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ON THE ASYMPTOTICS OF GREEN'S FUNCTIONS OF ELLIPTIC OPERATORS WITH CONSTANT COEFFICIENTS

by

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Abstract. — In this paper we discuss the following problem. Given an elliptic operator $P(D)$ with constant coefficients in $\mathbb{R}^n$ ($P(\xi) \neq 0$ in $\mathbb{R}^n$) and an infinite cone Γ in $\mathbb{R}^n$, give conditions which ensure that the corresponding Green's function $G(x)$ admits a nice asymptotic behavior as $|x| \to \infty$ in Γ. A solution to the problem is presented and some concrete applications are given. These are related to results by Evgrafov and Postnikov.

***Résumé* (Sur le comportement asymptotique des fonctions de Green des opérateurs elliptiques à coefficients constants)**

Dans cet article nous considérons le problème suivant. Étant donné un opérateur elliptique à coefficients constants, $P(D)$, dans $\mathbb{R}^n$ ($P(\xi) \neq 0$ dans $\mathbb{R}^n$), et un cône infini Γ dans $\mathbb{R}^n$, quelles sont les conditions pour que la fonction de Green associée $G(x)$ ait un bon comportement asymptotique lorsque $|x| \to \infty$ dans Γ ? Nous présentons une solution à ce problème ainsi que des applications. Ceci est relié à des travaux de Evgrafov et Postnikov.

1. Introduction

Let $P(D)$ be an elliptic operator with complex constant coefficients, of even order m, acting on functions on $\mathbb{R}^n$ ($D = (D_1, \cdots, D_n)$, $D_j = \frac{1}{i}\frac{\partial}{\partial x_j}$). Suppose that the polynomial $P(\xi) \neq 0$ for $\xi \in \mathbb{R}^n$. The Green's function $G(x)$ of $P(D)$ on $\mathbb{R}^n$ is given by

$$G(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \frac{e^{i\xi\cdot x}}{P(\xi)} d\xi, \quad x \in \mathbb{R}^n \setminus \{0\} \tag{1.1}$$

where the integral is understood in the distribution sense.

As is well known $G(x)$ is a smooth function on $\mathbb{R}^n \setminus \{0\}$ with a singularity at $x = 0$. $G(x)$ decays exponentially as $|x| \to \infty$.

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In this paper we propose to characterize a class of elliptic operators $P(D)$, $P(\xi) \neq 0$ on $\mathbb{R}^n$, possessing a Green's function with a nice asymptotic behavior as $|x| \to \infty$ ($x \in \mathbb{R}^n$ or, more generally, $x \in \Gamma$ where Γ is some infinite cone in $\mathbb{R}^n$). A prototype of such operators is the Helmholtz operator: $P = -\Delta - \lambda$, $\lambda \in \mathbb{C} \smallsetminus \{0\}$ whose Green's function $G_\lambda(x)$ has the following well known asymptotic formula (derived classically from the asymptotic formula for the Bessel functions). For $0 < \pm \arg \lambda \leqslant \pi$:

$$G_\lambda(x) = c_\pm \lambda^{(n-3)/4} |x|^{-(n-1)/2} e^{\pm i \lambda^{1/2}|x|} \left(1 + O\left(1/|x|\right)\right) \tag{1.2}$$

as $|x| \to \infty$ where $c_\pm = \frac{1}{2}(2\pi)^{-(n-1)/2} e^{\mp i\pi(n-3)/2}$. (Formula (1.2) is also valid for $G_{\lambda \pm i0}(x)$, $\lambda > 0$).

We mention some known results on asymptotic behavior of Green's functions of higher order elliptic operators. First we mention the following results which apply to a class of elliptic operators with constant coefficients different from the class of operators we study here. Suppose that $P(D)$ is positively elliptic: $P(\xi)$ is real for $\xi \in \mathbb{R}^n$, $P(\xi) > 0$ for large $|\xi|$. Suppose further that the set: $M = \{\xi \in \mathbb{R}^n : P(\xi) = 0\}$ is a non-empty connected C^∞ manifold, $P'(\xi) \neq 0$ on M. In this case there are two distinguished Green's functions defined by

$$G_\pm(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \frac{e^{i\xi \cdot x}}{P(\xi) \pm i0} d\xi. \tag{1.3}$$

If the manifold M is strictly convex it was shown by Vainberg [**5**] that the Green's functions $G_\pm(x)$ possess asymptotic formulas of the form:

$$G_\pm(x) = a_\pm(x) e^{\pm iK(x)} \left(1 + O\left(1/|x|\right)\right) \tag{1.4}$$

as $|x| \to \infty$ where $K(x)$ is some real, smooth, convex homogeneous function of degree 1 and $a_\pm(x)$ are certain smooth nowhere zero homogeneous functions of degree $-(n-1)/2$ on $\mathbb{R}^n \smallsetminus \{0\}$ (K and $a_\pm$ admit explicit expressions in terms of the manifold M).

