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DUAL ELLIPTIC PLANES

by

Jean-Claude Sikorav

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***Abstract.* —** An elliptic plane is a complex projective plane V equipped with an elliptic structure E in the sense of Gromov (generalization of an almost complex structure), which is tamed by the standard symplectic form. The space V^* of surfaces of degree 1 tangent to E (E-lines) is again a complex projective plane. We define on V^* a structure of elliptic plane E^*, such that to each E-curve one can associate its dual in V^*, which is an E^*-curve. Also, the bidual (V^{**}, E^{**}) is naturally isomorphic to (V, E).

***Résumé* (Plans elliptiques duaux). —** Un plan elliptique est un plan projectif complexe équipé d'une structure elliptique E au sens de Gromov (généralisation d'une structure quasi-complexe), qui est positive par rapport à la forme symplectique standard. L'espace V^* des surfaces de degré un tangentes à E (E-droites) est de nouveau un plan projectif complexe. Nous définissons sur V^* une structure de plan elliptique E^*, telle qu'à toute E-courbe on peut associer sa duale dans V^*, qui est une E^*-courbe. En outre, le bidual (V^{**}, E^{**}) est naturellement isomorphe à (V, E).

Introduction

Let V be a smooth oriented 4-manifold, which is a rational homology $\mathbb{CP}^2$ (*i.e.* $b_2(V) = 1$), and let J be an almost complex structure on V which is homologically equivalent to the standard structure J_0 on $\mathbb{CP}^2$. This means that there is an isomorphism $H^*(V) \to H^*(\mathbb{CP}^2)$ (rational coefficients) which is positive on H^4 and sends the Chern class $c_1(J)$ to $c_1(J_0)$.

By definition, a *J-line* is a J-holomorphic curve (or *J-curve*) of degree 1. By the *positivity of intersections* [**McD2**], it is an embedded sphere. We denote by V^* the set of J-lines.

Now assume that J is *tame*, *i.e.* positive with respect to some symplectic form ω, and also that V^* is nonempty. Then M. Gromov [**G**, 2.4.A] (*cf.* also [**McD1**]) has

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proved that by two distinct points $x, y \in V$ there passes a unique J-line $L_{x,y} \in V^*$, depending smoothly on (x, y); also, for any given $P \in \mathrm{Gr}_1^J(TV)$, the Grassmannian of J-complex lines in TV, there exists a unique J-line $L_P \in V^*$ tangent to P. Furthermore, V is oriented diffeomorphic to $\mathbb{CP}^2$, ω is isomorphic to $\lambda\omega_0$ for some positive λ so that J is homotopic to J_0. Finally, V^* has a natural structure of compact oriented 4-manifold; although it is not explicitly stated in [**G**], the above properties of V^* imply that it is also oriented diffeomorphic to $\mathbb{CP}^2$.

***Remark* (J. Duval)**. — The dependence of L_P upon P is continuous but not smooth. However, when p is fixed, the map $P \in \mathrm{Gr}_1^J(T_pV) \approx \mathbb{CP}^1 \mapsto L_P$ has quasiconformal components in any smooth chart of V^* given by intersections with two J-lines. For more details, see [**D**, p. 4-5].

Later, Taubes [**T1, T2**] proved that the hypothesis that V^* be nonempty is unnecessary, so that all the above results hold when J is tame. We shall call (V, J) an *almost complex projective plane*.

Following [**G**, 2.4.E], these facts can be extended to the case of an *elliptic structure* on V, *i.e.* one replaces $\mathrm{Gr}_1^J(TV)$ by a suitable submanifold E of the Grassmannian of oriented 2-planes $\widetilde{\mathrm{G}r}_2(TV)$. Such a structure is associated to a *twisted almost complex structure* J, which is a fibered map from TV to itself satisfying $J_v^2 = -\mathrm{Id}$ but such that J_v is not necessarily linear.

An elliptic structure on V gives rise to a notion of *E-curve*, *i.e.* a surface $S \subset V$ (not necessarily embedded or immersed) whose tangent plane at every point is an element of E (for the precise definitions, see section 2). It will be called *tame* if there exists a symplectic form ω strictly positive on each $P \in E$.

In Gromov's words, "all facts on J-curves extend to E-curves with an obvious change of terminology". In particular, let V be a rational homology $\mathbb{CP}^2$ equipped with a tame elliptic structure E so that (V, E) is homologically equivalent to $(\mathbb{CP}^2, \mathrm{Gr}_1^{\mathbb{C}}(T\mathbb{CP}^2))$. Then one can define the space V^* of E-lines (E-curves of degree 1), and prove that all the above properties still hold (see section 3). In particular, V and V^* are oriented diffeomorphic to $\mathbb{CP}^2$.

We shall call (V, E) with the above properties an *elliptic projective plane.* If $C \subset V$ is an E-curve, we define its *dual* $C^* \subset V^*$ by $C^* = \{L_{T_vC} \mid v \in C\}$. A more precise definition is given in section 4; one must require that no component of C be contained in an E-line. The main new result of this paper is then the following.

Theorem. — *Let (V, E) be an elliptic projective plane. Then there exists a unique elliptic structure $E^* \subset \widetilde{\mathrm{Gr}}_2(TV^*)$ on V^* with the following property: if $C \subset V$ is an E-curve which has no component contained in an E-line, then its dual $C^* \subset V^*$ is an E^*-curve.*

Furthermore, (V^*, E^*) is again an elliptic projective plane. Finally, the bidual (V^{**}, E^{**}) can be canonically identified with (V, E), and $C^{**} = C$ for every E-curve C.

If E comes from an almost complex structure, one may wonder if this is also the case for E^*, equivalently if the associated twisted almost complex structure J^* is linear on each fiber. Ben McKay has proved that this happens only if J is integrable, *i.e.* isomorphic to the standard complex structure on $\mathbb{CP}^2$: see the end of the Introduction.

The theorem above enables us to extend to J-curves in $\mathbb{CP}^2$ (for a tame J) some classical results obtained from the theory of dual algebraic curves. For instance, one immediately obtains the Plücker formulas, which restrict the possible sets of singularities of J-curves.

Such results could be interesting for the symplectic isotopy problem for surfaces in $\mathbb{CP}^2$ [**Sik2, Sh**]. And maybe also for the topology of a symplectic 4-manifold X, in view of the result of D. Auroux [**Aur**] showing that X is a branched covering of $\mathbb{CP}^2$, provided one could rule out negative cusps in the branch locus.

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This idea has also been discovered independently by Ben McKay, who made a very deep study of elliptic structures (which may exist in any even dimension for V) from the point of view of exterior differential systems (see the references at the end and also his web site). He uses the terminology "generalized Cauchy-Riemann equations" and "generalized pseudoholomorphic curves".

In particular, he proved that the submanifold E giving the structure is equipped with a canonical almost complex structure. He also gave a positive answer to a conjecture that I had made (see Section 5): if the elliptic structures on V and on its dual V^* are both almost complex, then they are integrable (and thus V is isomorphic to $\mathbb{CP}^2$ with the standard complex structure).

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Structure of the paper. — In section 1, we study elliptic surfaces in the Grassmannian of oriented 2-planes of a 4-dimensional real vector space. In section 2 we study elliptic structures on a 4-manifold, *i.e.* fibrations in elliptic surfaces in the tangent spaces. In section 3 we define and study elliptic projective planes. Most of the statements and all the ideas in these three sections are already in Gromov's paper (see especially [**G**, 2.4.E and 2.4.A]), except what regards singularities, where we give more precise results in the vein of [**McD2**] and [**MW**].

In section 4 we prove the main result.

In section 5 we give a special case of a more general general result of McKay: a tame almost complex structure J on $V = \mathbb{CP}^2$ such that the elliptic structure on V^* is not almost complex (and thus V^* has no natural almost complex structure).

Finally in section 6 we prove the Plücker formulas for E-curves and in particular for J-curves.

1. Elliptic surfaces in a Grassmannian

1.A. Definition. Associated complex lines. — Let T be an oriented real vector space of dimension 4. We denote by $G(T) = \widetilde{\mathrm{Gr}}_2(T)$ the Grassmannian of oriented 2-planes. Recall that for each $P \in G(T)$, the tangent plane $T_P G(T)$ is canonically identified with $\mathrm{Hom}(P, T/P)$.

