mathrm{H}^d_x(\mathcal{O}_X)^* \arrow[u, "\simeq"']
\end{tikzcd}$$

commutes.

The following punctual duality of Kashiwara [**Ka**], § 5 can be deduced from propositions 3.4.3 and 3.4.5 in a similar way we did in 3.3.6.

3.4.6. Proposition. — *Let $\mathcal{M}^\bullet$ be a bounded complex of left $\mathcal{D}_X$-modules with holonomic cohomology. Then, the punctual duality morphism*

$$\boldsymbol{\xi}_{\mathcal{M}^\bullet}(x) : \mathbf{DR}_x(\mathcal{M}^\bullet) \longrightarrow \mathbf{R}\,\mathrm{Hom}^\bullet_{\mathbb{C}}(\mathbf{S}_{!x}(\mathcal{M}^\bullet), i^!\mathbb{C}_X)$$

is an isomorphism for each point $x \in X$.

3.4.7. Corollary. — *Let $\mathcal{M}^\bullet$ be a bounded complex of $\mathcal{D}_X$-modules with holonomic cohomology. Then*

$$\mathrm{Ext}^i_{\mathcal{D}_{X,x}}(\mathcal{O}_{X,x},\mathcal{M}^\bullet_x) \simeq \mathrm{Ext}^{d-i}_{\mathcal{D}_{X,x}}(\mathcal{M}^\bullet_x, \mathrm{H}^d_x(\mathcal{O}_X))^*$$

for each $i \in \mathbb{Z}$ and for each $x \in X$.

Finally, according to corollary 3.4.2, we deduce that the morphism

$$(\boldsymbol{\xi}_{\mathcal{M}^\bullet})_x : \mathbf{DR}(\mathcal{M}^\bullet)_x \longrightarrow \mathbf{S}(\mathcal{M}^\bullet)^\vee_x$$

is an isomorphism for each $x \in X$, and the proof of theorem 3.1.1 is finished.

3.4.8. Remark. — Actually, proposition 3.4.6 and corollary 3.4.7 do not match exactly the statement in [**Ka**], § 5. The relation between both results becomes clear by considering the dual complex $(\mathcal{M}^\bullet)^*$ (*cf.* 3.5.3). Anyway, the point is to prove that the punctual duality morphism $\boldsymbol{\xi}_{\mathcal{M}^\bullet}(x)$ induced by the (formal) duality morphism $\boldsymbol{\xi}_{\mathcal{M}^\bullet}$ coincides with the isomorphism in *loc. cit.*

3.5. Some Complements

3.5.1. In a similar way we defined the duality morphism in 2.2.1, we find for every bounded complex of left $\mathcal{D}_X$-modules $\mathcal{M}^\bullet$ with coherent cohomology a natural morphism

$$\boldsymbol{\eta}_{\mathcal{M}^\bullet} : \mathbf{S}(\mathcal{M}^\bullet) \longrightarrow \mathbf{DR}(\mathcal{M}^\bullet)^\vee$$

by composing the natural morphism (*cf.* A.2)

$$\boldsymbol{\eta} : \mathbf{R}\,\mathcal{H}om^\bullet_{\mathcal{D}_X}(\mathcal{M}^\bullet,\mathcal{O}_X) \longrightarrow \mathbf{R}\,\mathcal{H}om^\bullet_{\mathbb{C}_X}(\mathbf{DR}(\mathcal{M}^\bullet),\mathbf{DR}(\mathcal{O}_X))$$

with the isomorphism induced by $\boldsymbol{\kappa}$ (2).

Call $\boldsymbol{\eta}^\vee_{\mathcal{M}^\bullet} := \mathbf{R}\,\mathit{Hom}^\bullet_X(\boldsymbol{\eta}_{\mathcal{M}^\bullet}, \mathbb{C}_X)$ and $\boldsymbol{\beta}_{\mathbf{DR}(\mathcal{M}^\bullet)}$ the biduality morphism corresponding to $\mathbf{DR}(\mathcal{M}^\bullet)$ (*cf.* 1.1.2). According to A.3 we have $\boldsymbol{\xi}_{\mathbf{DR}(\mathcal{M}^\bullet)} = \boldsymbol{\eta}^\vee_{\mathcal{M}^\bullet} \circ \boldsymbol{\beta}_{\mathbf{DR}(\mathcal{M}^\bullet)}$, and we obtain the following corollary of the LDT.

3.5.2. Corollary. — *For every bounded complex of left $\mathcal{D}_X$-modules $\mathcal{M}^\bullet$ with holonomic cohomology, the natural morphism*

$$\boldsymbol{\eta}_{\mathcal{M}^\bullet} : \mathbf{S}(\mathcal{M}^\bullet) \longrightarrow \mathbf{DR}(\mathcal{M}^\bullet)^\vee$$

is an isomorphism (in the derived category).

3.5.3. For every complex of left $\mathcal{D}_X$-modules $\mathcal{M}^\bullet$, its *(internal) dual* is defined by $(\mathcal{M}^\bullet)^* := \mathit{Hom}_{\mathcal{O}_X}(\omega_X, \mathbf{D}(\mathcal{M}^\bullet))[d]$, which is again a complex of left $\mathcal{D}_X$-modules (*cf.* [**Ca**], 1.1). The internal duality induces a self-(anti)equivalence of the derived category $D^b_c(\mathcal{D}_X)$.

The isomorphism $\boldsymbol{\alpha}$ of 2.1.2 induces natural isomorphisms $\mathcal{O}_X^* \simeq \mathcal{O}_X$ and

$$\mathbf{S}((\mathcal{M}^\bullet)^*) \xrightarrow{\ \simeq\ } \mathbf{DR}(\mathcal{M}^\bullet)$$

for every bounded complex of left $\mathcal{D}_X$-modules $\mathcal{M}^\bullet$ with coherent cohomology.

