Séminaires & Congrès
10, 2005, p. 209–232

A SURVEY ON ALEXANDER POLYNOMIALS OF PLANE CURVES

by

Mutsuo Oka

Abstract. — In this paper, we give a brief survey on the fundamental group of the complement of a plane curve and its Alexander polynomial. We also introduce the notion of θ-Alexander polynomials and discuss their basic properties.

***Résumé* (Un état des lieux sur les polynômes d'Alexander des courbes planes)**
Dans cet article, nous donnons un bref état des lieux sur le groupe fondamental du complémentaire d'une courbe plane et son polynôme d'Alexander. Nous introduisons de plus la notion de polynôme d'Alexander de type θ et discutons leurs propriétés élémentaires.

1. Introduction

For a given hypersurface $V \subset \mathbf{P}^n$, the fundamental group $\pi_1(\mathbf{P}^n - V)$ plays a crucial role when we study geometrical objects over $\mathbf{P}^n$ which are branched over V. By the hyperplane section theorem of Zariski [**51**], Hamm-Lê [**16**], the fundamental group $\pi_1(\mathbf{P}^n - V)$ can be isomorphically reduced to the fundamental group $\pi_1(\mathbf{P}^2 - C)$ where $\mathbf{P}^2$ is a generic projective subspace of dimension 2 and $C = V \cap \mathbf{P}^2$. A systematic study of the fundamental group was started by Zariski [**50**] and further developments have been made by many authors. See for example Zariski [**50**], Oka [**31–33**], Libgober [**22**]. For a given plane curve, the fundamental group $\pi_1(\mathbf{P}^2 - C)$ is a strong invariant but it is not easy to compute. Another invariant which is weaker but easier to compute is the Alexander polynomial $\Delta_C(t)$. This is related to a certain infinite cyclic covering space branched over C. Important contributions are done by Libgober, Randell, Artal, Loeser-Vaquié, and so on. See for example [**1, 2, 7, 9, 10, 13, 14, 20, 24, 26, 29, 41, 43, 44, 46, 47**]

2000 ***Mathematics Subject Classification.*** — 14H30,14H45, 32S55.
Key words and phrases. — θ-Alexander polynomial, fundamental group.

The main purpose of this paper is to give a survey for the study of the fundamental group and the Alexander polynomial (§§ 2,3). However we also give a new result on θ-Alexander polynomials in section 4.

In section two, we give a survey on the fundamental group of the complement of plane curves. In section three, we give a survey for the Alexander polynomial. It turns out that the Alexander polynomial does not tell much about certain non-irreducible curves. A possibility of a replacement is *the characteristic variety* of the multiple cyclic covering. This theory is introduced by Libgober [**23**].

Another possibility is *the Alexander polynomial set* (§ 4). For this, we consider the infinite cyclic coverings branched over C which correspond to the kernel of arbitrary surjective homomorphism $\theta : \pi_1(\mathbf{C}^2 - C) \to \mathbf{Z}$ and we consider the *θ-Alexander polynomial*. Basic properties are explained in the section 4.

2. Fundamental groups

The description of this section is essentially due to the author's lecture at School of Singularity Theory at ICTP, 1991.

2.1. van Kampen Theorem.— Let $C \subset \mathbf{P}^2$ be a projective curve which is defined by $C = \{[X,Y,Z] \in \mathbf{P}^2 \mid F(X,Y,Z) = 0\}$ where $F(X,Y,Z)$ is a reduced homogeneous polynomial $F(X,Y,Z)$ of degree d. The first systematic studies of the fundamental group $\pi_1(\mathbf{P}^2 - C)$ were done by Zariski [**49–51**] and van Kampen [**18**]. They used so called *pencil section method* to compute the fundamental group. This is still one of the most effective method to compute the fundamental group $\pi_1(\mathbf{P}^2 - C)$ when C has many singularities.

Let $\ell(X,Y,Z)$, $\ell'(X,Y,Z)$ be two independent linear forms. For any $\tau = (S,T) \in \mathbf{P}^1$, let $L_\tau = \{[X,Y,Z] \in \mathbf{P}^2 \mid T\ell(X,Y,Z) - S\ell'(X,Y,Z) = 0\}$. The family of lines $\mathcal{L} = \{L_\tau \mid \tau \in \mathbf{P}^1\}$ is called the *pencil* generated by $L = \{\ell = 0\}$ and $L' = \{\ell' = 0\}$. Let $\{B_0\} = L \cap L'$. Then $B_0 \in L_\tau$ for any τ and it is called *the base point* of the pencil. We assume that $B_0 \notin C$. L_τ is called a *generic line* (resp. *non-generic line*) of the pencil for C if L_τ and C meet transversally (resp. non-transversally). If L_τ is not generic, either L_τ passes through a singular point of C or L_τ is tangent to C at some smooth point. We fix two generic lines L_{τ_0} and L_{τ_∞}. Hereafter we assume that τ_∞ is the point at infinity ∞ of $\mathbf{P}^1$ (so $\tau_\infty = \infty$) and we identify $\mathbf{P}^2 - L_\infty$ with the affine space $\mathbf{C}^2$. We denote the affine line $L_\tau - \{B_0\}$ by L_τ^a. Note that $L_\tau^a \cong \mathbf{C}$. The complement $L_{\tau_0} - L_{\tau_0} \cap C$ (resp. $L_{\tau_0}^a - L_{\tau_0}^a \cap C$) is topologically S^2 minus d points (resp. $(d+1)$ points). We usually take $b_0 = B_0$ as the base point in the case of $\pi_1(\mathbf{P}^2 - C)$. In the affine case $\pi_1(\mathbf{C}^2 - C)$, we take the base point b_0 on L_{τ_0} which is sufficiently near to B_0 but $b_0 \neq B_0$. Let us consider two free groups

$$F_1 = \pi_1(L_{\tau_0} - L_{\tau_0} \cap C, b_0) \quad \text{and} \quad F_2 = \pi_1(L_{\tau_0}^a - L_{\tau_0}^a \cap C, b_0).$$

of rank $d-1$ and d respectively. We consider the set

$$\Sigma := \{\tau \in \mathbf{P}^1 \mid L_\tau \text{ is a non-generic line}\} \cup \{\infty\}.$$

We put ∞ in Σ so that we can treat the affine fundamental group simultaneously. We recall the definition of the action of the fundamental group $\pi_1(\mathbf{P}^1 - \Sigma, \tau_0)$ on F_1 and F_2. We consider the blowing up $\widetilde{\mathbf{P}^2}$ of $\mathbf{P}^2$ at B_0. $\widetilde{\mathbf{P}^2}$ is canonically identified with the subvariety

$$W = \{((X,Y,Z),(S,T)) \in \mathbf{P}^2 \times \mathbf{P}^1 \mid T\ell(X,Y,Z) - S\ell'(X,Y,Z) = 0\}$$

through the first projection $p : W \to \mathbf{P}^2$. Let $q : W \to \mathbf{P}^1$ be the second projection. The fiber $q^{-1}(s)$ is canonically isomorphic to the line L_s. Let $E = \{B_0\} \times \mathbf{P}^1 \subset W$. Note that E is the exceptional divisor of the blowing-up $p : W \to \mathbf{P}^2$ and $q|_E : E \to \mathbf{P}^1$ is an isomorphism. We take a tubular neighbourhood N_E of E which can be identified with the normal bundle of E. As the projection $q|N_E \to \mathbf{P}^1$ gives a trivial fibration over $\mathbf{P}^1 - \{\infty\}$, we fix an embedding $\phi : \Delta \times (\mathbf{P}^1 - \{\infty\}) \to N_E$ such that $\phi(0,\eta) = (B_0,\eta)$, $\phi(1,\tau_0) = (b_0,\tau_0)$ and $q(\phi(t,\eta)) = \eta$ for any $\eta \in \mathbf{P}^1 - \{\infty\}$. Here $\Delta = \{t \in \mathbf{C}; |t| \leqslant 1\}$. In particular, this gives a section of q over $\mathbf{C} = \mathbf{P}^1 - \{\infty\}$ by $\eta \mapsto b_{0,\eta} := \phi(1,\eta) \in L_\eta^a$. We take $b_{0,\eta}$ as the base point of the fiber L_η^a. Let $\widetilde{C} = p^{-1}(C)$. The restrictions of q to $\widetilde{C}$ and $\widetilde{C} \cup E$ are locally trivial fibrations by Ehresman's fibration theorem [**48**]. Thus the restrictions $q_1 := q|_{(W-\widetilde{C})}$ and $q_2 := q|_{(W-\widetilde{C}\cup E)}$ are also locally trivial fibrations over $\mathbf{P}^1 - \Sigma$. The generic fibers of q_1, q_2 are homeomorphic to $L_{\tau_0} - C$ and $L_{\tau_0}^a - C$ respectively. Thus there exists canonical action of $\pi_1(\mathbf{P}^1 - \Sigma, \tau_0)$ on F_1 and F_2. We call this action *the monodromy action* of $\pi_1(\mathbf{P}^1 - \Sigma, \tau_0)$. For $\sigma \in \pi_1(\mathbf{P}^1 - \Sigma, \tau_0)$ and $g \in F_1$ or F_2, we denote the action of σ on g by g^σ. The relations in the group F_ν

$$(R_1) \qquad \langle g^{-1}g^\sigma = e \mid g \in F_\nu, \sigma \in \pi_1(\mathbf{P}^1 - \Sigma, \tau_0)\rangle, \quad \nu = 1,2$$

are called *the monodromy relations*. The normal subgroup of F_ν, $\nu = 1,2$ which are normally generated by the elements $\{g^{-1}g^\sigma, \mid g \in F_\nu\}$ are called *the groups of the monodromy relations* and we denote them by N_ν for $\nu = 1,2$ respectively. The original van Kampen Theorem can be stated as follows. See also [**5, 6**].

