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ON ARRANGEMENTS OF THE ROOTS OF A HYPERBOLIC POLYNOMIAL AND OF ONE OF ITS DERIVATIVES

by

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Abstract. — We consider real monic *hyperbolic* polynomials in one real variable, *i.e.* polynomials having only real roots. Call *hyperbolicity domain* Π of the family of polynomials $P(x,a) = x^n + a_1x^{n-1} + \cdots + a_n$, $a_i, x \in \mathbf{R}$, the set $\{a \in \mathbf{R}^n \mid P \text{ is hyperbolic}\}$. The paper studies a stratification of Π defined by the arrangement of the roots of P and $P^{(k)}$, where $2 \leqslant k \leqslant n-1$. We prove that the strata are smooth contractible semi-algebraic sets.

***Résumé* (Sur les arrangements des racines d'un polynôme hyperbolique et d'une de ses dérivées)**

Nous considérons des polynômes moniques *hyperboliques* à une variable réelle, c'est-à-dire des polynômes dont toutes les racines sont réelles. Définissons le *domaine d'hyperbolicité* Π de la famille de polynômes $P(x,a) = x^n + a_1x^{n-1} + \cdots + a_n$, $a_i, x \in \mathbf{R}$, comme l'ensemble $\{a \in \mathbf{R}^n \mid P \text{ est hyperbolique}\}$. L'article étudie la stratification de Π définie par l'arrangement des racines de P et de $P^{(k)}$, où $2 \leqslant k \leqslant n-1$. Nous montrons que les strates sont des ensembles lisses, contractibles et semi-algébriques.

1. Introduction

In the present paper we consider real monic *hyperbolic* (resp. *strictly hyperbolic)* polynomials in one real variable, *i.e.* polynomials having only real (resp. only real distinct) roots. If a polynomial is (strictly) hyperbolic, then so are all its non-trivial derivatives.

Consider the family of polynomials $P(x,a) = x^n + a_1x^{n-1} + \cdots + a_n$, $a_i, x \in \mathbf{R}$. Call *hyperbolicity domain* Π the set $\{a \in \mathbf{R}^n \mid P \text{ is hyperbolic}\}$. The paper studies a stratification of Π defined by the *configuration* (we write sometimes *arrangement*) of the roots of P and $P^{(k)}$, where $2 \leqslant k \leqslant n-1$. The study of this stratification began in [**KoSh**], see also [**Ko1**] and [**Ko2**] for the particular cases $n = 4$ and $n = 5$.

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Properties of Π were proved in [**Ko3**] and [**Ko4**], the latter two papers use results of V.I. Arnold (see [**Ar**]), A.B. Givental (see [**Gi**]) and I. Meguerditchian (see [**Me1**] and [**Me2**]).

Notation 1. — Denote by $x_1 \leqslant \cdots \leqslant x_n$ the roots of P and by $\xi_1 \leqslant \cdots \leqslant \xi_{n-k}$ the ones of $P^{(k)}$. We write sometimes $x_i^{(k)}$ instead of ξ_i if the index k varies. Denote by $y_1 < \cdots < y_q$ the *distinct* roots of P and by $m_1, \ldots, m_q$ their multiplicities (hence, $m_1 + \cdots + m_q = n$).

The classical Rolle theorem implies that one has the following chain of inequalities:

$$x_i \leqslant \xi_i \leqslant x_{i+k}, \quad i = 1, \ldots, n-k \tag{1}$$

Definition 2. — A *configuration vector (CV)* of *length* n is a vector whose components are either positive integers (sometimes indexed by the letter a, their sum being n) or the letter a. The integers equal the multiplicities of the roots of P, the letters a indicate the positions of the roots of $P^{(k)}$; m_a means that a root of P of multiplicity $m < k$ coincides with a simple root of $P^{(k)}$. A CV is called *a priori admissible* if inequalities (1) hold for the configuration of the roots of P and $P^{(k)}$ defined by it.

Remark 3. — If a root of P of multiplicity $< k$ is also a root of $P^{(k)}$, then it is a simple root of $P^{(k)}$, see Lemma 4.2 from [**KoSh**]. By definition "a root of multiplicity 0" means "a non-root".

Example 4. — For $n = 8$, $k = 3$ the CV $(1, a, 1, 2_a, a, a, 4)$ (which is a priori admissible) means that the roots x_j and ξ_i are situated as follows: $x_1 < \xi_1 < x_2 < x_3 = x_4 = \xi_2 < \xi_3 < \xi_4 < x_5 = \cdots = x_8 = \xi_5$. The multiplicity 4 is not indexed with a because it is $> k$, *i.e.* it automatically implies $x_5 = \cdots = x_8 = \xi_5$.

Definition 5. — Given a hyperbolic polynomial P call *roots of class B* (resp. *roots of class A*) its roots of multiplicity $< k$ which coincide with roots of $P^{(k)}$ (resp. all its other roots). In a CV the roots of class B correspond to multiplicities indexed by a.

Definition 6. — For a given CV $\vec{v}$ call *stratum* of Π (defined by $\vec{v}$) its subset of polynomials P with configuration of the roots of P and $P^{(k)}$ defined by $\vec{v}$.

The aim of the present paper is to prove the following

Theorem 7. — *All strata of this stratification are smooth contractible real semi-algebraic sets. Their closures are real algebraic varieties.*

The theorem is proved in Section 5. That the strata mentioned above define a true stratification is shown in Remark 15.

Remark 8. — It is shown in [**KoSh**], Theorem 4.4, that every a priori admissible CV defines a non-empty connected stratum. The essentially new result of the present paper is the proof not only of connectedness but of contractibility. In [**Ko5**] the notion of a priori admissible CV is generalized in the case of not necessarily hyperbolic polynomials and it is shown there that all such CVs are realizable by the arrangements of the real roots of polynomials P and of their derivatives $P^{(k)}$ (the position and multiplicity of the complex roots is not taken into account there).