For higher order elliptic operators $P(D)$ such that $P(\xi) \neq 0$ on $\mathbb{R}^n$ (the class of operators which interests us here) an asymptotic formula for the Green's function was established by Evgrafov and Postnikov [**1**] for a rather special class of operators. The main result in [**1**], for the elliptic Green's function, can be formulated as follows.

Theorem 1.1. — *Let $P_0(D)$ be an elliptic operator on $\mathbb{R}^n$. Suppose that the form $P_0(\xi)$ is a positive homogeneous polynomial of even degree m on $\mathbb{R}^n \smallsetminus \{0\}$. Write $P_0(\xi)$ in the form:*

$$P_0(\xi) = \sum_{|\alpha|=m} a_\alpha \binom{m}{\alpha} \xi^\alpha.$$

Suppose that $P_0(\xi)$ verifies the following

Condition S **(Strong convexity condition)**

$$\sum_{|\alpha|=|\beta|=m/2} a_{\alpha+\beta} X_\alpha X_\beta > 0 \quad \text{in } \mathbb{R}^N \smallsetminus \{0\} \tag{1.5}$$

where N denotes the number of multi-indices $\alpha = (\alpha_1, \cdots, \alpha_n)$ of order $|\alpha| = m/2$ and $\{X_\alpha\}_{|\alpha|=m/2}$ stands for a generic point in $\mathbb{R}^N$.

Under these conditions the Green's function $G_\lambda(x)$ of $P_0(D) - \lambda$ verifies for $0 < \pm \arg \lambda < \pi$ an asymptotic formula of the form:

$$G_\lambda(x) = c_\pm \lambda^{\frac{n+1}{2m}-1} a(x) e^{\pm i\lambda^{1/m} Q_0(x)} \left(1 + O\left(1/|x|\right)\right) \tag{1.6}$$

as $|x| \to \infty$, uniformly in λ in any compact. Here $c_\pm$ are constants ($c_+ = \overline{c}_-$), $a(x)$ is a positive smooth homogeneous function of degree $-(n-1)/2$, and $Q_0(x)$ is a positive convex homogeneous function of degree 1 given by

$$Q_0(x) = \sup_{P_0(\xi)=1} \langle x, \xi \rangle,$$

(a more explicit expression of (1.6) *is given in §4, formula* (4.2)*).*

Note that in view of the homogeneity of $P_0(\xi)$ (1.6) can also be viewed as an asymptotic formula in λ (as λ tends suitably to infinity for a fixed $x \neq 0$).

Condition S is a strong convexity restriction. It was shown in [**1**] that Condition S implies in particular that the polynomial $P_0(\xi)$ is strictly convex, i.e.:

$$\operatorname{Hess} P_0(\xi) > 0 \quad \text{ for } \xi \in \mathbb{R}^n \smallsetminus \{0\}. \tag{1.7}$$

In this connection note that under the assumption that the weaker condition (1.7) holds it can be shown that the asymptotic formula (1.6) is valid for the Green's functions $G_{\lambda \pm i0}(x)$ for $\lambda \in \mathbb{R}_+$. This follows from the explicit form of formula (1.4).

The asymptotic formula (1.6) is deduced in [**1**] from an asymptotic formula for the Green's function $G(x,t)$ of the parabolic operator $\partial/\partial t + P_0(D)$ as $t \to +0$. It was conjectured in [**1**] that this last asymptotic formula and consequently that the asymptotic formula (1.6) for $G_\lambda(x)$ should hold when Condition S is replaced by the weaker condition (1.7). In a later publication [**2**] it was shown by the authors that this conjecture is false for the Green's function of the parabolic operator.

In this paper we shall consider the following general problem. Find sufficient and necessary conditions in order that the Green's function $G(x)$ of a given elliptic operator $P(D)$, with $P(\xi) \neq 0$ on $\mathbb{R}^n$, possesses an asymptotic formula of the form:

$$G(x) = a(x) e^{iA(x)} (1 + o(1)) \tag{1.8}$$

as $|x| \to \infty$ in some infinite open cone Γ, where $A(x)$ is a smooth homogeneous function of degree 1 and $a(x)$ is a smooth homogeneous function of degree $-(n-1)/2$ in Γ.