Theorem 1 **([18])**. — *The following canonical sequences are exact.*

$$1 \to N_1 \to \pi_1(L_{\tau_0} - L_{\tau_0} \cap C, b_0) \to \pi_1(\mathbf{P}^2 - C, b_0) \to 1$$
$$1 \to N_2 \to \pi_1(L_{\tau_0}^a - L_{\tau_0}^a \cap C, b_0) \to \pi_1(\mathbf{C}^2 - C, b_0) \to 1$$

Here 1 *is the trivial group. Thus the fundamental groups* $\pi_1(\mathbf{P}^2 - C, b_0)$ *and* $\pi_1(\mathbf{C}^2 - C, b_0)$ *are isomorphic to the quotient groups* F_1/N_1 *and* F_2/N_2 *respectively.*

For a group G, we denote the commutator subgroup of G by $D(G)$. The relation of the fundamental groups $\pi_1(\mathbf{P}^2 - C, b_0)$ and $\pi_1(\mathbf{C}^2 - C, b_0)$ are described by the following. Let $\iota : \mathbf{C}^2 - C \to \mathbf{P}^2 - C$ be the inclusion map.

Lemma 2 **([30])**. — *Assume that L_∞ is generic.*

(1) *We have the following central extension.*

$$1 \longrightarrow \mathbf{Z} \xrightarrow{\gamma} \pi_1(\mathbf{C}^2 - C, b_0) \xrightarrow{\iota_\#} \pi_1(\mathbf{P}^2 - C, b_0) \longrightarrow 1$$

A generator of the kernel Ker $\iota_\#$ *of* $\iota_\#$ *is given by a lasso* ω *for* L_∞.

(2) *Furthermore, their commutator subgroups coincide i.e.,* $D(\pi_1(\mathbf{C}^2 - C)) = D(\pi_1(\mathbf{P}^2 - C))$.

Proof. — A loop ω is called *a lasso* for an irreducible curve D if ω is homotopic to a path written as $\ell \circ \tau \circ \ell^{-1}$ where τ is the boundary circle of a normal small disk of D at a smooth point and ℓ is a path connecting the base point and τ [**35**]. For the assertion (1), see [**30**]. We only prove the second assertion. Assume that C has r irreducible components of degree $d_1, \ldots, d_r$. The restriction of the homomorphism $\iota_\#$ gives a surjective morphism $\iota_\# : D(\pi_1(\mathbf{C}^2 - C)) \to D(\pi_1(\mathbf{P}^2 - C))$. If there is a $\sigma \in \mathrm{Ker}\ \iota_\# \cap D(\pi_1(\mathbf{C}^2 - C))$, σ can be written as $\gamma(\omega)^a$ for some $a \in \mathbf{Z}$. As ω corresponds to $(d_1, \ldots, d_r)$ in the homology $H_1(\mathbf{C}^2 - C) \cong \mathbf{Z}^r$, σ corresponds to $(ad_1, \ldots, ad_r)$. As σ is assumed to be in the commutator group, this must be trivial. That is, $a = 0$. □

2.2. Examples of monodromy relations. — We recall several basic examples of the monodromy relations. Let C be a reduced plane curve of degree d.

We consider a model curve $C_{p,q}$ which is defined by $y^p - x^q = 0$ and we study $\pi_1(\mathbf{C}^2 - C_{p,q})$. For this purpose, we consider the pencil lines $x = t$, $t \in \mathbf{C}$. We consider the local monodromy relations for σ, which is represented by the loop $x = \varepsilon(2\pi i t)$, $0 \leqslant t \leqslant 1$. We take local generators $\xi_0, \xi_1, \ldots, \xi_{p-1}$ of $\pi_1(L_\varepsilon, b_0))$ as in Figure 1. Every loops are counter-clockwise oriented. It is easy to see that each point of $C_{p,q} \cap L_\varepsilon$ are rotated by the angle $2\pi \times q/p$. Let $q = mp + q'$, $0 \leqslant q' < p$. Then the monodromy relations are:

$$(R_1) \qquad \xi_j (= \xi_j^\sigma) = \begin{cases} \omega^m \xi_{j+q'} \omega^{-m}, & 0 \leqslant j < p - q' \\ \omega^{m+1} \xi_{j+q'-p} \omega^{-(m+1)}, & p - q' \leqslant j \leqslant p-1 \end{cases}$$

$$(R_2) \qquad \omega = \xi_{p-1} \cdots \xi_0.$$

The last relation in (R_1) can be omitted as it follows from the other relations.

$$\begin{aligned} \xi_{p-1} &= \omega(\xi_{p-2} \cdots \xi_0)^{-1} \\ &= \omega\omega^m \xi_{q'}^{-1} \omega^{-m} \ldots \omega^m \xi_{p-1}^{-1} \omega^{-m} \omega^{m+1} \xi_0^{-1} \omega^{-m-1} \cdots \omega^{m+1} \xi_{q'-2}^{-1} \omega^{-m-1} \\ &= \omega^{m+1} \xi_{q'-1} \omega^{-m-1}. \end{aligned}$$

For the convenience, we introduce two groups $G(p,q)$ and $G(p,q,r)$.

$$G(p,q) := \langle \xi_1, \ldots, \xi_p, \omega \mid R_1, R_2 \rangle, \quad G(p,q,r) := \langle \xi_1, \ldots, \xi_p, \omega \mid R_1, R_2, R_3 \rangle$$

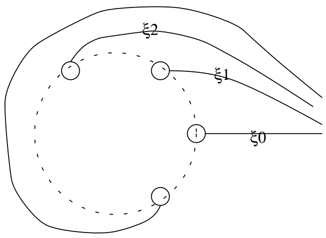

FIGURE 1. Generators

where R_3 is the vanishing relation of the big circle $\partial D_R = \{|y| = R\}$:

$$(R_3) \qquad \omega^r = e.$$

Now the above computation gives the following.

Lemma 3. — *We have $\pi_1(\mathbf{C}^2 - C_{p,q}, b_0) \cong G(p,q)$ and $\pi_1(\mathbf{P}^2 - C_{p,q}, b_0) \cong G(p,q,1)$.*

The groups of $G(p,q)$ and $G(p,q,r)$ are studied in [**12, 32**]. For instance, we have

Theorem 4 **([32])**

(i) *Let $s = \gcd(p,q)$, $p_1 = p/s$, $q_1 = q/s$. Then ω^{q_1} is the center of $G(p,q)$.*

(ii) *Put $a = \gcd(q_1, r)$. Then ω^a is in the center of $G(p,q,r)$ and has order r/a and the quotient group $G(p,q,r)/ < \omega^a >$ is isomorphic to $\mathbf{Z}_{p/s} * \mathbf{Z}_a * F(s-1)$.*

Corollary 5 **([32])**. — *Assume that $r = q$. Then $G(p,q,q) = \mathbf{Z}_{p_1} * \mathbf{Z}_{q_1} * F(s-1)$. In particular, if $\gcd(p,q) = 1$, $G(p,q,q) \cong \mathbf{Z}_p * \mathbf{Z}_q$.*

Let us recall some useful relations which follow from the above model.

(I) Tangent relation. — Assume that C and L_0 intersect at a simple point P with intersection multiplicity p. Such a point is called a *flex* point of order $p-2$ if $p \geqslant 3$ ([**50**]). This corresponds to the case $q = 1$. Then the monodromy relation gives $\xi_0 = \xi_1 = \cdots = \xi_{p-1}$ and thus $G(p,1) \cong \mathbf{Z}$. As a corollary, Zariski proves that *the fundamental group $\pi_1(\mathbf{P}^2 - C)$ is abelian if C has a flex of order $\geqslant d-3$.* In fact, if C has a flex of order at least $d-3$, the monodromy relation is given by $\xi_0 = \cdots = \xi_{d-2}$. On the other hand, we have one more relation $\xi_{d-1} \ldots \xi_0 = e$. In particular, considering the smooth curve defined by $C_0 = \{X^d - Y^d = Z^d\}$, we get that $\pi_1(\mathbf{P}^2 - C)$ is abelian for a smooth plane curve C, as C can be joined to C_0 by a path in the space of smooth curves of degree d.

(II) Nodal relation. — Assume that C has an ordinary double point (*i.e.*, a node) at the origin and assume that C is defined by $x^2 - y^2 = 0$ near the origin. This is the case when $p = q = 2$. Then as the monodromy relation, we get the commuting relation: $\xi_1\xi_2 = \xi_2\xi_1$. Assume that C has only nodes as singularities. The commutativity of

$\pi_1(\mathbf{P}^2 - C)$ was first asserted by Zariski [**50**] and is proved by Fulton-Deligne [**11, 15**]. See also [**28, 40, 41**].

(III) Cuspidal relation. — Assume that C has a cusp at the origin which is locally defined by $y^2 - x^3 = 0$ $(p = 2, q = 3)$. Then monodromy relation is: $\xi_1\xi_2\xi_1 = \xi_2\xi_1\xi_2$. This relation is known as the generating relation of the braid group B_3 (Artin [**3**]). Similarly in the case $p = 3$, $q = 2$, we get the relation $\xi_1 = \xi_3$, $\xi_1\xi_2\xi_1 = \xi_2\xi_1\xi_2$.

2.3. First Homology. — Let X be a path-connected topological space. By the theorem of Hurewicz, $H_1(X, \mathbf{Z})$ is isomorphic to the the quotient group of $\pi_1(X)$ by the commutator subgroup (see [**45**]). Now assume that C is a projective curve with r irreducible components $C_1, \dots, C_r$ of degree $d_1, \dots, d_r$ respectively. By Lefschetz duality, we have the following.

