ADE SURFACE SINGULARITIES, CHAMBERS AND TORIC VARIETIES

by

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Abstract. — We study the link between the positive divisors supported on the exceptional divisor of the minimal resolution of a rational double point and the root systems of Dynkin diagrams. Then, we calculate the toric variety corresponding to the fundamental Weyl chamber.

Résumé (Singularités ADE des surfaces, chambres et variétés toriques). — Nous étudions le lien entre les diviseurs positifs à support sur le diviseur exceptionnel de la résolution minimale d’un point double rationnel et les systèmes de racine des diagrammes de Dynkin. Puis, nous calculons la variété torique correspondant à la chambre fondamentale de Weyl.

1. Introduction

A singularity of a normal analytic surface is rational if the geometric genus of the surface doesn’t change by a resolution of the singularity. These singularities are rather simple among surface singularities since they are absolutely isolated and their resolutions have some nice combinatoric properties. A classification of rational singularities is done by the dual graph of the minimal resolution according to their multiplicities (see [11] for details and related references).

First, DuVal observed that the dual graph of the minimal resolution of a rational singularity of multiplicity 2, called rational double point, with algebraically closed field is one of the Dynkin diagrams $A_n$, $D_n$, $E_6$, $E_7$ and $E_8$, briefly ADE diagrams (see [2] or [4]). This means that the intersection matrix associated to the dual graph of the minimal resolution of a rational double point is the same as the Cartan matrix of the corresponding Dynkin diagram.

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The negative definiteness of the intersection matrix of the exceptional divisor of a resolution of a normal surface singularity permits us to study on a set of certain positive divisors supported on the exceptional divisor, which will be called the semigroup of Lipman. By using this set, we can associate a toric variety with a weighted graph whose intersection matrix is negative definite (see [1]).

In this work, motivated by a question appeared in [9], we give a geometric construction of the roots of an ADE diagram, listed in [3] (see Planche I,IV,VI,VII). Following [14], we observe that the semigroup of Lipman associated with an ADE diagram is the same as the fundamental Weyl chamber of the corresponding root system. In the last section, using [1], we describe the toric variety corresponding to the fundamental Weyl chamber of an ADE diagram (see [1], [15]).

2. Rational Singularities

Let $S$ be a germ at $\xi$ of a complex two dimensional normal space with a singularity at $\xi$. A resolution of $S$ is a complex nonsingular surface with a proper map $\pi : X \to S$ such that its restriction to $X - \pi^{-1}(\xi)$ is an isomorphism and $X - \pi^{-1}(\xi)$ is dense in $X$. A resolution $\pi : X \to S$ is called minimal resolution if any other resolution $\pi' : X' \to S$ factorizes by $\pi$. It is well known that the exceptional divisor $E = \pi^{-1}(\xi)$ of $\pi$ is connected and of dimension 1 (see [7], theorem V.5.2). Let us denote by $E_1, \ldots, E_n$ the irreducible components of $E$. The intersection matrix $M(E)$ associated with $E$ is defined by the intersection $(E_i \cdot E_j)$ of the components $E_i$ and $E_j$, which is the intersection number of $E_i$ and $E_j$ if $i \neq j$, and the first Chern class of the normal bundle to $E_i$ if $i = j$. It is a negative definite matrix (see [13]).

Let $G$ denote the free abelian group generated by the irreducible components of $E$:

$$G = \{ \sum_{i=1}^{n} m_i E_i, \ m_i \in \mathbb{Z} \}.$$ 

The elements of $G$ are called the divisors supported on $E$. The support of a divisor $Y = \sum m_i E_i$ is the set of the components for which $m_i \neq 0$. In the free abelian group $G$, the intersection matrix $M(E)$ defines a symmetric bilinear form. We shall denote $(Y \cdot Z)$ the value of this bilinear form on a pair $(Y, Z)$ of elements in $G$. An element of $G$ in which all the coefficients are non-negative and at least one is positive, is called a positive divisor.

**Theorem 2.1 (see [2]).** — The singularity $\xi$ of $S$ is a rational singularity if and only if the arithmetic genus $\frac{1}{2}(Y \cdot Y + \sum_{i=1}^{n} m_i (w_i - 2)) + 1$ of each positive divisor $Y = \sum_{i=1}^{n} m_i E_i$ in $G$ is $\leq 0$ where $w_i = -(E_i \cdot E_i)$.