Notation 9. — We denote by $D(i,j)$ the *discriminant set* $\{a \in \mathbf{R}^n \mid \mathrm{Res}(P^{(i)}, P^{(j)})=0\}$ (recall that for $a \in \Pi$ one has $\mathrm{Res}(P^{(i)}, P^{(j)}) = 0$ if and only if $P^{(i)}$ and $P^{(j)}$ have a common root).

Let $a_0 \in D(0,k) \cap \mathrm{Int}\,\Pi$ be such that for a_0 there hold exactly s equalities of the form $x_j^{(k)} = x_i$, with s different indices j and s different indices i.

Proposition 10. — *In a neighbourhood of the point a_0 the set $D(0,k)$ is locally the union of s smooth hypersurfaces intersecting transversally at a_0.*

All propositions are proved in Section 4. The proposition can be generalized in the following way. Suppose that at a point a_0 lying in the interior of Π there hold exactly s equalities $x_j^{(k_i)} = x_i$, with s different indices i and s different couples (k_i, j).

Proposition 11. — *In a neighbourhood of the point a_0 these s equalities define s smooth hypersurfaces intersecting transversally at a_0.*

Remark 12. — It is shown in [**Ko3**] that for each q-tuple of positive integers m_j with sum n the subset T of Π (we call it a *stratum* of Π defined by the *multiplicity vector* $(m_1, \dots, m_q)$, not by a CV) consisting of polynomials with distinct roots y_i, of multiplicities m_i, is a smooth variety of dimension q in $\mathbf{R}^n$.

Denote by T a stratum of Π defined by a multiplicity vector. Fix a point $G \in T$. Suppose that at G there are s among the roots y_j which are of class B. Suppose that one has $m_i < k$ for all i. The condition $m_i < k$ implies that all points from $D(0,k) \cap T$ close to G result from roots of $P^{(k)}$ coinciding with roots of P of class B.

Proposition 13. — *In a neighbourhood of the point G the set $D(0,k) \cap T$ is locally the union of s smooth codimension 1 subvarieties of T intersecting transversally at G.*

Remarks 14. — 1) A stratum of Π of codimension $\kappa \leqslant k$ defined by κ equalities of the form $x_i = \xi_j$ (*i.e.* P has no multiple root) has a tangent space transversal to the space $Oa_{n-\kappa+1} \dots a_n$. Indeed, the roots ξ_j depend smoothly on $a_1, \dots, a_{n-k}$, and the conditions $P(\xi_j, a) = 0$ allow one to express $a_{n-\kappa+1}, \dots, a_n$ as smooth functions of $a_1, \dots, a_{n-\kappa}$ (use Vandermonde's determinant with distinct arguments $\xi_1, \dots, \xi_\kappa$). It would be nice to prove or disprove the statements:

A) this property holds without the assumption $\kappa \leqslant k$ and that P has no multiple root;

B) the limit of the tangent space to a stratum, when a stratum in its closure is approached, exists and is transverse to the space $Oa_{n-\kappa+1} \dots a_n$.

For $n = 4$ and $n = 5$ this seems to be true, see [**Ko1**] and [**Ko2**]. The statements would be a generalization of such a transversality property of the strata of Π defined by multiplicity vectors, not by CVs (proved in [**Ko3**], Theorem 1.8; see Remark 12). Outside Π the first statement is not true − for $n = 4$, $a_1 = 0$, the discriminant set $D(0, 2)$ has a Whitney umbrella singularity at the origin and there are points where its tangent space is parallel to Oa_4; this can be deduced from [**Ko1**] (see Section 3 and Lemma 29 in it).

2) In [**KoSh**], [**Ko1**] and [**Ko2**] a stratification of Π defined by the arrangement of all roots of $P, P', \ldots, P^{(n-1)}$ is considered (the initial idea to consider this stratification belongs to B.Z. Shapiro). The results of the present paper cannot be transferred directly to that case for two reasons:

a) for $n \geqslant 4$ not all arrangements consistent with (1) are realized by hyperbolic polynomials and it is not clear how to determine for any $n \in \mathbf{N}^*$ the realizable ones (*e.g.* for $n = 4$ only 10 out of 12 such arrangements are realized, see [**KoSh**] or [**Ko1**]; for $n = 5$ only 116 out of 286, see [**Ko1**]); the reason for this is clear − a monic polynomial has only n coefficients that can be varied whereas there are $n(n + 1)/2$ roots of $P, P', \ldots, P^{(n-1)}$;

b) for $n \geqslant 4$ there are *overdetermined strata*, *i.e.* strata on which the number of equalities between any two of the roots of $P, P', \ldots, P^{(n-1)}$ is greater than the codimension of the stratum.

In Section 3 we prove two technical lemmas (and their corollaries) used in the proof of the theorem and the propositions. Section 2 is devoted to the dimension of a stratum and its relationship with the CV defining it. The above propositions are just the first steps in the study of the set $D(0, 1) \cup D(0, k)$ (and, more generally, of the set $D(0, 1) \cup \cdots \cup D(0, n - 1)$) at a point of Π.

2. Configuration vectors and dimensions of strata

In this section we recall briefly results some of which are from [**KoSh**]:

1) Call *excess of multiplicity* of a CV the sum $m = \sum(m_j - 1)$ taken over all multiplicities m_j of distinct roots of P. A *stratum* of codimension i is defined by a CV which has exactly $i - m$ letters a as indices, *i.e.* the polynomial P has exactly $i - m$ distinct roots of class B.

2) A stratum of codimension i is locally a smooth real algebraic variety of dimension $n - i$ in $\mathbf{R}^n$.