Proposition 6. — *$H_1(\mathbf{P}^2 - C, \mathbf{Z})$ is isomorphic to $\mathbf{Z}^{r-1} \times (\mathbf{Z}/d_0\mathbf{Z})$ where $d_0 = \gcd(d_1, \dots, d_r)$. In particular, if C is irreducible ($r = 1$), the fundamental group is a cyclic group of order d_1.*

2.4. Relation with Milnor Fibration. — Let $F(X, Y, Z)$ be a reduced homogeneous polynomial of degree d which defines $C \subset \mathbf{P}^2$. We consider the Milnor fibration of F [**25**] $F : \mathbf{C}^3 - F^{-1}(0) \to \mathbf{C}^*$ and let $M = F^{-1}(1)$ be the Milnor fiber. By the theorem of Kato-Matsumoto [**19**], M is path-connected. We consider the following diagram where the vertical map is the restriction of the Hopf fibration.

$$\begin{tikzcd}
& \mathbf{C}^* \arrow[d, "i"] \arrow[dr, "j"] & \\
M \arrow[r, hook, "\iota"] \arrow[dr, "p"'] & \mathbf{C}^3 - F^{-1}(0) \arrow[r, "F"] \arrow[d, "q"] & \mathbf{C}^* \\
& \mathbf{P}^2 - C &
\end{tikzcd}$$

Proposition 7 **([30])**

(I) *The following conditions are equivalent.*

(i) $\pi_1(\mathbf{P}^2 - C)$ *is abelian.*

(ii) $\pi_1(\mathbf{C}^3 - F^{-1}(0))$ *is abelian.*

(iii) $\pi_1(M)$ *is abelian and the first monodromy of the Milnor fibration $h_* : H_1(M) \to H_1(M)$ is trivial.*

(II) *Assume that C is irreducible. Then $\pi_1(M)$ is isomorphic to the commutator subgroup of $\pi_1(\mathbf{P}^2 - C)$. In particular, $\pi_1(\mathbf{P}^2 - C)$ is abelian if and only if M is simply connected.*

2.5. Degenerations and fundamental groups. — Let C be a reduced plane curve. The total Milnor number $\mu(C)$ is defined by the sum of the local Milnor numbers $\mu(C,P)$ at singular points $P \in C$. We consider an analytic family of reduced projective curves $C_t = \{F_t(X,Y,Z) = 0\}$, $t \in U$ where U is a connected open set with $0 \in \mathbf{C}$ and $F_t(X,Y,Z)$ is a homogeneous polynomial of degree d for any t. We assume that C_t, $t \neq 0$ have the same configuration of singularities so that they are topologically equivalent but C_0 obtain more singularities, *i.e.*, $\mu(C_t) < \mu(C_0)$. We call such a family *a degeneration of C_t at $t = 0$* and we denote this, for brevity, as $C_t \to C_0$. Then we have the following property about the fundamental groups.

Theorem 8. — *There is a canonical surjective homomorphism for $t \neq 0$:*

$$\varphi : \pi_1(\mathbf{P}^2 - C_0) \longrightarrow \pi_1(\mathbf{P}^2 - C_t).$$

In particular, if $\pi_1(\mathbf{P}^2 - C_0)$ is abelian, so is $\pi_1(\mathbf{P}^2 - C_t)$.

Proof. — Take a generic line L which cuts C_0 transversely. Let N be a neighborhood of C_0 so that $\iota : \mathbf{P}^2 - N \hookrightarrow \mathbf{P}^2 - C_0$ is a homotopy equivalence. For instance, N can be a regular neighborhood of C_0 with respect to a triangulation of $(\mathbf{P}^2, C_0)$. Take sufficiently small $t \neq 0$ so that $C_t \subset N$. Then taking a common base point at the base point of the pencil, we define φ as the composition:

$$\pi_1(\mathbf{P}^2 - C_0, b_0) \xrightarrow{\iota_\#^{-1}} \pi_1(\mathbf{P}^2 - N, b_0) \longrightarrow \pi_1(\mathbf{P}^2 - C_t, b_0)$$

We can assume that C_t and L intersect transversely for any $t \leqslant \varepsilon$ and $L - L \cap N \hookrightarrow L - L \cap C_t$ is a homotopy equivalence for $0 \leqslant t \leqslant \varepsilon$. Then the surjectivity of φ follows from the following commutative diagram

$$\begin{tikzcd}
\pi_1(L - L \cap C_0, b_0) \arrow[d, "a"'] & \pi_1(L - L \cap N, b_0) \arrow[l, "\alpha'"'] \arrow[r, "\beta'"] \arrow[d, "b"] & \pi_1(L - L \cap C_t, b_0) \arrow[d, "c"] \\
\pi_1(\mathbf{P}^2 - C_0, b_0) & \pi_1(\mathbf{P}^2 - N, b_0) \arrow[l, "\alpha"'] \arrow[r, "\beta"] & \pi_1(\mathbf{P}^2 - C_t, b_0)
\end{tikzcd}$$

where the vertical homomorphisms a, c are surjective by Theorem 1 and α, α', β' are canonically bijective. Thus β is also surjective. Thus define $\varphi : \pi_1(\mathbf{P}^2 - C_0, b_0) \to \pi_1(\mathbf{P}^2 - C_t, b_0)$ by the composition $\alpha^{-1} \circ \beta$. The second assertion is immediate from the first assertion. This completes the proof. ◻

Applying Theorem 8 to the degeneration $C_t \cup L \to C_0 \cup L$, we get

Corollary 9. — *There is a a surjective homomorphism: $\pi_1(\mathbf{C}^2 - C_0) \to \pi_1(\mathbf{C}^2 - C_t)$.*

Corollary 10. — *Let C_t, $t \in \mathbf{C}$ be a degeneration family. Assume that we have a presentation*

$$\pi_1(\mathbf{P}^2 - C_0) \cong \langle g_1, \ldots, g_d \mid R_1, \ldots, R_s \rangle$$

Then $\pi_1(\mathbf{P}^2 - C_t)$, $t \neq 0$ can be presented by adding a finite number of other relations.

***Corollary 11* (Sandwich isomorphism).** — *Assume that we have two degeneration families $C_t \to C_0$ and $D_s \to D_0$ such that $D_1 = C_0$. Assume that the composition*

$$\pi_1(\mathbf{P}^2 - D_0) \longrightarrow \pi_1(\mathbf{P}^2 - D_s) = \pi_1(\mathbf{P}^2 - C_0) \longrightarrow \pi_1(\mathbf{P}^2 - C_t)$$

is an isomorphism. Then we have isomorphisms

$$\pi_1(\mathbf{P}^2 - D_0) \cong \pi_1(\mathbf{P}^2 - D_t) \quad \text{and} \quad \pi_1(\mathbf{P}^2 - C_0) \cong \pi_1(\mathbf{P}^2 - C_t).$$

Example 12. — Assume that C is a sextic of tame torus type whose configuration of singularity is not $[C_{3,9} + 3A_2]$. For the definition of the singularity $C_{3,9}$, we refer to [**38, 42**]. First we can degenerate a generic sextic of torus type C_{gen} into C. Secondly we can degenerate C into a tame sextic $C_{\max}$ of torus type with maximal configuration (or with $C_{3,8}+3A_2$). In [**38**], it is shown that $\pi_1(\mathbf{P}^2 - C_{gen}) \cong \pi_1(\mathbf{P}^2 - C_{\max}) \cong \mathbf{Z}_2 * \mathbf{Z}_3$. Thus we have $\pi_1(\mathbf{P}^2 - C) = \mathbf{Z}_2 * \mathbf{Z}_3$.

Example 13. — Assume that C is a reduced curve of degree d with n nodes as singularities with $n < \binom{d}{2}$. By a result of J. Harris [**17**], there is a degeneration C_t of $C = C_1$ so that C_0 obtains more nodes and C_0 has no other singularities. (This was asserted by Severi but his proof had a gap.) Repeating this type of degenerations, one can deform a given nodal curve C to a reduced curve C_0 with $\binom{d}{2}$ nodes, which is a union of d generic lines. On the other hand, $\pi_1(\mathbf{P}^2 - C_0)$ is abelian by Corollary 16 below. Thus we have

***Theorem 14* ([11, 17, 28, 41, 50]).** — *Let C be a nodal curve. Then $\pi_1(\mathbf{P}^2 - C)$ is abelian.*

2.6. Product formula.

— Assume that C_1 and C_2 are reduced curves of degree d_1 and d_2 respectively which intersect transversely and let $C := C_1 \cup C_2$. We take a generic line L_∞ for C and we consider the the corresponding affine space $\mathbf{C}^2 = \mathbf{P}^2 - L_\infty$.

***Theorem 15* (Oka-Sakamoto [39]).** — *Let $\varphi_k : \mathbf{C}^2 - C \to \mathbf{C}^2 - C_i,\ k = 1, 2$ be the inclusion maps. Then the homomorphism*

$$\varphi_{1\#} \times \varphi_{2\#} : \pi_1(\mathbf{C}^2 - C) \longrightarrow \pi_1(\mathbf{C}^2 - C_1) \times \pi_1(\mathbf{C}^2 - C_2)$$

is isomorphic.

Corollary 16. — *Assume that $C_1, \ldots, C_r$ are the irreducible components of C and $\pi_1(\mathbf{P}^2 - C_j)$ is abelian for each j and they intersect transversely so that $C_i \cap C_j \cap C_k = \varnothing$ for any distinct three i, j, k. Then $\pi_1(\mathbf{P}^2 - C)$ is abelian.*

2.7. Covering transformation. — Assume that C is a reduced curve defined by $f(x,y)=0$ in the affine space $\mathbf{C}^2 := \mathbf{P}^2 - L_\infty$. The line at infinity L_∞ is assumed to be generic so that we can write

$$f(x,y)=\prod_{i=1}^{d}(y-\alpha_i x)+(\text{lower terms}),\quad \alpha_1,\dots,\alpha_d\in\mathbf{C}^*,\ \alpha_j\neq\alpha_k,\ j\neq k.$$

Take positive integers $n \geqslant m \geqslant 1$. We assume that the origin O is not on C and the coordinate axes $x=0$ and $y=0$ intersect C transversely and $C\cap\{x=0\}$ and $C\cap\{y=0\}$ has no point on L_∞. Consider the doubly branched cyclic covering

$$\Phi_{m,n}:\mathbf{C}^2\longrightarrow\mathbf{C}^2,\ (x,y)\longmapsto(x^m,y^n).$$

Put $f_{m,n}(x,y) := f(x^m,y^n)$ and put $\mathcal{C}_{m,n}=\{f_{m,n}(x,y)=0\}=\Phi_{m,n}^{-1}(C)$. The topology of the complement of $\mathcal{C}_{m,n}(C)$ depends only on C and m,n. We will call $\mathcal{C}_{m,n}(C)$ as *a generic (m,n)-fold covering transform* of C.