Assume that $\pi : X \to S$ is a resolution of a normal surface singularity which is not necessarily rational. Let $f$ be an element of the maximal ideal $M$ of $O_{S, \xi}$. Then the divisor $(\pi^*f)$ of $f$ on $X$ is written as $(\pi^*f) = Y + T_f$ where $Y$ is a positive divisor supported on the exceptional divisor $E$ of $\pi$ and $T_f$, called the strict transform of $f$.
by \( \pi \), intersects \( E \) in finitely many point at most. Since \( ((\pi^* f) \cdot E_i) = 0 \) for all \( i \), we obtain \( (Y \cdot E_i) \leq 0 \) for all \( i \). The inverse is true when the singularity \( \xi \) is rational. We mean that, if \( Y \) is a positive divisor on \( X \) such that \( (Y \cdot E_i) = -(T \cdot E_i) \) for all \( i \), then there exists a function \( f \) in \( \mathcal{M} \) such that \( \pi^* f = Y + T \) (see [2]). Now, as in [12] (see section 18), let us consider the set

\[
\mathcal{E}^+(E) = \{ Y \in \mathcal{G} \mid (Y \cdot E_i) \leq 0 \text{ for all } i \}
\]

By [18], this set is not empty. It is an additive semigroup: For \( Y_1, Y_2 \in \mathcal{E}^+(E) \), we have \( Y_1 + Y_2 \in \mathcal{E}^+(E) \).

**Definition 2.2.** — The set \( \mathcal{E}^+(E) \) is called the semigroup of Lipman.

Since \( E \) is connected, for all \( Y = \sum m_i E_i \) in \( \mathcal{E}^+(E) \), we have \( m_i \geq 1 \) for all \( i \). A partial order on \( \mathcal{E}^+(E) \) is defined as follows: For two elements \( Y_1 = \sum_{i=1}^n a_i E_i \) and \( Y_2 = \sum_{i=1}^n b_i E_i \) of \( \mathcal{E}^+(E) \), we say \( Y_1 \leq Y_2 \) if \( a_i \leq b_i \) for all \( i \). The smallest element of this set is called the fundamental cycle of the resolution \( \pi \). The proposition 4.1 in [10], gives the following algorithm to construct the fundamental cycle of a given \( E \):

Let us denote by \( Z \) the fundamental cycle of \( \pi \). Consider \( Z_1 = \sum_{i=1}^n E_i \). If \( (Z_1 \cdot E_i) \leq 0 \) for all \( i \), then \( Z_1 = Z \). If else, there exists an \( E_i \) such that \( (Z_1 \cdot E_i) > 0 \); in this case, we put \( Z_2 = Z_1 + E_i \) and we see whether \( (Z_2 \cdot E_i) \leq 0 \) for all \( i \). The term \( Z_j \) (\( j \geq 1 \)), of the sequence satisfies, either \( (Z_j \cdot E_i) \leq 0 \) for all \( i \), then we put \( Z = Z_j \), or there is an irreducible component \( E_i \) such that \( (Z_j \cdot E_i) > 0 \); then we put \( Z_{j+1} = Z_j + E_i \). Thus the fundamental cycle of \( \pi \) is the first cycle \( Z_k \) of this sequence such that \( (Z_k \cdot E_i) \leq 0 \) for all \( i \). By the same method, we can construct all other elements of \( \mathcal{E}^+(E) \) (see [14] or [17]).

The following result of Artin characterize what an exceptional divisor of a resolution of a rational singularity looks like:

**Theorem 2.3 (see [2]).** — A singularity of a normal analytic surface in \( \mathbb{C}^N \) is rational if and only if the arithmetic genus of the fundamental cycle of the exceptional divisor of a resolution of the singularity vanishes.

This gives:

**Corollary 2.4 (see [2]).** — The exceptional divisor of any resolution of a rational singularity is normal crossing, with each \( E_i \) nonsingular and of genus zero, and any two distinct components intersect transversally at most in one point.

A proof of this corollary can be found also in [17].

Then the dual graph associated with the exceptional divisor of a resolution of a rational singularity, in which each \( E_i \) is represented by a vertex and each intersection point is represented by an edge between the vertices corresponding to the intersecting components, is a tree. Each vertex in the dual graph is weighted by \( -(E_i \cdot E_i) \). Conversely, with a given weighted graph, by plumbing, we can associate a configuration.
of curves embedded in a nonsingular surface and, if such a configuration of curves satisfies theorem 2.3, its contraction gives a rational singularity of a normal analytic surface (see \[6\], \[11\]).