3) In what follows we say a stratum of codimension i to be of dimension $n-i-2$. We decrease its dimension in $\mathbf{R}^n$ by 2 to factor out the possible shifting of the variable x by a constant and the one-parameter group of transformations $x \mapsto \exp(t)x$, $a_j \mapsto \exp(jt)a_j$, $t \in \mathbf{R}$; both of them leave CVs unchanged. This allows one to consider the

family P only for $a_1=0$, $a_2=-1$ (if $a_1=0$, then there are no hyperbolic polynomials for $a_2>0$ and for $a_2=0$ the only one is x^n).

4) In accordance with the convention from 3), it can be deduced from 1) that the CVs defining strata of dimension δ are exactly the ones in which the polynomial P has $\delta+2$ distinct roots of class A, *i.e.* these are CVs having $\delta+2$ components which are multiplicities of roots of P not indexed by the letter a.

5) A point of a stratum of codimension $i>1$ defined by a CV $\vec{v}$ belongs to the closure of any stratum of codimension $i-1$ whose CV $\vec{w}$ is obtained from $\vec{v}$ by means of one of the following three operations:

i) if $\vec{v}=(A,l_a,B)$, $l\leqslant k-1$, A and B are non-void, then $\vec{w}=(A,l,a,B)$ or $\vec{w}=(A,a,l,B)$;

ii) if $\vec{v}=(A,r_a,B)$, $r\leqslant k-1$, A and B are non-void, then $\vec{w}=(A,r',r''_a,B)$ or $\vec{w}=(A,r'_a,r'',B)$, $r'>0$, $r''>0$, $r'+r''=r$.

iii) if $\vec{v}=(A,r,B)$, then $\vec{w}=(A,C,B)$ where C is a CV defining a stratum of dimension 0 in $\mathbf{R}^r$, see 4).

6) It follows from the definition of the codimension of a stratum that the three possibilities *i)*, *ii)* and *iii)* from 5) are the only ones to increase by 1 the dimension of a stratum S when passing to a stratum containing S in its closure. Indeed, one has to increase by 1 the number of roots of class A, see 4). If to this end one has to change the number or the multiplicities of the roots of class B, then there are no possibilities other than *i)* and *ii)*. If not, then exactly one root x_i of class A must bifurcate, the roots stemming from it and the roots of $P^{(k)}$ close to x_i must define an a priori admissible CV (they must satisfy conditions (1)), and among these roots there must be exactly two of class A. Hence, the bifurcating roots must define a CV of dimension 0 in $\mathbf{R}^r$, see 4).

Remark 15. — The strata define a true stratification in the sense that they are connected components of differences of closed sets of a filtration. Indeed, the filtration is defined by the codimension of the strata. Contractibility (hence, connectedness) follows from Theorem 7. To obtain a stratum as a difference of closed sets one can represent it as the difference between its closure Z and the closure of the union of all strata of lower dimension belonging to Z.

3. Two technical lemmas and their corollaries

For a monic strictly hyperbolic polynomial P of degree n consider the roots $x_j^{(k)}$ of $P^{(k)}$ as functions of the roots x_i of P. Hence, these functions are smooth because the roots $x_j^{(k)}$ are simple, see Remark 3.

Lemma 16. — *For $i=1,\dots,n$, $k=1,\dots,n-1$, $j=1,\dots,n-k$ one has $\partial(x_j^{(k)})/\partial(x_i)>0$.*

Proof. — 1°. Set $x_i = c$, $P = (x-c)Q(x)$, $\deg Q = n-1$, $\xi = x_j^{(k)}$. We prove that for $k=1$ one has $\partial\xi/\partial c > 0$. One has $(\xi - c)Q'(\xi) + Q(\xi) = 0$. Hence,

$$\left(\frac{\partial\xi}{\partial c} - 1\right) Q'(\xi) + (\xi - c)Q''(\xi)\frac{\partial\xi}{\partial c} + Q'(\xi)\frac{\partial\xi}{\partial c} = 0, \ \ i.e.$$

$$\frac{\partial\xi}{\partial c} = \frac{Q'(\xi)}{(\xi-c)Q''(\xi) + 2Q'(\xi)} = \frac{Q'(\xi)}{P''(\xi)}.$$

As $Q(\xi) = P(\xi)/(\xi - c)$, one has

$$\frac{\partial\xi}{\partial c} = \frac{(\xi-c)P'(\xi) - P(\xi)}{(\xi-c)^2 P''(\xi)} = -\frac{P(\xi)}{(\xi-c)^2 P''(\xi)} \tag{2}$$

For a strictly hyperbolic monic polynomial the signs of $P(\xi)$ and $P''(\xi)$ are opposite and $\xi \neq c$. This proves the lemma for $k=1$.

2°. For $k > 1$ use induction on k. Considering the roots of $P^{(k+1)}$ as functions of the ones of $P^{(k)}$ one can write

$$\frac{\partial(x_j^{(k+1)})}{\partial c} = \sum_{\nu=1}^{n-k} \frac{\partial(x_j^{(k+1)})}{\partial(x_\nu^{(k)})}\,\frac{\partial(x_\nu^{(k)})}{\partial c} \tag{3}$$

and observe that all factors in the right hand-side are > 0. The lemma is proved. □

Remark 17. — The roots $x_j^{(k)}$ are C^1-smooth functions of the roots x_i (one can forget for a moment that $x_1 \leqslant \cdots \leqslant x_n$ and assume that $(x_1, \ldots, x_n) \in \mathbf{R}^n$ and the claim is true for not necessarily strictly hyperbolic polynomials; however, in order to define correctly $x_j^{(k)}$ one has to impose the condition $x_1^{(k)} \leqslant \cdots \leqslant x_{n-k}^{(k)}$). Indeed, it suffices to prove this for $k=1$ (because in the same way one proves that the roots of $P^{(\nu+1)}$ are C^1-smooth functions of the roots of $P^{(\nu)}$ for $\nu = 1, \ldots, n-2$ etc.). For $k=1$ the claim can be deduced from equality (2) – the fraction in the right hand-side has a finite limit for $\xi \to c$ (this limit depends on the order of c as a zero of P) and for ξ close to c it is a function continuous in c. We leave the details for the reader.