The plan of this paper is as follows. In section 2 we describe some notions and preliminary results needed in the sequel. Our main theorem giving necessary and sufficient conditions for (1.8) to hold is discussed in section 2. In section 3 we describe applications of the main theorem to Green's functions of the operator $P_0(D) - \lambda$ where $P_0(D)$ is the operator in Theorem 1.1 with Condition S replaced by the condition that $P_0(\xi)$ is strictly convex. The main applications consist in giving necessary and sufficient conditions on the complex zeros of $P_0(\zeta) - \lambda$ in order that the Green's function $G_\lambda(x)$ will possess a nice asymptotic expansion.

In conclusion we observe that this paper is a revised version of a lecture given at the Journées Jean Leray on the occasion of the inauguration of the Laboratoire de Mathématiques Jean Leray at the University of Nantes. This is an expository paper with indications of proofs of the main results.

2. Preliminaries

In the following $P(D)$ denotes an elliptic operator with complex constant coefficients, of even order m, such that $P(\xi) \neq 0$ for $\xi \in \mathbb{R}^n$. $G(x)$ denotes the Green's function defined by (1.1).

With the polynomial $P(\zeta)$, $\zeta \in \mathbb{C}^n$, associate *norm functions* $K_P^*(x)$ and $K_P(x)$ on $\mathbb{R}^n$ defined as follows. For any unit vector $\theta \in \mathbb{R}^n$ set:

$$r(\theta) = \min\{t \in \mathbb{R}_+ : P(\xi + it\theta) = 0 \quad \text{for some } \xi \in \mathbb{R}^n\}.$$

Define

$$K_P^*(x) = \frac{|x|}{r\,(x/|x|)} \quad \text{for } x \in \mathbb{R}^n \smallsetminus \{0\}, \tag{2.1}$$

$K_P^*(0) = 0$, and set:

$$\Omega^* = \{x \in \mathbb{R}^n : K_P^*(x) < 1\}. \tag{2.1'}$$

Ω^* is a bounded open connected set in $\mathbb{R}^n$ containing the origin. Furthermore, since Ω^* is a connected component of the set: $\{\eta \in \mathbb{R}^n : P(\xi + i\eta) \neq 0, \forall\, \xi \in \mathbb{R}^n\}$ it follows by a known theorem that Ω^* is convex (see [**3**, p. 43]). Thus $K_P^*(x)$ is a convex homogeneous function of degree 1, $K_P^*(x) > 0$ for $x \neq 0$. Next define:

$$K_P(x) = \sup_{\xi \neq 0} \frac{\langle x, \xi\rangle}{K_P^*(\xi)} = \sup_{\xi \in \partial\Omega^*} \langle x, \xi\rangle. \tag{2.2}$$

It is well known that $K_P(x)$, referred to as the *polar* of $K_P^*(x)$, is a positive convex homogeneous function of degree 1. Set:

$$\Omega = \{x \in \mathbb{R}^n : K_P(x) < 1\}.$$

Clearly, Ω is a convex open set containing the origin. The convexity of $K_P^*(x)$ implies that $K_P^*(x)$ is also the polar of $K_P(x)$, i.e.:

$$K_P^*(x) = \sup_{\xi \in \partial\Omega} \langle x, \xi\rangle. \tag{2.2'}$$

Next, observe that the Green's function of $P(D)$ verifies the following estimate:

$$|G(x)| \leqslant C|x|^m e^{-K_P(x)} \quad \text{for } |x| \geqslant 1, \tag{2.3}$$

C some constant.

We indicate the proof of the essentially known estimate (2.3). Pick a function $\chi(t) \in C^\infty(\mathbb{R})$ such that $\chi \equiv 0$ for $t \leqslant 1/2$, $\chi \equiv 1$ for $t \geqslant 1$. Set: $G_1(x) = \chi(|x|)G(x)$. Then $P(D)G_1 = f$ where $f \in C_0^\infty(\mathbb{R}^n)$. By Fourier transform:

$$G(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \frac{\widehat{f}(\xi)}{P(\xi)} e^{i\xi\cdot x} d\xi \quad \text{for } |x| \geqslant 1. \tag{2.4}$$

