*Séminaires & Congrès*
**10**, 2005, p. 127–138

# DETERMINATION OF LIPSCHITZ STRATIFICATIONS FOR THE SURFACES $y^a = z^b x^c + x^d$

*by*

Dwi Juniati & David Trotman

***Abstract.*** — We determine Lipschitz stratifications for the family of surfaces $y^a = z^b x^c + x^d$, where $a, b, c, d$ are positive integers.

***Résumé*** **(Détermination de stratifications de Lipschitz pour les surfaces $y^a = z^b x^c + x^d$)**
Nous déterminons des stratifications de Lipschitz pour la famille de surfaces $y^a = z^b x^c + x^d$, où $a, b, c, d$ sont des entiers positifs.

## 1. Introduction and previous results

R. Thom and H. Whitney suggested the use of stratifications as a method of understanding the geometric structure of singular analytic spaces. The stronger the conditions imposed on the stratified set, the better the understanding of the geometry of this set. One of the strongest conditions is the $L$-regularity introduced by T. Mostowski in 1985 [**3**]. Mostowski introduced the notion of Lipschitz stratification and proved its existence for complex analytic sets. The existence of Lipschitz stratification for subanalytic sets and real analytic sets was later proved by A. Parusinski [**5**], [**6**], [**7**]. The Lipschitz stratifications ensure bi-Lipschitz triviality of the stratified set along each stratum, and bi-Lipschitz homeomorphisms preserve sets of measure zero, order of contact, and Lojasiewicz exponents. The $L$-conditions are preserved after intersection with generic wings, that is $L$-regularity implies $L^*$-regularity [**2**]; this was one of the conditions required of a good equisingularity notion by B. Teissier in his foundational 1974 Arcata paper [**8**].

In this paper, we check Mostowski's conditions for almost all members ($\sim 99\%$) of the family of surfaces with two strata given by $(V - Oz, Oz)$ where $V$ is the algebraic surface

$$\{(x, y, z) \mid y^a = z^b x^c + x^d\}$$

in $\mathbb{R}^3$ or $\mathbb{C}^3$, with $a, b, c, d$ positive integers.

**2000** ***Mathematics Subject Classification.*** — 14B05, 14J10, 14J17, 14P05, 32S60, 58A35.
***Key words and phrases.*** — Stratification, Lipschitz, real algebraic surfaces.

The character $C$ in this paper will stand for various constants.

We begin by recalling the definition of Lipschitz stratification due to Mostowski [**3**].

Let $X$ be a closed subanalytic subset of an open subset of $\mathbb{R}^n$. By a *stratification* of $X$ we shall mean a family $\Sigma = \{S^j\}_{j=l}^m$ of closed subanalytic subsets of $X$ defining a filtration:

$$X = S^m \supset S^{m-1} \supset \cdots \supset S^l \neq \varnothing$$

and $\overset{\circ}{S^j} = S^j \setminus S^{j-1}$, for $j = l, l+1, \ldots, m$ (where $S^{l-1} = \varnothing$), is a smooth manifold of pure dimension $j$ or empty. We call the connected components of $\overset{\circ}{S^j}$ the *strata* of $\Sigma$. We denote the function measuring distance to $S^j$ by $d_j$, so that $d_j(q) = \mathrm{dist}(q, S^j)$. Set $d_{l-1} \equiv 1$, by convention (this will be used in the definitions below).

***Definition 1.1***. — Let $\gamma > 1$ be a fixed constant. A *chain* for a point $q \in \overset{\circ}{S^j}$ is a (strictly) decreasing sequence of indices $j = j_1, j_2, \ldots, j_r = l$ such that each $j_s$, $s \geqslant 2$, is the greatest integer less than $j_{s-1}$ for which

$$d_{j_s - 1}(q) \geqslant 2\gamma^2 d_{j_s}(q).$$

For each $j_s \in \{j_1, \ldots, j_r\}$ choose a point $q_{j_s} \in \overset{\circ}{S^{j-s}}$ such that $q_{j_1} = q$ and $|q - q_{j_s}| \leqslant \gamma d_{j_s}(q)$.

If there is no confusion, we will call the sequence of points $(q_{j_s})$ a chain of $q$.

For $q \in \overset{\circ}{S^j}$, let $P_q : \mathbb{R}^n \to T_q\overset{\circ}{S^j}$ be the orthogonal projection to the tangent space $T_q\overset{\circ}{S^j}$ and let $P_q^\perp = I - P_q$ be the orthogonal projection onto the normal space $(T_q\overset{\circ}{S^j})^\perp$.

***Definition 1.2***. — A stratification $\Sigma = \{S^j\}_{j=l}^m$ of $X$ is said to be a Lipschitz stratification (or to satisfy the $L$-conditions) if for some constant $C > 0$ and for every chain $q = q_{j_1}, q_{j_2}, \ldots, q_{j_r}$ and every $k$, $2 \leqslant k \leqslant r$,

$$(L1) \qquad |P_q^\perp P_{q_{j_2}} \cdots P_{q_{j_k}}| \leqslant C|q - q_{j_2}|/d_{j_k - 1}(q)$$

and for each $q' \in \overset{\circ}{S^{j-1}}$ such that $|q - q'| \leqslant (1/2\gamma) d_{j_1 - 1}(q)$,

$$(L2) \qquad |(P_q - P_{q'})P_{q_{j_2}} \cdots P_{q_{j_k}}| \leqslant C|q - q'|/d_{j_k - 1}(q)$$

and

$$(L3). \qquad |P_q - P_{q'}| \leqslant C|q - q'|/d_{j_1 - 1}(q)$$

## 2. Classifications and Calculations

In this section we give diagrams showing when $L-$regularity holds for the stratification with two strata given by $(V - Oz, Oz)$ where $V$ is the germ of the algebraic surface

$$\{(x, y, z) \mid y^a = z^b x^c + x^d\}$$

in $\mathbb{R}^3$ or $\mathbb{C}^3$, and $a, b, c, d$ are positive integers.