By definition, an *elliptic surface* in $G(T)$ is a smooth, closed, connected and embedded surface X such that for every $P \in X$ one has

$$T_P X \smallsetminus \{0\} \subset \mathrm{Isom}_+(P, T/P).$$

Lemma. — *Let P_1, P_2, P_3 be three oriented real planes ($\mathbb{R}$-vector spaces of dimension 2), and*

$$\phi : P_1 \longrightarrow \mathrm{Hom}(P_2, P_3)$$

be a linear map such that $\phi(P_1 \smallsetminus \{0\}) \subset \mathrm{Isom}_+(P_2, P_3)$. Then there exists unique complex structures j_1, j_2, j_3 on P_1, P_2, P_3, making them complex lines, compatible with the orientations, and such that the restriction $\phi : P_1 \to \mathrm{im}(\phi)$ is a complex isomorphism onto $\mathrm{Isom}_{\mathbb{C}}(P_2, P_3)$, i.e.

$$(*) \qquad \phi(p_1) \circ j_2 = j_3 \circ \phi(p_1), \quad \phi(j_1 p_1) = \phi(p_1) \circ j_2 = j_3 \circ \phi(p_1).$$

Proof of the Lemma. — We prove the uniqueness first. Let j_1, j_2, j_3 have the desired properties. Let (p_1^1, p_1^2) be an oriented base of P_1, and let

$$u = \phi(p_1^1)^{-1} \phi(p_1^2) \in \mathrm{GL}_+(P_2).$$

The hypothesis implies that u has eigenvalues $a \pm ib$ with $b > 0$. Replacing p_1^2 by $(p_1^2 - a p_1^1)/b$, we can obtain that these eigenvalues are $\pm i$.

Note that u belongs to the plane $P = \phi(p_1^1)^{-1}[\mathrm{im}(\phi)] \subset \mathrm{End}(P_2)$. This plane is generated by Id and $j_2 = \phi(p_1^1)^{-1} \phi(j_1 p_1^1)$, thus the fact that u has eigenvalues $\pm i$ implies $j_2 = \varepsilon u$ with $\varepsilon = \pm 1$.

Thus $j_1 p_1^1 = \varepsilon p_1^2$, and since $(p_1^1, j_1 p_1^1)$ and (p_1^1, p_1^2) are both oriented bases of P_1, we have $\varepsilon = 1$, thus

$$\begin{aligned} &j_2 = \phi(p_1^1)^{-1} \phi(p_1^2), \\ &j_1(p_1^1) = p_1^2, \quad j_1(p_1^2) = -p_1^1, \\ &j_3 = \phi(p_1^2) \circ \phi(p_1^1)^{-1}. \end{aligned}$$

This proves the uniqueness.

Conversely, it is easy to see that these formulas define complex structures compatible with the orientations, and that $(*)$ is satisfied. □

Applying this lemma, we obtain complex structures on $T_P X$, P, T/P, making them complex lines. We shall denote by

– $j_{X,P}$ the structure on $T_P X$,
– j_P and $j_P^{\perp}$ the structures on P and T/P.

By the integrability of almost complex structures on surfaces, X inherits a well-defined structure of Riemann surface.

1.B. Elliptic surfaces and complex structures. — The first example of elliptic surface is a Grassmannian $\mathrm{Gr}_1^J(T)$ of complex J-lines for a positive complex structure J on T.

We now prove that every elliptic surface is deformable to such a $\mathrm{Gr}_1^J(T)$. More precisely, denote by $\mathcal{J}(T)$ the space of positive complex structures, and $\mathcal{E}(T)$ the space of elliptic surfaces. Then the embedding $\mathcal{J}(T) \to \mathcal{E}(T)$ just defined admits a retraction by deformation. In particular, X is always diffeomorphic to $\mathbb{CP}^1$ and thus biholomorphic to $\mathbb{CP}^1$.

To prove this, we fix a Euclidean metric on V and replace $\mathcal{J}(T)$ by the subspace $\mathcal{J}_0(T)$ of isometric structures, to which it retracts by deformation. The space of 2-vectors $\Lambda^2 T$ has a decomposition $\Lambda^2 T = \Lambda^2_+ T \oplus \Lambda^2_- T$ into self-dual and antiself-dual vectors. The Grassmannian $G(T)$ is identified with $S^2_+ \times S^2_- \subset \Lambda^2_+ T \times \Lambda^2_- T$ by sending a plane P to $(\sqrt{2}(x \wedge y)_+, \sqrt{2}(x \wedge y)_-)$ where (x, y) is any positive orthonormal basis. We denote by $P = \phi(u_+, u_-)$ the plane associated to (u_+, u_-). Identifying T/P with $P^{\perp}$, the canonical isomorphism

$$T_{u_+} S^2_+ \times T_{u_-} S^2_- \longrightarrow \mathrm{Hom}(P, P^{\perp})$$

sends (α_+, α_-) to A such that

$$A.\xi = *(\xi \wedge (\alpha_+ + \alpha_-)).$$

This can be seen by working in a unitary oriented basis of T, (e_1, e_2, e_3, e_4) such that $u_\pm = \frac{1}{\sqrt{2}}(e_1 \wedge e_2 \pm e_3 \wedge e_4)$. This leads to unitary oriented bases of $T_{u_+} S^2_+$ and $T_{u_-} S^2_-$:

$$v^\pm = \frac{1}{\sqrt{2}}(e_1 \wedge e_3 \mp e_2 \wedge e_4), \quad w^\pm = \frac{1}{\sqrt{2}}(e_1 \wedge e_4 \pm e_2 \wedge e_3).$$

Still working in these bases, one gets

$$\det A = -\|\alpha_+\|^2 + \|\alpha_-\|^2,$$

(beware the signs!). Thus an elliptic structure is given by a surface $X \subset S^2_+ \times S^2_-$ such that the projections $p_\pm : X \to S^2_\pm$ satisfy

– dp_- is an isomorphism at all points of X,
– $\|dp_+ \circ (dp_-)^{-1}\| < 1$ at all points of X.

Since X is closed and connected, p_- is a diffeomorphism from X to S^2_- and X is the set of points $(a(u), u)$, where $a : S^2_- \to S^2_+$ is a smooth contraction.

Thus $\mathcal{E}(T)$ is homeomorphic to the space of smooth contractions from S^2 to itself.

Proposition ([**McK2**, Proposition 1]). — *A contraction of S^2 has an image contained in an open hemisphere.*

Corollary. — *The space* $\mathrm{Contr}(S^2)$ *of smooth contractions from S^2 to itself retracts by deformation to the space of constant maps.*

Proof of the proposition. — Let h be a contraction of S^2. Then h must map some pair of antipodal points to the same point, otherwise the map

$$x \in S^2 \longmapsto h(x) - h(-x) \in \mathbb{R}^2$$

would satisfy $f(-x) = -f(x)$ and $f(x) \neq 0$ for all x, contradicting the Borsuk-Ulam theorem.

If $h(x) = h(-x) = y$, then $h(S^2) = h(H_x) \cup h(H_{-x})$ where H_x is the closed hemisphere centered on x. Since h is a contraction, this implies that $h(S^2)$ is contained in the interior of H_y. □

Proof of the corollary. — We define $x(h)$ as the unique point at which the function

$$m_h(x) = \min_{y \in S^2} \langle h(x), y\rangle$$

attains its maximum $M(h)$. The uniqueness comes from the fact that $M(h) > 0$ by the proposition, and also the fact that $h(S^2)$ is contained in the interior of $H_{x(h)}$. It is easy to see that $x(h)$ is continuous in h. Then the obvious retraction of $H_{x(h)}$ onto its center will give the desired retraction of $\mathrm{Contr}(S^2)$.

Since a constant map $S^2_- \to S^2_+$ corresponds to a Grassmannian $\mathrm{Gr}^J_1(T)$ for some $J \in \mathcal{J}_0(T)$, this gives the retraction by deformation from $\mathcal{E}(T)$ to $\mathcal{J}_0(T)$. □

Remark. — The proof of the corollary given in the preprint was wrong!

1.C. Twisted complex structure associated to an elliptic surface. — Let $X \subset G(T)$ be an elliptic surface. Then we have the

Proposition. — *The space* $T \smallsetminus \{0\}$ *is the disjoint union* $\bigcup_{P \in X} P \smallsetminus \{0\}$.