3.5.4. Corollary. — *For every bounded complex of left $\mathcal{D}_X$-modules $\mathcal{M}^\bullet$ with holonomic cohomology, there exist natural isomorphisms*

$$\mathbf{S}((\mathcal{M}^\bullet)^*) \xrightarrow{\ \simeq\ } \mathbf{S}(\mathcal{M}^\bullet)^\vee, \quad \mathbf{DR}((\mathcal{M}^\bullet)^*) \xrightarrow{\ \simeq\ } \mathbf{DR}(\mathcal{M}^\bullet)^\vee.$$

3.5.5. Definition. — A bounded constructible complex $\mathcal{K}^\bullet \in D^b_c(\mathbb{C}_X)$ satisfies the *support conditions* if it is concentrated in degrees $[0, d]$ and if $\dim \operatorname{supp} h^i\mathcal{K}^\bullet \leqslant d - i$ for each $i = 0, \dots, d$. If both $\mathcal{K}^\bullet$ and its dual $(\mathcal{K}^\bullet)^\vee$ satisfy the support conditions we say that $\mathcal{K}^\bullet$ is a *perverse sheaf.*

The full subcategory of $D^b_c(\mathbb{C}_X)$ whose objects are the perverse sheaves is known to be *abelian* (*cf.* [**B-B-D**]).

If $\mathcal{M}$ is a holonomic $\mathcal{D}_X$-module, according to [**Ka**] we know that $\mathbf{S}(\mathcal{M})$ and $\mathbf{DR}(\mathcal{M})$ satisfy the support conditions (*cf.* also [**M-N1**], prop. 3). The LDT gives us the following result.

3.5.6. Proposition. — *If $\mathcal{M}$ is a holonomic $\mathcal{D}_X$-module, the complexes $\mathbf{S}(\mathcal{M})$ and $\mathbf{DR}(\mathcal{M})$ are perverse sheaves.*

Appendix

In this Appendix we have collected some results on the extension of some functors, natural transformations and commutative diagrams to the category of complexes. A complete reference for these constructions is [**De2**], 1.1 (see also Erratum in [**SGA 4$\frac{1}{2}$**], p. 312). We have extracted from there (some of) the results we need and, for the ease of the reader, we have stated them in a very concrete way.

A.1. Let $\mathcal{R}_X$ be a sheaf of rings on a topological space X, let $\mathcal{R}_X^0$ be a sheaf of rings contained in its center and let R^0 the global sections of $\mathcal{R}_X^0$. The functors

$$\begin{aligned} \operatorname{Hom}^\bullet_{\mathcal{R}_X}(-,-) &: C(\mathcal{R}_X)\times C(\mathcal{R}_X) \longrightarrow C(R^0),\\ \mathcal{H}om^\bullet_{\mathcal{R}_X}(-,-) &: C(\mathcal{R}_X)\times C(\mathcal{R}_X) \longrightarrow C(\mathcal{R}_X^0)\\ -\overset{\bullet}{\otimes}_{\mathcal{R}_X} - &: C({}_r\mathcal{R}_X)\times C(\mathcal{R}_X) \longrightarrow C(\mathcal{R}_X^0)\end{aligned}$$

are defined with the usual conventions.

Given two complexes of left $\mathcal{R}_X$-modules $\mathcal{F}^\bullet, \mathcal{I}^\bullet$, the complex $\mathcal{A}^\bullet = \operatorname{Hom}^\bullet_{\mathcal{R}_X}(\mathcal{F}^\bullet,\mathcal{I}^\bullet)$ is defined by $\mathcal{A}^n = \prod_{q-p=n}\operatorname{Hom}_{\mathcal{R}_X}(\mathcal{F}^p,\mathcal{I}^q)$ and the differential $d_{\mathcal{A}}(\underline{h}) = d_{\mathcal{I}}\circ\underline{h} - (-1)^{\deg\underline{h}}\underline{h}\circ d_{\mathcal{F}}$. The complex $\mathcal{H}om^\bullet_{\mathcal{R}_X}(\mathcal{F}^\bullet,\mathcal{I}^\bullet)$ is defined in a similar way.

Given a complex of right (resp. left) $\mathcal{R}_X$-modules $\mathcal{N}^\bullet$ (resp. $\mathcal{M}^\bullet$), the complex $\mathcal{B}^\bullet = \mathcal{N}^\bullet\overset{\bullet}{\otimes}_{\mathcal{R}_X}\mathcal{M}^\bullet$ is defined by $\mathcal{B}^n = \bigoplus_{j+k=n}\mathcal{N}^j\otimes_{\mathcal{R}_X}\mathcal{M}^k$ and the differential $d_{\mathcal{B}}(y\otimes x) = (d_{\mathcal{N}}y)\otimes x + (-1)^{\deg y}y\otimes(d_{\mathcal{M}}x)$. The action of these functors on morphisms are defined in the direct way (no signs are involved).
The complex $\mathcal{G}^\bullet = \mathcal{F}^\bullet[1]$ is defined by $\mathcal{G}^n = \mathcal{F}^{n+1}$ and $d_{\mathcal{G}^\bullet} = -d_{\mathcal{F}^\bullet}$.

We have derived functors

$$\begin{aligned}\mathbf{R}\operatorname{Hom}^\bullet_{\mathcal{R}_X}(-,-) &: D^*(\mathcal{R}_X)\times D^+(\mathcal{R}_X)\longrightarrow D^\star(R^0)\\ \mathbf{R}\,\mathcal{H}om^\bullet_{\mathcal{R}_X}(-,-) &: D^*(\mathcal{R}_X)\times D^+(\mathcal{R}_X)\longrightarrow D^\star(\mathcal{R}_X^0)\end{aligned}$$

for $* = \star = \varnothing$ or $* = -, \star = +$, and

$$-\overset{\bullet}{\otimes}_{\mathcal{R}_X} - : D^-({}_r\mathcal{R}_X)\times D^-(\mathcal{R}_X)\to D^-(\mathcal{R}_X^0)$$

(*cf.* [**Ha**], II, § 3, § 4; see also [**Sp**] in order to avoid boundedness conditions on complexes).

A.2. Given three complexes $\mathcal{F}^\bullet, \mathcal{I}^\bullet, \mathcal{J}^\bullet$ of left $\mathcal{R}_X$-modules, we define a natural morphism in $C(\mathcal{R}_X^0)$

$$\xi^\bullet : \mathcal{H}om^\bullet_{\mathcal{R}_X}(\mathcal{F}^\bullet,\mathcal{I}^\bullet)\longrightarrow \mathcal{H}om^\bullet_{\mathcal{R}_X^0}(\mathcal{H}om^\bullet_{\mathcal{R}_X}(\mathcal{I}^\bullet,\mathcal{J}^\bullet),\mathcal{H}om^\bullet_{\mathcal{R}_X}(\mathcal{F}^\bullet,\mathcal{J}^\bullet))$$

in the following way

$$\xi^\bullet(\underline{h})(\underline{a}) = (-1)^{(\deg\underline{h})(\deg\underline{a})}\underline{a}\circ\underline{h}.$$

In a similar way we define a natural morphism

$$\eta^\bullet : \mathcal{H}om^\bullet_{\mathcal{R}_X}(\mathcal{F}^\bullet,\mathcal{I}^\bullet)\longrightarrow \mathcal{H}om^\bullet_{\mathcal{R}_X^0}(\mathcal{H}om^\bullet_{\mathcal{R}_X}(\mathcal{J}^\bullet,\mathcal{F}^\bullet),\mathcal{H}om^\bullet_{\mathcal{R}_X}(\mathcal{J}^\bullet,\mathcal{I}^\bullet))$$

by putting $\eta^\bullet(\underline{h})(\underline{b}) = \underline{h}\circ\underline{b}$.