Diagrams classifying Whitney $a$-regularity and $b$-regularity for this family of algebraic surfaces were obtained by the second author ([**9**] and [**10**]), while the finer classification needed for the Kuo-Verdier $w$-regularity can be found in the thesis of L. Noirel [**4**].

**Calculations.** — It is easy to see that the $(L1)$ condition in the definition of Lipschitz stratification of Mostowski implies the Kuo-Verdier $(w)$ condition, and in our case (where there are only two strata) is actually equivalent to $(w)$. So we only need to study cases when $(w)$ is satisfied in the classification obtained by L. Noirel. And for these we only need to check condition $(L2)$ (for $k = 2$) and $(L3)$. Here we illustrate the method of obtaining the classification with details of the calculations deciding several branches of Diagram 2 (see the end of the paper). The essential technique in checking $(L2)$ and $(L3)$ is to apply the mean value theorem to compare values at the two points $q$ and $q'$ whose distance apart is controlled.

Complete detailed calculations for the case of Lipschitz stratifications can be found in the first author's thesis [**1**].

(2.1) If $a = 1$, $V$ is the graph of a smooth function, hence $V$ is a smooth submanifold of $\mathbb{R}^3$, and consequently is $(L)$ regular.

The Lipschitz condition of Mostowski is stronger than $(b)$-regularity and $(w)$-regularity, so using the classifications of Trotman in [**9**] and Noirel in [**4**] we have:

(2.2) If $a \neq 1$, $d > c$, and $b$ is odd, the conditions $(b)$ and $(w)$ do not hold, and hence the stratification is not $(L)$ regular.

(2.3) If $a \neq 1$, $d > c$, $b$ is even and $(d - c)$ is odd, the conditions $(b)$ and $(w)$ do not hold, and hence the stratification is not $(L)$ regular.

(2.4) If $a \neq 1$, $d > c$, $b$ is even, $(d-c)$ is even and $a(d-c) > b|d-a|$, the condition $(w)$ does not hold, and hence the stratification is not $(L)$ regular.

(2.5) If $a \neq 1$, $d > c$, $b$ is even, $(d - c)$ is even, $a(d - c) \leqslant b|d - a|$, and $c < a \leqslant d$, the conditions $(b)$ and $(w)$ do not hold, and hence the stratification is not $(L)$ regular.

**Notation.** — Let $f$ and $g$ be two real valued functions defined on the same set. We will write $f \ll g$ if there exists a function $\alpha$ which tends to 0 at 0, such that $f \leqslant \alpha g$. We also will write $f \approx g$ if there is a positive constant $C$ such that $f \leqslant Cg$ and $g \leqslant Cf$. For functions defined on $\mathbb{R}^3$, we will use this notation for their restrictions to $V$.

***Proposition 2.1***. — *Consider the stratification with two strata $(V - Oz,\ Oz)$ of the germ $V = \{(x, y, z) \mid y^a = z^b x^c + x^d\}$ in $\mathbb{R}^3$, where $a, b, c, d$ are positive integers. Suppose $c \geqslant d$. Then the stratification is Lipschitz.*

*Proof.* — Let $V = \{(x, y, z) \mid f(x, y, z) = -y^a + z^b x^c + x^d\}$, so that $\operatorname{Sing} V = Oz$.

Let $(q, q_{j_2})$ be a chain, with $q = (x, y, z) \in \operatorname{Reg} V$ and $q_{j_2} \in O_z$. We shall write $\partial_q f$ for the gradient of $f$ at $q$ from now on.

$$\text{Then} \quad \partial_q f = (cx^{c-1}z^b + dx^{d-1}, -ay^{a-1}, bx^c z^{b-1}).$$

Then $d_{j_2-1}(q) = 1$, and $d_{j_1-1}(q) = d(q, O_z)$. By the definition of orthogonal projection, we have the following inequalities:

$$|P_q - P_{q'}| = |P_q^{\perp} - P_{q'}^{\perp}|, \quad \text{and} \quad P_q^{\perp}(v) = \Big\langle v, \frac{\partial_q f}{\|\partial_q f\|} \Big\rangle \frac{\partial_q f}{\|\partial_q f\|}.$$

Suppose $c \geqslant d$. In this case $(w)$-regularity holds, so that to verify that the stratification is Lipschitz it is enough to check $(L2)$ and $(L3)$ in the definition of Lipschitz stratification, because for a stratification composed of two strata, $(w)$ is equivalent to $(L1)$.

We work in a neighbourhood of the origin. By definition of $V$, we have $y = \pm x^{d/a}\big(z^b x^{c-d} + 1\big)^{1/a}$ and we see that $y \approx x^{s\frac{d}{a}}$.

*I. Case $a < d$.* — We have that $y \approx x^{d/a}$ and $a < d$, hence $|y| = \circ|x|$ and $d(q, O_z) \approx |x|$. Take $q' = (x', y', z') \in V - Oz$, such that $|q - q'| \leqslant \frac{1}{2} d(q, O_z)$. That is:

$$|q - q'| = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2} \leqslant \frac{1}{2}|x|.$$

It is easy to see that $xx' > 0$. We shall treat separately the cases $yy' < 0$ and $yy' \geqslant 0$.