Proof. — We use the representation $X = \{P_u \mid u \in S^2_-\}$, with $P_u = \phi((a(u), u)$, $a : S^2_- \to S^2_+$ being a smooth contraction.

Let $\xi \in T \smallsetminus \{0\}$ be given. Then

$$\xi \in P_u \Longleftrightarrow \xi \wedge (a(u) + u) = 0 \Longleftrightarrow \xi \wedge (a(u) - u) = 0.$$

We can identify $\xi^\perp \subset T$ with Λ^2_+ via $u \mapsto *(\xi \wedge u)$. Then

$$*(\xi \wedge ((a(u) - u)) = u + b(u),$$

where $b : S^2_- \to S^2_-$ is a smooth contraction. Thus there exists a unique $u \in S^2_-$ such that $-b(u) = u$, *i.e.* a unique $P = P_u$ in X containing ξ.

Thus if u, v are distinct points in S^2_-, we have $P_u \cap P_v = \{0\}$. In fact P_u and P_v are *positively* transverse (first occurrence of the positivity of intersections): indeed, the inequality $\|da\| < 1$ implies $\|a(u) - a(v)\|^2 < \|u - v\|^2$ *i.e.*

$$\langle a(u), a(v)\rangle - \langle u, v\rangle > 0,$$

which is precisely the positive transversality of P_u and P_v. $\square$

This enables us to put together the j_P, $P \in X$, to obtain a map $J : T \to T$ with the following properties:

– $J^2 = -\mathrm{Id}$,

– J is continuous, and homogeneous of degree 1,

– J is smooth away from 0 (by homogeneity, it is not differentiable at 0 except if it is linear),

– for every $x \in T \smallsetminus \{0\}$, $J(x)$ is linearly independent of x, and J is linear on the plane $\langle x, J(x)\rangle$.

Conversely, given J satisfying (i)-(iv), we can define a smooth surface $X \subset G(T)$ by

$$X = \{\langle x, J(x)\rangle \mid x \in T \smallsetminus \{0\}\}.$$

A straightforward computation gives that X is elliptic if and only if

– for every $x \in T \smallsetminus \{0\}$ and $\xi \in \langle x, J(x)\rangle^{\perp} \smallsetminus \{0\})$, $(x, J(x), \xi, dJ_x.\xi)$ is an oriented basis of T. Clearly, J is linear if and only if it is a complex structure on T. In that case, we say that X is *linear*, or is associated to a complex structure on V.

1.D. Local form of an elliptic surface. — Let X be an elliptic surface in $G(T)$, and fix $P \in X$. Identify T with $\mathbb{C}^2$ such that

(i) P is sent to the horizontal plane $H = \mathbb{C} \times \{0\}$,

(ii) the identifications $P \leftrightarrow H$ and $T/P \leftrightarrow \mathbb{C}^2/H$ are complex-linear for the complex structures defined in 1.A.

Then X is given near P by a family of planes of the following form:

$$P_\lambda = \{(\delta z, \delta w) \in \mathbb{C}^2 \mid \delta w = \lambda \delta z + h(\lambda)\overline{\delta z}\},$$

where h is a smooth germ $(\mathbb{C}, 0) \to (\mathbb{C}, 0)$ such that $h(0) = 0$. The tangent space $T_{P_\lambda}X$ is identified with the image of

$$\xi \in \mathbb{C} \longmapsto \xi \mathrm{Id} + (dh(\lambda) \cdot \xi)\sigma \in \mathrm{End}_{\mathbb{R}}(\mathbb{C}),$$

where σ is the complex conjugation. Thus the ellipticity translates to the inequality $\|dh\| < 1$. Property (ii) becomes $dh(0) = 0$.

Remark. — Denote by J the complex structure on T. Properties (i) and (ii) imply that $P \in \mathrm{Gr}_1^J T$ and $T_P\mathrm{Gr}_1^J(TV) = T_P X$, *cf.* [**G**, 2.4.E].

2. Elliptic structure on a 4-manifold. Solutions of E, E-maps and E-curves

2.A. Definition. Twisted almost complex structure. — Let V be an oriented 4-manifold, and let $G = \widetilde{\mathrm{Gr}}_2(TV)$ be the Grassmannian of oriented tangent 2-planes, which is fibered over V with fiber $G_v = \widetilde{\mathrm{Gr}}_2(T_vV)$.

By definition, an *elliptic structure* on V is a smooth compact submanifold $E \subset G$ of dimension 6, transversal to the fibration $G \to V$, such that each fiber E_v is an elliptic surface in $\widetilde{\mathrm{Gr}}_2(T_vV)$.

Denote by $\mathcal{E}(TV)$ the space of elliptic structures on V and $\mathcal{J}(TV)$ the space of positive almost complex structures on V. The map $J \mapsto \mathrm{Gr}_1^J(TV)$ gives a natural embedding from $\mathcal{J}(TV)$ to $\mathcal{E}(TV)$. These are both spaces of sections of a bundle on V, with respective fibers $\mathcal{E}(T_vV)$ and $\mathcal{J}(T_vV)$. Since $\mathcal{E}(T_vV)$ retracts by deformation to $\mathcal{J}(T_vV)$, $\mathcal{E}(TV)$ retracts by deformation to $\mathcal{J}(TV)$. In particular, every elliptic structure defines a unique homotopy class of almost complex structures on V. Thus the Chern class $c_1(E) = c_1(TV, J) \in H^2(V, \mathbb{Z})$ is well defined.

Finally, the twisted structures J_v, $v \in V$, can be put together to give a *twisted almost complex structure* on V, *i.e.* a fiber-preserving map $J : TV \to TV$ such that all the J_v have the properties (i)-(v) of 1.C. It is continuous on TV [in fact locally Lipschitz], and smooth away from the zero section. Conversely, a map J with all these properties clearly defines an elliptic structure.

Clearly, J is linear if and only if it is an almost complex structure on V. In that case, we say that E is *linear*, or is associated to an almost complex structure on V.

In the remainder of this section we consider an oriented 4-manifold V equipped with an elliptic structure $E \subset G = \widetilde{\mathrm{Gr}}_2(TV)$. If S is an oriented surface and $f : S \to V$ is an immersion, we denote by $\gamma_f : S \to G$ the associated Gauss map.

2.B. Immersed solutions. Local equation as a graph. — By definition, an immersed *solution of* E is a $\mathcal{C}^1$ immersion $f : S \to V$ where S is an oriented surface and $\gamma_f(S) \subset E$.

Let $v \in V$ and $P \in E_v$ be fixed. We describe a local equation for germs of immersed solutions of E which are tangent to P at v, or more generally which have a tangent close enough to P. Choose a local chart $(V, v) \to (\mathbb{C}^2, 0)$ such that the properties of 1.D are satisfied for E_v and P. Then the elliptic structure on E near H is given by a family of planes

$$P_{z,w,\lambda} = \{(\delta z, \delta w) \in \mathbb{C}^2 \mid \delta w = \lambda \delta z + h(z, w, \lambda)\overline{\delta z}\}.$$

Here (z, w, λ) belongs to a neighbourhood of 0 in $\mathbb{C}^3$, and $P_{z,w,\lambda}$ represents a plane tangent at the point of coordinates (z, w). The map h is a smooth germ $(\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$

such that

$$\begin{cases} \|D_3h(z,w,\lambda)\| < 1 \quad (\forall\,(z,w,\lambda)), \\ D_3h(0,0,0) = 0. \end{cases} \tag{1}$$

A germ of surface $S \subset V$ passing through P with a tangent plane close enough to P, can be written as a graph $w = f(z)$, where $f : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ satisfies

$$\frac{\partial f}{\partial \overline{z}} = h\Big(z, f(z), \frac{\partial f}{\partial z}\Big). \tag{2}$$

Remark. — This equation with the property $\|D_3h\| < 1$ is the general (resolved) form of an elliptic equation $\mathbb{C} \to \mathbb{C}$, *cf.* [**V**]. It implies the existence of a local immersed solution of E with any given tangent plane, and even with an arbitrary "compatible" k-jet (an easy proof can be given by a suitable implicit function theorem, modifying slightly the proofs given in Chapter V or VI of [**AL**]), and also that each solution is of class $\mathcal{C}^\infty$.

2.C. Conformal parametrization, E-maps. — Let $f : S \to V$ be an immersed solution of E. Since every tangent plane $P_z = df_z(T_zS)$ has a well-defined complex structure j_{P_z}, this induces a canonical almost complex structure j_f on S, *i.e.* a natural structure of Riemann surface. In other words, every (immersed) solution of E admits a natural conformal parametrization.