If $\mathcal{F}^\bullet = \mathcal{R}_X$, we have an obvious identification (no signs are involved) between the identity functor of $C(\mathcal{R}_X)$ and $\mathcal{H}om^\bullet_{\mathcal{R}_X}(\mathcal{R}_X,-)$, and then we obtain a natural "biduality morphism"

$$\beta^\bullet : \mathcal{I}^\bullet \longrightarrow \mathcal{H}om^\bullet_{\mathcal{R}_X^0}(\mathcal{H}om^\bullet_{\mathcal{R}_X}(\mathcal{I}^\bullet,\mathcal{J}^\bullet),\mathcal{J}^\bullet)$$

given by $\beta^\bullet(h)(\underline{a}) = (-1)^{(\deg h)(\deg\underline{a})}\underline{a}(h)$.

Given three complexes $\mathcal{F}^\bullet, \mathcal{I}^\bullet, \mathcal{J}^\bullet$ of left $\mathcal{R}_X$-modules, call $\mathcal{K}^\bullet = \mathcal{H}om^\bullet_{\mathcal{R}_X}(\mathcal{F}^\bullet, \mathcal{F}^\bullet)$, $\mathcal{L}^\bullet = \mathcal{H}om^\bullet_{\mathcal{R}_X}(\mathcal{F}^\bullet, \mathcal{I}^\bullet)$, $\mathcal{M}^\bullet = \mathcal{H}om^\bullet_{\mathcal{R}_X}(\mathcal{I}^\bullet, \mathcal{F}^\bullet)$, $\mathcal{G}^\bullet = \mathcal{H}om^\bullet_{\mathcal{R}^0_X}(\mathcal{L}^\bullet, \mathcal{K}^\bullet)$ and $\eta^\bullet : \mathcal{M}^\bullet \to \mathcal{G}^\bullet$, $(\eta^\bullet)^* = \mathcal{H}om^\bullet_{\mathcal{R}^0_X}(\eta^\bullet, \mathcal{K}^\bullet)$,

$$\beta^\bullet : \mathcal{L}^\bullet \longrightarrow \mathcal{H}om^\bullet_{\mathcal{R}^0_X}(\mathcal{H}om^\bullet_{\mathcal{R}_X}(\mathcal{L}^\bullet, \mathcal{K}^\bullet), \mathcal{K}^\bullet) = \mathcal{H}om^\bullet_{\mathcal{R}^0_X}(\mathcal{G}^\bullet, \mathcal{K}^\bullet)$$

the natural morphisms defined above.

A.3. Lemma. — *With the above notations, the equation $(\eta^\bullet)^* \circ \beta^\bullet = \xi^\bullet$ holds.*

A.4. Assume that $\mathcal{R}^0_X$ is the constant sheaf associated with a field K. Then, the natural morphism $\xi^\bullet$ induces another one

$$\boldsymbol{\xi} : \mathbf{R}\,\mathcal{H}om^\bullet_{\mathcal{R}_X}(\mathcal{F}^\bullet, \mathcal{G}^\bullet) \longrightarrow \mathbf{R}\,\mathcal{H}om^\bullet_{\mathcal{R}^0_X}(\mathbf{R}\,\mathcal{H}om^\bullet_{\mathcal{R}_X}(\mathcal{G}^\bullet, \mathcal{H}^\bullet), \mathbf{R}\,\mathcal{H}om^\bullet_{\mathcal{R}_X}(\mathcal{F}^\bullet, \mathcal{H}^\bullet))$$

for $\mathcal{F}^\bullet \in D^-(\mathcal{R}_X)$ and $\mathcal{G}^\bullet, \mathcal{H}^\bullet \in D^+(\mathcal{R}_X)$. For that[(2)], take a bounded below injective resolution $\mathcal{G}^\bullet \to \mathcal{I}^\bullet$ and a bounded below injective Godement resolution $\mathcal{H}^\bullet \to \mathcal{J}^\bullet$, *i.e.* $\mathcal{J}^p = \Delta_*\mathcal{J}^p_0$ where Δ is the identity map from the space X, endowed with the discrete topology, to X, and the $\mathcal{J}^p_0$ are injective sheaves of $\Delta^{-1}\mathcal{R}_X$-modules. We then have $\mathbf{R}\,\mathcal{H}om^\bullet_{\mathcal{R}_X}(\mathcal{F}^\bullet, \mathcal{G}^\bullet) = \mathcal{H}om^\bullet_{\mathcal{R}_X}(\mathcal{F}^\bullet, \mathcal{I}^\bullet)$, $\mathbf{R}\,\mathcal{H}om^\bullet_{\mathcal{R}_X}(\mathcal{G}^\bullet, \mathcal{H}^\bullet) = \mathcal{H}om^\bullet_{\mathcal{R}_X}(\mathcal{G}^\bullet, \mathcal{J}^\bullet)$ and $\mathbf{R}\,\mathcal{H}om^\bullet_{\mathcal{R}_X}(\mathcal{F}^\bullet, \mathcal{H}^\bullet) = \mathcal{H}om^\bullet_{\mathcal{R}_X}(\mathcal{F}^\bullet, \mathcal{J}^\bullet)$. The last complex is a complex of injective sheaves of K-vector spaces because

$$\begin{aligned}\mathcal{H}om^n_{\mathcal{R}_X}(\mathcal{F}^\bullet, \mathcal{J}^\bullet) &= \prod_{p\in\mathbb{Z}} \mathcal{H}om_{\mathcal{R}_X}(\mathcal{F}^p, \mathcal{J}^{p+n}) \\ &= \prod_{p\in\mathbb{Z}} \mathcal{H}om_{\mathcal{R}_X}(\mathcal{F}^p, \Delta_*\mathcal{J}^{p+n}_0) = \Delta_*\Big(\prod_{p\in\mathbb{Z}} \mathcal{H}om_{\Delta^{-1}\mathcal{R}_X}(\Delta^{-1}\mathcal{F}^p, \mathcal{J}^{p+n}_0)\Big),\end{aligned}$$