Noting that $\widehat{f}(\zeta)$ is an entire function in $\zeta \in \mathbb{C}^n$ which decays rapidly as $|\zeta| \to \infty$ in any tube: $|\operatorname{Im}\zeta| \leqslant R$, it follows by complex integration that in the integral (2.4) the domain of integration $\mathbb{R}^n$ can be shifted to the domain $\mathbb{R}^n + i(1 - 1/|x|)\omega^*$ where ω^* is any point in $\partial\Omega^*$. An easy estimation of the resulting integral yields:

$$|G(x)| \leqslant C|x|^m e^{-\langle\omega^*, x\rangle} \quad \text{for } |x| \geqslant 1, \tag{2.5}$$

C some constant independent of x or ω^*. Minimizing the r.h.s. of (2.5) with respect to ω^* yields (2.3).

The following (essentially well known) proposition shows that the estimate (2.5) is quite precise in the exponential factor.

Proposition 2.1. — *Suppose that $G(x)$ verifies an estimate of the form:*

$$|G(x)| \leqslant C|x|^N e^{-Q(x)} \quad \textit{for } |x| \geqslant 1,$$

where $Q(x)$ is some continuous homogeneous function of degree 1 on $\mathbb{R}^n \smallsetminus \{0\}$. Then

$$Q(\omega) \leqslant K_P(\omega)$$

at all points $\omega \in \partial\Omega$ which are extremal points of $\overline{\Omega}$.

We conclude this section with some notions and definitions related to the boundaries of the conjugate convex sets Ω and Ω^*.

Let Γ be an infinite open convex cone in $\mathbb{R}^n$ with vertex at the origin. Consider the boundary set:

$$\partial\Omega_\Gamma := \partial\Omega \cap \Gamma. \tag{2.6}$$

Assume that $\partial\Omega_\Gamma$ is a C^2 manifold with a positive Gaussian curvature at every point (so that $K_P(x)$ is a C^2 function and $\operatorname{Hess} K_P(x)^2 > 0$ in Γ). Define:

$$\Gamma^* = \{x \in \mathbb{R}^n \smallsetminus \{0\} : x/|x| = K_P'(y)/|K_P'(y)| \quad \text{for some } y \in \Gamma\},$$

(here $K_P'(y) := \nabla K_P(y)$). Γ^* is an open convex cone which we shall refer to as the polar to Γ with respect to the "norm" $K_P(x)$. One finds readily that for $x \in \Gamma^*$:

$$K_P^*(x) = \langle x, \omega(x)\rangle \tag{2.7}$$

where $\omega(x)$ is the unique point in $\partial\Omega_\Gamma$ such that $K_P'(\omega(x))$ is in the direction of x. From (2.7) it follows that $K_P^*(x)$ is a C^2 function in Γ^* and setting:

$$\partial\Omega^*_{\Gamma^*} = \partial\Omega^* \cap \Gamma^* \tag{2.6*}$$

it follows that $\partial\Omega^*_{\Gamma^*}$ is a C^2 manifold having a positive Gaussian curvature at every point. Also, in analogy to (2.7):

$$K_P(x) = \langle x, \omega^*(x)\rangle \quad \text{for } x \in \Gamma, \tag{2.7*}$$

where $\omega^*(x)$ is the unique point in $\partial\Omega^*_{\Gamma^*}$ such that $\nabla K^*(\omega^*(x))$ is in the direction of x. These considerations show that the gradient map:

$$\nabla : \partial\Omega_\Gamma \ni \omega \longrightarrow K'_P(\omega) = \omega^* \in \partial\Omega^*_{\Gamma^*} \tag{2.8}$$

is a $1-1$ C^1 map from $\partial\Omega_\Gamma$ onto $\partial\Omega^*_{\Gamma^*}$, with an inverse given by the map:

$$\partial\Omega^*_{\Gamma^*} \ni \omega^* \longrightarrow \nabla K^*_P(\omega^*) \in \partial\Omega_\Gamma.$$

Definition 2.1. — A point $\omega \in \partial\Omega_\Gamma$ and its image $\omega^* \in \partial\Omega^*_{\Gamma^*}$ under the map (2.8) will be referred to as conjugate points (with respect to K_P).

3. The main theorem

We shall present in this section a solution to the following problem on the asymptotic behavior of Green's functions mentioned in §1.

Problem. — *Given the elliptic operator $P(D)$ and an infinite open convex cone Γ (as above, $0 \notin \Gamma$) give conditions which ensure that the Green's function $G(x)$ admits in Γ an asymptotic behavior of the form:*

$$G(x) = a(x)e^{iA(x)}(1 + o(1)) \tag{3.1}$$

as $x \to \infty$ in Γ, where $A(x)$ is a C^2 homogeneous function of degree 1 in Γ and $a(x)$ is a C^2 homogeneous function of degree $-\frac{n-1}{2}$, $a(x) \neq 0$.