**Example 2.5.** — A configuration of curves associated with an ADE diagram is contracted to a rational singularity of a normal analytic surface.

Moreover, we have:

**Proposition 2.6 (see [2]).** — Let \(\pi : X \rightarrow S\) be the minimal resolution of the rational singularity \(\xi\) of \(S\). Then the multiplicity of \(S\) at \(\xi\) equals \(-(Z \cdot Z)\) where \(Z\) is the fundamental cycle of \(\pi\).

Recall that the minimal resolution is characterized by \((E_i \cdot E_i) \leq -2\) for all irreducible components \(E_i\) of the exceptional divisor. A rational double point is a rational singularity for which the fundamental cycle of the minimal resolution satisfies \((Z \cdot Z) = -2\). We know that a rational double point of a surface is defined by the power series with the form \(f(x, y) + z^2 = 0\). By using the results given above, we deduce:

**Proposition 2.7 (see [2] or [4]).** — A normal analytic surface singularity is a rational double point if and only if the exceptional divisor of the minimal resolution of the singularity is a configuration of curves associated with one of the ADE diagrams.

**3. Root systems of rational double points**

There is a well known construction of ADE diagrams starting from a semisimple Lie algebra. In this section, we are interested in the inverse of that construction, as suggested in [9]. We will see that, using the geometry of a Dynkin diagram, we can obtain the roots of the corresponding semisimple Lie algebra. This gives a partial answer to the question of Ito and Nakamura (see [9], p. 194).

Let \(V\) be an euclidean space endowed with a positive definite symmetric bilinear form \((,\)\). A reflection \(s\) on \(V\) is an orthogonal transformation \(s : V \rightarrow V\) such that, for \(v \in V\), \(s(v) = -v\) and it fixes pointwise the hyperplane \(H_v = \{ u \in V | (u, v) = 0\}\) of \(V\). We can describe the reflection by the formula \(s_v(u) = u - \frac{2(u, v)}{(v, v)}v\).

**Definition 3.1.** — A subset \(R\) of \(V\) is called a root system if

(i) it is finite, generates \(V\) and doesn’t contain 0,
(ii) for every \(v \in R\), there exists a unique reflection \(s_v\) such that \(s_v(R) = R\),
(iii) for every \(v \in R\), the only multiples of \(v\) in \(R\) are \(\pm v\),
(iv) for \(u, v \in R\), we have \(\frac{2(u, v)}{(v, v)} \in \mathbb{Z}\).

The finite group generated by the reflections is called the Weyl group. See [8] for more details.
In what follows, $E$ will denote a configuration of curves associated with an ADE diagram, called ADE configuration. Now, following [14], (see p.158), we want to establish the relation between the root systems and the semigroup of Lipman of $E$. Denote by $E_1, \ldots , E_n$ the irreducible components of $E$. We know that $(E_i \cdot E_j)$ equals $-2$ if $i = j$ and equals 0 or 1 if $i \neq j$ (see [4] or [12]). Now, consider the following subset of $\mathcal{G}$:

$$R(E) = \{ Y \in \mathcal{G} \mid (Y \cdot Y) = -2 \}.$$

**Proposition 3.2 (see [14]).** — The set $R(E)$ is a root system.

Replacing the inner product in the definition above by the symmetric bilinear form defined by the intersection matrix $M(E)$, we can see that $R(E)$ satisfies the conditions of the definition above.

We will call root divisors the elements of $R(E)$. By definition, $E_1, \ldots , E_n$ and $-E_1, \ldots , -E_n$ are root divisors but $E_i - E_j$ is not a root divisor since $(E_i - E_j \cdot E_i - E_j) \neq -2$ for any $i \neq j$. Let us denote $B = \{ E_1, \ldots , E_n \}$. We can see that $B$ is a vector space basis of $R(E)$ in $\mathcal{G} \otimes \mathbb{R}$ and every element $Y$ in $R(E)$ can be written as the sum of $E_i$’s with coefficients all nonnegative or all nonpositive (compare with [8], pp. 47-48). If we denote by $R^+(E)$ the set of the elements of $R(E)$ with coefficients all nonnegative, then we have $R(E) = R^+(E) \cup (-R^+(E))$.