and so

$$\begin{aligned}&\mathbf{R}\,\mathcal{H}om^\bullet_{\mathcal{R}^0_X}(\mathbf{R}\,\mathcal{H}om^\bullet_{\mathcal{R}_X}(\mathcal{G}^\bullet, \mathcal{H}^\bullet), \mathbf{R}\,\mathcal{H}om^\bullet_{\mathcal{R}_X}(\mathcal{F}^\bullet, \mathcal{H}^\bullet)) \\ &\qquad = \mathcal{H}om^\bullet_{\mathcal{R}^0_X}(\mathcal{H}om^\bullet_{\mathcal{R}_X}(\mathcal{G}^\bullet, \mathcal{J}^\bullet), \mathcal{H}om^\bullet_{\mathcal{R}_X}(\mathcal{F}^\bullet, \mathcal{J}^\bullet)) \\ &\qquad\qquad = \mathcal{H}om^\bullet_{\mathcal{R}^0_X}(\mathcal{H}om^\bullet_{\mathcal{R}_X}(\mathcal{I}^\bullet, \mathcal{J}^\bullet), \mathcal{H}om^\bullet_{\mathcal{R}_X}(\mathcal{F}^\bullet, \mathcal{J}^\bullet)).\end{aligned}$$

The morphism $\boldsymbol{\xi}$ then comes from the natural morphism

$$\xi^\bullet : \mathcal{H}om^\bullet_{\mathcal{R}_X}(\mathcal{F}^\bullet, \mathcal{I}^\bullet) \longrightarrow \mathcal{H}om^\bullet_{\mathcal{R}^0_X}(\mathcal{H}om^\bullet_{\mathcal{R}_X}(\mathcal{I}^\bullet, \mathcal{J}^\bullet), \mathcal{H}om^\bullet_{\mathcal{R}_X}(\mathcal{F}^\bullet, \mathcal{J}^\bullet)).$$

In a similar way the natural morphisms $\beta^\bullet$ and $\eta^\bullet$ induce other ones

$$\boldsymbol{\beta} : \mathcal{G}^\bullet \longrightarrow \mathbf{R}\,\mathcal{H}om^\bullet_{\mathcal{R}^0_X}(\mathbf{R}\,\mathcal{H}om^\bullet_{\mathcal{R}_X}(\mathcal{G}^\bullet, \mathcal{H}^\bullet), \mathcal{H}^\bullet)$$

for $\mathcal{G}^\bullet, \mathcal{H}^\bullet \in D^+(\mathcal{R}_X)$ and

$$\boldsymbol{\eta} : \mathbf{R}\,\mathcal{H}om^\bullet_{\mathcal{R}_X}(\mathcal{F}^\bullet, \mathcal{G}^\bullet) \longrightarrow \mathbf{R}\,\mathcal{H}om^\bullet_{\mathcal{R}^0_X}(\mathbf{R}\,\mathcal{H}om^\bullet_{\mathcal{R}_X}(\mathcal{H}^\bullet, \mathcal{F}^\bullet), \mathbf{R}\,\mathcal{H}om^\bullet_{\mathcal{R}_X}(\mathcal{H}^\bullet, \mathcal{G}^\bullet))$$

for $\mathcal{F}^\bullet, \mathcal{G}^\bullet \in D^+(\mathcal{R}_X)$ and $\mathcal{H}^\bullet \in D^-(\mathcal{R}_X)$.

(2) I owe this argument to Z. Mebkhout.

The natural morphisms $\xi^\bullet, \beta^\bullet, \eta^\bullet, \boldsymbol{\xi}, \boldsymbol{\beta}, \boldsymbol{\eta}$ are "cocontractions" in the (co)sense of [**De2**], 1.1.9.

A.5. For each $m \in \mathbb{Z}$, we have natural isomorphisms

$$\mathcal{H}om^\bullet_{\mathcal{R}_X}(\mathcal{I}^\bullet[-m], \mathcal{J}^\bullet) \xrightarrow[\simeq]{\eta^\bullet_{1,m}} \mathcal{H}om^\bullet_{\mathcal{R}_X}(\mathcal{I}^\bullet, \mathcal{J}^\bullet)[m] \xleftarrow[\simeq]{\eta^\bullet_{2,m}} \mathcal{H}om^\bullet_{\mathcal{R}_X}(\mathcal{I}^\bullet, \mathcal{J}^\bullet[m])$$

given by $\eta^\bullet_{1,m}(\underline{a}) = (-1)^{m \deg \underline{a}}\underline{a}, \quad \eta^\bullet_{2,m}(\underline{b}) = \underline{b}$.

Let $\mathcal{F}^\bullet, \mathcal{I}^\bullet, \mathcal{J}^\bullet$ be three complexes of left $\mathcal{R}_X$-modules and let m be an integer. Call $\mathcal{A}^\bullet = \mathcal{H}om^\bullet_{\mathcal{R}_X}(\mathcal{I}^\bullet, \mathcal{J}^\bullet), \mathcal{B}^\bullet = \mathcal{H}om^\bullet_{\mathcal{R}_X}(\mathcal{F}^\bullet, \mathcal{J}^\bullet), \mathcal{A}^\bullet_m = \mathcal{H}om^\bullet_{\mathcal{R}_X}(\mathcal{I}^\bullet, \mathcal{J}^\bullet[m]), \mathcal{B}^\bullet_m = \mathcal{H}om^\bullet_{\mathcal{R}_X}(\mathcal{F}^\bullet, \mathcal{J}^\bullet[m])$ and

$$\Lambda^\bullet_m : \mathcal{H}om^\bullet_{\mathcal{R}^0_X}(\mathcal{A}^\bullet_m, \mathcal{B}^\bullet_m) \xrightarrow{\simeq} \mathcal{H}om^\bullet_{\mathcal{R}^0_X}(\mathcal{A}^\bullet, \mathcal{B}^\bullet)$$

the isomorphism obtained by composing

$$\begin{aligned}
&\mathcal{H}om^\bullet_{\mathcal{R}^0_X}(\mathcal{A}^\bullet_m, \mathcal{B}^\bullet_m) \xrightarrow{(\eta^\bullet_{2,m})^*} \mathcal{H}om^\bullet_{\mathcal{R}^0_X}(\mathcal{A}^\bullet[m], \mathcal{B}^\bullet_m)\\
&\quad \xrightarrow{(\eta^\bullet_{2,m})_*} \mathcal{H}om^\bullet_{\mathcal{R}^0_X}(\mathcal{A}^\bullet[m], \mathcal{B}^\bullet[m]) \xrightarrow{\eta^\bullet_{1,-m}} \mathcal{H}om^\bullet_{\mathcal{R}^0_X}(\mathcal{A}^\bullet, \mathcal{B}^\bullet[m])[-m]\\
&\qquad \xrightarrow{\eta^\bullet_{1,m}} \mathcal{H}om^\bullet_{\mathcal{R}^0_X}(\mathcal{A}^\bullet, \mathcal{B}^\bullet)[m][-m] = \mathcal{H}om^\bullet_{\mathcal{R}^0_X}(\mathcal{A}^\bullet, \mathcal{B}^\bullet).
\end{aligned}$$