We describe a solution to the problem under the following regularity assumption on the function K_P.

Condition R. — $K_P(x)$ is a C^2 function in Γ verifying

$$\operatorname{Hess} K_P(x)^2 > 0 \quad \text{in } \Gamma.$$

Note that if $K_P(x)$ is a C^2 function in Γ then the convexity of $K_P(x)$ implies that $\operatorname{Hess} K_P(x)^2 \geqslant 0$ in Γ. It is also easy to see that Condition R is equivalent to each of the following conditions.

Condition R$_1$. — The set $\partial\Omega_\Gamma := \partial\Omega \cap \Gamma$ is a C^2 manifold possessing a positive Gaussian curvature at every point.

Condition R$_2$. — *The set $\partial\Omega^*_{\Gamma^*} := \partial\Omega^* \cap \Gamma^*$ is a C^2 manifold possessing a positive Gaussian curvature at every point.*

Theorem 3.1. — *Assume $K_P(x)$ satisfies Condition R in a cone Γ. Then*

(i) In order that $G(x)$ will have the asymptotic behavior (3.1) in Γ it is necessary that the following condition hold:

Condition A. — For any point $\omega_0^* \in \partial\Omega_{\Gamma^*}^*$ the equation $P(\xi + i\omega_0^*) = 0$ has a unique solution $\xi = \xi_0$ in $\mathbb{R}^n$. Moreover, the zero $\xi_0 + i\omega_0^*$ of $P(\zeta)$ is simple in the direction ω_0^* in the sense that

$$\frac{d}{ds} P(\xi_0 + s\omega_0^*)|_{s=i} \neq 0. \tag{3.2}$$

(ii) In order that $G(x)$ will possess the asymptotics (3.1) in Γ it is sufficient that Condition A and Condition B (described below) should hold.

To describe Condition B assume that Condition A holds. Denote by $\mathbb{R}^{n-1}_{\omega_0^*}$ the subspace in $\mathbb{R}^n$ orthogonal to ω_0^*. By the analytic implicit function theorem the equation

$$P(\xi_0 + \xi' + s\omega_0^*) = 0$$

has a unique solution $s = s(\xi') \in \mathbb{C}$ for $\xi' \in \mathbb{R}^{n-1}_{\omega_0^*}, |\xi'| < \delta, \delta > 0$ sufficiently small, such that $s(0) = i; s(\xi')$ real analytic in ξ'. Now from the definition of Ω^* it follows that $P(\xi + s\omega_0^*) \neq 0$ for $0 \leqslant \operatorname{Im} s < 1, \forall \xi \in \mathbb{R}^n$. Hence, it follows from the above that $\operatorname{Im} s(\xi') \geqslant 1$ for $|\xi'| < \delta, \operatorname{Im} s(0) = 1$.

Condition B. — The following holds:

$$\det \operatorname{Hess} s(\xi')|_{\xi'=0} \neq 0, \xi' \in \mathbb{R}^{n-1}_{\omega_0^*}.$$

Remark. — Under the sufficient conditions in Theorem 3.1 one finds that the functions $a(x)$ and $A(x)$ are C^∞ functions in $\Gamma \smallsetminus \{0\}$. Also, (3.1) can be replaced by an asymptotic infinite series expansion.

We give some indications of the proof of the necessity part in the statement of the theorem.

Thus assume that the asymptotic relation (3.1) holds in Γ. Noting that by Condition R all points of $\partial\Omega_\Gamma$ are extremal points of $\overline{\Omega}$ it follows from Proposition 2.1 and the estimate (2.3), that

$$\operatorname{Im} A(x) = K_P(x). \tag{3.3}$$

To prove that condition A is necessary we shall make use of the formula:

$$\frac{1}{P(\xi)} = \int_{\mathbb{R}^n} e^{-i\xi\cdot x} G(x) dx. \tag{3.4}$$

Now, pick a point $\omega_0^* \in \partial\Omega_{\Gamma^*}^*$. Since Condition R holds, it follows from (2.8), and Definition 2.1, that ω_0^* is the conjugate (w.r.t. K_P) of a unique point $\omega_0 \in \partial\Omega_\Gamma$ and that $\omega_0^* = K_P'(\omega_0)$. Using the estimate (2.3) on $G(x)$, noting that by (2.2)