Corollary 18. — *For a (not necessarily strictly) hyperbolic polynomial one has* $\partial(x_j^{(k)})/\partial(x_i) \geqslant 0$ *for* i, j, k *as in the lemma.*

The corollary is automatic.

Corollary 19. — *For a monic strictly hyperbolic polynomial one has* $0 < \partial(x_j^{(k)})/\partial(x_i) < (n-k)/n$ *(for* i, j, k *as in the lemma) and* $\sum_{j=1}^{n-k} \partial(x_j^{(k)})/\partial(x_i) = (n-k)/n$.

Proof. — By Vieta's formulas one has $x_1 + \cdots + x_n = -a_1$, $x_1^{(k)} + \cdots + x_{n-k}^{(k)} = -\frac{n-k}{n}a_1$. As $\partial(x_j^{(k)})/\partial(x_i) > 0$ for all j, one has

$$\frac{\partial(x_j^{(k)})}{\partial(x_i)} < \frac{\partial(x_1^{(k)} + \cdots + x_{n-k}^{(k)})}{\partial(x_i)} = \frac{(n-k)}{n}\,\frac{\partial(x_1 + \cdots + x_n)}{\partial(x_i)} = \frac{n-k}{n}$$

which proves the corollary. □

Remark 20. — In the above corollary one sums up w.r.t. the index j. When summing up w.r.t. i one obtains the equality

$$\sum_{i=1}^{n} \frac{\partial(x_j^{(k)})}{\partial(x_i)} = 1 \tag{4}$$

Indeed, if the roots x_i are functions of one real parameter (say, τ), then one has the equality $\sum_{i=1}^{n} \frac{\partial(x_j^{(k)})}{\partial(x_i)} \dot{x}_i = \dot{x}_j^{(k)}$ where $\dot{x}_i$ stands for $dx_i/d\tau$. When one has $\dot{x}_i = 1$ for all i, *i.e.* the variable x is shifted with constant speed 1, then one has $\dot{x}_j^{(k)} = 1$ for all k, j and one gets (4). One needs not suppose the roots x_i distinct.

In the case of a not strictly hyperbolic polynomial P consider the roots $x_j^{(k)}$ as functions of the distinct roots y_i of P (their multiplicities remain fixed).

Lemma 21. — *For $i = 1, \dots, q$, $k = 1, \dots, n-1$ one has $\partial(x_j^{(k)})/\partial(y_i) \geqslant 0$ with equality exactly if $x_j^{(k)}$ is a root of P of multiplicity $\geqslant k$ (hence, of multiplicity $\geqslant k+1$) and $x_j^{(k)} \neq y_i$.*

Proof. — 1°. The proof follows the same ideas as the proof of Lemma 16. Set $\xi = x_j^{(k)}$, $P = (x-c)^s Q(x)$ where $c = y_i$, $s = m_i$ for some i, $1 \leqslant i \leqslant q$.

Let first $k = 1$. One has $(\xi - c)^s Q'(\xi) + s(\xi - c)^{s-1} Q(\xi) = 0$. Hence,

$$\begin{aligned} s\left(\frac{\partial\xi}{\partial c} - 1\right)(\xi - c)^{s-1} Q'(\xi) &+ (\xi - c)^s Q''(\xi) \frac{\partial\xi}{\partial c} \\ &+ s(\xi - c)^{s-1} Q'(\xi) \frac{\partial\xi}{\partial c} + s(s-1)(\xi - c)^{s-2}\left(\frac{\partial\xi}{\partial c} - 1\right) Q(\xi) = 0, \ \textit{i.e.} \end{aligned}$$

$$\begin{aligned} \frac{\partial\xi}{\partial c} &= \frac{s(s-1)Q(\xi) + s(\xi - c)Q'(\xi)}{s(s-1)Q(\xi) + 2s(\xi - c)Q'(\xi) + (\xi - c)^2 Q''(\xi)} \\ &= \frac{s((\xi - c)P'(\xi) - P(\xi))}{(\xi - c)^2 P''(\xi)}. \end{aligned}$$

If $\xi = c$, *i.e.* $s > 1$, then $\partial\xi/\partial c = 1$. If not, then $\partial\xi/\partial c = -P(\xi)/(\xi - c)^2 P''(\xi)$. Either $P(\xi) = P'(\xi) = 0$ and in this case $\partial\xi/\partial c = 0$ whatever the multiplicity of ξ as a root of P is, or $P(\xi) \neq 0$, $P(\xi)$ and $P''(\xi)$ have opposite signs and $\partial\xi/\partial c > 0$. This proves the lemma for $k = 1$.

2°. For $k > 1$ use induction on k. Consider the roots of $P^{(k+1)}$ as functions of the ones of $P^{(k)}$. Then there holds (3). All factors in the right hand-side are $\geqslant 0$.

One has $\partial(x_j^{(k+1)})/\partial c = 0$ exactly if in every summand in the right hand-side of (3) at least one of the two factors is 0. This is the case if $\xi = x_j^{(k+1)}$ is a root of P of multiplicity $\geqslant k+1$ and $\xi \neq c$. Indeed, in this case one has $\partial(x_j^{(k+1)})/\partial(x_\nu^{(k)}) = 0$ if $x_j^{(k+1)} \neq x_\nu^{(k)}$ and $\partial(x_\nu^{(k)})/\partial c = 0$ if $x_j^{(k+1)} = x_\nu^{(k)}$ (and, hence, $x_\nu^{(k)} \neq c$).

If ξ is a root of P of multiplicity $\geqslant k+1$ and $\xi = c$, then one has $\partial(x_j^{(k+1)})/\partial c = 1$.