Call

$$\begin{aligned}
\xi^\bullet &: \mathcal{H}om^\bullet_{\mathcal{R}_X}(\mathcal{F}^\bullet, \mathcal{I}^\bullet) \longrightarrow \mathcal{H}om^\bullet_{\mathcal{R}^0_X}(\mathcal{H}om^\bullet_{\mathcal{R}_X}(\mathcal{I}^\bullet, \mathcal{J}^\bullet), \mathcal{H}om^\bullet_{\mathcal{R}_X}(\mathcal{F}^\bullet, \mathcal{J}^\bullet))\\
{}'\xi^\bullet &: \mathcal{H}om^\bullet_{\mathcal{R}_X}(\mathcal{F}^\bullet, \mathcal{I}^\bullet) \longrightarrow \mathcal{H}om^\bullet_{\mathcal{R}^0_X}(\mathcal{H}om^\bullet_{\mathcal{R}_X}(\mathcal{I}^\bullet, \mathcal{J}^\bullet[m]), \mathcal{H}om^\bullet_{\mathcal{R}_X}(\mathcal{F}^\bullet, \mathcal{J}^\bullet[m]))
\end{aligned}$$

the natural morphisms.

A.6. Lemma. — *With the above notations, the equality $\Lambda^n_m \circ {}'\xi^n = (-1)^{mn}\xi^n$ holds for every $n \in \mathbb{Z}$.*

A.7. For each $d \in \mathbb{Z}$, we have obvious natural isomorphisms (there is no signs involved)

$$\operatorname{Hom}_{K(\mathcal{R}_X)}(\mathcal{I}^\bullet[-d], \mathcal{J}^\bullet) \xleftarrow[\simeq]{\nu^d_1} h^d \operatorname{Hom}^\bullet_{\mathcal{R}_X}(\mathcal{I}^\bullet, \mathcal{J}^\bullet) \xrightarrow[\simeq]{\nu^d_2} \operatorname{Hom}_{K(\mathcal{R}_X)}(\mathcal{I}^\bullet, \mathcal{J}^\bullet[d]).$$

Call $\boldsymbol{\nu}^d : h^d \mathbf{R}\operatorname{Hom}^\bullet_{\mathcal{R}_X}(\mathcal{I}^\bullet, \mathcal{J}^\bullet) \xrightarrow{\simeq} \operatorname{Hom}_{D(\mathcal{R}_X)}(\mathcal{I}^\bullet, \mathcal{J}^\bullet[d])$ the induced "derived" isomorphism.

A.8. Lemma. — *Given three complexes $\mathcal{F}^\bullet, \mathcal{I}^\bullet, \mathcal{J}^\bullet$ of left $\mathcal{R}_X$-modules and an integer $d \in \mathbb{Z}$, the following diagrams are commutative:*

$$\begin{tikzcd}
h^d \operatorname{Hom}^\bullet_{\mathcal{R}_X}(\mathcal{F}^\bullet, \mathcal{I}^\bullet) \arrow[r, "h^d\xi^\bullet"] \arrow[d, "\nu_2^d"'] & h^d \operatorname{Hom}^\bullet_{\mathcal{R}_X^0}(\mathit{Hom}^\bullet_{\mathcal{R}_X}(\mathcal{I}^\bullet, \mathcal{J}^\bullet), \mathit{Hom}^\bullet_{\mathcal{R}_X}(\mathcal{F}^\bullet, \mathcal{J}^\bullet)) \arrow[d, "(-1)^d(\eta^\bullet_{1,-d})^* \circ \nu_1^d"] \\
\operatorname{Hom}_{K(\mathcal{R}_X)}(\mathcal{F}^\bullet, \mathcal{I}^\bullet[d]) \arrow[r, "\mathit{Hom}^\bullet_{\mathcal{R}_X}(-,\mathcal{J}^\bullet)"] & \operatorname{Hom}_{K(\mathcal{R}_X^0)}(\mathit{Hom}^\bullet_{\mathcal{R}_X}(\mathcal{I}^\bullet[d], \mathcal{J}^\bullet), \mathit{Hom}^\bullet_{\mathcal{R}_X}(\mathcal{F}^\bullet, \mathcal{J}^\bullet)),
\end{tikzcd}$$

$$\begin{tikzcd}
h^d \operatorname{Hom}^\bullet_{\mathcal{R}_X}(\mathcal{F}^\bullet, \mathcal{I}^\bullet) \arrow[r, "h^d\xi^\bullet"] \arrow[d, "\nu_1^d"'] & h^d \operatorname{Hom}^\bullet_{\mathcal{R}_X^0}(\mathit{Hom}^\bullet_{\mathcal{R}_X}(\mathcal{I}^\bullet, \mathcal{J}^\bullet), \mathit{Hom}^\bullet_{\mathcal{R}_X}(\mathcal{F}^\bullet, \mathcal{J}^\bullet)) \arrow[d, "((\eta^\bullet_{1,d})^{-1})_* \circ \nu_2^d"] \\
\operatorname{Hom}_{K(\mathcal{R}_X)}(\mathcal{F}^\bullet[-d], \mathcal{I}^\bullet) \arrow[r, "\mathit{Hom}^\bullet_{\mathcal{R}_X}(-,\mathcal{J}^\bullet)"] & \operatorname{Hom}_{K(\mathcal{R}_X^0)}(\mathit{Hom}^\bullet_{\mathcal{R}_X}(\mathcal{I}^\bullet, \mathcal{J}^\bullet), \mathit{Hom}^\bullet_{\mathcal{R}_X}(\mathcal{F}^\bullet[-d], \mathcal{J}^\bullet))
\end{tikzcd}$$