$K_P(x) \geqslant \langle \omega_0^*, x\rangle$ for $x \in \mathbb{R}^n$, it follows from (3.4) by analytic continuation that $P(\xi + it\omega_0^*) \neq 0$ for $0 \leqslant t < 1, \xi \in \mathbb{R}^n$, and that

$$\frac{1}{P(\xi + it\omega_0^*)} = \int_{\mathbb{R}^n} e^{-i\xi\cdot x} e^{t\omega_0^*\cdot x} G(x)dx. \tag{3.5}$$

We shall consider the behavior of the r.h.s. of (3.5) as $t \uparrow 1$. To this end observe that since $\partial\Omega_\Gamma$ is a C^2 manifold having everywhere a positive Gaussian curvature the inequality: $K_P(x) \geqslant \langle \omega_0^*, x\rangle, \forall\, x \in \mathbb{R}^n$, can be sharpened as follows:

$$(1 - \varepsilon(x))K_P(x) \geqslant \langle \omega_0^*, x\rangle \quad \text{for } x \in \mathbb{R}^n \smallsetminus \{0\} \tag{3.6}$$

where $\varepsilon(x)$ is some continuous function on $\mathbb{R}^n \smallsetminus \{0\}$, homogeneous of degree 0, verifying:

$$\varepsilon(\omega) \geqslant c|\omega - \omega_0|^2 \quad \text{for} \omega \in \partial\Omega, \tag{3.6'}$$

c some positive constant. Using (3.1), together with (3.3) and (3.6), (3.6') to estimate the integral (3.5) one finds (via integration by parts) that

$$\frac{1}{P(\xi + it\omega_0^*)} = o\left(\frac{1}{1-t}\right) \quad \text{as } t \uparrow 1 \tag{3.7}$$

for any fixed $\xi \in \mathbb{R}^n, \xi \neq \xi_0$ where $\xi_0 = \operatorname{Re} A'(\omega_0)$. One also proves that

$$\frac{1}{P(\xi_0 + it\omega_0^*)} = O\left(\frac{1}{1-t}\right) \quad \text{as } t \uparrow 1. \tag{3.7'}$$

It thus follows that $P(\xi + i\omega_0^*) \neq 0$ for $\xi \neq \xi_0$. On the other hand, since $\omega_0^* \in \partial\Omega^*$ it follows (by the definition of Ω^*) that $P(\xi + i\omega_0^*) = 0$ for some $\xi \in \mathbb{R}^n$. Hence, $\xi = \xi_0$ is the unique zero of the equation: $P(\xi + i\omega_0^*) = 0, \xi \in \mathbb{R}^n$. This and (3.7') establish the necessity of Condition A.

As for the proof of the sufficiency part of the theorem, showing that if Conditions A and B (as well as Condition R) hold then $G(x)$ verifies in Γ an asymptotic formula of the form (3.1), we just remark that the proof uses the method of stationary phase in the general case where the phase function is complex (see [4], p. 220). The method of stationary phase is applied to a "main term" of $G(x)$ for $x \to \infty$ along the ray: $x = t\omega_0, t > 0 (\omega_0$ the conjugate of $\omega_0^*)$. The main term is obtained by the residuum theorem starting with the (distribution sense) formula:

$$G(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \frac{e^{i\xi\cdot x}}{P(\xi + s\omega_0^*)} d\xi$$

valid for any $s \in \mathbb{C}, 0 \leqslant \operatorname{Im} s < 1$.

4. Applications

In this section we give applications of the main theorem to problems of asymptotics of Green's functions of higher order elliptic operators with constant coefficients described in the Introduction. Following the notation used in §1, we denote by $P_0(D)$

an elliptic operator with constant coefficients such that $P_0(\xi)$ is a homogeneous polynomial of even degree $m, P_0(\xi) > 0$ for $\xi \in \mathbb{R}^n \setminus \{0\}$. We shall assume in addition that $P_0(\xi)$ is strictly convex:

$$\text{Hess}\, P_0(\xi) > 0 \quad \text{for } \xi \in \mathbb{R}^n \setminus \{0\}. \tag{4.1}$$