If $yy' < 0$, then as $xx' > 0$, it follows that $|y| + |y'| \leqslant C|q - q'|$. For a point $q = (x, y, z)$, we have:

$$\begin{aligned}
\Big|\frac{\partial f}{\partial x}\Big| &= |cx^{c-1}z^b + dx^{d-1}| \approx |x^{d-1}| \quad \text{because } c \geqslant d \\
\Big|\frac{\partial f}{\partial y}\Big| &= |ay^{a-1}| \approx |x^{d-d/a}| \\
\Big|\frac{\partial f}{\partial z}\Big| &= |bx^c z^{b-1}|.
\end{aligned}$$

As $a < d$, in some neighbourhood of 0 we have:

$$\Big|\frac{\partial f}{\partial x}\Big| \ll \Big|\frac{\partial f}{\partial y}\Big| \quad \text{and} \quad \Big|\frac{\partial f}{\partial z}\Big| \ll \Big|\frac{\partial f}{\partial y}\Big|.$$

Calculating,

$$\begin{aligned}
\Big|\frac{bx^c z^{b-1}\partial_q f}{\|\partial_q f\|^2}\Big| &= \Big|\frac{bx^c z^{b-1}(cx^{c-1}z^b + dx^{d-1}, -ay^{a-1}, bx^c z^{b-1})}{(cx^{c-1}z^b + dx^{d-1})^2 + (-ay^{a-1})^2 + (bx^c z^{b-1})^2}\Big| \\
&\approx \Big|\frac{x^c z^{b-1}}{y^{a-1}}\Big|
\end{aligned}$$

since $|\partial f/\partial x| \ll |\partial f/\partial y|$ and $|\partial f/\partial z| \ll |\partial f/\partial y|$ in some neighbourhood of 0, and

$$\begin{aligned}\Big|\frac{x^c z^{b-1}}{y^{a-1}y}\Big| = \Big|\frac{x^c z^{b-1}}{z^b x^c + x^d}\Big| < \Big|\frac{x^c z^{b-1}}{x^d}\Big| \\ \approx \Big|x^{c-d}z^{b-1}\Big| < 1 \quad \text{in some neighbourhood of 0.}\end{aligned}$$

Then

$$\begin{aligned}\Big|(P_q - P_{q'})P_{q_{j_2}}\Big| &= \Big|(P_q - P_{q'})(0,0,1)\Big| = \Big|(P_q^{\perp} - P_{q'}^{\perp})(0,0,1)\Big| \\ &= \Big|\frac{bx^c z^{b-1}\partial_q f}{\|\partial_q f\|^2} - \frac{bx'^c z'^{b-1}\partial_{q'} f}{\|\partial_{q'} f\|^2}\Big| \\ &\leqslant |y| + |y'| < C|q-q'|.\end{aligned}$$

We have shown that $(L2)$ holds when $yy' < 0$.

Now we check $(L3)$ when $yy' < 0$. Take the basis vectors: $(1,0,0),(0,1,0)$ and $(0,0,1)$. Calculating:

$$\begin{aligned}\Big|\frac{(cx^{c-1}z^b + dx^{d-1})\partial_q f}{\|\partial_q f\|^2}\Big| &= \Big|\frac{(cx^{c-1}z^b + dx^{d-1})(cx^{c-1}z^b + dx^{d-1}, -ay^{a-1}, bx^c z^{b-1})}{(cx^{c-1}z^b + dx^{d-1})^2 + (-ay^{a-1})^2 + (bx^c z^{b-1})^2}\Big| \\ &\approx \Big|\frac{cx^{c-1}z^b + dx^{d-1}}{y^{a-1}}\Big|\end{aligned}$$

since $|\partial f/\partial x| \ll |\partial f/\partial y|$ and $|\partial f/\partial z| \ll |\partial f/\partial y|$ in some neighbourhood of 0, and

$$\Big|\frac{cx^{c-1}z^b + dx^{d-1}}{ay^{a-1}}\Big|\frac{|x|}{|y|} \text{ is bounded in some neighbourhood of 0.}$$

Hence

$$\begin{aligned}\Big|(P_q - P_{q'})(1,0,0)\Big| &= \Big|(P_q^{\perp} - P_{q'}^{\perp})(1,0,0)\Big| \\ &= \Big|\frac{(cx^{c-1}z^b + dx^{d-1})\partial_q f}{\|\partial_q f\|^2} - \frac{(cx'^{c-1}z'^b + dx'^{d-1})\partial_{q'} f}{\|\partial_{q'} f\|^2}\Big| \\ &\leqslant \Big|\frac{y}{x}\Big| + \Big|\frac{y'}{x'}\Big| \leqslant C\frac{|q-q'|}{d(q,O_z)}.\end{aligned} \tag{i}$$

We do the same for the basis vector $(0,1,0)$. Because $|\partial f/\partial x| \ll |\partial f/\partial y|$ and $|\partial f/\partial z| \ll |\partial f/\partial y|$ in a neighbourhood of 0,

$$\Big|\frac{ay^{a-1}\partial_q f}{\|\partial_q f\|^2}\Big| \quad \approx \quad \Big|\frac{(y^{a-1})^2}{\|\partial_q f\|^2}\Big|.$$

Also

$$\begin{aligned}|(P_q - P_{q'})(0,1,0)| &= |(P_q^{\perp} - P_{q'}^{\perp})(0,1,0)| \\ &= \Big|\frac{ay^{a-1}\partial_q f}{\|\partial_q f\|^2} - \frac{ay'^{a-1}\partial_{q'} f}{\|\partial_{q'} f\|^2}\Big|.\end{aligned}$$

Note that the majoration of the $x$-coordinate of the vector in the line above has already been done in the calculation for $(1,0,0)$. The majoration for the $z$-coordinate has also been done above in checking $(L2)$. So we just need to consider the $y$-coordinate:

$$\Big|\frac{(y^{a-1})(y^{a-1})}{\|\partial_q f\|^2} - \frac{((y')^{a-1})((y')^{a-1})}{\|\partial_{q'} f\|^2}\Big|.$$