**Proposition 3.3 (see [14]).** — Let $Z = \sum_{i=1}^n a_i E_i$ be the fundamental cycle of $E$. Then, for each root divisor $Y = \sum_{i=1}^n m_i E_i$ in $R(E)$, we have $m_1 \leq a_1, \ldots , m_n \leq a_n$.

The fundamental cycle is called the highest (or biggest) root divisor in $R(E)$.

**Proof.** — Since $E$ is the exceptional divisor of the minimal resolution of a rational double point, we have $(Z \cdot Z) = -2$. So $Z \in R(E)$. Assume that there is a positive divisor $Y$ in $R(E)$ such that $Y > Z$ and $(Y \cdot Y) = -2$. So we have $Y = Z + D$ where $D$ is a positive divisor. This gives $(Y \cdot Y) = (Z \cdot Z) + 2(Z \cdot D) + (D \cdot D)$. Thus $2(Z \cdot D) = -(D \cdot D)$. Since $Z$ is the fundamental cycle, we have $(Z \cdot E_i) \leq 0$ for all $i$, so $(Z \cdot D) \leq 0$. This implies $D = 0$. \hfill $\square$

Hence, we can calculate the highest root divisor by the algorithm of Laufer given in the preceding section. The following proposition gives an algorithm to construct all elements of $R(E)$ from $Z$ by using $B$:

**Theorem 3.4.** — Let $R^+(E) = \{ Y_0, \ldots , Y_k \}$ with $Y_k = Z$. Then, for each $j = 0, \ldots , k - 1$, there exists an element $Y_i$ in $R^+(E)$ such that $(Y_i \cdot E_i) = k_i < 0$ and $Y_j = Y_i + k_i E_i$ for some $i$. Inversely, for each $E_i$ in $B$ such that $(Y_i \cdot E_i) = k_i < 0$, $Y_i + k_i E_i$ is a root divisor in $R(E)$.

**Proof.** — The existence of at least one irreducible component $E_i$ in each $Y_j$ such that $(Y_j \cdot E_i) < 0$ is due to negative definiteness of the intersection matrix. Then, theorem follows from the fact that $(Y_i + k_i E_i) \cdot (Y_i + k_i E_i) = -2$. \hfill $\square$
(Compare the root divisors obtained by the theorem with the roots given in [3] (see Planche I,IV,V,VI,VII.).)

In particular, we have:

**Corollary 3.5.** — The divisor $Y = \sum_{i=1}^{n} E_i$ (i.e. $m_i = 1$ for all $i$) is an element of $R^+(E)$.

**Proof.** — It follows from theorem 3.4. □

Now, for each $E_i \in B$, consider the hyperplane $H_i = \{ P \in \mathbb{R}^n \mid (P \cdot E_i) = 0 \}$. It divides $\mathbb{R}^n$ into two parts such that:

$$H_i^+ := \{ D \in \mathbb{R}^n \mid (D \cdot E_i) > 0 \} \quad \text{and} \quad H_i^- := \{ D \in \mathbb{R}^n \mid (D \cdot E_i) < 0 \}.$$

We have $H_i^+ = -H_i^-$. A connected component of $\mathbb{R}^n - \bigcup_{i=1}^{n} H_i$ is called a (Weyl) chamber and the chamber defined by $C(E) := \bigcap_{E_i \in B} (H_i^-)$ is called the fundamental (Weyl) chamber (see [8]). Thus the closure of $C(E)$,

$$\overline{C}(E) := \{ D \in \mathbb{R}^n \mid (D \cdot E_i) \leq 0 \},$$

is a closed convex cone. Then:

**Remark 3.6.** — Let $E$ be an ADE configuration. The semigroup $E^+(E)$ of Lipman is the fundamental chamber $\overline{C}(E)$. In particular, the highest root of $R(E)$ belongs to $\overline{C}(E)$.

### 4. Toric varieties

The fundamental chamber, or equivalently the semigroup of Lipman, of an ADE configuration defines a polyhedral cone in $\mathbb{R}^n$. In this section, by using [1], we will construct the toric variety corresponding to that cone.