We denote by $G_\lambda(x)$ the Green's function of the operator $P_0(D) - \lambda$ for $\lambda \in \mathbb{C} \setminus \overline{\mathbb{R}}_+$ (given by the corresponding formula (1.1)). We recall that under the strong convexity assumption (1.5) it was established in [**1**] that $G_\lambda(x)$ verifies an asymptotic formula of the form (1.6) for all λ verifying: $0 < |\arg \lambda| < \pi$. In the following we discuss the validity of (1.6) under the weaker assumption (4.1). We shall write (1.6) in its more explicit form. To this end we introduce some notation. We set:

$$M = \{\xi \in \mathbb{R}^n : P_0(\xi) = 1\}.$$

M is a compact, smooth, strictly convex manifold. For any $x \in \mathbb{R}^n \setminus \{0\}$ we denote by $\xi(x)$ the unique point on M such that $P_0'(\xi(x))$ is in the direction of x. $\xi(x)$ is a smooth homogeneous function of degree 0 on $\mathbb{R}^n \setminus \{0\}$. We set:

$$Q_0(x) = \sup_{\xi \in M} \langle x, \xi \rangle = \langle x, \xi(x) \rangle,$$

$$\Delta(x) = \det \left(\frac{\partial^2}{\partial \xi_i \partial \xi_j} P_0(\xi(x)) \right).$$

Under the above conditions and notation we have:

Theorem 4.1. — *There exists a number $\alpha, 0 < \alpha \leqslant \pi$, such that for any λ in the sectors: $0 \leqslant \pm \arg \lambda < \alpha$, the Green's function $G_\lambda(x) (G_{\lambda \pm i0}(x)$ if $\arg \lambda = 0)$ verifies in $\mathbb{R}^n$ the asymptotic formula:*

$$G_\lambda(x) = c_\pm \lambda^{\frac{n+1}{2m}-1} \Delta(x)^{-\frac{1}{2}} Q_0(x)^{-\frac{n-1}{2}} e^{\pm i \lambda^{1/m} Q_0(x)} \left(1 + O\left(1/|x|\right)\right) \tag{4.2}$$

as $|x| \to \infty$. Here the principal branch of the powers of λ are taken, and

$$c_\pm = (2\pi)^{-\frac{n-1}{2}} (m-1)^{\frac{1}{2}} m^{\frac{n-2}{2}} e^{\mp i\pi \frac{n-3}{4}}.$$

Theorem 4.1 bis. — *A necessary and sufficient condition that $G_\lambda(x)$ verifies* (4.2) *for some $\lambda, 0 < |\arg \lambda| < \pi$, is that (with $\gamma = \arg \lambda$) the following holds:*

$$P_0\left(\xi + t e^{i\frac{\gamma}{m}} \eta\right) - e^{i\gamma} \neq 0 \tag{4.3}$$

for any $\xi \in \mathbb{R}^n, \eta \in M, 0 < t \leqslant 1$, except when $t = 1$ and $\xi = 0$ (any $\eta \in M$).

The proof of the theorems is based on Theorem 3.1 Here are some indications of the proof. First note (as before) that the validity of (4.2) for $G_{\lambda \pm i0}(x), \lambda \in \mathbb{R}_+$, follows by applying the relevant formula (1.4). Hence in view of the homogeneity of $P_0(\xi)$ it would suffice to prove Theorem 4.1 for λ of the form: $\lambda = e^{i\gamma}, 0 < \gamma < \pi$. Set:

$$P(D) = P_0(D) - e^{i\gamma}.$$

In order to find whether Theorem 3.1 is applicable to the operator P, consider the complex roots of the polynomial $P(\zeta)$. Clearly: $P(e^{i\frac{\gamma}{m}}\eta) = 0$ for any $\eta \in M$. On the other hand one can show, using the strict convexity of the manifold M, that there exists a number $\alpha, 0 < \alpha < \pi$, such that for any γ verifying $0 < \gamma < \alpha$ the following holds:

$$P\left(\xi + te^{i\frac{\gamma}{m}}\eta\right) \neq 0$$

for any $\xi \in \mathbb{R}^n, \eta \in M, 0 < t \leqslant 1$ except when $t = 1$ and $\xi = 0$. Assume from now on that $0 < \gamma < \alpha$. The last observations on the complex zeros of $P(\zeta)$ can be used to compute the "norm function" $K_P^*(x)$ defined by (2.1). It follows that

$$K_P^*(x) = \frac{1}{\sin(\gamma/m)} P_0(x)^{\frac{1}{m}} \tag{4.4}$$

and that $K_P(x)$, defined by (2.2), is given here by:

$$K_P(x) = \sin\left(\frac{\gamma}{m}\right)\left(P_0(x)^{\frac{1}{m}}\right)^* = \sin\left(\frac{\gamma}{m}\right) Q_0(x).$$

From the strict convexity of $P_0(\xi)$ it follows further that $K_P(x)^2$ is a smooth strictly convex function on $\mathbb{R}^n \setminus \{0\}$.