Let

$$\alpha = \frac{(y^{a-1})^2}{\|\partial_q f\|^2}.$$

Then it suffices to show that $\alpha$ is Lipschitz. Calculation shows that:

$$x\frac{\partial\alpha}{\partial x}, \quad x\frac{\partial\alpha}{\partial y}, \quad \text{and} \quad x\frac{\partial\alpha}{\partial z}$$

are bounded in a neighbourhood of 0, and we can apply the mean value theorem to show there exists a constant $C$ such that:

$$\text{(ii)} \qquad |(P_q - P_{q'})(0,1,0)| \leqslant C\frac{|q-q'|}{d(q,O_z)}.$$

By (i) and (ii) we can conclude that $(L3)$ holds when $yy' < 0$.

Now let $yy' \geqslant 0$ and $xx' > 0$ ($q$ and $q'$ are in a connected set). We shall check $(L2)$:

$$|(P_q - P_{q'})P_{q_{j_2}}| = |(P_q - P_{q'})(0,0,1)| = |(\xi_1, \xi_2, \xi_3)|$$

with $\xi_i = g_i(q) - g_i(q')$, where $g_i : \mathbb{R}^3 \to \mathbb{R}$, $q \mapsto g_i(q)$, and

$$g_1 = \frac{\partial f/\partial z \cdot \partial f/\partial x}{\|\partial_q f\|^2}, \quad g_2 = \frac{\partial f/\partial z \cdot \partial f/\partial y}{\|\partial_q f\|^2}, \quad g_3 = \frac{(\partial f/\partial z)^2}{\|\partial_q f\|^2}.$$

It is not hard to show that $\partial g_i/\partial x$, $\partial g_i/\partial y$ and $\partial g_i/\partial z$ are bounded in some neighbourhood of 0, for $i = 1,2,3$.

Note that $q$ and $q'$ can be joined by a curve of length comparable to $|q-q'|$. Then we can apply the mean value theorem, and say there exists a constant $C_i$ such that $|\xi_i| \leqslant C_i|q-q'|$, $\forall\, i = 1,2,3$.

We conclude that there exists a constant $C > 0$ such that $|(P_q - P_{q'})P_{q_{j_2}}| \leqslant C|q-q'|$, *i.e.* condition $(L2)$ holds when $yy' \geqslant 0$.

To check condition $(L3)$ when $yy' \geqslant 0$, we use basis vectors $v_1 = (1,0,0)$, and $v_2 = (0,1,0)$.

For $v_1 = (1,0,0)$, we have:

$$|(P_q - P_{q'})(v_1)| = \left|\frac{\frac{\partial f}{\partial x}(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})}{\|\partial_q f\|^2} - \frac{\frac{\partial f}{\partial x'}(\frac{\partial f}{\partial x'}, \frac{\partial f}{\partial y'}, \frac{\partial f}{\partial z'})}{\|\partial_{q'} f\|^2}\right|.$$

Let

$$m_1 = \frac{(\partial f/\partial x)^2}{\|\partial_q f\|^2}, \quad m_2 = \frac{\partial f/\partial x \cdot \partial f/\partial y}{\|\partial_q f\|^2}, \quad \text{and} \quad m_3 = \frac{\partial f/\partial x \cdot \partial f/\partial z}{\|\partial_q f\|^2}.$$

Calculating, we can show that:

$$x\frac{\partial m_i}{\partial x}, \quad x\frac{\partial m_i}{\partial y}, \quad \text{and} \quad x\frac{\partial m_i}{\partial z}$$

are bounded in some neighbourhood of 0, for $i = 1, 2, 3$.

Then we can apply the mean value theorem to show there exists a constant $C$ such that:

$$|(P_q - P_{q'})(v_1)| \leqslant C\frac{|q - q'|}{d(q, O_z)}. \tag{1}$$

For $v_2 = (0, 1, 0)$, we have:

$$|(P_q - P_{q'})(v_2)| = \left|\frac{\frac{\partial f}{\partial y}(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})}{\|\partial_q f\|^2} - \frac{\frac{\partial f}{\partial y'}(\frac{\partial f}{\partial x'}, \frac{\partial f}{\partial y'}, \frac{\partial f}{\partial z'})}{\|\partial_{q'} f\|^2}\right|.$$

Let

$$n_1 = \frac{\partial f/\partial y \cdot \partial f/\partial x}{\|\partial_q f\|^2}, \quad n_2 = \frac{(\partial f/\partial y)^2}{\|\partial_q f\|^2}, \quad \text{and} \quad n_3 = \frac{\partial f/\partial y \cdot \partial f/\partial z}{\|\partial_q f\|^2}.$$

It is easy to show that:

$$x\frac{\partial n_i}{\partial x}, \quad x\frac{\partial n_i}{\partial y}, \quad \text{and} \quad x\frac{\partial n_i}{\partial z}$$

are bounded in some neighbourhood of 0, for $i = 1, 2, 3$.

Then we can apply the mean value theorem to show there exists a constant $C$ such that:

$$|(P_q - P_{q'})(v_2)| \leqslant C\frac{|q - q'|}{d(q, O_z)}. \tag{2}$$

By (1) and (2) we see that $(L3)$ holds when $yy' \geqslant 0$.

*II. Case $a \geqslant d$.* — Supposing that $y \approx x^{d/a}$ and $a \geqslant d$, then $|x| = \circ|y|$ or $|x| \approx |y|$, and $d(q, O_z) \approx |y|$.