We start by recalling what a toric variety is. Let $N$ be a lattice which is isomorphic to $\mathbb{Z}^n$. Let $\sigma$ be a rational polyhedral cone in the real vector space $N_{\mathbb{R}} = N \otimes \mathbb{R}$ which contains no line through the origin. Denote by $M = \text{Hom}(N, \mathbb{Z})$ the dual lattice of $N$. The dual cone $\hat{\sigma}$ is the set of vectors in $M_{\mathbb{R}}$ which are nonnegative on $\sigma$. The semigroup $S_{\sigma} := \hat{\sigma} \cap M = \{ u \in M \mid (u, v) \geq 0, \text{ for all } v \}$ is finitely generated. We denote by $\chi^u$ the element in the algebra $\mathbb{C}[S_{\sigma}]$ corresponding to the element $u$ of $S_{\sigma}$.

Each element of $\mathbb{C}[S_{\sigma}]$ is in the form of a finite sum $\sum a_i \chi^{u_i}$ for $a_i \in \mathbb{C}$ and $u_i \in S_{\sigma}$.

The variety $\text{Spec} \mathbb{C}[S_{\sigma}]$ is an affine toric variety (see [5] for more details).

Here we want to find the toric variety $\text{Spec} \mathbb{C}[\hat{\sigma} \cap M]$ when $\sigma$ is defined by $\overline{C}(E)$ where $E$ is an ADE configuration. Notice that $\overline{C}(E)$ satisfies the conditions on the cone by which we construct an affine toric variety above. In order to construct the toric variety corresponding to $\overline{C}(E)$, we first need to find the generators of $\overline{C}(E)$, which are the generators of $E^+(E)$: Consider $F'_i$ such that $(F'_i \cdot E_j) = -\delta_{ij}$. We obtain $F'_i = \sum_{i=1}^{n} m_{ij} E_j$ with $m_{ij} \in \mathbb{Q}^+$. The divisor $F_i$ such that $F'_i = k_i \cdot F_i$ is a
positive divisor where \(k_i\) denotes the least common factor of the denominators of the coefficients \(m_{ij}\), \((j = 1, \ldots, n)\).

**Theorem 4.1 (see [1]).** — With the preceding notation, \(F_1, \ldots, F_n\) belong to \(\mathcal{E}^+(E)\) and they generate the cone \(\mathcal{E}^+(E)\) over \(\mathbb{Q}^+\).

**Proof.** — By construction, \(F_1, \ldots, F_n\) belong to \(\mathcal{E}^+(E)\). We will show that each element in \(\mathcal{E}^+(E)\) can be written as a linear combination of the elements \(F_i\)'s with coefficients in \(\mathbb{Q}^+\).

Let \(G\) be the semigroup generated by \(F_1, \ldots, F_n\) with coefficients in \(\mathbb{Q}^+\) and \(\mathcal{G}\) be as defined before. We need to show that \(\mathcal{E}^+(E) = \mathcal{G} \cap G\): Let \(Y = \sum_{i=1}^{n} m_i E_i\) be an element of \(\mathcal{E}^+(E)\). Consider \(M(E) \cdot (m_1, \ldots, m_n)^t = (y_1, \ldots, y_n)^t\) where \(M(E)\) is the intersection matrix (same as the Cartan matrix multiplied by \(-1\)) of \(E\). Notice that \((Y \cdot E_i) = y_i\) so \(y_i < 0\) for all \(i\). Let \(D = \sum_{i=1}^{n} d_i E_i\) be an element of \(G\). So, \(d_i \in \mathbb{Q}^+\) for all \(i\). Assume \(M(E) \cdot (d_1, \ldots, d_n)^t = (0, \ldots, 0, -1, 0, \ldots, 0)^t\) where the entry \(-1\) is in the \(i\)-th row. The fact that \(M(E)\) is an invertible matrix gives \(Y = -\sum_{i=1}^{n} d_i y_i E_i\). So, the coefficient \(-d_i y_i\) is in \(\mathbb{Q}^+\) for all \(i\). This says \(Y \in \mathcal{G} \cap G\).

Now we will see the inclusion \(\mathcal{G} \cap G \subset \mathcal{E}^+(E)\): Let \(D \in \mathcal{G} \cap G\). This means \(D = \sum_{j=1}^{n} b_j F_j\) with \(b_j \in \mathbb{Q}^+\). Consider \((D \cdot E_i) = \sum_{j=1}^{n} b_j (F_j \cdot E_i)\). Since \(F_j\), \((j = 1, \ldots, n)\), is an element of \(\mathcal{E}^+(E)\) and \(b_j \in \mathbb{Q}^+\) for all \(j\), we have \((D \cdot E_i) \leq 0\) for all \(i\). Hence \(D \in \mathcal{E}^+(E)\). Then \(\mathcal{E}^+(E) = \mathcal{G} \cap G\). □

**Definition 4.2.** — The elements \(F_1, \ldots, F_n\) are called the generators of \(\mathcal{E}^+(E)\) (or \(\mathcal{C}(E)\)).