A.9. Given four complexes $\mathcal{P}^\bullet, \mathcal{M}^\bullet, \mathcal{J}^\bullet, \mathcal{D}^\bullet$ of left $\mathcal{R}_X$-modules, a complex $\mathcal{F}^\bullet$ of $\mathcal{R}_X^0$-modules and a complex $\mathcal{Q}^\bullet$ of $(\mathcal{R}_X, \mathcal{R}_X)$-bimodules, we define natural morphisms

$$\begin{aligned}
\lambda_1^\bullet &: \mathit{Hom}^\bullet_{\mathcal{R}_X}(\mathcal{P}^\bullet, \mathcal{Q}^\bullet) \overset{\bullet}{\otimes}_{\mathcal{R}_X} \mathcal{M}^\bullet \longrightarrow \mathit{Hom}^\bullet_{\mathcal{R}_X}(\mathcal{P}^\bullet, \mathcal{Q}^\bullet \overset{\bullet}{\otimes}_{\mathcal{R}_X} \mathcal{M}^\bullet)\\
\lambda_2^\bullet &: \mathit{Hom}^\bullet_{\mathcal{R}_X}(\mathcal{Q}^\bullet \overset{\bullet}{\otimes}_{\mathcal{R}_X} \mathcal{M}^\bullet, \mathcal{J}^\bullet) \longrightarrow \mathit{Hom}^\bullet_{\mathcal{R}_X}(\mathcal{M}^\bullet, \mathit{Hom}^\bullet_{\mathcal{R}_X}(\mathcal{Q}^\bullet, \mathcal{J}^\bullet))\\
\mu_0^\bullet &: \mathit{Hom}^\bullet_{\mathcal{R}_X^0}(\mathcal{D}^\bullet, \mathcal{F}^\bullet) \overset{\bullet}{\otimes}_{\mathcal{R}_X} \mathcal{M}^\bullet \longrightarrow \mathit{Hom}^\bullet_{\mathcal{R}_X^0}(\mathit{Hom}^\bullet_{\mathcal{R}_X}(\mathcal{M}^\bullet, \mathcal{D}^\bullet), \mathcal{F}^\bullet)
\end{aligned}$$

by

$$\begin{aligned}
\lambda_1^\bullet(\underline{h} \otimes x)(z) &= (-1)^{(\deg x)(\deg z)} \underline{h}(z) \otimes x\\
\lambda_2^\bullet(\underline{a})(v)(u) &= (-1)^{(\deg v)(\deg u)} \underline{a}(u \otimes v)\\
\mu_0^\bullet(\underline{b} \otimes w)(\underline{c}) &= (-1)^{(\deg w)(\deg \underline{c})} \underline{b}(\underline{c}(w)).
\end{aligned}$$

A.10. Lemma. — *Given three complexes $\mathcal{P}^\bullet, \mathcal{M}^\bullet, \mathcal{J}^\bullet$ of left $\mathcal{R}_X$-modules and a complex $\mathcal{Q}^\bullet$ of $(\mathcal{R}_X, \mathcal{R}_X)$-bimodules, the following diagram*

$$\begin{tikzcd}
\mathit{Hom}^\bullet_{\mathcal{R}_X}(\mathcal{P}^\bullet, \mathcal{Q}^\bullet \overset{\bullet}{\otimes}_{\mathcal{R}_X} \mathcal{M}^\bullet) \arrow[r, "\xi^\bullet"] & \mathit{Hom}^\bullet_{\mathcal{R}_X^0}(\mathit{Hom}^\bullet_{\mathcal{R}_X}(\mathcal{Q}^\bullet \overset{\bullet}{\otimes}_{\mathcal{R}_X} \mathcal{M}^\bullet, \mathcal{J}^\bullet), \mathit{Hom}^\bullet_{\mathcal{R}_X}(\mathcal{P}^\bullet, \mathcal{J}^\bullet)) \\
\mathit{Hom}^\bullet_{\mathcal{R}_X}(\mathcal{P}^\bullet, \mathcal{Q}^\bullet) \overset{\bullet}{\otimes}_{\mathcal{R}_X} \mathcal{M}^\bullet \arrow[r, "\xi^\bullet \otimes Id"] \arrow[u, "\lambda_1^\bullet"] & \mathit{Hom}^\bullet_{\mathcal{R}_X^0}(\mathit{Hom}^\bullet_{\mathcal{R}_X}(\mathcal{Q}^\bullet, \mathcal{J}^\bullet), \mathit{Hom}^\bullet_{\mathcal{R}_X}(\mathcal{P}^\bullet, \mathcal{J}^\bullet)) \overset{\bullet}{\otimes}_{\mathcal{R}_X} \mathcal{M}^\bullet \arrow[u, "(\lambda_2^\bullet)^* \circ \mu_0^\bullet"']
\end{tikzcd}$$

is commutative. In particular, if $\mathcal{Q}^\bullet = \mathcal{R}_X$*, then we obtain a natural commutative diagram*

$$\begin{array}{ccc} \mathcal{H}om^\bullet_{\mathcal{R}_X}(\mathcal{P}^\bullet,\mathcal{M}^\bullet) & \xrightarrow{\ \xi^\bullet\ } & \mathcal{H}om^\bullet_{\mathcal{R}^0_X}(\mathcal{H}om^\bullet_{\mathcal{R}_X}(\mathcal{M}^\bullet,\mathcal{J}^\bullet),\mathcal{H}om^\bullet_{\mathcal{R}_X}(\mathcal{P}^\bullet,\mathcal{J}^\bullet)) \\ {\scriptstyle\lambda^\bullet}\big\uparrow & & \big\uparrow{\scriptstyle\mu^\bullet} \\ \mathcal{H}om^\bullet_{\mathcal{R}_X}(\mathcal{P}^\bullet,\mathcal{R}_X)\overset{\bullet}{\otimes}_{\mathcal{R}_X}\mathcal{M}^\bullet & \xrightarrow{\ \xi^\bullet\otimes Id\ } & \mathcal{H}om^\bullet_{\mathcal{R}^0_X}(\mathcal{J}^\bullet,\mathcal{H}om^\bullet_{\mathcal{R}_X}(\mathcal{P}^\bullet,\mathcal{J}^\bullet))\overset{\bullet}{\otimes}_{\mathcal{R}_X}\mathcal{M}^\bullet. \end{array}$$