The above considerations show that the operator $P_0(D) - e^{i\gamma}$ verifies Condition R as well as the main part of Condition A of Theorem 3.1, with $\Gamma = \mathbb{R}^n \setminus \{0\}$. A straight forward computation shows that (3.2) and Condition B also hold. Applying Theorem 3.1 one finds that the Green's function $G_{e^{i\gamma}}(x)$ has an asymptotic formula of the form:

$$G_{e^{i\gamma}}(x) = a_\gamma(x) e^{iA_\gamma(x)} \left(1 + O\left(1/|x|\right)\right)$$

as $|x| \to \infty$, where $a_\gamma(x)$ is a homogeneous function of degree $-\frac{n-1}{2}$ and the phase function $A_\gamma(x)$ is a homogeneous function of degree 1 verifying:

$$\operatorname{Im} A_\gamma(x) = K_P(x) = \sin\left(\frac{\gamma}{m}\right) Q_0(x).$$

Finally, using a more complete information on the asymptotic formula in Theorem 3.1 (not given in this paper) one finds that $A_\gamma(x) = e^{i\frac{\gamma}{m}} Q_0(x)$ and that the explicit asymptotic formula (4.2), with $\lambda = e^{i\gamma}$, holds.

Theorem 4.1 bis is a straightforward application of Theorem 3.1. The sufficiency part of the theorem follows in exactly the same manner as in the indicated proof of the asymptotics in Theorem 4.1.

For the necessity part of the theorem observe that (with $\lambda = e^{i\gamma}, P = P_0 - e^{i\gamma}$) the asymptotics (4.2) and (3.3) imply that $K_P(x) = \sin(\gamma/m) Q_0(x)$ and thus its polar $K_P^*(x)$ is given by (4.4). This and (2.1) imply that (4.3) must hold for $0 < t < 1$. Furthermore, since $P_0(e^{i\gamma/m}\eta) - e^{i\gamma} = 0$ for any $\eta \in M$, the necessity of Condition A (in Theorem 3.1, when (4.2) holds, implies that (4.3) must also hold for $t = 1$ if $\xi \neq 0$.

We conclude by considering the asymptotic expansion of $G_\lambda(x)$ for λ a negative number when $P_0(D)$ is an operator of order $m > 2$. In this case it was shown in [**1**], under the strong convexity assumption (1.5), that $G_\lambda(x)$ admits a two terms asymptotic expansion which in our (different) notation can be written in the following form. Set $\lambda = -\rho, \rho > 0$. Then:

$$\begin{aligned}(4.5)\qquad & \rho^{1-\frac{n+1}{2m}}\Delta(x)^{\frac{1}{2}}Q_0(x)^{\frac{n-1}{2}}G_{-\rho}(x) \\ &= \left(1+O\left(\frac{1}{|x|}\right)\right)c_+^0\exp\left(ie^{i\frac{\pi}{m}}\rho^{\frac{1}{m}}Q_0(x)\right) \\ &\quad + \left(1+O\left(\frac{1}{|x|}\right)\right)c_-^0\exp\left(-ie^{-i\frac{\pi}{m}}\rho^{\frac{1}{m}}Q_0(x)\right),\end{aligned}$$

as $|x| \to \infty$ where $c_\pm^0 = c_\pm \exp(\pm\pi i\frac{n+1-2m}{2m})$. Now, Theorem 3.1 which deals with asymptotic expansions of Green's functions involving a single phase function can easily be generalized to include asymptotic expansions involving sum of several terms with different phase functions. Using this generalization one derives necessary and sufficient conditions for the validity of the expansion (4.5) under the assumption that $P_0(\xi)$ satisfies (4.1) (but not necessarily (1.5)). One obtains the following:

Theorem 4.2. — *Under the convexity condition* (4.1) *on* $P_0(\xi)(m > 2)$*, a necessary and sufficient condition for the asymptotic expansion* (4.5) *to hold is that:*

$$P_0\left(\xi + it\sin\left(\frac{\pi}{m}\right)\eta\right) + 1 \neq 0$$

for any $\xi \in \mathbb{R}^n, \eta \in M$ *and* $0 < t \leqslant 1$*, except when* $t = 1$ *and* $\xi = \pm\cos(\pi/m)\eta$.

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