If ξ is a root of P of multiplicity $\leqslant k$, then it is not a root of $P^{(k)}$. Hence, $\partial(x_j^{(k+1)})/\partial(x_\nu^{(k)}) > 0$ for all ν. At least one of the factors $\partial(x_\nu^{(k)})/\partial c$ is > 0 (*i.e.* for at least one ν). Indeed, if c is a root of P of multiplicity $\geqslant k+1$, then this is true for the root $x_\nu^{(k)}$ which equals c (by inductive assumption). If c is a root of P of multiplicity $\leqslant k$, then there exists a simple root $x_\nu^{(k)}$ of $P^{(k)}$ (this follows from Rolle's theorem applied k times). Hence, $x_\nu^{(k)}$ is a root of P of multiplicity $\leqslant k-1$, and for this root one has $\partial(x_\nu^{(k)})/\partial c > 0$.

The lemma is proved. □

Corollary 22. — *For a monic hyperbolic polynomial one has* $0 \leqslant \partial(x_j^{(k)})/\partial(y_i) \leqslant (n-k)/n$ *(for* i,j,k *as in the lemma) and* $\sum_{j=1}^{n-k} \partial(x_j^{(k)})/\partial(y_i) = (n-k)/n$.

The corollary is proved by analogy with Corollary 19.

4. Proofs of the propositions

Proof of Proposition 10. — We first prove the smoothness. The roots $x_j^{(k)}$ are smooth functions of the coefficients $a_1, \ldots, a_{n-k}$. The condition $P(x_j^{(k)}, a) = 0$ allows one to express a_n as a smooth function of $a_1, \ldots, a_{n-1}$. Hence, this equation defines locally a smooth hypersurface in $\mathbf{R}^n$.

To prove the transversality assume first that the indices are changed so that $i = j = 1, \ldots, s$. It suffices to prove that the "Jacobian" matrix $\left\{\partial(x_j - x_j^{(k)})/\partial(x_\nu)\right\}$, $j, \nu = 1, \ldots, s$, is of maximal rank (in the true Jacobian matrix one has $\nu = 1, \ldots, n$, not $\nu = 1, \ldots s$). Its diagonal entries equal $1 - \partial(x_j^{(k)})/\partial(x_j)$ while its non-diagonal ones equal $-\partial(x_j^{(k)})/\partial(x_\nu)$. Corollary 19 implies that the matrix is diagonally dominated – for ν fixed its diagonal entry (which is positive) is greater than the sum of the absolute values of its non-diagonal entries (which are all negative). Hence, the matrix is non-degenerate. □

Proof of Proposition 11. — The proof of the smoothness is done as in the proof of Proposition 10. To prove the transversality assume again that $i = j = 1, \ldots, s$ and consider again the "Jacobian" matrix $\left\{\partial(x_j - x_j^{(k_j)})/\partial(x_\nu)\right\}$, $j, \nu = 1, \ldots, s$. As in the previous proof we show that the matrix is diagonally dominated, hence, non-degenerate. However, the numbers k_j are not necessarily the same and therefore we fix j (hence, k_j as well) and we change ν. By equality (4), one has

$$\sum_{\nu=1}^{s} \frac{\partial(x_j - x_j^{(k_j)})}{\partial(x_\nu)} = 1 - \sum_{\nu=1}^{s} \frac{\partial(x_j^{(k_j)})}{\partial(x_\nu)} \geqslant 1 - \sum_{\nu=1}^{n} \frac{\partial(x_j^{(k_j)})}{\partial(x_\nu)} = 0$$

and the case of equality has to be excluded because the smallest and the greatest root of P are not among the roots $x_1, \ldots, x_s$ and all partial derivatives are strictly positive, see Lemma 16. The last inequality implies that the matrix is diagonally dominated. □

Proof of Proposition 13. — The proof is almost a repetition of the one of Proposition 10. The only difference is that the Jacobian matrix looks like this: $\left\{\partial(y_j - m_\nu x_j^{(k)})/\partial(y_\nu)\right\}$ (recall that y_ν, of multiplicity m_ν, are the distinct roots of P). □

5. Proof of Theorem 7

1°. Smoothness is proved in [**KoSh**], Proposition 4.5; it is evident that the strata are semi-algebraic sets and that their closures are algebraic varieties. So one has to prove only contractibility. Assume that $a_1 = 0$, $a_2 = -1$.

To prove contractibility of the strata represent each stratum T of dimension $\delta \geqslant 1$ as a fibration whose fibres are one-dimensional varieties with the following properties:

a) the fibres are phase curves of a smooth vectorfield without zeros defined on T; hence, each fibre can be smoothly parametrized by $\tau \in (0, 1)$; this is proved in 2°–4°;

b) the limits for $\tau \to 1$ of the points of the fibres exist and they belong to a finite union $\mathcal{U}$ of strata of lower dimension; we call the limits *endpoints*; the proof of this is given in 3°–5°;

c) the union $\mathcal{U}$ is a contractible set (proved in 7°–8°);

d) each point of the union $\mathcal{U}$ is the endpoint of some fibre (proved in 6°).

Thus the union $\mathcal{U}$ is a retract of the given stratum and contractibility of $\mathcal{U}$ implies the one of the stratum. Contractibility of the strata of dimension 0 will be proved directly (in 7°).

2°. A shift γ_1 and a rescaling γ_2 of the x-axis fix the smallest root of P at 0 and the greatest one at 1. Set $\gamma = \gamma_2 \circ \gamma_1$.

Notation 23. — Denote by Δ the set of monic hyperbolic polynomials obtained from the stratum T by applying the transformation γ to each point of T.

Remark 24. — The set Δ (like T) is a smooth variety of dimension δ. The transformation γ defines a diffeomorphism $\overline{T} \to \overline{\Delta}$ while γ^{-1} defines a diffeomorphism $\overline{\Delta} \to \overline{T}$; this can be deduced from the conditions $a_1 = 0$, $a_2 = -1$.