A.11. Let $i: F \hookrightarrow X$ be a closed immersion and denote $\mathcal{R}_F = i^{-1}\mathcal{R}_X$, $\mathcal{R}^0_F = i^{-1}\mathcal{R}^0_X$. Given left $\mathcal{R}_X$-modules $\mathcal{I},\mathcal{J}$, there are well known canonical natural morphisms

$$\begin{aligned} f &: i^{-1}\,\mathcal{H}om_{\mathcal{R}_X}(\mathcal{I},\mathcal{J}) \longrightarrow \mathcal{H}om_{\mathcal{R}_F}(i^{-1}\mathcal{I}, i^{-1}\mathcal{J}) \\ n &: i^{-1}\,\mathcal{H}om_{\mathcal{R}_X}(\mathcal{I},\mathcal{J}) \longrightarrow \mathcal{H}om_{\mathcal{R}_F}(i^!\mathcal{I}, i^!\mathcal{J}) \\ g &: i^!\,\mathcal{H}om_{\mathcal{R}_X}(\mathcal{I},\mathcal{J}) \xrightarrow{\ \simeq\ } \mathcal{H}om_{\mathcal{R}_F}(i^{-1}\mathcal{I}, i^!\mathcal{J}). \end{aligned}$$

They induce natural morphisms $f^\bullet, n^\bullet, g^\bullet$ at the level of complexes in the obvious way (no signs are involved).

Given two complexes $\mathcal{I}^\bullet,\mathcal{J}^\bullet$ of left $\mathcal{R}^0_X$-modules, consider the following natural morphisms

$$\begin{aligned} n^\bullet &: i^{-1}\,\mathcal{H}om^\bullet_{\mathcal{R}^0_X}(\mathcal{I}^\bullet,\mathcal{J}^\bullet) \longrightarrow \mathcal{H}om^\bullet_{\mathcal{R}^0_F}(i^!\mathcal{I}^\bullet, i^!\mathcal{J}^\bullet), \\ \beta^\bullet &: i^{-1}\,\mathcal{H}om^\bullet_{\mathcal{R}^0_X}(\mathcal{I}^\bullet,\mathcal{J}^\bullet) \longrightarrow \mathcal{H}om^\bullet_{\mathcal{R}^0_F}(\mathcal{H}om_{\mathcal{R}^0_F}(i^{-1}\,\mathcal{H}om^\bullet_{\mathcal{R}^0_X}(\mathcal{I}^\bullet,\mathcal{J}^\bullet), i^!\mathcal{J}^\bullet), i^!\mathcal{J}^\bullet), \\ (i^!\beta^\bullet)^* &: \mathcal{H}om^\bullet_{\mathcal{R}^0_F}(i^!\,\mathcal{H}om^\bullet_{\mathcal{R}^0_X}(\mathcal{H}om^\bullet_{\mathcal{R}^0_X}(\mathcal{I}^\bullet,\mathcal{J}^\bullet),\mathcal{J}^\bullet), i^!\mathcal{J}^\bullet) \longrightarrow \mathcal{H}om^\bullet_{\mathcal{R}^0_F}(i^!\mathcal{I}^\bullet, i^!\mathcal{J}^\bullet) \end{aligned}$$

the morphism induced by $i^!\beta^\bullet : i^!\mathcal{I}^\bullet \to i^!\,\mathcal{H}om^\bullet_{\mathcal{R}^0_X}(\mathcal{H}om^\bullet_{\mathcal{R}^0_X}(\mathcal{I}^\bullet,\mathcal{J}^\bullet),\mathcal{J}^\bullet)$ and

$$\begin{aligned} (g^\bullet)^* &: \mathcal{H}om^\bullet_{\mathcal{R}^0_F}(\mathcal{H}om^\bullet_{\mathcal{R}^0_F}(i^{-1}\,\mathcal{H}om^\bullet_{\mathcal{R}^0_X}(\mathcal{I}^\bullet,\mathcal{J}^\bullet), i^!\mathcal{J}^\bullet), i^!\mathcal{J}^\bullet) \\ &\qquad \xrightarrow{\ \simeq\ } \mathcal{H}om^\bullet_{\mathcal{R}^0_F}(i^!\,\mathcal{H}om^\bullet_{\mathcal{R}^0_X}(\mathcal{H}om^\bullet_{\mathcal{R}^0_X}(\mathcal{I}^\bullet,\mathcal{J}^\bullet),\mathcal{J}^\bullet), i^!\mathcal{J}^\bullet) \end{aligned}$$

the isomorphism induced by

$$g^\bullet : i^!\,\mathcal{H}om^\bullet_{\mathcal{R}^0_X}(\mathcal{H}om^\bullet_{\mathcal{R}^0_X}(\mathcal{I}^\bullet,\mathcal{J}^\bullet),\mathcal{J}^\bullet) \xrightarrow{\ \simeq\ } \mathcal{H}om^\bullet_{\mathcal{R}^0_F}(i^{-1}\,\mathcal{H}om^\bullet_{\mathcal{R}^0_X}(\mathcal{I}^\bullet,\mathcal{J}^\bullet), i^!\mathcal{J}^\bullet).$$

A.12. Lemma. — *With the above notations, the equality* $n^\bullet = (i^!\beta^\bullet)^* \circ (g^\bullet)^* \circ \beta^\bullet$ *holds.*

Given three complexes $\mathcal{F}^\bullet,\mathcal{I}^\bullet,\mathcal{J}^\bullet$ of left $\mathcal{R}_X$-modules, consider the following natural morphisms

$$\begin{aligned} i^{-1}\xi^\bullet &: i^{-1}\,\mathcal{H}om^\bullet_{\mathcal{R}_X}(\mathcal{F}^\bullet,\mathcal{I}^\bullet) \longrightarrow i^{-1}\,\mathcal{H}om^\bullet_{\mathcal{R}^0_X}(\mathcal{H}om^\bullet_{\mathcal{R}_X}(\mathcal{I}^\bullet,\mathcal{J}^\bullet),\mathcal{H}om^\bullet_{\mathcal{R}_X}(\mathcal{F}^\bullet,\mathcal{J}^\bullet)) \\ f^\bullet &: i^{-1}\,\mathcal{H}om^\bullet_{\mathcal{R}_X}(\mathcal{F}^\bullet,\mathcal{I}^\bullet) \longrightarrow \mathcal{H}om^\bullet_{\mathcal{R}_F}(i^{-1}\mathcal{F}^\bullet, i^{-1}\mathcal{I}^\bullet) \\ \xi^\bullet &: \mathcal{H}om^\bullet_{\mathcal{R}_F}(i^{-1}\mathcal{F}^\bullet, i^{-1}\mathcal{I}^\bullet) \longrightarrow \mathcal{H}om^\bullet_{\mathcal{R}^0_F}(\mathcal{H}om^\bullet_{\mathcal{R}_F}(i^{-1}\mathcal{I}^\bullet, i^!\mathcal{J}^\bullet),\mathcal{H}om^\bullet_{\mathcal{R}_F}(i^{-1}\mathcal{F}^\bullet, i^!\mathcal{J}^\bullet)) \end{aligned}$$