Let $q' = (x', y', z') \in \operatorname{Reg} V$, such that $|q - q'| \leqslant \frac{1}{2}d(q, O_z)$. This means that:

$$|q - q'| = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2} \leqslant \frac{1}{2}|y|.$$

It is easy to see that $yy' > 0$, and that $|x| < |q - q'|$ and $|x'| < |q - q'|$. We shall treat separately the cases $xx' < 0$ and $xx' \geqslant 0$.

First let $xx' < 0$, and $yy' > 0$, then $|x| + |x'| \leqslant C|q - q'|$. For a point $q = (x, y, z)$, we have:

$$\begin{aligned}
\left|\frac{\partial f}{\partial x}\right| &= |cx^{c-1}z^b + dx^{d-1}| \approx |x^{d-1}| \quad \text{because } c \geqslant d \\
\left|\frac{\partial f}{\partial y}\right| &= |ay^{a-1}| \approx |(z^b x^c + x^d)^{(a-1)/a}| \approx |x^{d-d/a}| \\
\left|\frac{\partial f}{\partial z}\right| &= |bx^c z^{b-1}|.
\end{aligned}$$

If $a > d$, then in some neighbourhood of 0 we have:

$$\left|\frac{\partial f}{\partial y}\right| \ll \left|\frac{\partial f}{\partial x}\right| \quad \text{and} \quad \left|\frac{\partial f}{\partial z}\right| \ll \left|\frac{\partial f}{\partial x}\right|,$$

while if $a = d$, then

$$\left|\frac{\partial f}{\partial y}\right| \approx \left|\frac{\partial f}{\partial x}\right| \quad \text{and} \quad \left|\frac{\partial f}{\partial z}\right| \ll \left|\frac{\partial f}{\partial x}\right|.$$

Using the same method as in the case when $a < d$, we can show that the $L$ conditions hold.

In the case when $xx' \geqslant 0$, and $yy' > 0$, we can use the mean value theorem and the same method as for $a < d$, to check that the $L$ conditions hold. This completes the proof of the Proposition. □

***Proposition 2.2***. — *Let two strata be given by $(V - Oz,\ Oz)$ where the germ $V = \{(x,y,z) \mid y^a = z^b x^c + x^d\}$ in $\mathbb{R}^3$, and $a,b,c,d$ are positive integers. Suppose $a(d-c) \leqslant b\,|d-a|$. The condition (L2) in the definition of Lipschitz stratification is not satisfied in the cases below:*

*(1) $c < d < a$, and $d, c$ are even,*
*(2) $c < d < a$, and $b < 2(d-c)$,*
*(3) $c < d$, $a < d$, and $a$ is even,*
*(4) $a \leqslant c < d$, and $2a(d-c) > db$.*

*Proof*. — Let V $= \{(x,y,z) \mid f(x,y,z) = -y^a + z^b x^c + x^d\}$, so $\operatorname{Sing} V \subset Oz$.

Case (1). Let $q = (t, 2^{1/a}t^{d/a}, t^{(d-c)/b})$, $q' = (-t, 2^{1/a}t^{d/a}, t^{(d-c)/b})$, and $q_{j_2} = (0, 0, t^{(d-c)/b})$, for positive $t$ near 0. We have:

$$|q - q'| = 2t \quad \text{and} \quad d_{j_1-1}(q) = d(q, O_z) = \sqrt{t^2 + t^{2d/a}}.$$

Then there is $\gamma > 1$ such that $|q - q'| \leqslant \frac{1}{2\gamma} d_{j_1-1}(q)$, because $d < a$. Now,

$$\partial_q f = (cx^{c-1}z^b + dx^{d-1}, -ay^{a-1}, bx^c z^{b-1}),$$

Also $d_{j_2-1}(q) = 1$, $|P_q - P_{q'}| = |P_q^\perp - P_{q'}^\perp|$, and $P_q^\perp(v) = \langle v, \partial_q f/\|\partial_q f\|\rangle \partial_q f/\|\partial_q f\|$. Then

$$\begin{aligned} |(P_q - P_{q'})P_{q_{j_2}}(v)| &= |(P_q - P_{q'})(0,0,1)| = |(P_q^\perp - P_{q'}^\perp)(0,0,1)| \\ &= \left|\frac{bx^c z^{b-1}\partial_q f}{\|\partial_q f\|^2} - \frac{bx'^c z'^{b-1}\partial_{q'} f}{\|\partial_{q'} f\|^2}\right| \\ &\approx \left|\frac{t^{d-d-c/b}t^{d-1}}{(t^{d-1})^2 + (t^{d-d/a})^2 + (t^{(d-(d-c)/b})^2}\right| \approx \left|\frac{t^{d-(d-c)/b}}{t^{d-1}}\right|. \end{aligned}$$

We have $|(P_q - P_{q'})P_{q_{j_2}}| \approx t^{1-(d-c)/b}$ and $|q - q'|/d_{j_2-1}(q) \approx t$, so $(L2)$ fails.

Case (2). Let $q = (t, 2^{1/a}t^{d/a}, t^{(d-c)/b})$, $q' = (t, 2^{1/a}t^{d/a}, -t^{(d-c)/b})$, and $q_{j_2} = (0, 0, t^{(d-c)/b})$, for positive $t$ near 0. With the same technique as above we have that $|(P_q - P_{q'})P_{q_{j_2}}| \approx t^{1-(d-c)/b}$ and $|q - q'|/d_{j_2-1}(q) \approx t^{(d-c)/b}$, and by hypothesis $b < 2(d-c)$ so $(L2)$ fails.