If S is a Riemann surface, we say that an immersion $f : S \to V$ is a *conformal solution* of E if it is a solution and j_f is the canonical almost structure on S. This is equivalent to the equation

$$df_z \circ i = J_{f(z)} \circ df_z. \tag{3}$$

We can now eliminate the immersion condition and define an *E-map* as a $\mathcal{C}^1$ map $f : S \to V$, where S is a Riemann surface, which satisfies (3).

Note that since J is only Lipschitz, the fact that E-maps are smooth is not completely obvious at this stage. But the arguments of [**AL**, chap. V or VI] imply that if f is a nonconstant local E-map, then df has only isolated zeros and the Gauss map γ_f can be extended continuously. And also that there exist E-immersions with an arbitrary given 2-jet, satisfying suitable compatibility conditions.

2.D. E-maps as pseudoholomorphic maps. — $\Theta_P = d\pi_P^{-1}(P)$, where $P \in E \subset G$ and $\pi : G \to V$ is the natural projection. It is characterized by the property that every Gauss map $\gamma_f : S \to E$ associated to an E-map (not locally constant), is tangent to Θ. The construction of [**G**, section 2.4] can be generalized to give the

Proposition. — *There exists a unique almost complex structure $\widetilde{J}$ on Θ, such that every Gauss map $\gamma : S \to E$ associated to an E-map is $\widetilde{J}$-holomorphic (or is a $\widetilde{J}$-map), i.e. it is of class $\mathcal{C}^1$, tangent to Θ and satisfies*

$$d\gamma_z \circ i = \widetilde{J}_{\gamma(z)} \circ d\gamma_z. \tag{4}$$

Proof. — For every $X \in \Theta_P \setminus F_P$ there is an E-map $f : S \to V$ and a vector $u \in T_zS$ such that $d\gamma_z(u) = X$ where γ is the Gauss map associated to f. Thus necessarily

$$\widetilde{J}_P.X = d\gamma_z(iu),$$

which implies the uniqueness.

To prove the existence, it suffices to show that $(d\gamma_z(u) = 0 \Rightarrow d\gamma_z(iu) = 0)$: this follows from (3) by differentiation. □

Remarks

(i) McKay [**McK1, McK2**] has explained how to define a canonical structure $\widehat{J}$ on TE extending $\widetilde{J}$. It can be characterized by the existence of local coordinates (z, w) on V and (z, w, λ) on E as before, with the additional properties

$$D_1h(0,0,0) = D_2h(0,0,0) = 0 \text{ (thus } Dh(0,0,0) = 0),$$
$$\operatorname{tr}(D_3^2 h)(0,0,0) = 0.$$

In other words, h is of the form

$$h(z, w, \lambda) = a\lambda^2 + b\overline{\lambda}^2 + O(|\lambda|(|z| + |w|)) + O(|z|^3 + |w|^3 + |\lambda|^3).$$

(ii) The differential $d\pi_P : T_PG \to P$ is complex linear on Θ_P, and $\widetilde{J}_{P|F_P} = j_{E_v,P}$ with the notation of 1.A.

Let $\gamma : S \to E$ be a $\widetilde{J}$-map whose image is not locally contained in a fiber. then the map $f = \pi \circ \gamma$ satisfies (3) and is not locally constant. Thus its Gauss map γ_f is well-defined and one has $\gamma_f = \gamma$. Thus $(f \mapsto \gamma_f)$ gives a bijection between E-maps and $\widetilde{J}$-maps not locally contained in a fiber.

Corollary. — *Every E-map is smooth.*

Proof. — We know already that γ is smooth away from singularities of f. If z is such a singularity, then since γ satisfies (4) away from z and is continuous at z, it is smooth everywhere. Note that it implies that every E-map is smooth. □

2.E. General E-curves, compactness theorem. — Using the conformal parametrization, we can now define an E-curve as a $\widetilde{J}$-curve $\widetilde{C}$ in E which is "almost transverse" to F. To make this definition precise, one has

(i) to choose a definition of $\widetilde{J}$-curve, for instance a stable $\widetilde{J}$-curve in the sense of Kontsevich.

(ii) to say what "almost transverse" means: essentially that no nonconstant component has an image contained in a fiber, or that it is transverse to F except on a finite subset.

One then obtains topological spaces of E-curves. We shall not give any details here since the only spaces of E-curves we shall consider will consist of curves which are embedded in V. The space of embedded E-curves will be considered as a subspace of the space of smooth surfaces in V: recall that this is a smooth Fréchet manifold whose tangent space at S is the space of sections of the normal bundle $N(S,V) = T_S V/TS$.

We shall also use the following notions of individual E-curves:

– a *primitive* (or irreducible) E-curve is the image $C = f(S)$ where S is a closed and connected Riemann surface, f is an E-map which does not factor $f = f_1 \circ \pi$ with π a nontrivial holomorphic covering, As in the case of J-curves, the image determines (S, f) up to isomorphism.

– an E-*cycle* (à la Barlet) $C = \sum n_i C_i$ where the C_i are distinct primitive E-curves and the n_i are positive integers,

– analogous local versions of these.

One expects a compactness theorem for E-curves, analogous to the one for pseudoholomorphic curves: roughly speaking, it should say that (if V is compact) a set of E-curves is relatively compact if their areas in V are uniformly bounded. If one replaces "areas in V" by "areas in E of the Gauss maps", then such a result follows from

– the compactness theorem for $\widehat{J}$ on TE (*cf.* the remark (i) in 2.D),

– the fact that the conditions "tangent to Θ" and "almost transverse to F" are closed conditions (the last one, under suitable homological assumptions).

However, it is not clear that an area bound in V gives an area bound in E. Gromov [**G**, 2.4.E] says that the Schwarz lemma is still valid for E-curves under an area bound in V, but I do not understand the proof. Anyhow, here we shall only need the compactness theorem for E-lines, which we shall prove in section 3.

2.F. Singularities of E-curves and positivity of intersections. — Here we extend to E-curves the result of M. Micallef and B. White [**MW**] (see also [**Sik1**]): we prove that a E-curve, possibly non reduced, is $\mathcal{C}^1$-equivalent to a germ of standard holomorphic curve in $\mathbb{C}^2$. It implies the positivity of intersections for E-curves, in particular the genus and intersection formulas.

Such a result could be proved by showing that such a surface is *quasiminimizing* in the sense of [**MW**], but we prefer to use more complex-analytic arguments as in [**Sik1**].

We use the chart of 2.B to write the equation in intrinsic form, *i.e.* as a graph over the tangent space. If the curve is non singular, this is the equation (2) where h satisfies (1). Now, consider a germ of non-immersed E-map $F : (\mathbb{C}, 0) \to (\mathbb{C}^2, 0)$ with horizontal tangent at the origin. Then equation (3) and the similarity principle [**Sik1**] (proposition 2; *cf.* also [**McD2**]) give the existence of $a \in \mathbb{C}^*$ and $k \in \mathbb{N}$, $k \geqslant 2$, such that

$$f(z) = (az^k, 0) + O_{2,1-}(z^{k+1}).$$

Here we use a notation from [**MW, Sik1**]: $g(z) = O_{2,1-}(z^{k+1})$ means

$$\begin{cases} g(z) = O(z^{k+1}), dg(z) = O(z^k), d^2g(z) = O(z^{k-1}), \\ (\forall\, \alpha < 1) \quad d^2g \text{ is } \alpha\text{-Hölder with Hölder constant } O(|z|^{k-1-\alpha}). \end{cases}$$

Thus we can reparametrize the curve by setting $pr_1 \circ F(z) = t^k$, where $z \mapsto t$ is a $\mathcal{C}^1$ local diffeomorphism. We obtain a map $t \mapsto (t^k, F(t))$ where F is of class $\mathcal{C}^{2,1-} = \cap_{\alpha<1}\mathcal{C}^{1,\alpha}$, with $F(t) = O_{2,1-}(t^{k+1})$ (*cf.* [**Sik1**, proposition 4]). The fact that the image, viewed locally as a graph satisfies (1), means that F satisfies

$$\frac{\partial F}{\partial \bar{t}} = k\bar{t}^{k-1} h\Big(t^k, F(t), \frac{1}{kt^{k-1}}\frac{\partial F}{\partial t}\Big). \tag{5}$$

Note that this makes sense also at the origin.