Now, let \(N\) be a lattice generated by \(E_1, \ldots, E_n\) and \(M\) be its dual lattice generated by \(E_1^*, \ldots, E_n^*\) such that \((E_i^* \cdot E_j) = \delta_{ij}\). Let \(N'\) be the lattice generated by \(F_1, \ldots, F_n\) and \(M'\) be its dual lattice generated by \(F_1^*, \ldots, F_n^*\) such that \((F_i^* \cdot F_j) = \delta_{ij}\). Since \(N'\) is a subgroup of \(N\) of finite index, we have:

**Theorem 4.3 (see [5]).** — With preceding notation, we have \(\mathbb{C}[^* \cap M] = \mathbb{C}[M']^{N/N'}\).

This means that the affine toric variety \(\text{Spec} \mathbb{C}[^* \cap M]\) is the quotient \(\mathbb{C}^n/G\) where \(G\) is the finite group \(N/N'\).

Now, let us see the construction method of the affine toric variety corresponding to \(\mathcal{C}(E)\) when \(E\) is associated with the diagram \(A_2\). For this, it is enough to describe the finite group \(N/N'\) and to see the action of this group on \(\mathbb{C}[M']\): It is well known that the intersection matrix \(M(E)\) associated with \(A_2\) is \(\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}\). From the formula \((F_i^* \cdot F_j) = -\delta_{ij}\) given above, we find the generators of \(\mathcal{E}^+(E)\) as \(F_1 = 2E_1 + E_2\) and \(F_2 = E_1 + 2E_2\).

Consider the lattice \(N = \langle E_1, E_2 \rangle\) and its sublattice \(N' = \langle F_1, F_2 \rangle\) and, denote by \(M = \langle E_1^*, E_2^* \rangle\) and \(M' = \langle F_1^*, F_2^* \rangle\) the dual lattices of \(N\) and \(N'\) respectively. It is easy to see that \(F_1^* = \frac{1}{2}(-2E_1^* + E_2^*)\) and \(F_2^* = \frac{1}{2}(E_1^* - 2E_2^*)\). Notice that...
\det M(E) = 3 and \( M' \) is generated by the rows of the intersection matrix multiplied by \(-1/\det M(E)\).

Now, let us describe the finite group \( N/N' \):

**Proposition 4.4.** — The group \( N/N' \) is generated by \( E_1 \) and \( E_2 \) over \( \mathbb{Z} \) with \( \text{ord}(E_i) = 3 \) for \( i = 1, 2 \) where \( E_i = E_i + N' \) and \( \text{ord}(E_i) \) is the order of \( E_i \) in \( N/N' \).

**Proof.** — Let \( F = F + N' \in N/N' \). This says \( F \in N' \) if and only if there exist \( a_i \in \mathbb{Z} \) such that \( F = a_1 E_1 + a_2 E_2 \). Hence there exist the \( b_i \in \mathbb{Z} \) such that \( F = b_1 E_1 + b_2 E_2 \).

Using the generators \( F_i \) obtained above, we find \((2a_1 + a_2)E_1 + (a_1 + 2a_2)E_2 = b_1 E_1 + b_2 E_2 \). For \( b_2 = 0 \), we find \( \text{ord}(E_1) \) as the smallest \( b_1 \in \mathbb{Z} \) satisfying this equation, so \( b_1 = 1 \). For \( b_1 = 0 \), we find \( b_2 = \text{ord}(E_2) = 3 \). Therefore the proposition follows. 

Let us denote \( E_i \) by \( \eta_i \) for \( i = 1, 2 \). We have \( \eta_i = (\exp 2\pi i)^{1/3} \) such that \( \eta_1^3 = 1 \).

Denote \( \mathbb{C}[\sigma \cap M] = \mathbb{C}[x_1, x_2] \) and \( \mathbb{C}[M'] = \mathbb{C}[u_1, u_2] \) with \( u_1 = x_1^{2/3} x_2^{-1/3} \) and \( u_2 = x_1^{-1/3} x_2^{2/3} \). The action of \( N/N' \) on the coordinates of \( \mathbb{C}[M'] \) (see p. 34 in [5]) gives:

\[
\eta_1(u_1, u_2) = (\eta_1^2 u_1, \eta_1^{-1} u_2) \quad \text{and} \quad \eta_2(u_1, u_2) = (\eta_2^{-1} u_1, \eta_2^2 u_2).
\]

Using proposition 4.4, we find:

**Theorem 4.5.** — With the preceding notation, the ring of invariants \( \mathbb{C}[M']^{N/N'} \) is generated by \( u_1^3, u_1^2 u_2, u_1 u_2^2, u_2^3 \).