Case (3). Let $q = (t, 2^{1/a}t^{d/a}, t^{(d-c)/b})$, $q' = (t, -2^{1/a}t^{d/a}, t^{(d-c)/b})$, and $q_{j_2} = (0, 0, t^{(d-c)/b})$, for positive $t$ near 0. With the same technique as above we have $|(P_q - P_{q'})P_{q_{j_2}}| \approx t^{d/a-(d-c)/b}$ and $|q - q'|/d_{j_2-1}(q) \approx t^{d/a}$, and by hypothesis $d/a - (d-c)/b < d/a$ so $(L2)$ fails.

Case (4). Let $q = (t, 2^{1/a}t^{d/a}, t^{(d-c)/b})$, $q' = (t, 2^{1/a}t^{d/a}, -t^{(d-c)/b})$, and $q_{j_2} = (0, 0, t^{(d-c)/b})$, for positive $t$ near 0. With the same technique as above we have $|(P_q - P_{q'})P_{q_{j_2}}| \approx t^{d/a-(d-c)/b}$ and $|q - q'|/d_{j_2-1}(q) \approx t^{(d-c)/b}$, and by hypothesis $2a(d-c) > db$ so $(L2)$ fails. This completes the proof of the Proposition. □

Now we determine when the stratification is Lipschitz supposing that $d > c$, $b$ and $d - c$ are even, and that $a$ is odd, $a \geqslant 3$. To ensure that $(L1)$ holds (*i.e.* $(w)$) we need that $a(d-c) \leqslant b|d-a|$.

We can replace the equation $y^a = z^b x^c + x^d$ by $y = (z^b x^c + x^d)^{1/a}$ to simplify the calculation. Let $\phi(x; z) = y - (x^c z^b + x^d)^{1/a}$. Then $V = \{(x, y, z) \mid \phi(x, y, z) = 0\}$. Let $q = (x, y, z)$ and $q' = (x', y', z')$, so $\partial_q\varphi = (\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial z})$, and $\partial_{q'}\varphi = (\frac{\partial\varphi}{\partial x'}, \frac{\partial\varphi}{\partial y'}, \frac{\partial\varphi}{\partial z'})$. We know that $d_{j_2-1}(q) = 1$. Then

$$|(P_q - P_{q'})P_{q_{j_2}}| = \left| \frac{\frac{\partial\varphi}{\partial z}(\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial z})}{\|\partial_q\varphi\|^2} - \frac{\frac{\partial\varphi}{\partial z'}(\frac{\partial\varphi}{\partial x'}, \frac{\partial\varphi}{\partial y'}, \frac{\partial\varphi}{\partial z'})}{\|\partial_{q'}\varphi\|^2} \right|.$$

Let $h_1 = \partial\varphi/\partial z \cdot \partial\varphi/\partial x/\|\partial_q\varphi\|^2$, $h_2 = \partial\varphi/\partial z/\|\partial_q\varphi\|^2$, and $h_3 = (\partial\varphi/\partial z)^2/\|\partial_q\varphi\|^2$.

If $\partial h_1/\partial x$ and $\partial h_1/\partial z$ are bounded near 0, then by using the mean value theorem, there is a constant $C$ for $q$, $q'$ near 0, such that:

$$\left| \frac{\partial\varphi/\partial z \cdot \partial\varphi/\partial x}{\|\partial_q\varphi\|^2} - \frac{\partial\varphi/\partial z' \cdot \partial\varphi/\partial x'}{\|\partial_{q'}\varphi\|^2} \right| \leqslant C|q - q'|.$$

Similarly for $h_2$ and $h_3$, if $\partial h_2/\partial x$, $\partial h_2/\partial z$, $\partial h_3/\partial x$, $\partial h_3/\partial z$, are bounded near 0, then there is a constant C such that:

$$\left| \frac{\partial\varphi/\partial z}{\|\partial_q\varphi\|^2} - \frac{\partial\varphi/\partial z'}{\|\partial_{q'}\varphi\|^2} \right| \leqslant C|q - q'|, \quad \text{and} \quad \left| \frac{(\partial\varphi/\partial z)^2}{\|\partial_q\varphi\|^2} - \frac{(\partial\varphi/\partial z')^2}{\|\partial_{q'}\varphi\|^2} \right| \leqslant C|q - q'|.$$

This means that there exists a constant $C$, such that for $q$ and $q'$ near 0, we have:

$$(L2). \qquad |(P_q - P_{q'})P_{q_{j_2}}| \leqslant C|q - q'|$$

To verify the $(L3)$ condition in the definition of Lipschitz stratification, we use the vector basis $v_1 = (1, 0, 0), v_2 = (0, 1, 0)$, and $v_3 = (0, 0, 1)$. If $c \geqslant a$, $d_{j_1-1} = d(q, Oz) \approx x$. For $v_1 = (1, 0, 0)$, we have:

$$|(P_q - P_{q'})(v_1)| = \left| \frac{\frac{\partial\varphi}{\partial x}(\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial z})}{\|\partial_q\varphi\|^2} - \frac{\frac{\partial\varphi}{\partial x'}(\frac{\partial\varphi}{\partial x'}, \frac{\partial\varphi}{\partial y'}, \frac{\partial\varphi}{\partial z'})}{\|\partial_{q'}\varphi\|^2} \right|.$$

Let $k_1 = (\partial\varphi/\partial x)^2/\|\partial_q\varphi\|^2$, $k_2 = \partial\varphi/\partial x/\|\partial_q\varphi\|^2$, and $k_3 = \partial\varphi/\partial x \cdot \partial\varphi/\partial z/\|\partial_q\varphi\|^2$.