Next, we show that the "difference" of two such germs satisfies the similarity principle. More precisely, we have the following generalization of [**Sik1**, prop. 5]:

Proposition. — *Assume that F and G satisfy* (5) *with the same value of k, and are not identical as germs. Then there exists $a \in \mathbb{C}^*$ and $\ell \in \mathbb{N}^*, \ell \geqslant k$ such that*

$$F(t) - G(t) = at^\ell + O_{1,1-}\left(t^{\ell+1}\right).$$

Proof. — Set $u = F - G$, and take the difference of the two equations on F and G. Using Taylor's integral formula and setting

$$\gamma(t,s) = \left(t^k, G(t) + su(t), \frac{1}{kt^{k-1}}\Big(\frac{\partial G}{\partial t} + s\frac{\partial u}{\partial t}\Big)\right),$$

we get

$$\frac{\partial u}{\partial \bar{t}} = A(t)\cdot u(t) + B(t).\frac{\partial u}{\partial t},$$

where

$$A(t) = \int_0^1 k\bar{t}^{k-1} D_2h(\gamma(t,s))ds, \quad B(t) = \int_0^1 k\bar{t}^{k-1} D_3h(\gamma(t,s))ds.$$

The properties of h, F and G imply that A and B are of class $\mathcal{C}^{1,1-}$, and $\|B\|_{L^\infty} < 1$. Then the proposition follows from a variant of proposition 2 in [**Sik1**]. □

Finally, one proceeds exactly as in [**Sik1**] (inspired by [**MW**]) to deduce from this proposition the

Proposition. — *Let E be a germ of elliptic structure on $\mathbb{C}^2$ near* 0 *such that the horizontal plane $H = \mathbb{C} \times \{0\}$ belongs to E_0. Let $f_i : (\mathbb{C}, 0) \to (\mathbb{C}^2, 0)$, $i = 1, \ldots, r$, be germs of E-maps, all tangent to H at* 0. *Then there exist*

– a local $\mathcal{C}^1$ diffeomorphism $\phi : (\mathbb{C}^2, 0) \mapsto (\mathbb{C}^2, 0)$, with support in an arbitrarily small sector

$$S_\varepsilon = \{(x,y) \in \mathbb{C}^2 \mid |y| \leqslant \varepsilon|x|\},$$

– local diffeomorphisms $u_i : (\mathbb{C}, 0) \mapsto (\mathbb{C}, 0)$, tangent to the identity,

such that all the maps $\phi \circ f_i \circ u_i$ are holomorphic.

If the tangents to the f_i are not the same, we cannot in general expect to find a differentiable chart on V in which the image becomes holomorphic: there is an obstruction already at the linear algebraic level. However, by superposing the diffeomorphisms given by the proposition we easily obtain a Lipschitz chart:

Theorem. — *Let $f_i : (\mathbb{C}, 0) \to (V, v)$, $i = 1, \ldots, r$, be germs of E-maps through the same point. Then there exists a germ of Lipschitz oriented homeomorphism $\phi : (V, v) \to (\mathbb{C}^2, 0)$ such that all the maps $\phi \circ f_i \circ u_i$ are holomorphic.*

Proof. — We may assume that $(V, v) = (\mathbb{C}^2, 0)$. Let (P_j), $j = 1, \ldots, r$, be the different tangent planes to the f_i at v, and let $I_j \subset \{1, \ldots, r\}$ be the indices corresponding to the branches with tangent P_j.

The P_j are not complex linear in general, but there exists a Lipschitz oriented homeomorphism h of $\mathbb{C}^2$ such that the $h(P_j)$ are complex linear, thus there exist a family of complex linear A_j such that

$$(\forall\, j) \quad A_j h(P_j) = \mathbb{C} \times \{0\}.$$

Furthermore, we may assume that h is smooth [even linear] on a sector S_j around P_j.

Then we can apply the proposition to $A_j h f_i$, $i \in I_j$: there exists a $\mathcal{C}^1$ diffeomorphism $\phi_j : \mathbb{C}^2 \to \mathbb{C}^2$, such that $\phi_j A_j h f_i$ is holomorphic with horizontal tangent for $i \in I_j$. Moreover, we may assume that the support of ϕ_j is contained in $A_j h(S_j)$.

Then the desired homeomorphism is given by

$$\phi = \begin{cases} A_j^{-1} \circ \phi_j \circ A_j \circ h \text{ on } S_j, \\ h \text{ elsewhere.} \end{cases} \qquad \square$$

From this theorem, one deduces the positivity of intersections. More precisely, one can define

– a local intersection index $(C, C')_v \in \mathbb{N}^*$ for two germs C and C' of E-cycles at the point v without common component. It is equal to 1 if and only if C and C' are smooth at v, with distinct tangents

– a local self-intersection number $\delta_v(C) \in \mathbb{N}$ (number of double points in a generic deformation equitopological at the source). It is equal to 0 if and only if C is smooth at v.

One then has the intersection and genus (or “adjunction”) formulas

Theorem

(i) *If C and C' are two E-cycles without common components, then $C \cap C'$ (intersection of the supports) is finite and the homological intersection is given by*

$$C \cdot C' = \sum_{v \in C \cap C'} (C, C')_v.$$

(ii) *If C is an irreducible E-curve of genus g, then it has a finite number of singularities, and its genus is given by*

$$g = \frac{C \cdot C - c_1(E) \cdot C}{2} + 1 - \sum_v \delta_v.$$

Assume that (V, E) is homologically equivalent to $(\mathbb{CP}^2, E_0)$. Thus there is a well defined degree map $H_2(V; \mathbb{Z}) \to \mathbb{Z}$ (an isomorphism modulo torsion, but not necessarily an isomorphism at this stage). Define an *E-line* as a primitive E-curve $C \subset V$ of degree 1. Then since an E-line satisfies $C.C = 1$ and $c_1(E) \cdot C = 3$, we get the

Corollary. — *Every E-line is an embedded sphere, and two distinct E-lines intersect transversely in one unique point.*

2.G. Linearization of the equation of E-curves; automatic genericity

We consider here embedded E-curves, *i.e.* smooth surfaces $S \subset V$ satisfying $\widetilde{S} \subset E$ where $\widetilde{S}$ is the Gaussian lift. Following [**G**, 2.4.E] we linearize this "equation" at S, obtaining an equation $\overline{\partial}_E f = 0$ where $\overline{\partial}_E$ acts on sections $f : S \to N = T_S V/TS$, the normal bundle, with values in $\Omega^{0,1}_J(S, N)$. Here $J = j^{\perp}_{TS}$ (*cf.* 1.A) is the natural complex structure on N.

One can obtain explicitly this equation by using the equation (2) in local coordinates. The compatibility of the complex structure on $\mathbb{C}^2$ with the structure on N means that $D_3 h(z, 0, 0) \equiv 0$. Thus the linearization of (2) has locally the form

$$\frac{\partial f}{\partial \overline{z}} - D_2 h \cdot f = 0,$$

i.e. the operator $\overline{\partial}_E$ has the form $\overline{\partial}_E = \overline{\partial} + R$ where $\overline{\partial}$ is associated to a holomorphic structure on N and R is of order 0. Thus one can apply to it the arguments of [**G**, 2.1.C] (*cf.* also [**HLS**]):

Proposition. — *If $c_1(N) > 2g - 2$, i.e. $c_1(E).S > 0$, then $\overline{\partial}_E$ is onto. Thus the space M_A of connected embedded E-curves in the class $A \in H_2(V; \mathbb{Z})$, if nonempty, is a smooth manifold if $c_1(E).A > 0$. Its real dimension is*

$$2(c_1(N) \cdot S + 1 - g) = 2A \cdot A + (c_1(E) \cdot A - A \cdot A) = A \cdot A + c_1(E) \cdot A.$$

Also, M_A is oriented since the homotopy $\ker(\overline{\partial} + R_t)$ gives it a natural homotopy class of almost complex structure.