3°. Recall that y_i denotes the distinct roots of P. We construct (see 4°–5°) the speeds $\dot{y}_i$ on Δ which amounts to constructing a vectorfield defined on Δ. Therefore the fibration from 1° can be defined by means of the phase curves of a vectorfield defined on T (to this end one has to apply γ^{-1}). We leave the technical details for the reader.

Remark 25. — It follows from our construction (see in particular part 3) of Lemma 26) that these two vectorfields can be continuously extended respectively on $\overline{\Delta}$ and $\overline{T}$.

Along a phase curve of the vectorfield, all roots of P of class A except one (in particular, the smallest and the greatest one) do not change their position and multiplicity; the rest of the roots of P do not change their multiplicity. The limits (forwards and backwards) of the points of the phase curves exist when the boundary of Δ is approached. At these limit points, if a confluence of roots of P occurs, then the multiplicities of the coinciding roots are added. The images under γ^{-1} of the forward limits are the endpoints (see b) from 1°).

Denote by P_σ ($\sigma \in \mathbf{R}$) a family of monic hyperbolic polynomials represented by the points of a given phase curve in Δ. We prove in 4° that there exists $\sigma_0 > 0$ such that for $\sigma \in [0, \sigma_0)$ one has $P_\sigma \in \Delta$ (hence, $\gamma^{-1}(P_\sigma) \in T$) while $P_{\sigma_0} \notin \Delta$ (hence, $\gamma^{-1}(P_{\sigma_0}) \notin T$). The polynomial P_{σ_0} represents the forward limit point of the given phase curve. We set $\dot{y}_i = dy_i/d\sigma$.

4°. Change for convenience (in 4°–6°) the indices of the distinct roots y_i of P and of the roots ξ_i of $P^{(k)}$. Choose a root of class A different from the smallest and the greatest one. Denote it by y_1. Denote by $y_2, \ldots, y_d$ the roots of class B and by $\xi_2, \ldots, \xi_d$ the roots of $P^{(k)}$ which are equal to them.

Set $\dot{y}_1 = 1$. We look for speeds $\dot{y}_i$ for which one has $\dot{y}_i = \dot{\xi}_i$, $i = 2, \ldots, d$. Hence, one would have $y_i = \xi_i$, $i = 2, \ldots, d$, and the multiplicities of the roots of P do not change for $\sigma > 0$ close to 0. This means that for all such values of σ for which the order of the union of roots of P and $P^{(k)}$ is preserved, the point $\gamma^{-1}(P_\sigma)$ belongs to T. The value σ_0 (see 3°) corresponds to the first moment when a confluence of roots of P or of a root of P and a root of $P^{(k)}$ occurs (such a confluence occurs at latest for $\sigma = 1$ because $\dot{y}_1 = 1$ while the smallest and the greatest roots of P remain equal respectively to 0 and 1).

Lemma 26. — *1) One can define the speeds $\dot{y}_i$, $i = 2, \ldots, d$, in a unique way so that $\dot{y}_i = \dot{\xi}_i$, $i = 2, \ldots, d$.*

2) For these speeds one has $0 \leqslant \dot{y}_i \leqslant 1$.

3) The speeds are continuous and bounded on $\overline{\Delta}$ and smooth on Δ.

The lemma is proved after the proof of the theorem.

Remark 27. — The lemma implies property a) of the fibration from 1°. The absence of stationary points in the vectorfield on Δ results from $\dot{y}_i \geqslant 0$, $\dot{y}_1 = 1$ which implies that $\dot{a}_1 < 0$. As γ^{-1} is a diffeomorphism, the vectorfield on T has no stationary points either.

5°. The lemma implies that for $\sigma = \sigma_0$ one or several of the following things happen:

– a root ξ_{i_0} of $P^{(k)}$ which is not a root of P becomes equal to a root y_{j_0} of P of class A different from y_1, from the smallest and from the greatest one; for $\sigma \in [0, \sigma_0)$ one has $\xi_{i_0} < y_{j_0}$; this is the contrary to what happens in *i)* from 5) of Section 2;

– the root y_1 becomes equal to a root ξ_{i_1} of $P^{(k)}$ (and eventually to y_{i_1} if y_{i_1} is a root of class B); for $\sigma \in [0, \sigma_0)$ one has $y_1 < \xi_{i_1}$ and ξ_{i_1} is not a root of P; this is the contrary to what happens in *i)* or *ii)* from 5) of Section 2;

– the root y_1 becomes equal to a root y_{i_2} of class A; for $\sigma \in [0, \sigma_0)$ one has $y_1 < y_{i_2}$; there might be roots of $P^{(k)}$ (and eventually roots of P of class B) between y_1 and y_{i_2}; this is the contrary to what happens in *iii)* from 5) of Section 2.

Remarks 28. — 1) If the CV allows the third possibility (*i.e.* if the third possibility leads to no contradiction with condition (1) and with Section 2), then it does not allow the second or the first one with $j_0 = i_2$. Indeed, if the third possibility exists, then between y_1 and y_{i_2} there must be $\mu - k$ roots of $P^{(k)}$ counted with the multiplicities where μ is the sum of the multiplicities of y_1, y_{i_2} and of all roots of P (if any) between them; if the second possibility exists as well, then for $\sigma = \sigma_0$ there must be $\mu - k$ roots of $P^{(k)}$ strictly between y_1 and y_{i_2} which means that for $\sigma < \sigma_0$ there were $\mu - k + 1$ of them (one must add the root ξ_{i_1}) – a contradiction. In the same way one excludes the first possibility with $j_0 = i_2$.

2) If the third possibility takes place, then y_{i_2} is the first to the right w.r.t. y_1 of the roots of class A because these roots do not change their positions.