If $n>m$, $\mathcal{C}_{m,n}(C)$ has one singularity at $\rho_\infty=[1;0;0]$ and the local equation at ρ_∞ takes the following form:

$$\prod_{i=1}^{d}(\zeta^n-\alpha_i\xi^{n-m})+(\text{higher terms}),\quad \zeta=Y/X,\ \xi=Z/X$$

In the case $m=n$, we have no singularity at infinity. We denote the canonical homomorphism $(\Phi_{m,n})_\#:\pi_1(\mathbf{C}^2-\mathcal{C}_{m,n}(C))\to\pi_1(\mathbf{C}^2-C)$ by $\phi_{m,n}$ for simplicity.

Theorem 17 (**[34]**). — *Assume that $n\geqslant m\geqslant 1$ and let $\mathcal{C}_{m,n}(C)$ be as above. Then the canonical homomorphism*

$$\phi_{m,n}:\pi_1(\mathbf{C}^2-\mathcal{C}_{m,n}(C))\longrightarrow\pi_1(\mathbf{C}^2-C)$$

is an isomorphism and it induces a central extension of groups

$$1\longrightarrow\mathbf{Z}/n\mathbf{Z}\xrightarrow{\iota}\pi_1(\mathbf{P}^2-\mathcal{C}_{m,n}(C))\xrightarrow{\widetilde{\phi_{m,n}}}\pi_1(\mathbf{P}^2-C)\longrightarrow 1$$

The kernel of $\widetilde{\phi_{m,n}}$ is generated by an element ω' in the center and $\widetilde{\phi_{m,n}}(\omega')$ is homotopic to a lasso ω for L_∞ in the target space. The restriction of $\widetilde{\phi_{m,n}}$ gives an isomorphism of the respective commutator groups $\widetilde{\phi_{m,n}}_\#:D(\pi_1(\mathbf{P}^2-\mathcal{C}_{m,n}(C)))\to D(\pi_1(\mathbf{P}^2-C))$. We have also the exact sequence for the first homology groups:

$$1\longrightarrow\mathbf{Z}/n\mathbf{Z}\longrightarrow H_1(\mathbf{P}^2-\mathcal{C}_{m,n}(C))\xrightarrow{\overline{\Phi}_{m,n}}H_1(\mathbf{P}^2-C)\longrightarrow 1$$

Corollary 18

(1) *$\pi_1(\mathbf{P}^2-\mathcal{C}_{m,n}(C))$ is abelian if and only if $\pi_1(\mathbf{P}^2-C)$ is abelian.*

(2) *Assume that C is irreducible. Put*

$$F(x,y,z)=z^d f(x/z,y/z),\quad F_{m,n}(x,y,z)=z^{dn}f_{m,n}(x/z,y/z)$$

Let $M_{m,n}$ and M be the Minor fibers of $F_{m,n}$ and F respectively. Then we have an isomorphism of the respective fundamental groups: $\pi_1(M_{m,n}) \cong \pi_1(M)$.

For a group G, we consider the following condition : $Z(G) \cap D(G) = \{e\}$ where $Z(G)$ is the center of G. This is equivalent to the injectivity of the composition: $Z(G) \to G \to H_1(G)$. When this condition is satisfied, we say that G satisfies *homological injectivity condition of the center* (or (H.I.C)-condition in short). A pair of reduced plane curves of a same degree $\{C, C'\}$ is called *a Zariski pair* if there is an bijection $\alpha : \Sigma(C) \to \Sigma(C')$ of their singular points so that the two germs (C, P), $(C', \alpha(P))$ are topologically equivalent for each $P \in \Sigma(C)$ and the fundamental group of the complement $\pi_1(\mathbf{P}^2 - C)$ and $\pi_1(\mathbf{P}^2 - C')$ are not isomorphic. (This definition is slightly stronger than that in [**34**].)

Corollary 19 **([34])**. — *Let $\{C, C'\}$ be a Zariski pair and assume that $\pi_1(\mathbf{P}^2 - C')$ satisfies (H.I.C)-condition. Then for any $n \geqslant m \geqslant 1$, $\{\mathcal{C}_{m,n}(C), \mathcal{C}_{m,n}(C')\}$ is a Zariski pair.*

See also Shimada [**44**].

3. Alexander polynomial

Let X be a topological space which has a homotopy type of a finite CW-complex and assume that we have a surjective homomorphism: $\phi : \pi_1(X) \to \mathbf{Z}$. Let t be a generator of $\mathbf{Z}$ and put $\Lambda = \mathbf{C}[t, t^{-1}]$. Note that Λ is a principal ideal domain. Consider an infinite cyclic covering $p : \widetilde{X} \to X$ such that $p_{\#}(\pi_1(\widetilde{X})) = \mathrm{Ker}\ \phi$. Then $H_1(\widetilde{X}, \mathbf{C})$ has a structure of Λ-module where t acts as the canonical covering transformation. Thus we have an identification:

$$H_1(\widetilde{X}, \mathbf{C}) \cong \Lambda/\lambda_1 \oplus \cdots \oplus \Lambda/\lambda_n$$

as Λ-modules. We normalize the denominators so that λ_i is a polynomial in t with $\lambda_i(0) \neq 0$ for each $i = 1, \ldots, n$. The Alexander polynomial $\Delta(t)$ is defined by the product $\prod_{i=1}^{n} \lambda_i(t)$.

The classical one is the case $X = S^3 - K$ where K is a knot. As $H_1(S^3 - K) = \mathbf{Z}$, we have a canonical surjective homomorphism $\phi : \pi_1(S^3 - K) \to H_1(S^3 - K, \mathbf{Z})$ induced by the Hurewicz homomorphism. The corresponding Alexander polynomial is called *the Alexander polynomial of the knot K*.

In our situation, we consider a plane curve C defined by a homogeneous polynomial $F(X, Y, Z)$ of degree d. Unless otherwise stated, we always assume that the line at infinity L_∞ is generic for C and we identify the complement $\mathbf{P}^2 - L_\infty$ with $\mathbf{C}^2$. Let $\phi : \pi_1(\mathbf{C}^2 - C) \to \mathbf{Z}$ be the canonical homomorphism induced by the composition

$$\pi_1(\mathbf{C}^2 - C) \xrightarrow{\xi} H_1(\mathbf{C}^2 - C, \mathbf{Z}) \cong \mathbf{Z}^r \xrightarrow{s} \mathbf{Z}$$

where ξ is the Hurewicz homomorphism and s is defined by $s(a_1,\dots,a_r)=\sum_{i=1}^r a_i$ and r is the number of irreducible components of C. We call s the summation homomorphism.

Let $\widetilde{X}\to \mathbf{C}^2-C$ be the infinite cyclic covering corresponding to Ker ϕ. The corresponding Alexander polynomial is called the *generic Alexander polynomial* of C and we denote it by $\Delta_C(t)$ or simply $\Delta(t)$ if no ambiguity is likely. It does not depend on the choice of the generic line at infinity L_∞. Let $M=F^{-1}(1)\subset \mathbf{C}^3$ be the Milnor fiber of F. The monodromy map $h: M\to M$ is defined by the coordinatewise multiplication of $\exp(2\pi i/d)$. Randell showed in [**43**] the following important theorem.

Theorem 20. — *The Alexander polynomial $\Delta(t)$ is equal to the characteristic polynomial of the monodromy $h_*: H_1(M)\to H_1(M)$. Thus the degree of $\Delta(t)$ is equal to the first Betti number $b_1(M)$.*

Lemma 21. — *Assume that C has r irreducible components. Then the multiplicity of the factor $(t-1)$ in $\Delta(t)$ is $r-1$.*

Proof. — As $h^d=\mathrm{id}_M$, the monodromy map $h_*: H_1(M)\to H_1(M)$ has a period d. This implies that h_* can be diagonalized. Assume that ρ is the multiplicity of $(t-1)$ in $\Delta(t)$. Consider the Wang sequence:

$$H_1(M)\xrightarrow{h_*-\mathrm{id}} H_1(M)\longrightarrow H_1(E)\longrightarrow H_0(M)\longrightarrow 0$$

where $E:=S^5-V\cap S^5$ and $V=F^{-1}(0)$. Then we get $b_1(E)=\rho+1$. On the other hand, by Alexander duality, we have $H_1(E)\cong H^3(S^5,V\cap S^5)$ and $b_1(E)=r$. Thus we conclude that $\rho=r-1$. $\square$

The following Lemma describes the relation between the generic Alexander polynomial and local singularities.

***Lemma 22* (Libgober [20])**. — *Let $P_1,\dots,P_k$ be the singular points of C and let $\Delta_i(t)$ be the characteristic polynomial of the Milnor fibration of the germ (C,P_i). Then the generic Alexander polynomial $\Delta(t)$ divides the product $\prod_{i=1}^k\Delta_i(t)$*

***Lemma 23* (Libgober [20])**. — *Let d be the degree of C. Then the Alexander polynomial $\Delta(t)$ divides the Alexander polynomial at infinity $\Delta_\infty(t)$ which is given by $(t^d-1)^{d-2}(t-1)$. In particular, the roots of Alexander polynomial are d-th roots of unity.*

The last assertion also follows from Theorem 20 and the periodicity of the monodromy.