Now assume that (V, E) is homologically equivalent to $(\mathbb{CP}^2, E_0)$. Then the space of E-lines is the disjoint union of the M_A for all $A \in H_2(V; \mathbb{Z})$ of degree 1. For such classes, we have $c_1(E) \cdot A = 3$ thus the proposition applies:

Corollary. — *The space of E-lines V^*, if nonempty, is naturally a smooth oriented 4-manifold.*

Extensions. — Since $c_1(E) \cdot A = 3$ we can still impose on S a condition of complex codimension 1 or 2, and keep the automatic genericity (*cf.* for instance [**B**]). In particular:

(i) Let $\mathcal{L}_v^*$ be the space of E-lines through a given $v \in V$: it is an oriented smooth surface in V^* when nonempty. Note that it can be identified with an open subset of the projective line $G_1^J(T_vV)$. Also, $\mathcal{L}_v^*$ depends smoothly on v.

(ii) Let $\mathcal{L}_{v,w}^*$ be the space of E-lines through two given points $v, w \in V$: when nonempty, it is a point $L_{v,w}$ which depends smoothly on (v, w). This is the case for some open subset $U_1 \subset V \times V \smallsetminus \Delta_V$.

(iii) Let $\mathcal{L}_P^* =$ be the space of E-lines with a given tangent plane $P \in E$: again, when nonempty, it is a point L_P which depends continuously on P. This is the case for some open subset $U_2 \subset E$.

3. Tame elliptic projective planes

By definition, a *tame elliptic projective plane* (V, E) is a 4-manifold equipped with a tame elliptic structure, homologically equivalent to $(\mathbb{CP}^2, E_0)$. Note that we do not require a priori V to be diffeomorphic to $\mathbb{CP}^2$.

3.A.

Proposition. — *Let (V, E) be a tame elliptic projective plane. Then*

(i) *by two distinct points $x, y \in V$ there passes a unique E-line $L_{v,w}$, and for any given $P \in E$ there exists a unique E-line L_P tangent to P.*

(ii) *V is oriented diffeomorphic to $\mathbb{CP}^2$, E is homotopic to E_0, and any taming ω is isomorphic to $\lambda\omega_0$ for some $\lambda > 0$.*

Proof. — Assume first that the space V^* of E-lines is nonempty.

(i) It suffices to prove that V^* is compact: this will imply that the open sets U_1 and U_2 defined at the end of 2.G are also closed, so $U_1 = V \times V \smallsetminus \Delta_V$ and $U_2 = E$, which proves (i).

The compactness of V^* will follow from the compactness theorem for $\widehat{J}$. First, there exists a taming Ω for $\widehat{J}$: as usual in the theory of symplectic bundles, we set $\Omega = \pi^*\omega + \alpha$ where α is a closed 2-form on E which is positive on every fiber. Such a form exists since $H^2(E) \to H^2(V)$ is onto: this is true since it holds for in the standard case $V = \mathbb{CP}^2$, $E = E_0$ and that our case is homologically standard.

Furthermore, let $A \in H_2(E; \mathbb{Z})$ be the homology class of the Gauss lift of E-lines. Then A is Ω-*indecomposable*, *i.e.* not equal to a sum $A = A_1 + A_2$ with $\omega(A_i) > 0$. This can again be seen in the standard situation [in that case, a holomorphic curve C in the class A is always a section $s(L)$ over a line L in $\mathbb{CP}^2$; if C is not the Gaussian lift of L, then there exists $v_0 \in \mathbb{CP}^2 \smallsetminus L$ such that $s(v)$ is the tangent to the line $[v_0v]$ for every $v \in L$].

The Ω-indecomposability and the compactness theorem of [G] imply that the space M of rational $\widehat{J}$-curves in the class A is compact, and since V^* is homeomorphic to a closed subset of M it is also compact.

(ii) Fix three lines L_0, L_1, L_∞. We deform E so that it remains tamed by ω, the L_i are still E-lines and E comes from a complex structure isomorphic to the standard one near L_∞^0. This is possible, using Darboux-Givental and the contraction $\mathcal{E}_\omega \to \mathcal{J}_\omega$.

We shall find a diffeomorphism $\phi : V \to \mathbb{CP}^2$ which sends them to the x-axis L_0^0, the y-axis L_1^0 and the line at infinity L_∞^0.

Let v_0, v_1 be the intersections $L_0 \cap L_\infty$, $L_1 \cap L_\infty$. Let $v \in V \setminus L_\infty$. Then the E-lines $v_0 v$ and $v_1 v$ meet L_0 and L_1 in $x(v)$ and $y(v)$ respectively. Identifying L_0 with L_0^0, L_1 with L_1^0, we define $\phi(v)$ to be the intersection of $v_0 x(v)$ and $v_1 y(v)$. We obtain thus a smooth map $\phi : V \setminus L_\infty \to \mathbb{CP}^2 \setminus L_\infty^0$

Exchanging the roles of V and $\mathbb{CP}^2$, we obtain $\psi : \mathbb{CP}^2 \setminus L_\infty^0 \to V \setminus L_\infty$ which is the inverse of ϕ. Since everything is standard near L_∞, one can extend ϕ to L_∞ and ψ to L_∞^0.

The fact that ω is isomorphic to $\lambda\omega_0$ results from Moser's lemma.

Finally, we prove that V^* is indeed nonempty. This follows from the almost complex case proved by Taubes [**T1**] [**T2**], since E can be deformed among tame elliptic structures to an almost complex structure, and V^* remains a fixed compact manifold during the deformation. □

3.B. Conversely, as shown by Gromov, one has the

Proposition ([**G**, 2.4.A']). — *Assume that V^* is compact and nonempty. Then there exists a taming symplectic form ω.*

Proof. — Using a positive volume form ν on V^* (identified with a smooth measure $d\nu$), define a 2-form ω by Crofton's formula:

$$\int_S \omega = \int_{V^*} \mathrm{Int}(S, L)\, d\nu(L)$$

for every oriented surface $S \subset V$. Here, $\mathrm{Int}(S, L)$ is the algebraic intersection number, which is defined for almost all $L \in V^*$. □

Let us give a more explicit definition of ω. First, fix $v \in V$ and denote by $\mathcal{L}_v^* \subset V^*$ the subset of E-lines containing a given $v \in V$, which is a submanifold diffeomorphic to $\mathbb{CP}^1$.

Proposition. — *There is a canonical isomorphism*

$$\nu_{v,L} : N_v L \longrightarrow N_L \mathcal{L}_v^*$$

between normal bundles.

Proof. — Choose an E-line $L^{\perp}$ different from L at v. If $\delta L \in T_L V^*$, let (L_t) be a path such that $\frac{d}{dt}_{t=0} L_t = \delta L$, and set

$$\phi(\delta L) = \left(\frac{d}{dt}\right)_{|t=0} (L_t \cap L^{\perp}) \in T_v V.$$

Dually, choose a point $w \in L$ different from v. If $\delta v \in T_v V$, let (v_t) be a path such that $\frac{d}{dt}_{t=0} v_t = \delta v$, and set

$$\psi(\delta v) = \left(\frac{d}{dt}\right)_{|t=0} L_{v_t, w} \in T_L V^*.$$

Then clearly, ϕ induces the desired isomorphism $\nu_{v,L}$ and ψ its inverse.

We can now define a morphism

$$s : T_v V \longrightarrow \Gamma(\mathcal{L}_v^*, N(\mathcal{L}_v^*, V^*))$$

by composing

$$T_v V \longrightarrow N_v L \xrightarrow{\nu_{v,L}} N_L \mathcal{L}_v^* \ , \quad L \in \mathcal{L}_v^*.$$

Thus for $X, Y \in T_v V$ and $L \in \mathcal{L}_v^*$, $s(X)(L)$ and $s(Y)(L)$ are elements of $N_L \mathcal{L}_v^*$. Lifting them to $\widetilde{X}, \widetilde{Y} \in T_L V^*$, we see that

$$\nu(\widetilde{X}, \widetilde{Y}) = \iota_{\widetilde{X}} \iota_{\widetilde{Y}} \nu \in \Lambda^2 T_L^* V^*$$

(interior products) is independent of the lifts. Varying $L \in \mathcal{L}_v^*$, we obtain a 2-form on $\mathcal{L}_v^*$ which we denote by $\nu(s(X), s(Y))$, and we set

$$\omega(X, Y) = \int_{\mathcal{L}_v^*} \nu(s(X), s(Y)).$$

It is easy to see that it is positive on E and satisfies Crofton's formula. This formula implies that ω is closed, which proves the proposition. $\square$

4. Proof of the main result

4.A. Definition of the dual structure. — We set

$$E^* = \{T_L \mathcal{L}_v^* \mid v \in V, L \in \mathcal{L}_v^*\}.$$

This is clearly a submanifold fibered over V^*, the fiber at L being

$$E_L^* = \{T_L \mathcal{L}_v^* \mid v \in L\}.$$

It is equipped with a natural distribution of codimension 2, $\Theta_{P^*} = d\pi^{*-1}(P^*)$. Note also that E^* is naturally diffeomorphic to E via $\phi : T_v L \mapsto T_L \mathcal{L}_v^*$, in fact both are naturally diffeomorphic to the incidence variety

$$I = \{(v, L) \in V \times V^* \mid v \in L\} = \{(v, L) \in V \times V^* \mid L \in \mathcal{L}_v^*\}.$$

This variety is equipped with two natural fibrations $p : I \to V$, $p^* : I \to V^*$. It also has one natural distribution. Indeed, by differentiating the condition $(v(t) \in L(t))$, one obtains the

Proposition. — *If* $(\delta v, \delta L) \in T_{v,L} I$, *then* $(\delta v \in T_v L \Leftrightarrow \delta L \in T_L \mathcal{L}_v^*)$.