3) Part 1) of these remarks implies that if the CV allows several possibilities of the above three types, with different possible indices i_0, j_0, i_1, i_2 to happen, then they can happen independently and simultaneously (all of them or any part of them). These possibilities can be expressed analytically as conditions (we call them *equalities* further in the text) of the form $y_i = \xi_j$ or $y_{i_1} = y_{i_2}$ for $\sigma = \sigma_0$ while for $\sigma < \sigma_0$ there holds $y_i > \xi_j$ or $y_i < \xi_j$ or $y_{i_1} > y_{i_2}$.

4) Property b) of the fibration from 1° follows from 1) – 3); the CVs of the strata from $\mathcal{U}$ are obtained by replacing certain inequalities between roots by the corresponding equalities in the sense of 3) from these remarks.

6°. Denote by $\mathcal{U}'$ the set of images under γ of strata of Π (we call these images *strata* of $\mathcal{U}'$) whose CVs are obtained from the one of T by replacing some or all inequalities by the corresponding equalities, see part 4) of Remarks 28.

Consider the vectorfield defined on $\Delta \cup \mathcal{U}'$ by the conditions $\dot{y}_1 = -1$ and $\dot{y}_i = \dot{\xi}_i$, $i = 2, \ldots, d$. On each stratum of $\mathcal{U}'$, when defining the vectorfield, some of the multiple roots of P and/or $P^{(k)}$ should be considered as several coinciding roots of given multiplicities. What we are doing resembles an attempt "to reverse the phase curves of the already constructed vectorfield on Δ" (and it is the case on Δ) but we have not proved yet that each point of each stratum of $\mathcal{U}'$ is a limit point of a phase curve of that vectorfield and that each point of $\mathcal{U}'$ belongs to $\overline{\Delta}$. Notice that due to the definition of the vectorfield each phase curve stays in $\Delta \cup \mathcal{U}'$ on some time interval.

Each phase curve of the vectorfield defines a family P_σ of polynomials. It is convenient to choose as parameter again $\sigma \in [0, \sigma_0]$ where the point of the family belongs to $\mathcal{U}'$ for $\sigma = \sigma_0$.

Lemma 29. — *For $\sigma < \sigma_0$ and close to σ_0 the point of the family P_σ belongs to Δ.*

The lemma is proved after the proof of Lemma 26. It follows from the lemma that $\mathcal{U}' \subset \overline{\Delta}$. Hence, one can set $\mathcal{U} = \gamma^{-1}(\mathcal{U}')$ and property d) of the fibration follows.

7°. There remains to be proved that the fibration possesses property c). To this end prove first that all strata of dimension 0 are contractible, *i.e.* connected. Recall that a hyperbolic polynomial from a stratum of dimension 0 has exactly two distinct roots of class A – the smallest and the greatest one (see 4) of Section 2).

The strata of dimension 0 whose CVs contain only two multiplicities are connected. Indeed, the uniqueness of such monic polynomials up to transformations γ, see 2°, is obvious – they equal $x^{m_1}(x-1)^{n-m_1}$.

Prove the uniqueness up to a transformation γ of all polynomials defining strata of dimension 0 by induction on q (the number of distinct roots of P). For $q = 2$ the uniqueness is proved above. Denote by A_i parts (eventually empty) of the CV which are maximal packs of consecutive letters a.

Deduce the uniqueness of the stratum V defined by the CV

$$\vec{v} = (m_1, A_1, (m_2)_a, A_2, (m_3)_a, A_3, \ldots, (m_{q-1})_a, A_{q-1}, m_q)$$

from the uniqueness of the stratum W defined by the CV

$$\vec{w} = (m_1, A_1', (m_2)_a, A_2', (m_3)_a, A_3, \ldots, (m_{q-1})_a, A_{q-1}, m_q)$$

We denote again the distinct roots of P by $0 = y_1 < \cdots < y_q = 1$ (and we change the indices of the roots ξ_i so that on V, $\xi_2, \ldots, \xi_{q-1}$ be equal respectively to $y_2, \ldots, y_{q-1}$). The part A_1' (resp. A_2') contains one letter a more than A_1 (resp. one letter a less than A_2). Possibly A_2' can be empty.

To do this construct a one-parameter family P_σ (depending on $\sigma \in [0, \sigma_0]$) of polynomials joining the two strata (for $\sigma = 0$ we are on V, for $\sigma = \sigma_0$ we are on W); these polynomials belong to the one-dimensional stratum Z defined by the CV

$$\vec{z} = (m_1, A_1', m_2, A_2, (m_3)_a, A_3, \ldots, (m_{q-1})_a, A_{q-1}, m_q)$$

For the root y_2 one has $\dot{y}_2 = 1$. One defines $\dot{y}_i$, $i = 3, \ldots, q-1$ so that $\dot{\xi}_i = \dot{y}_i$. This condition defines them in a unique way (see Lemma 26) and there exists a unique $\sigma_0 > 0$ for which one obtains $\vec{w}$ as CV (this follows from the uniqueness of W – the ratio $(y_2 - y_1)/(y_2 - y_q) = y_2/(y_2 - 1)$ increases strictly with σ which implies the uniqueness of σ_0).

Remark 30. — One has $P_\sigma \in V$ only for $\sigma = 0$, and for $\sigma > 0$ one has $y_2 > \xi_2$. This can be proved by full analogy with Lemma 29.

For $\sigma = \sigma_0$ no confluence of roots of P or of P and $P^{(k)}$ other than the one of y_2 with the left most root of A_2 can take place. This can be deduced by a reasoning similar to the one from part 1) of Remarks 28.

On the other hand, one can reverse the speeds, *i.e.* for the polynomial defining the CV $\vec{w}$ one can set $\dot{y}_2 = -1$, $\dot{\xi}_i = \dot{y}_i$, $i = 3, \ldots, q-1$ and deform it continuously into a polynomial defining the CV $\vec{v}$; the deformation passes through polynomials from the stratum Z. This means that the polynomials defining the strata V and W can be obtained from the family P_σ. The uniqueness of the strata of dimension 0 is proved.