3.1. Fox calculus. — Suppose that $\phi : \pi_1(X) \to \mathbf{Z}$ is a given surjective homomorphism. Assume that $\pi_1(X)$ has a finite presentation as

$$\pi_1(X) \cong \langle x_1, \dots, x_n \mid R_1, \dots, R_m \rangle$$

where R_i is a word of $x_1, \dots, x_n$. Thus we have a surjective homomorphism $\psi : F(n) \to \pi_1(X)$ where $F(n)$ is a free group of rank n, generated by $x_1, \dots, x_n$. Consider the group ring of $F(n)$ with $\mathbf{C}$-coefficients $\mathbf{C}[F(n)]$. The Fox differential

$$\frac{\partial}{\partial x_j} : \mathbf{C}[F(n)] \longrightarrow \mathbf{C}[F(n)]$$

is $\mathbf{C}$-linear map which is characterized by the property

$$\frac{\partial}{\partial x_j} x_i = \delta_{i,j}, \quad \frac{\partial}{\partial x_j}(uv) = \frac{\partial u}{\partial x_j} + u \frac{\partial v}{\partial x_j}, \ u, v \in \mathbf{C}[F(n)]$$

The composition $\phi \circ \psi : F(n) \to \mathbf{Z}$ gives a ring homomorphism $\gamma : \mathbf{C}[F(n)] \to \mathbf{C}[t, t^{-1}]$. *The Alexander matrix* A is $m \times n$ matrix with coefficients in $\mathbf{C}[t, t^{-1}]$ and its (i, j)-component is given by $\gamma(\partial R_i / \partial x_j)$. Then it is known that the Alexander polynomial $\Delta(t)$ is given by the greatest common divisor of $(n-1)$-minors of A ([**8**]). The following formula will be useful.

$$\frac{\partial}{\partial x_j} \omega^k = (1 + \omega + \cdots + \omega^{k-1}) \frac{\partial}{\partial x_j} \omega, \quad \frac{\partial}{\partial x_j} \omega^{-k} = -\omega^{-k} \frac{\partial}{\partial x_j} \omega^k$$

Example 24. — We gives several examples.

(1) Consider the trivial case $\pi_1(X) = \mathbf{Z}$ and ϕ is the canonical isomorphism. Then $\pi_1(X) \cong \langle x_1 \rangle$ (no relation) and $\Delta(t) = 1$. More generally assume that $\pi_1(X) = \mathbf{Z}^r$ with $\phi(n_1, \dots, n_r) = n_1 + \cdots + n_r$. Then

$$\pi_1(X) = \langle x_1, \dots, x_r \mid R_{i,j} = x_i x_j {x_i}^{-1} {x_j}^{-1}, \ 1 \leqslant i < j \leqslant r \rangle$$

As we have

$$\gamma\left(\frac{\partial}{\partial x_\ell} R_{i,j}\right) = \begin{cases} 1 - t & \ell = i \\ t - 1 & \ell = j \\ 0 & \ell \neq i, j \end{cases},$$

we have $\Delta(t) = (t-1)^{r-1}$.

Definition 25. — We say that Alexander polynomial of a curve C is trivial if $\Delta(t) = (t-1)^{r-1}$ where r is the number of the irreducible components of C.

(2) Let $C = \{y^2 - x^3 = 0\}$ and $X = \mathbf{C}^2 - C$. Then

$$\pi_1(X) = \langle x_1, x_2 \mid x_1 x_2 x_1 = x_2 x_1 x_2 \rangle.$$

is known as the braid group $B(3)$ of three strings and the Alexander polynomial is given by $\Delta(t) = t^2 - t + 1$.

(3) Let us consider the curve $C := \{y^2 - x^5 = 0\} \subset \mathbf{C}^2$ and put $X = \mathbf{C}^2 - C$. Then by § 2.2,

$$\pi_1(X) \cong G(2,5) = \langle x_0, x_1 \mid x_0(x_1x_0)^2 x_1^{-1}(x_1x_0)^{-2}\rangle$$

In this case, we get

$$\Delta(t) = t^4 - t^3 + t^2 - t + 1 = \frac{(t^{10}-1)(t-1)}{(t^2-1)(t^5-1)}.$$

3.2. Degeneration and Alexander polynomial. — We consider a degeneration $C_t \to C_0$. By Corollary 10 and Fox calculus, we have

Theorem 26. — *Assume that we have a degeneration family of reduced curves $\{C_s \mid s \in U\}$ at $s = 0$. Let $\Delta_s(t)$ be the Alexander polynomial of C_s. Then $\Delta_s(t)|\Delta_0(t)$ for $s \neq 0$.*

***Corollary 27* (Sandwich principle)**. — *Suppose that we have two families of degeneration $C_s \to C_0$ and $D_\tau \to D_0$ such that $D_1 = C_0$ and assume that $\Delta_{D_0}(t)$ and $\Delta_{C_s}(t)$, $s \neq 0$ coincide. Then we have also $\Delta_{C_s}(t) = \Delta_{C_0}(t)$, $s \neq 0$.*

3.3. Explicit computation of Alexander polynomials. — Let C be a given plane curve of degree d defined by $f(x,y) = 0$ and let $\Sigma(C)$ be the singular locus of C and let $P \in \Sigma(C)$ be a singular point. Consider an embedded resolution of C, $\pi : \widetilde{U} \to U$ where U is an open neighbourhood of P in $\mathbf{P}^2$ and let $E_1, \dots, E_s$ be the exceptional divisors. Let us choose (u,v) be a local coordinate system centered at P and let k_i and m_i be the order of zero of the canonical two form $\pi^*(du \wedge dv)$ and $\pi^* f$ respectively along the divisor E_i. We consider an ideal of $\mathcal{O}_P$ generated by the function germ ϕ such that the pull-back $\pi^*\phi$ vanishes of order at least $-k_i + [km_i/d]$ along E_i and we denote this ideal by $\mathcal{J}_{P,k,d}$. Namely

$$\mathcal{J}_{P,k,d} = \{\phi \in \mathcal{O}_P, (\pi^*\phi) \geqslant \textstyle\sum_i (-k_i + [km_i/d])E_i\}$$

Let us consider the canonical homomorphisms induced by the restrictions:

$$\sigma_{k,P} : \mathcal{O}_P \longrightarrow \mathcal{O}_P/\mathcal{J}_{P,k,d}, \quad \sigma_k : H^0(\mathbf{P}^2, \mathcal{O}(k-3)) \longrightarrow \bigoplus_{P \in \Sigma(C)} \mathcal{O}_P/\mathcal{J}_{P,k,d}$$

where the right side of σ_k is the sum over singular points of C. We define two invariants:

$$\rho(P,k) = \dim_{\mathbf{C}} \mathcal{O}_P/\mathcal{J}_{P,k,d}, \quad \rho(k) = \sum_{P \in \Sigma(C)} \rho(P,k)$$

Let ℓ_k be the dimension of the cokernel σ_k. Then the formula of Libgober [**21**] and Loeser-Vaquié [**24**], combined with a result of Esnault and Artal [**1, 14**], can be stated as follows.

Lemma 28. — *The polynomial $\Delta(t)$ is written as the product*

$$\widetilde{\Delta}(t) = \prod_{k=1}^{d-1} \Delta_k(t)^{\ell_k}, \quad k = 1, \dots, d-1$$

where

$$\Delta_k = \big(t - \exp(2k\pi i/d)\big)\big(t - \exp(-2k\pi i/d)\big).$$

Note that for the case of sextics $d = 6$, the above polynomials take the form:

$$\Delta_5(t) = \Delta_1(t) = t^2 - t + 1, \quad \Delta_4(t) = \Delta_2(t) = t^2 + t + 1, \quad \Delta_3(t) = (t+1)^2.$$

3.4. Triviality of the Alexander Polynomials. — We have seen that the Alexander polynomial is trivial if C is irreducible and $\pi_1(\mathbf{P}^2 - C)$ is abelian. However this is not a necessary condition, as we will see in the following. Let $F(X, Y, Z)$ be the defining homogeneous polynomial of C and let $M = F^{-1}(1) \subset \mathbf{C}^3$ the Milnor fiber of F.

Theorem 29. — *Assume that C is an irreducible curve. The Alexander polynomial $\Delta(t)$ of C is trivial if and only if the first homology group of the Milnor fiber $H_1(M)$ is at most a finite group.*

Proof. — By Theorem 20, the first Betti number of M is equal to the degree of $\Delta(t)$. □

Corollary 30. — *Assume that $\pi_1(\mathbf{P}^2 - C)$ is a finite group. Then the Alexander polynomial is trivial.*

Proof. — This is immediate from Theorem 7 as $D(\pi_1(\mathbf{P}^2 - C)) = \pi_1(M)$ and it is a finite group under the assumption. □

3.5. Examples

(1) *(Zariski's three cuspidal quartic,* [**50**]*)* Let Z_4 be a quartic curve with three A_2-singularities. The corresponding moduli space is irreducible. Then the fundamental groups are given by [**33, 50**] as

$$\pi_1(\mathbf{C}^2 - Z_4) \cong \langle \rho, \xi \mid \rho\,\xi\,\rho = \xi\,\rho\,\xi,\ \rho^2 = \xi^2 \rangle$$
$$\pi_1(\mathbf{P}^2 - Z_4) \cong \langle \rho, \xi \mid \rho\,\xi\,\rho = \xi\,\rho\,\xi,\ \rho^2\xi^2 = e \rangle$$

Then by an easy calculation, $\Delta(t) = 1$. This also follows from Theorem 29 as $\pi_1(\mathbf{P}^2 - Z_4)$ is a finite group of order 12 by Zariski [**50**]. By Theorem 17, the generic covering transform $\mathcal{C}_{n,n}(Z_4)$ has also a trivial Alexander polynomial for any n.

(2) *(Libgober's criterion)* Assume that for any singularity P of C, the characteristic polynomial of (C, P) does not have any root which is a d-th root of unity. Then by Lemma 22 and Lemma 23, the Alexander polynomial is trivial. For example, an irreducible curve C with only A_2 or A_1 as singularity has a trivial Alexander polynomial if the degree d is not divisible by 6.