Thus one can define the distribution $D \subset TI$ by

$$D_{v,L} = dp^{-1}(T_v L) = dp^{*-1}(T_L L_v^*).$$

We then have a commutative triangle

$$\begin{tikzcd} & (I,D) \arrow[dl, "\gamma"'] \arrow[dr, "\gamma^*"] & \\ (E,\Theta) \arrow[rr, "\phi"] & & (E^*,\Theta^*) \end{tikzcd}$$

where $\gamma(v,L) = T_v L$ and $\gamma^*(v,L) = T_L L_v^*$.

4.B. Proof that E^* is elliptic. — Let us fix (v,L) such that $v \in L$, and define

$$P = T_v L, \quad P^* = T_L \mathcal{L}_v^*.$$

Then there are natural embeddings

$$i : T_P E_v \longrightarrow \mathrm{Hom}(T_v L, N_v L),$$
$$i^* : T_{P^*} E_L^* \longrightarrow \mathrm{Hom}(T_L \mathcal{L}_v^*, N_L \mathcal{L}_v^*).$$

By the ellipticity of E, $i(p)$ is an oriented isomorphism if $p \neq 0$. We want to prove the same property for $i^*(p^*)$. This will follow from

– the existence of canonical isomorphisms $T_P E_v \approx P^*, : T_{P^*} E_L^* \approx P$, $N_v L \approx N_L \mathcal{L}_v^*$ (this last we know already); thus i and i^* become morphisms $P^* \to \mathrm{Hom}(P,N)$ and $P \to \mathrm{Hom}(P^*,N)$,

– the formula

$$i^*(p)(p^*) = i(p^*)(p).$$

To prove this, we define local charts on V, V^* and G:

1) We start with a chart $\Phi : V \to T_v L \times N_v L$, such that

$$\begin{cases} \Phi(L) = T_v L \times \{0\}, \\ \Phi(L^\perp) = \{0\} \times N_v L, \\ \mathrm{pr}_1 \circ d\Phi_{v|T_v L} = \mathrm{Id}, \\ \mathrm{pr}_2 \circ d\Phi_v = \text{natural projection}. \end{cases}$$

2) We define a chart $\Psi : V^* \to T_L \mathcal{L}_v^* \times N_L \mathcal{L}_v^*$ such that $\Psi^{-1}(\alpha, 0)$ passes through v and $\Psi^{-1}(0,\beta)$ is "horizontal". More precisely, $\Psi^{-1}(\alpha,\beta)$ is given in the chart Φ by an equation

$$y = f_{\alpha,\beta}(x), \quad x \in T_v L, \quad y \in N_v L,$$

such that

$$\begin{cases} f_{0,\beta}(x) = \nu_{v,L}(\beta), \\ f_{\alpha,0}(0) = 0, \\ \dfrac{\partial f_{\alpha,0}}{\partial x}(0) = i(\alpha). \end{cases}$$

In the last equation, $\alpha \in T_L\mathcal{L}_v^* = P^*$ is interpreted as an element of T_PE_v, so that $i(\alpha) \in \mathrm{Hom}(T_vL, N_vL)$.

3) Finally, let $P'^* \in \widetilde{\mathrm{Gr}}_2(T_LV^*)$ close to P^*, we define $\chi(P'^*) \in \mathrm{Hom}(T_L\mathcal{L}_v^*, N_L\mathcal{L}_v^*)$ as the unique h such that

$$d\Psi_L(P'^*) = \mathrm{graph}(h).$$

End of the proof. — Let $w = \Phi^{-1}(x,0)$ be an element of L close to v. Then $\Psi^{-1}(\alpha,\beta) \in \mathcal{L}_w^*$ if and only if $f_{\alpha,\beta}(x) = 0$, thus $T_L\mathcal{L}_w^*$ is given by

$$\left\{(\delta\alpha,\delta\beta) \mid \frac{\partial f_{\alpha,0}(x)}{\partial\alpha}|_{\alpha=0}.\delta\alpha + \frac{\partial f_{0,\beta}(x)}{\partial\beta}|_{\beta=0}.\delta\beta = 0\right\},$$

i.e.

$$\partial\frac{f_{\alpha,0}(x)}{\partial\alpha}|_{\alpha=0}.\delta\alpha + \nu_{v,L}(\delta\beta) = 0.$$

In other words

$$\chi(T_L\mathcal{L}_w^*) = -\nu_{v,L}^{-1}\left(\frac{\partial f_{\alpha,0}(x)}{\partial\alpha}|_{\alpha=0}\right).$$

Thus, the tangent space of E_L^* at $T_L\mathcal{L}_v^*$ is identified with the image of the morphism

$$i^* = \nu_{v,L}^{-1} \circ \frac{\partial^2 f_{\alpha,0}(x)}{\partial x\partial\alpha}|_{(\alpha,x)=(0,0)} : T_vL \longrightarrow \mathrm{Hom}(T_L\mathcal{L}_v^*, N_L\mathcal{L}_v^*).$$

Since $\dfrac{\partial f_{\alpha,0}(x)}{\partial x}|_{x=0} = i(\alpha)$, one has

$$\frac{\partial^2 f_{\alpha,0}(x)}{\partial x\partial\alpha}|_{(\alpha,x)=(0,0)} = i,$$

thus

$$i^*(\xi)(\delta\alpha) = i(\delta\alpha)(\xi), \quad (\xi,\delta\alpha) \in T_vL \times T_L\mathcal{L}_v^*.$$

Since E_v is elliptic, $i(\delta\alpha)$ is invertible and orientation-preserving if $\delta\alpha \neq 0$. Thus one can identify the oriented planes $T_L\mathcal{L}_v^*$, T_vL and $N_L\mathcal{L}_v^*$ with $\mathbb{C}$ so that $i(\delta\alpha)$ is the multiplication by $\delta\alpha$. Then $i^*(\xi)$ is the multiplication by ξ, thus it is invertible and orientation-preserving if $\xi \neq 0$, which means that E_L^* is elliptic. □

4.C. Proof that V^* is oriented diffeomorphic to $\mathbb{CP}^2$. — One could prove it similarly to the proof for V. The simplest proof however is to remark that the space of elliptic structures on $V = \mathbb{CP}^2$ which are tamed by ω_0 is contractible. For each E in this space, we obtain an oriented manifold V_E^* which varies smoothly with E, thus keeps the same oriented diffeomorphism type. Since for E_0 associated to J_0 one has $V_{E_0}^* = \mathbb{CP}^{2*}$, the standard dual projective plane, this proves the result.

4.D. Tameness of E^* and identification $(V^*)^* = V$. — For each $v \in V$, the surface $\mathcal{L}_v^* \subset V^*$ is an E^*-curve of degree 1, *i.e.* an E^*-line. Moreover, for two distinct points $L, L' \in V^*$ there exists a unique $v \in L \cap L'$, equivalently a unique $\mathcal{L}_v^*$ containing L and L': this means that the E^*-lines are precisely the $\mathcal{L}_v^*$, and thus that E^* is tame and $V^{**} = V$. The equivalence $(v \in L \Leftrightarrow L \in \mathcal{L}_v^*)$ implies that E^{**} is identified to E.