If $x\partial k_1/\partial x$, and $x\partial k_1/\partial z$ are bounded near 0, by using the mean value theorem, there is a constant $C$ such that for $q$, $q'$ near 0:

$$\left| \frac{(\partial\varphi/\partial x)^2}{\|\partial_q\varphi\|^2} - \frac{(\partial\varphi/\partial x')^2}{\|\partial_{q'}\varphi\|^2} \right| \leqslant C\frac{|q - q'|}{d_{j_1-1}(q)}.$$

Similarly with $k_2$ and $k_3$, if $x\partial k_2/\partial x$, $x\partial k_2/\partial z$, $x\partial k_3/\partial x$, $x\partial k_3/\partial z$, are bounded near 0, then there is a constant $C$ such that:

$$\left|\frac{\partial\varphi/\partial x}{\|\partial_q\varphi\|^2} - \frac{\partial\varphi/\partial x'}{\|\partial_{q'}\varphi\|^2}\right| \leqslant C\frac{|q-q'|}{d_{j_1-1}(q)} \quad \text{and} \quad \left|\frac{\frac{\partial\varphi}{\partial x}\cdot\frac{\partial\varphi}{\partial z}}{\|\partial_q\varphi\|^2} - \frac{\frac{\partial\varphi}{\partial x'}\cdot\frac{\partial\varphi}{\partial z'}}{\|\partial_{q'}\varphi\|^2}\right| \leqslant C\frac{|q-q'|}{d_{j_1-1}(q)}.$$

This means that there exists a constant $C$, such that for $q$ and $q'$ near 0, we have:

$$(L3)(\mathrm{i}) \qquad |(P_q - P_{q'})(v_1)| \leqslant C\frac{|q-q'|}{d_{j_1-1}(q)}.$$

For $v_2 = (0,1,0)$, we have:

$$|(P_q - P_{q'})(v_2)| = \left|\frac{\frac{\partial\varphi}{\partial y}(\frac{\partial\varphi}{\partial x},\frac{\partial\varphi}{\partial y},\frac{\partial\varphi}{\partial z})}{\|\partial_q\varphi\|^2} - \frac{\frac{\partial\varphi}{\partial y'}(\frac{\partial\varphi}{\partial x'},\frac{\partial\varphi}{\partial y'},\frac{\partial\varphi}{\partial z'})}{\|\partial_{q'}\varphi\|^2}\right|.$$

It is clear that if $(L2)$ and $(L3)(i)$ are verified, then we have

$$(L3)(\mathrm{ii}) \qquad |(P_q - P_{q'})(v_2)| \leqslant C\frac{|q-q'|}{d_{j_1-1}(q)}.$$

For $v_3 = (0,0,1)$, we have:

$$|(P_q - P_{q'})(v_3)| = \left|\frac{\frac{\partial\varphi}{\partial z}(\frac{\partial\varphi}{\partial x},\frac{\partial\varphi}{\partial y},\frac{\partial\varphi}{\partial z})}{\|\partial_q\varphi\|^2} - \frac{\frac{\partial\varphi}{\partial z'}(\frac{\partial\varphi}{\partial x'},\frac{\partial\varphi}{\partial y'},\frac{\partial\varphi}{\partial z'})}{\|\partial_{q'}\varphi\|^2}\right|.$$

It is clear that if $(L2)$ is verified, then we have

$$(L3)(\mathrm{iii}) \qquad |(P_q - P_{q'})(v_3)| \leqslant C\frac{|q-q'|}{d_{j_1-1}(q)}.$$

By calculation of differentials, we find that if $\partial\varphi/\partial x$, $\partial\varphi/\partial z$, $\partial^2\varphi/\partial x\partial z$, and $\partial^2\varphi/\partial z^2$ are bounded near 0, then

$$\frac{\partial h_1}{\partial x},\ \frac{\partial h_1}{\partial z},\ x\frac{\partial k_2}{\partial x},\ x\frac{\partial k_2}{\partial z},\ x\frac{\partial k_3}{\partial x},\ x\frac{\partial k_3}{\partial z},\ x\frac{\partial k_1}{\partial x},\ x\frac{\partial h_1}{\partial z},\ x\frac{\partial k_2}{\partial x},\ x\frac{\partial k_2}{\partial z},\ x\frac{\partial k_3}{\partial x},\ x\frac{\partial k_3}{\partial z},$$

are all bounded near 0. And we have that:

- $\partial\varphi/\partial x$ is bounded if $c \geqslant a$, $d \geqslant a$,
- $\partial\varphi/\partial z$ is bounded if $b \geqslant a$, or if $a(d-c) \leqslant d$,
- $\partial^2\varphi/\partial z^2$ is bounded if $b \geqslant 2a$, or if $2a(d-c) \leqslant d$,
- $\partial^2\varphi/\partial x\partial z$ is bounded if $b \geqslant a$, $c \geqslant a$ or if $2a(d-c) \leqslant d-a$.

In the complex case, if $a = 1$ the $L$-conditions hold, while if $a > 1$ and $d > c$, Whitney's condition $(b)$ is not satisfied ([**9**], [**10**]), and so the $L$-conditions fail. When $d \leqslant c$, we can prove that the $L$-conditions are satisfied by the mean value theorem.

*Acknowledgements*. — We are grateful to the referee for pertinent comments on a previous version of this paper. We also thank Claudio Murolo and Guillaume Valette for help with the preparation of the manuscript.