(3) *(Curves of torus type)* A curve C of degree d is called of (p,q)-*torus type* if its defining polynomial $F(X,Y,Z)$ is written as

$$F(X,Y,Z)=F_1(X,Y,Z)^p+F_2(X,Y,Z)^q$$

where F_i is a homogeneous polynomial of degree d_i, $i=1,2$ so that $d=pd_1=qd_2$. Assume that (1) two curves $F_1=0$ and $F_2=0$ intersect transversely at d_1d_2 distinct points, and (2) the singularities of C are only on the intersection $F_1=F_2=0$. We say C is a generic curve of (p,q)-torus type if the above conditions are satisfied. Let $\mathcal{M}$ be the space $\{(F_1,F_2)\mid \text{degree } F_1=d_1, \text{degree}\, F_2=d_2\}$ and let $\mathcal{M}'$ be the subspace for which the conditions (1) and (2) are satisfied. Then by an easy argument, $\mathcal{M}$ is an affine space of dimension $\binom{d_1+2}{2}+\binom{d_2+2}{2}$ and $\mathcal{M}'$ is a Zariski open subset of $\mathcal{M}$. Thus the topology of the complement of C does not depend on a generic choice of $C\in\mathcal{M}'$.

Fundamental groups $\pi_1(\mathbf{P}^2-C)$ and $\pi_1(\mathbf{C}^2-C)$ for a generic curve of (p,q)-torus type are computed as follows. Put $s=\gcd(p,q)$, $p_1=p/s$, $q_1=q/s$. As $d_1p=d_2q$, we can write $d_1=q_1\,m$ and $d_2=p_1\,m$.

Theorem 31

(a) ([**31, 32**]) *Then we have* $\pi_1(\mathbf{C}^2-C)\cong G(p,q)$ *and* $\pi_1(\mathbf{P}^2-C)\cong G(p,q,mq_1)$

(b) *The Alexander polynomial is the same as the characteristic polynomial of the Pham-Brieskorn singularity* $B_{p,q}$ *which is given by*

$$\Delta(t)=\frac{(t^{p_1q_1s}-1)^s\,(t-1)}{(t^p-1)\,(t^q-1)} \tag{1}$$

Proof. — The assertion for the fundamental group is proved in [**31, 32**]. For the assertion about the Alexander polynomial, we can use Fox calculus for small p and q. For example, assume that $p=2$, $q=3$ and $d_1=3$, $d_2=2$. Thus C is a sextic with six (2,3)-cusps. As

$$\pi_1(\mathbf{C}^2-C)=\langle x_0,x_1\mid x_0x_1x_0=x_1x_0x_1\rangle,$$

the assertion follows from 2 of Example 22.

Assume that $p=4$, $q=6$ and $d_1=3$, $d_2=2$. Then C has two irreducible component of degree 6:

$$C: f_3(x,y)^4\ -\ f_2(x,y)^6=(f_3(x,y)^2-f_2(x,y)^3)\,(f_3(x,y)^2+f_2(x,y)^3)$$

where deg $f_k(x,y)=k$. By Lemma 3, we have

$$\pi_1(\mathbf{C}^2-C)=\langle \xi_0,\dots,\xi_3\mid R_j, j=0,1,2\rangle$$

where using $\omega:=\xi_3\dots\xi_0$ the relations are given as

$$R_1:\xi_0\omega\xi_2^{-1}\omega^{-1},\quad R_2:\xi_1\omega\xi_3\omega^{-1},\quad R_3:\xi_2\omega^2\xi_0\omega^{-2}$$

Thus by an easy calculation, the Alexander matrix is given as

$$\begin{bmatrix} 1+t^4-t^3 & (t-1)\,t^2 & -\left(-t+t^3+1\right)t & t-1 \\ (t-1)\,t^3 & 1+t^3-t^2 & (t-1)\,t & t-t^4-1 \\ -\left(-t+t^4+1\right)t^3 & \left(t+t^5-t^4-1\right)t^2 & 1+t^2+t^6-t^5-t & t+t^5-t^4-1 \end{bmatrix}$$

and the Alexander polynomial is given by

$$\Delta(t) = (t-1)\,(t^2+1)\,(t^2+t+1)\,(t^2-t+1)\,(t^4-t^2+1)^2 = \frac{(t^{12}-1)^2(t-1)}{(t^4-1)(t^6-1)}$$

To prove the assertion for general p,q, we use the following result of Nemethi [**27**].

Lemma 32. — *Let $u := (h,g) : (\mathbf{C}^{n+1},O) \to (\mathbf{C}^2,O)$ and $P : (\mathbf{C}^2,O) \to (\mathbf{C},O)$ be germs of analytic mappings and assume that u defines an isolated complete intersection variety at O. Let D be the discriminant locus of u and $V(P) := \{P=0\}$. Consider the composition $f = P\circ u : (\mathbf{C}^{n+1},O) \to (\mathbf{C},O)$. Let M_f and M_P be the respective Minor fibers of f and P. Assume that $D\cap V(P) = \{O\}$ in a neighbourhood of O. Then the characteristic polynomial of the Minor fibration of f on $H_1(M_f)$ is equal to the characteristic polynomial of the Milnor fibration of P on $H_1(M_P)$, provided $n \geqslant 2$.*

Proof of the equality (1). — Let F_1, F_2 be homogeneous polynomials of degree q_1m, p_1m respectively and let $F = F_1^p + F_2^q$. We assume that F_1, F_2 are generic so that the singularities of $F=0$ are only the intersection $F_1 = F_2 = 0$ which are $p_1q_1m^2$ distinct points. Then by Lemma 32, the characteristic polynomial of the monodromy $h_* : H_1(M_F) \to H_1(M_F)$ of the Milnor fibration of F is equal to that of $P(x,y) := y^p + x^q$. Thus the assertion follows from [**4, 25**] and Theorem 20. □

3.6. Sextics of torus type. — Let us consider a sextic of torus type

$$C : f_2(x,y)^3 + f_3(x,y)^2 = 0, \quad \operatorname{degree} f_j = j,\ j = 2,3,$$

as an example. Assume that C is reduced and irreducible. A sextic of torus type is called *tame* if the singularities are on the intersection of the conic $f_2(x,y)=0$ and the cubic $f_3(x,y)=0$. A generic sextic of torus type is tame but the converse is not true. Then the possibility of Alexander polynomials for sextics of torus type is determined as follows.

Theorem 33 ([**36, 37**]). — *Assume that $C$is an irreducible sextic of torus type. The Alexander polynomial of C is one of the following.*

$$(t^2-t+1), \quad (t^2-t+1)^2, \quad (t^2-t+1)^3$$

Moreover for tame sextics of torus type, the Alexander polynomial is given by t^2-t+1 and the fundamental group of the complement in $\mathbf{P}^2$ is isomorphic to $\mathbf{Z}_2 * \mathbf{Z}_3$ except the case when the configuration is $[C_{3,9}, 3A_2]$. In the exceptional case, the Alexander polynomial is given by $(t^2-t+1)^2$.

3.7. Weakness of Alexander polynomial. — Let C_1 and C_2 be curves which intersect transversely. We take a generic line at infinity for $C_1 \cup C_2$. Theorem 15 says that

$$\pi_1(\mathbf{C}^2 - C_1 \cup C_2) \cong \pi_1(\mathbf{C}^2 - C_1) \times \pi_1(\mathbf{C}^2 - C_2)$$

which tell us that the fundamental group of the union of two curves keeps informations about each curves C_1, C_2. On the other hand, the Alexander polynomial of $C_1 \cup C_2$ keeps little information about each curves C_1, C_2. In fact, we have

Theorem 34. — *Assume that C_1 and C_2 intersect transversely and let $C = C_1 \cup C_2$. Then the generic Alexander polynomial $\Delta(t)$ of C is given by given by $(t-1)^{r-1}$ where r is the number of irreducible components of C.*

Proof. — Assume that $\pi_1(\mathbf{C}^2 - C_j)$, $j = 1, 2$ is presented as

$$\pi_1(\mathbf{C}^2 - C_1) = \langle g_1, \dots, g_{s_1} \mid R_1, \dots, R_{p_1} \rangle,\ \pi_1(\mathbf{C}^2 - C_2) = \langle h_1, \dots, h_{s_2} \mid S_1, \dots, S_{p_2} \rangle$$

Then by Theorem 15, we have

$$\begin{aligned}\pi_1(\mathbf{C}^2 - C) = \big\langle g_1, \dots, g_{s_1}, h_1, \dots, h_{s_2} \mid R_1, \dots, R_{p_1}, S_1, \dots, S_{p_2}, T_{i,j}, \\ 1 \leqslant i \leqslant s_1,\ 1 \leqslant j \leqslant s_2 \big\rangle\end{aligned}$$

where $T_{i,j}$ is the commutativity relation $g_i\, h_j g_i^{-1}\, h_j^{-1}$. Let

$$\gamma : \mathbf{C}[g_1, \dots, g_{s_1}, h_1, \dots, h_{s_2}] \longrightarrow \mathbf{C}[t, t^{-1}]$$

be the ring homomorphism defined before (§ 3.1). Put $g_{s_1+j} = h_j$ for brevity. Then the submatrix of the Alexander matrix corresponding to

$$\left(\gamma\Big(\frac{\partial T_{i,j}}{\partial g_k}\Big)\right), \quad \begin{array}{l}\{i = 1, \dots, s_1,\ j = s_2\} \text{ or } \{i = s_1,\ j = 1, \dots, s_2 - 1\}, \\ \text{and } 1 \leqslant k \leqslant s_1 + s_2 - 1\end{array}$$

is given by $(1-t) \times A$ where

$$A = \begin{pmatrix} E_{s_1} & 0 \\ K & -E_{s_2-1} \end{pmatrix}$$

and E_ℓ is the $\ell \times \ell$-identity matrix and K is a $(s_2 - 1) \times s_1$ matrix with only the last column is non-zero. Thus the determinant of this matrix gives $\pm(t-1)^{s_1+s_2-1}$ and the Alexander polynomial must be a factor of $(t-1)^{s_1+s_2-1}$. As the monodromy of the Milnor fibration of the defining homogeneous polynomial $F(X, Y, Z)$ of C is periodic, this implies that $h_* : H_1(M) \to H_1(M)$ is the identity map. Thus $\Delta(t) = (t-1)^{b_1}$ where b_1 is the first Betti number of M. On the other hand, $b_1 = r - 1$ by Lemma 21. ∎

4. Possible generalization: θ-Alexander polynomial

To cover the weakness of Alexander polynomials for reducible curves, there are two possible modifications. One is to consider the multiple cyclic coverings and their characteristic varieties (Libgober [**23**]). For the detail of this theory, we refer to the above paper of Libgober.