**Proof.** — Let \( u = u_1^{k_1} u_2^{k_2} \). By the action of the finite group \( N/N' \) on \( u \), we have:

\[
\eta_1(u) = \eta_1^{2k_1 - k_2} u \quad \text{and} \quad \eta_2(u) = \eta_2^{-k_1 + 2k_2} u
\]

Since the ring of invariants \( \mathbb{C}[M']^{N/N'} \) is determined by the smallest \( k_1 \) and \( k_2 \) satisfying \( \eta_i(u) = u \) for \( i = 1, 2 \), we obtain the following system of equations:

\[
2k_1 - k_2 = 3l_1 \quad \text{and} \quad -k_1 + 2k_2 = 3l_2,
\]

for some \( l_1 \) and \( l_2 \). Hence \( k_i = 0, 1, 2 \pmod{3} \) for \( i = 1, 2 \). When \( k_1 = 3 \) (resp. \( k_2 = 3 \)) we have \( k_2 = 0 \) (resp. \( k_1 = 0 \)); so, \( u_1^3 \) and \( u_2^3 \) are in \( \mathbb{C}[M']^{N/N'} \). When \( k_1 = 2 \), we obtain \( k_2 = 1 \); so, \( u_1^2 u_2 \in \mathbb{C}[M']^{N/N'} \). When \( k_1 = 1 \), we obtain \( k_2 = 2 \); so, \( u_1 u_2^2 \in \mathbb{C}[M']^{N/N'} \).

Now, we need to find the ideal, called toric ideal, whose zero set is the affine toric variety \( \text{Spec} \mathbb{C}[\sigma \cap M] \). For this, we use [16]. The idea is to identify a \( 2 \times 4 \) matrix \( \mathcal{A} = (m_1, \ldots, m_4) \) with \( m_i = (m_{i1}, m_{i2})^t \) to the generators \( u_{m_i} = u_1^{m_{i1}} u_2^{m_{i2}} \) given in theorem 4.5. By lemma 1.1 in [16], the toric ideal \( \mathcal{I}_\mathcal{A} \) is generated by \( z^{v_+} - z^{v_-} \) for all integer vectors \( v = v_+ - v_- \) in the kernel of \( \mathcal{A} \). Hence, we conclude:
Corollary 4.6. — Let $E$ be the configuration of curves associated with the diagram $A_2$. The affine toric variety $\text{Spec} \mathbb{C}[\sigma \cap M]$ corresponding to $\mathcal{C}(E)$ is defined as the zero set of the toric ideal

$$\mathcal{I}_A = \langle z_1z_3 - z_2^2, z_1z_4 - z_2z_3, z_2z_4 - z_3^2 \rangle$$

where $z_i = u^{m_i}$.

Proof. — The matrix $A$ corresponding to the generators given in theorem 4.5 is

$$
\begin{pmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 2 & 0 \\
0 & 1 & 2 & 3
\end{pmatrix}
$$

So, the vectors

$$
\begin{pmatrix}
1 \\
0 \\
1
\end{pmatrix}
- \begin{pmatrix}
2 \\
0 \\
0
\end{pmatrix},
\begin{pmatrix}
0 \\
1 \\
1
\end{pmatrix}
- \begin{pmatrix}
0 \\
2 \\
0
\end{pmatrix}
\text{ and }
\begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix}
- \begin{pmatrix}
0 \\
1 \\
1
\end{pmatrix}
$$

of the kernel of $A$ generate our toric ideal $\mathcal{I}_A$ (see lemma 1.1 and example 1.2.(a) in [16]).

Applying the same method to any ADE configuration, we can obtain the corresponding toric variety. The reader can find each of these types in detail in [1]. We remark that our interest for the construction method of the affine toric variety corresponding to $\mathcal{C}(E)$ is coming from [15]. One of the natural continuations is to explore the possibility of a relation between the invariants of the affine toric variety and those of the corresponding normal surface singularity.

References


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