4.E. Dual curves. — Let $\mathcal{J}$ be the restriction to TI of (J, J^*), where J and J^* are the twisted almost complex structures associated to E and E^*: it is an almost complex structure, whose images by γ and γ^* (notations of 4.A) are $\widetilde{J}$ and $\widetilde{J}^*$, the complex structures on Θ and Θ^* associated to E and E^*. Thus the map $\phi : (E, \Theta) \to (E^*, \Theta^*)$ is a $(\widetilde{J}, \widetilde{J}^*)$-biholomorphism.

Now let $C = f(S) \subset V$ be an irreducible E-curve (or an irreducible germ) not contained in an E-line. Let $\gamma : S \to E$ be the Gauss map, which is $\widetilde{J}$-holomorphic. Then $\gamma^* = \phi \circ \gamma : S \to E^*$ is $\widetilde{J}$-holomorphic and not locally constant, thus it is the Gauss map of an E^*-map $f^* : S \to V^*$. By definition, $C^* = f^*(S)$ is the *dual* curve of C: it is again an irreducible E^*-curve (or germ), not contained in an E^*-line, and of course one has $C^{**} = C$.

5. Nonlinearity of the elliptic structure on V^*

Here we construct a tame almost complex structure J on $V = \mathbb{CP}^2$ such that the elliptic structure E^* on V^* is non linear. Equivalently, the twisted almost complex structure is non linear.

In fact, McKay [**McK1**] proved that it is always the case if J is non integrable, thus solving a conjecture that I had made in the preprint. More precisely: *if J_L^* is linear, then the Nijenhuis torsion of J vanishes on L.* In particular, if E and E^* are both linear, then E is isomorphic to the standard elliptic structure on $\mathbb{CP}^2$, associated to the standard complex structure.

His proof uses the theory of exterior differential systems to define invariants whose vanishing characterizes the linearity or the integrability. Presumably, an example of the type given below could be shown to exist always as soon as J is non integrable, and thus we would obtain a more concrete proof of the result of McKay.

In our example, we impose on J the following properties:

– it is standard outside

$$U_0 = (\Delta(2) \smallsetminus \Delta(1)) \times \Delta(2) \subset \mathbb{C}^2 \subset \mathbb{CP}^2,$$

– it is ω_0-positive,

– for $\alpha \in \mathbb{C}$ small enough, the J-line $L(\alpha, \alpha)$ passing through the point $(0, \alpha)$ with the slope α has an intersection with U_0 given by the equation

$$y = f_{\alpha,\alpha}(x) = \alpha + \alpha x + \frac{1}{5}\rho(x) \cdot \overline{\alpha} x^2,$$

where $\rho : \mathbb{C} \to [0,1]$ takes the value 1 on $U_1 = \Delta(1) \times \Delta(1)$ and 0 outside U_0. Note that the factor $1/5$ guarantees that $\alpha \mapsto f_{\alpha,\alpha}(x)$ is an embedding for $|x| < 2$ near 0.

One can find such a J under the form

$$J(x,y) = \begin{pmatrix} i & 0 \\ b(x,y)\sigma & i \end{pmatrix}$$

where $b(x,y) \in \mathbb{C}$ and σ is the complex conjugation. Then $L(\alpha,\alpha) \cap U_0$ is J-holomorphic if and only if

$$b(x, f_{\alpha,\alpha}(x)) = \frac{\partial f_{\alpha,\alpha}}{\partial \overline{x}}(x).$$

Since $\alpha \mapsto f_{\alpha,\alpha}(x)$ is an embedding near 0 for $|x| < 2$ and the second member vanishes for $|x|$ close to 2, one can find a smooth $b(x,y)$ with support in U_0, satisfying the above equality for $|x| < 2$ and α small enough.

We now prove that J_L^* is not linear. Note that on U_1, we have $J = J_0$ and $L(\alpha,\alpha) \cap U_1$ is given by

$$y = f_{\alpha,\alpha}(x) = \alpha + \alpha x + \frac{1}{5}\overline{\alpha} x^2.$$

Let L be the J-line $L(0,0)$, which is the x-axis. Recall that for each $v \in L$ the subspace $T_L\mathcal{L}_v^* \subset T_L V^*$ is preserved by J_L^*, which is linear on it. The global linearity of J_L^* is equivalent to the following:

$$(\forall\, \xi, \eta \in T_L V^*) \quad \xi + \eta \in T_L\mathcal{L}_v^* \Longrightarrow J_L^*(\xi) + J_L^*(\eta) \in T_L\mathcal{L}_v^*.$$

Consider on L the points $v_0 = 0$ and $v_1 = \infty$. Then we have a direct sum $T_L V^* = T_L\mathcal{L}_0^* \oplus T_L\mathcal{L}_\infty^*$. Then fix $\alpha \neq 0$ and consider the path $t \in [0,1] \mapsto \gamma(t) = L(t\alpha, t\alpha) \in V^*$, and write its derivative at $t = 0$ as

$$\dot{\gamma} = \xi + \eta, \quad \xi \in T_L\mathcal{L}_0^*, \quad T_L\mathcal{L}_\infty^*.$$

It belongs to $T_L\mathcal{L}_v^*$ where $v = \lim_{t\to 0}(\gamma(t) \cap L)$. Identifying L with $\mathbb{CP}^1$, this means that v is the solution of the equation

$$\alpha + \alpha v + \frac{1}{5}\overline{\alpha} v^2 = 0.$$

If we change α to $i\alpha$, ξ and η are changed to $J^*(\xi)$ and $J^*(\eta)$ (essentially since J_0 is standard near 0 and ∞), thus $J_L^*(\xi) + J_L^*(\eta) \in T_L\mathcal{L}_w^*$ where w is the solution of

$$\alpha + \alpha w - \frac{1}{5}\overline{\alpha} w^2 = 0.$$

Thus $w \neq v$, which means that J_L^* is not linear.

6. Plücker formulas for E-curves

We follow the classical topological method in algebraic geometry, *cf.* for instance [**GH**, p. 279].

Let $C = f(S) \subset V$ be an irreducible E-curve, not contained in an E-line, and let $C^* \subset V^*$ be its dual. We compute the degree d^* of C^*, which is the number of intersection points of C^* with an E^*-line, *i.e.* the number of points of C such that the tangent line L_vC contains v. This number is to be interpreted algebraically, but for a generic v it is equal to the set-theoretic number.

Let L be an E-line disjoint from v, then the central projection $V \setminus \{v\} \to L$ along E-lines through v induces an "almost holomorphic" branched covering $C \to L$ of degree d, in the sense that each singularity has a model $z \to z^k$: this is a consequence of the positivity of intersections. Let S be the normalization of C, then the number of branch points of the induced covering $S \to L$ is $d^* + \kappa$ where κ is the algebraic number of cusps, *i.e.* the algebraic number of zeros of df if f is a parametrization of C. Thus we have the Hurwitz formula $2 - 2g = 2d - (d^* + \kappa)$, where g is the genus of C, *i.e.*

$$d^* = 2d + 2g - 2 - \kappa.$$

In particular, if C has only δ nodes and κ cusps, we have $2g - 2 = d(d-3) - 2\delta - 2\kappa$ thus we get the first Plücker formula

$$d^* = d(d-1) - 2\delta - 3\kappa.$$

As in the classical case, the other Plücker formulas follow from this and the genus formula, with the fact that an ordinary bitangent (resp. flex) of C corresponds to a node (resp. cusp) of C^*.

This implies restrictions on the possible sets of singularities going beyond the genus formula. For instance, if C has only nodes and cusps, then another form of Plücker formula is

$$\kappa = 2g - 2 + 2d - d^*.$$

If $d = 5$ and $g = 0$ we get $\kappa = 8 - d^*$, and since $d^* \geqslant 3$ we have $\kappa \leqslant 5$: not all 6 nodes of a generic rational curve can be transformed to cusps.

In general, if C is rational with only nodes and cusps, we get $\kappa = 2d - 2 - d^* < 3d$, which implies that the space of rational J-curves is, at the point C, a smooth manifold of the expected dimension (equal to $d(d+3)$ over $\mathbb{R}$): this follows from the generalization of the automatic genericity proved in [**B**].

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J.-C. Sikorav, École Normale Supérieure de Lyon, Unité de Mathématiques Pures et Appliquées, 46, allée d'Italie, F-69364 Lyon Cedex 7, France • *E-mail :* `sikorav@umpa.ens-lyon.fr`