Another possibility which we propose now is the following. Consider a plane curve with r irreducible components $C_1,\dots,C_r$ with degree $d_1,\dots,d_r$ respectively. We assume that the line at infinity is generic for C. For the generic Alexander polynomial, we have used the summation homomorphism s. This is not enough for reducible curves. We consider every possible surjective homomorphism $\theta:\pi_1(\mathbf{C}^2-C)\to\mathbf{Z}$ and the corresponding infinite cyclic covering $\pi_\theta: X_\theta\to\mathbf{C}^2-C$. The corresponding Alexander polynomial will be denoted by $\Delta_{C,\theta}(t)$ (or $\Delta_\theta(t)$ if no ambiguity is likely) and we call it *the generic θ-Alexander polynomial of C*. Note that a surjective homomorphism θ factors through the Hurewicz homomorphism, and a surjective homomorphism $\theta':H_1(\mathbf{C}^2-C)\cong\mathbf{Z}^r\to\mathbf{Z}$. On the other hand, θ' corresponds to a multi-integer $\boldsymbol{m}=(m_1,\dots,m_r)$ with $\gcd(m_1,\dots,m_r)=1$. So we denote θ as $\theta_{\boldsymbol{m}}$ hereafter. We denote the set of all Alexander polynomials by $\mathcal{A}(C)$

$$\mathcal{A}(C):=\{\Delta_\theta(t)\mid\theta:\pi_1(\mathbf{C}^2-C)\to\mathbf{Z}\text{ is surjective}\}$$

and we call $\mathcal{A}(C)$ *the Alexander polynomial set* of C. We say $\mathcal{A}(C)$ is *trivial* if $\mathcal{A}(C)=\{(t-1)^{r-1}\}$. It is easy to see that $\mathcal{A}(C)$ is a topological invariant of the complement $\mathbf{P}^2-C$.

***Theorem 35* (Main Theorem)**. — *The Alexander polynomial set is not trivial if there exists a component C_{i_0} for which the Alexander polynomial $\Delta_{C_{i_0}}(t)$ is not trivial.*

For the proof, we prepare several lemmas. First we define *the radical* $\sqrt{q(t)}$ of a polynomial $q(t)$ to be the generator of the radical $\sqrt{(q(t))}$ of the ideal $(q(t))$ in $\mathbf{C}[t]$.

Lemma 36. — *Assume that $\pi_1(\mathbf{P}^2-C)$ is abelian. Then $\mathcal{A}(C)$ is trivial.*

Proof. — Take the obvious presentation.

$$\pi_1(\mathbf{C}^2-C)=\langle g_1,\dots,g_r\mid T_{ij}=g_ig_jg_i^{-1}g_j^{-1},\ 1\leqslant i<j\leqslant r\rangle.$$

Take a surjective homomorphism $\theta_{\boldsymbol{m}}:\pi_1(\mathbf{C}^2-C)\to\mathbf{Z}$, $\boldsymbol{m}:=(m_1,\dots,m_r)$. Then the Alexander matrix is given by $r(r-1)/2$ raw vectors V_{ij} where V_{ij} has two non-zero coefficients. The i-th and j-th coefficients are given by $(1-t^{m_j})$ and $(t^{m_i}-1)$ respectively. Thus taking for example the minor corresponding to $\gamma_\theta(\partial T_{1j}/\partial g_k)$, $2\leqslant j,k\leqslant r$, we get $(t^{m_1}-1)^{r-1}$. Similarly we get $(t^{m_i}-1)^{r-1}$ for any i. This implies that $\Delta_\theta(t)=(t-1)^{r-1}$ as $\gcd(m_1,\dots,m_r)=1$. □

Lemma 37. — *Assume that C is a reduced curve of degree d with a non-trivial Alexander polynomial $\Delta_C(t)$. Assume that C' is irreducible, $\pi_1(\mathbf{C}^2 - C') \cong \mathbf{Z}$ and the canonical homomorphism $\pi_1(\mathbf{C}^2 - C \cup C') \to \pi_1(\mathbf{C}^2 - C) \times \pi_1(\mathbf{C}^2 - C')$ is isomorphic. Put $D = C \cup C'$. Then the Alexander polynomial set of D contains a polynomial $q(t)$ which is divisible by $\sqrt{\Delta_C(t)}$.*

Proof. — First we may assume that

$$\pi_1(\mathbf{C}^2 - C) = \langle g_1, \dots, g_k \mid R_1, \dots, R_\ell \rangle$$
$$\pi_1(\mathbf{C}^2 - D) = \langle g_1, \dots, g_k, h \mid R_1, \dots, R_\ell, T_j,\ 1 \leqslant j \leqslant k \rangle$$

where T_j is the commuting relation: $h g_j h^{-1} g_j^{-1}$. Consider the homomorphism

$$\theta : H_1(\mathbf{C}^2 - D) \longrightarrow \mathbf{Z}, \quad [g_j] \longmapsto t,\ [h] \longmapsto t^d.$$

Then the image of the differential of the relation T_j by the ring homomorphism

$$\gamma_\theta : \mathbf{C}(F(k+1)) \longrightarrow \mathbf{C}(\pi_1(\mathbf{C}^2 - D)) \longrightarrow \mathbf{C}[t, t^{-1}]$$

gives the raw vector v_j whose j-th component is $(t^d - 1)$, $(k+1)$-th component is $1 - t$. Thus the θ-Alexander matrix of D is given by

$$A' := \begin{pmatrix} A & O \\ (t^d - 1)E_k & (1-t)\vec{w} \end{pmatrix}, \quad O = {}^t(0, \dots, 0),\ \vec{w} = {}^t(1, \dots, 1)$$

where A is the Alexander matrix for C with respect to the summation homomorphism. Take $k \times k$ minor B of A'. If B contains at least a $(k-1) \times (k-1)$ minor of A, det B is a linear combination of the $(k-1)$-minors of A and therefore divisible by $\Delta_C(t)$. Assume that B does not contain such a minor. Then any $k \times k$ minor of B is divisible by $t^d - 1$. As $\sqrt{\Delta_C(t)}$ divides $t^d - 1$ by Proposition 20, we conclude that $\sqrt{\Delta_C(t)}$ divides $\Delta_{D,\theta}(t)$. ☐

Corollary 38. — *Assume that C is as in Lemma 37 with r irreducible components and let C' be a curve with $\pi_1(\mathbf{C}^2 - C') = \mathbf{Z}^s$ with s is the number of irreducible components of C'. Suppose that the canonical homomorphism $\pi_1(\mathbf{C}^2 - C \cup C') \to \pi_1(\mathbf{C}^2 - C) \times \pi_1(\mathbf{C}^2 - C')$ is isomorphic. Then $\Delta_{C \cup C', \theta}(t)$ is divisible by $\sqrt{\Delta_C(t)}$ for $\theta = \theta_{\boldsymbol{m}}$ where $\boldsymbol{m} = (\boldsymbol{u}, \boldsymbol{v}) \in \mathbf{Z}^r \times \mathbf{Z}^s$, $\boldsymbol{u} = (1, \dots, 1)$ and $\boldsymbol{v} = (d, \dots, d)$.*

Proof. — Suppose that we have the following presentation.

$$\pi_1(\mathbf{C}^2 - C) = \langle g_1, \dots, g_k \mid R_1, \dots, R_\ell \rangle$$

Then the presentation of $\pi_1(\mathbf{C}^2 - C \cup C')$ is given by

$$\pi_1(\mathbf{C}^2 - C \cup C') = \langle g_1, \dots, g_k, h_1, \dots, h_s \mid R_1, \dots, R_\ell, T_{j,\ell},\ 1 \leqslant j \leqslant k,\ 1 \leqslant \ell \leqslant s \rangle$$

where $T_{j,\ell}$ is the commuting relation: $h_\ell g_j h_\ell^{-1} g_j^{-1}$. Consider the homomorphism

$$\theta_{\boldsymbol{m}} : H_1(\mathbf{C}^2 - C \cup C') \cong \mathbf{Z}^r \times \mathbf{Z}^s \longrightarrow \mathbf{Z}, \quad (\boldsymbol{a}, \boldsymbol{b}) \longmapsto \sum_{i=1}^r a_i + d \sum_{j=1}^s b_j$$

Assume that $C' = C'_1 \cup \cdots \cup C'_s$ be the irreducible decomposition. Put $D_j = C'_1 \cup \cdots \cup C'_j$. We have a family of surjective homomorphisms $\theta_j : H_1(\mathbf{C}^2 - C \cup D_j) \to \mathbf{Z}$ which give the commutative diagram:

$$\begin{tikzcd} H_1(\mathbf{C}^2 - C \cup C') \arrow[r] \arrow[d, "\theta_{\boldsymbol{m}}"'] & H_1(\mathbf{C}^2 - C \cup D_j) \arrow[d, "\theta_j"] \\ \mathbf{Z} \arrow[r, equal] & \mathbf{Z} \end{tikzcd}$$

Then the assertion follows from Lemma 37, by showing that $C \cup D_j$ has a non-trivial θ_j-Alexander polynomial which is divisible by $\sqrt{\Delta_C(t)}$, by the inductive argument on $j = 1, \ldots, s$. $\square$

The following assertion is immediate from Corollary 10.