8°. Prove the contractibility of the set $\mathcal{U}$. Each of the strata of $\mathcal{U}$ is defined by a finite number of *equalities* (see part 3) of Remarks 28) which replace inequalities that hold in the CV defining the stratum T. For each stratum of $\mathcal{U}$ of dimension $p > 0$ one can construct a fibration in the same way as this was done for T and show that the stratum can be retracted to a finite subset of the strata from $\mathcal{U}$ which are all of dimension $< p$. Hence, $\mathcal{U}$ can be retracted on its only stratum of dimension 0 (it is defined by all equalities). By 7° this stratum is a point. Hence, $\mathcal{U}$ is contractible, T as well. □

Proof of Lemma 26. — 1°. Fix the index i of a root of class B. Recall that we denote by m_ν the multiplicity of the root y_ν. Set $G_{i,\nu} = (\partial(\xi_i)/\partial(y_\nu))$. One has

$$\dot{\xi}_i = \sum_{\nu=1}^{d} m_\nu G_{i,\nu} \dot{y}_\nu.$$

Hence, the condition $\dot{\xi}_i = \dot{y}_i$ for $i = 2, \ldots, d$ reads:

$$\dot{y}_i = \sum_{\nu=1}^{d} m_\nu G_{i,\nu} \dot{y}_\nu, \quad i = 2, \ldots, d \tag{5}$$

Further in the proof "vector" means "$(d-1)$-vector-column". Denote by V the vector with components $\dot{y}_i$. Hence, the last system can be presented in the form $V = GV + H$ $(*)$ or $(I - G)V = H$ where H is the vector with entries $m_1 G_{i,1}$, $2 \leqslant i \leqslant d$ (recall that $\dot{y}_1 = 1$) and G is the matrix with entries $G_{i,\nu}$, $i, \nu = 2, \ldots, d$.

2°. As in the proof of Proposition 10 one shows that the matrix $I - G$ is diagonally dominated. Hence, system (5) has a unique solution V. Moreover, its components are all non-negative. Indeed, one has $m_1 G_{i,1} \geqslant 0$ for $i = 2, \ldots, d$, all entries of the matrix G are non-negative (see Lemma 16 and Corollary 18), and one can present V as a convergent series $H + GH + G^2 H + \ldots$ whose terms are vectors with non-negative entries. This proves 1) and the left inequality of 2).

3°. To prove the right inequality of 2) denote by V_0 the vector whose components are units; write equation $(*)$ in the form $(V - V_0) = G(V - V_0) + H + GV_0 - V_0$ and observe that all components of the vector $H + GV_0 - V_0$ are non-positive (this can be deduced from Corollary 22). As in 2° we prove that the vector $V - V_0$ is with non-positive components. This proves the right inequality of 2).

4°. Boundedness and continuity of the speeds $\dot{y}_i$ on $\overline{\Delta}$ follows from the boundedness and continuity of G on $\overline{\Delta}$ (which is compact), and from the fact that the matrix $I - G$

is uniformly diagonally dominated for any point of $\overline{\Delta}$ (see Corollary 22). Smoothness of the speeds in Δ follows from the fact that the entries of G are smooth there – all roots $x_j^{(k)}$ are smooth functions of x_i inside Π, *i.e.* when x_i are distinct. □

Proof of Lemma 29. — 1°. We show that for $\sigma < \sigma_0$ and sufficiently close to σ_0 the CV of P_σ changes – at least one equality (see part 3) of Remarks 28) is replaced by the corresponding inequality. Hence, either the point of the phase curve belongs to Δ for all $\sigma < \sigma_0$ sufficiently close to σ_0 or it belongs to a stratum S of $\mathcal{U}'$ of higher dimension than the dimension of the initial one S_0. The same reasoning can be applied then to S instead of S_0 which will lead to the conclusion that the curve cannot stay on S for $\sigma \in (\sigma_0 - \varepsilon, \sigma_0]$ for any $\varepsilon > 0$ small enough. Hence, the curve passes through Δ for such ε.

2°. If for $\sigma = \sigma_0$ there occurs a confluence of two roots of P (w.r.t. $\sigma < \sigma_0$), then it is obvious that the CV has changed. So suppose that there occurs a confluence of a root y_{j_0} of P and of a root ξ_{i_0} of $P^{(k)}$ without a confluence of y_{j_0} with another root of P. Hence, y_{j_0} is a root of P of multiplicity $\leqslant k-1$.

By full analogy with Lemma 26, one proves that one has $-1 \leqslant \dot{y}_i \leqslant 0$ for all indices i of roots of class B.

3°. Suppose first that $j_0 = 1$. Show that one has $-1 < \dot{\xi}_{i_0} < 0$ which implies that the CV has changed (because $\dot{y}_1 = -1$). One has $\dot{\xi}_{i_0} = \sum_{j=1}^{q} m_j \frac{\partial \xi_{i_0}}{\partial y_j} \dot{y}_j$ with $\partial\xi_{i_0}/\partial y_j > 0$ for all j (see Lemma 21) and $-1 \leqslant \dot{y}_i \leqslant 0$. Moreover, one has $\dot{y}_i = 0$ for the smallest and for the greatest root of P. As $\sum_{j=1}^{q} m_j \partial\xi_{i_0}/\partial y_j = 1$ (see (4)), one has $-1 < \dot{\xi}_{i_0} < 0$.

4°. If $j_0 \neq 1$, then one has $\dot{y}_{j_0} = 0$ (because before the confluence y_{j_0} has been a root of class A). Like in 3° one shows that $-1 < \dot{\xi}_{i_0} < 0$. Hence, the CV changes again. □

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