and

$$\widetilde{n^\bullet} : i^{-1}\, Hom^\bullet_{\mathcal{R}^0_X}(Hom^\bullet_{\mathcal{R}_X}(\mathcal{I}^\bullet, \mathcal{J}^\bullet), Hom^\bullet_{\mathcal{R}_X}(\mathcal{F}^\bullet, \mathcal{J}^\bullet)) \longrightarrow Hom^\bullet_{\mathcal{R}^0_F}(Hom^\bullet_{\mathcal{R}_F}(i^{-1}\mathcal{I}^\bullet, i^!\mathcal{J}^\bullet), Hom^\bullet_{\mathcal{R}_F}(i^{-1}\mathcal{F}^\bullet, i^!\mathcal{J}^\bullet))$$

the morphism induced by

$$n^\bullet : i^{-1}\, Hom^\bullet_{\mathcal{R}^0_X}(Hom^\bullet_{\mathcal{R}_X}(\mathcal{I}^\bullet, \mathcal{J}^\bullet), Hom^\bullet_{\mathcal{R}_X}(\mathcal{F}^\bullet, \mathcal{J}^\bullet)) \longrightarrow Hom^\bullet_{\mathcal{R}^0_F}(i^!\, Hom^\bullet_{\mathcal{R}_X}(\mathcal{I}^\bullet, \mathcal{J}^\bullet), i^!\, Hom^\bullet_{\mathcal{R}_X}(\mathcal{F}^\bullet, \mathcal{J}^\bullet))$$

and the isomorphisms

$$i^!\, Hom^\bullet_{\mathcal{R}_X}(\mathcal{I}^\bullet, \mathcal{J}^\bullet) \xrightarrow[\simeq]{g^\bullet} Hom^\bullet_{\mathcal{R}_F}(i^{-1}\mathcal{I}^\bullet, i^!\mathcal{J}^\bullet),$$
$$i^!\, Hom^\bullet_{\mathcal{R}_X}(\mathcal{F}^\bullet, \mathcal{J}^\bullet) \xrightarrow[\simeq]{g^\bullet} Hom^\bullet_{\mathcal{R}_F}(i^{-1}\mathcal{F}^\bullet, i^!\mathcal{J}^\bullet).$$

A.13. Lemma. — *With the above notations, the equality* $\widetilde{n^\bullet} \circ (i^{-1}\xi^\bullet) = \xi^\bullet \circ f^\bullet$ *holds.*

A.14. Let $p : Y \to X$ be a continuous map between topological spaces and denote $\mathcal{R}_Y = p^{-1}\mathcal{R}_X$, $\mathcal{R}^0_Y = p^{-1}\mathcal{R}^0_X$. Given two left $\mathcal{R}_Y$-modules $\mathcal{A}, \mathcal{B}$, the well known natural morphism

$$h : p_*\, Hom_{\mathcal{R}_Y}(\mathcal{A}, \mathcal{B}) \longrightarrow Hom_{\mathcal{R}_X}(p_*\mathcal{A}, p_*\mathcal{B})$$

induces a natural morphism $h^\bullet$ at the level of complexes in the obvious way (no signs are involved).

Given three complexes of left $\mathcal{R}_Y$-modules $\mathcal{A}^\bullet, \mathcal{B}^\bullet, \mathcal{C}^\bullet$, consider the following natural morphisms

$$\begin{aligned}
p_*\xi^\bullet &: p_*\, Hom^\bullet_{\mathcal{R}_Y}(\mathcal{A}^\bullet, \mathcal{B}^\bullet) \longrightarrow p_*\, Hom^\bullet_{\mathcal{R}^0_Y}(Hom^\bullet_{\mathcal{R}_Y}(\mathcal{B}^\bullet, \mathcal{C}^\bullet), Hom^\bullet_{\mathcal{R}_Y}(\mathcal{A}^\bullet, \mathcal{C}^\bullet))\\
h^\bullet &: p_*\, Hom^\bullet_{\mathcal{R}_Y}(\mathcal{A}^\bullet, \mathcal{B}^\bullet) \longrightarrow Hom^\bullet_{\mathcal{R}_X}(p_*\mathcal{A}^\bullet, p_*\mathcal{B}^\bullet)\\
m^\bullet &: p_*\, Hom^\bullet_{\mathcal{R}^0_Y}(Hom^\bullet_{\mathcal{R}_Y}(\mathcal{B}^\bullet, \mathcal{C}^\bullet), Hom^\bullet_{\mathcal{R}_Y}(\mathcal{A}^\bullet, \mathcal{C}^\bullet))\\
&\qquad\longrightarrow Hom^\bullet_{\mathcal{R}^0_X}(p_*\, Hom^\bullet_{\mathcal{R}_Y}(\mathcal{B}^\bullet, \mathcal{C}^\bullet), Hom^\bullet_{\mathcal{R}_X}(p_*\mathcal{A}^\bullet, p_*\mathcal{C}^\bullet))
\end{aligned}$$

the morphism induced by using $h^\bullet$ twice, and

$$q^\bullet : Hom^\bullet_{\mathcal{R}_X}(p_*\mathcal{A}^\bullet, p_*\mathcal{B}^\bullet) \longrightarrow Hom^\bullet_{\mathcal{R}^0_X}(p_*\, Hom^\bullet_{\mathcal{R}_Y}(\mathcal{B}^\bullet, \mathcal{C}^\bullet), Hom^\bullet_{\mathcal{R}_X}(p_*\mathcal{A}^\bullet, p_*\mathcal{C}^\bullet))$$

the morphism induced by $\xi^\bullet$ and $h^\bullet$.

A.15. Lemma. — *With the above notations, the equality* $q^\bullet \circ h^\bullet = m^\bullet \circ (p_*\xi^\bullet)$ *holds.*

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