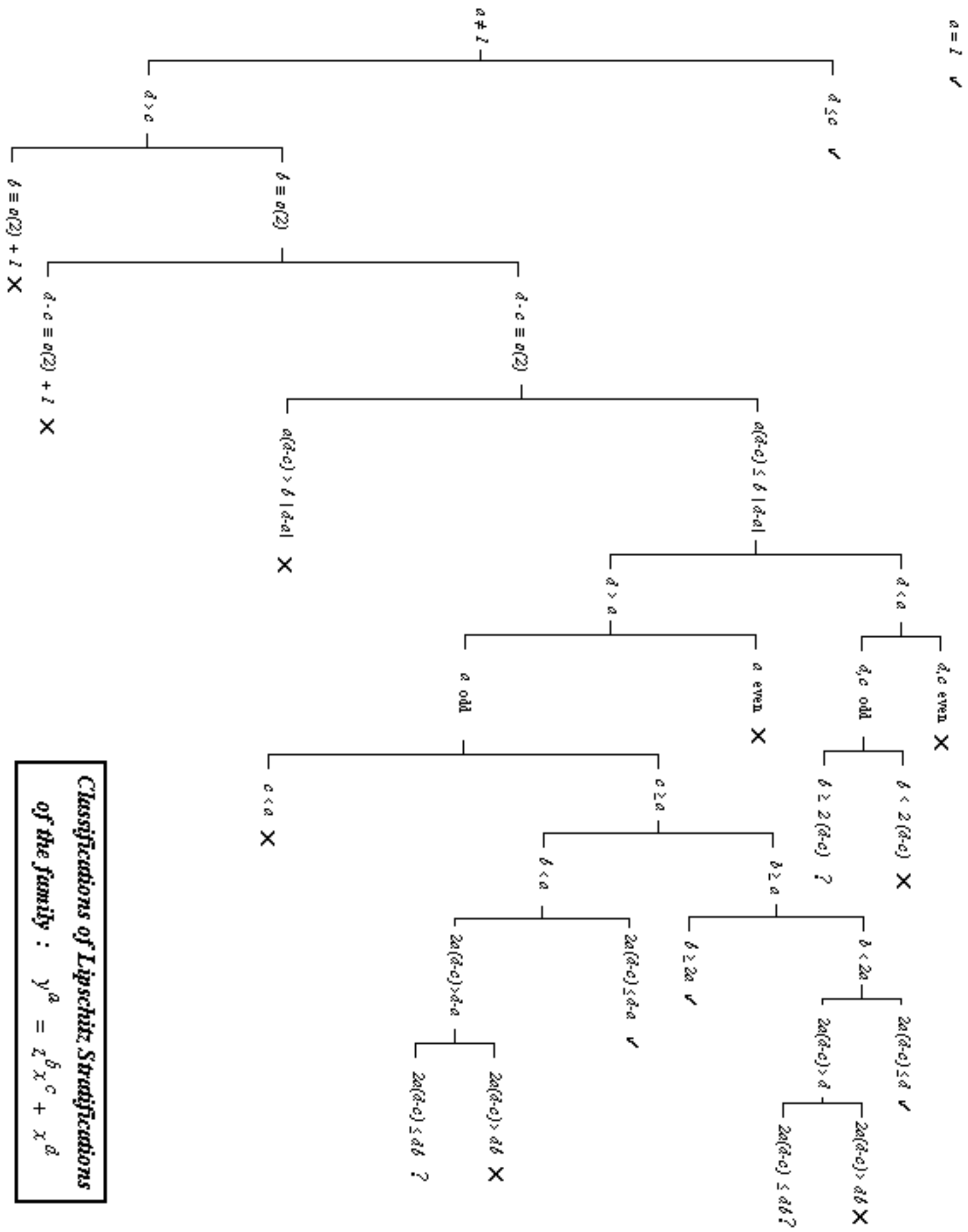


FIGURE 1. $L-$regularity in $\mathbb{R}^3$. Note: $\surd$ means that regularity holds, $\times$ means that regularity fails, $b \equiv 0(2)$ means that $b$ is even, etc.

## References


[1] D. Juniati – De la régularité Lipschitz des espaces stratifiés, thèse, Université de Provence, 2002.

[2] D. Juniati, D. Trotman & G. Valette – Lipschitz stratifications and generic wings, *J. London Math. Soc. (2)* **68** (2003), p. 133–147.

[3] T. Mostowski – *Lipschitz equisingularity*, Dissertationes Math., vol. 243, PWN, Warsaw, 1985.

[4] L. Noirel – Plongements sous-analytiques d'espaces stratifies de Thom-Mather, thèse, Université de Provence, 1996.

[5] A. Parusinski – Lipschitz properties of semi-analytic sets, *Ann. Inst. Fourier (Grenoble)* **38** (1988), p. 189–213.

[6] ———, Lipschitz stratification of real analytic sets, in *Singularities* (S. Lojasiewicz, ed.), Banach Center publications, vol. 20, PWN, Warsaw, 1988, p. 323–333.

[7] ———, Lipschitz stratification of subanalytic sets, *Ann. scient. Éc. Norm. Sup.* 4e *série* **27** (1994), no. 6, p. 661–696.

[8] B. Teissier – Introduction to equisingularity problems, in *Algebraic Geometry Symposium (Arcata, 1974)*, American Mathematical Society, Providence, RI, 1975, p. 593–632.

[9] D. Trotman – Whitney stratifications : faults and detectors, Ph.D. Thesis, Warwick, 1977.

[10] ———, On the canonical Whitney stratifications of algebraic hypersurface, in *Séminaire sur la géométrie algébrique réelle* (J.-J. Risler, ed.), Publ. Math., vol. 24, Univ. Paris VII, Paris, 1987, p. 123–152.



D. Juniati, Laboratoire d'Analyse, Topologie et Probabilités, UMR 6632 du CNRS, Université de Provence, 39 rue Joliot-Curie, 13453 Marseille, France • Dept. of Mathematics, Faculty of Natural Sciences, Universitas Negeri Surabaya (UNESA), Indonesia
*E-mail :* dwi_juniati@yahoo.com

D. Trotman, Laboratoire d'Analyse, Topologie et Probabilités, UMR 6632 du CNRS, Université de Provence, 39 rue Joliot-Curie, 13453 Marseille, France • *E-mail :* trotman@cmi.univ-mrs.fr