Lemma 39. — *Assume that we have a degeneration family $C_t \to C_0$. Take an arbitrary surjective homomorphism $\theta : \pi_1(\mathbf{C}^2 - C_t) \to \mathbf{Z}$. Let θ' be the composition*

$$\theta' : \pi_1(\mathbf{C}^2 - C_0) \longrightarrow \pi_1(\mathbf{C}^2 - C_t) \xrightarrow{\theta} \mathbf{Z}$$

Then we have the divisibility: $\Delta_{C_t,\theta} \mid \Delta_{C_0,\theta'}$.

Now we are ready to prove the Main theorem.

Proof of Theorem 35. — Assume that C has irreducible components $C_1, \ldots, C_r$ and assume that an irreducible component C_{i_0} has a non-trivial Alexander polynomial $\Delta_{C_{i_0}}(t)$. For simplicity, we assume $i_0 = 1$. Put $d_1 = \operatorname{degree} C_1$. Consider a degeneration family C_t, $t \in \mathbf{C}$ such that $C_0 = C$ and C_t has r irreducible components $C_{t,1}, \ldots, C_{t,r}$ and $C_{t,1} \equiv C_1$ and $C_{t,j}$, $j > 1$ is smooth for $t \neq 0$ and the intersection of $C_{t,1}, \ldots, C_{t,r}$ is transverse so that C_t has only A_1-singularities besides those of C_1 and $\pi_1(\mathbf{C}^2 - C_t) \cong \pi_1(\mathbf{C}^2 - C_1) \times \mathbf{Z}^{r-1}$. Consider the surjective homomorphism $\theta : H_1(\mathbf{C}^2 - C_t) \to \mathbf{Z}$ which is defined by $\theta(a_1, \ldots, a_r) = a_1 + d_1(a_2 + \cdots + a_r)$. By Corollary 38, $\Delta_{C_t,\theta}$ is divisible by $\sqrt{\Delta_{C_1}(t)}$ and thus $\Delta_{C_0,\theta}$ is divisible by $\sqrt{\Delta_{C_1}(t)}$. $\square$

Example 40. — Let us consider a generic sextic of torus type $C\colon f_2(x,y)^3 - f_3(x,y)^2 = 0$ with six A_2's. Assume $f_2(x,y)$ and $f_3(x,y)$ be generic polynomials of degree 2 and 3 respectively so that the conic $f_2(x,y) = 0$ and the cubic $f_3(x,y) = 0$ intersect transversely at 6 points. Assume that C' is a smooth curve degree d_1 which is transverse to C. Put $D = C \cup C'$. Then their fundamental groups are presented as

$$\pi_1(\mathbf{C}^2 - C) = \langle \xi, \rho \mid R_1 \rangle, \qquad R_1 = \xi\rho\xi\rho^{-1}\xi^{-1}\rho^{-1}$$
$$\pi_1(\mathbf{C}^2 - D) = \langle \xi, \rho, \alpha \mid R_1, T_1, T_2 \rangle, \qquad T_1 = \alpha\xi\alpha^{-1}\xi^{-1}, \quad T_2 = \alpha\rho\alpha^{-1}\rho^{-1}$$

We consider the homomorphism:

$$\theta : \pi_1(\mathbf{C}^2 - D) \longrightarrow \mathbf{Z}, \quad \xi, \rho \longmapsto t,\ \alpha \longmapsto t^6$$

The corresponding Alexander matrix is given by

$$\begin{pmatrix} t^2-t+1 & -(t^2-t+1) & 0 \\ t^6-1 & 0 & 1-t \\ 0 & t^6-1 & 1-t \end{pmatrix}$$

Then $\Delta_C(t) = t^2 - t + 1$ and $\Delta_{D,\theta}(t) = (t-1)\,(t^2-t+1)$.

Consider mutually coprime integers m, n and consider the homomorphism $\theta_{m,n} : \pi_1(\mathbf{C}^2 - D) \to \mathbf{Z}$ defined by $\theta_{m,n}(\xi) = \theta_{m,n}(\rho) = t^n$ and $\theta_{m,n}(\alpha) = t^m$. The corresponding Alexander matrix is given by

$$\begin{pmatrix} t^{2n}-t^n+1 & -(t^{2n}-t^n+1) & 0 \\ t^m-1 & 0 & 1-t^n \\ 0 & t^m-1 & 1-t^n \end{pmatrix}$$

Thus the corresponding Alexander polynomial is given by

$$\begin{cases} (t-1) & m \not\equiv 0 \text{ modulo } 6 \\ (t-1)\,(t^2-t+1) & m \equiv 0 \text{ modulo } 6 \end{cases}$$

This proves that $\mathcal{A}(D) = \{(t-1),\,(t-1)(t^2-t+1)\}$.

Example 41. — Next we consider a generic curve C of $(4,6)$-torus type of degree 12. Let $f_2(x,y)$ and $f_3(x,y)$ be as in previous Example. Then C can be defined by $f_2(x,y)^6 - f_3(x,y)^4 = 0$. It has two components $C_1 : f_2(x,y)^3 - f_3(x,y)^2 = 0$ and $C_2 : f_2(x,y)^3 + f_3(x,y)^2 = 0$. However two sextics intersect at 6 cusps with the same tangent cones so that their local intersection number is 6. Then the fundamental group is presented as

$$\pi_1(\mathbf{C}^2 - C) = \langle x_0, x_1, x_2, x_3 \mid R_1, R_2, R_3 \rangle$$
$$R_1 = x_0 \omega x_2^{-1} \omega^{-1}, \quad R_2 = x_1 \omega x_3^{-1} \omega^{-1}, \quad R_3 = x_2 \omega^2 x_0^{-1} \omega^{-2},\ \omega = x_3 \cdots x_0$$

The generic Alexander polynomial with respect to $\theta_n : \pi_1(\mathbf{C}^2 - C) \to \mathbf{Z}$, which is defined by

$$\theta_n : \quad x_0, x_2 \longmapsto t^n, \quad x_1, x_3 \longmapsto t$$

are given by

$$(t-1)\left(t^2+1\right)\left(t^2-t+1\right)\left(t^2+t+1\right)\left(t^4-t^2+1\right)^2, \quad n=1$$
$$(t-1)\,(t+1)\left(t^2-t+1\right)\left(t^6+t^3+1\right)\left(t^6-t^3+1\right)^2, \quad n=2$$
$$(t-1)\left(t^4+1\right)\left(t^2+t+1\right)\left(t^2-t+1\right)\left(t^4-t^2+1\right)\left(t^8-t^4+1\right)^2, \quad n=3$$

and so on. It seems that the Alexander set contains infinite number of polynomials.

Remark 42. — Let $C_1, \ldots C_r$ be irreducible components of C and let d_i be the degree of C_i. Consider irreducible homogeneous polynomials $F_j(X,Y,Z)$, $j = 1, \ldots r$ which define C_j and put $F = F_1 \cdots F_r$. Let $K = S^5 \cap F^{-1}(0)$. In the proof of Theorem 20, Randell proved that there is a canonical isomorphism

$$\psi : \pi_1(\mathbf{P}^2 - C \cup L_\infty) \longrightarrow \pi_1(S^5 - K)$$

Consider the surjective homomorphism $\theta = \theta_{\boldsymbol{m}} : \pi_1(\mathbf{C}^2 - C) \to \mathbf{Z}$ where $\boldsymbol{m} = (m_1, \ldots, m_r)$. We assume for simplicity that $m_1, \ldots, m_r$ are positive integers such that $\gcd(m_1, \ldots, m_r) = 1$. Then the isomorphism can be taken so that $\theta = g_\# \circ \psi$ where $g = F_\theta/|F_\theta| : S^5 - K \to S^1$ is the projection map of the Milnor fibration of the function $F_\theta = F_1^{m_1} \cdots F_r^{m_r}$.

$$\begin{tikzcd}
\pi_1(\mathbf{C}^2 - C) \arrow[r, "\theta"] \arrow[d, "\psi"'] & \mathbf{Z} \arrow[d, "\mathrm{id}"] \\
\pi_1(S^5 - K) \arrow[r, "g_\#"] & \mathbf{Z}
\end{tikzcd}$$

Thus we can generalize Theorem 20.

***Theorem 43* (Restated).** — *The Alexander polynomial $\Delta_\theta(t)$ is equal to the characteristic polynomial of the monodromy $h_* : H_1(M) \to H_1(M)$ for the polynomial $F_\theta = F_1^{m_1} \cdots F_r^{m_r}$.*

References

[1] E. Artal – Sur les couples de Zariski, *J. Algebraic Geom.* **3** (1994), p. 223–247.

[2] E. Artal Bartolo & J. Carmona Ruber – Zariski pairs, fundamental groups and Alexander polynomials, *J. Math. Soc. Japan* **50** (1998), no. 3, p. 521–543.

[3] E. Artin – Theory of braids, *Ann. of Math. (2)* **48** (1947), p. 101–126.

[4] E. Brieskorn – Beispiele zur Differentialtopologie von Singularitäten, *Invent. Math.* **2** (1966), p. 1–14.

[5] D. Chéniot – Le groupe fondamental du complémentaire d'une hypersurface projective complexe, in *Singularités à Cargèse*, Astérisque, vol. 7-8, Société Mathématique de France, Paris, 1973, p. 241–251.

[6] D. Chéniot – Le théorème de Van Kampen sur le groupe fondamental du complémentaire d'une courbe algébrique projective plane, in *Fonctions de plusieurs variables complexes, Sém. François Norguet, à la mémoire d'André Martineau*, Lect. Notes in Math., vol. 409, Springer, Berlin, 1974, p. 394–417.

[7] D.C. Cohen & A.I. Suciu – Characteristic varieties of arrangements, *Math. Proc. Cambridge Philos. Soc.* **127** (1999), no. 1, p. 33–53.

[8] R.H. Crowell & R.H. Fox. – *Introduction to Knot Theory*, Ginn and Co., Boston, Mass., 1963.

[9] A.I. Degtyarev – Alexander polynomial of a curve of degree six, *J. Knot Theory Ramifications* **3** (1994), p. 439–454.

[10] ———, Quintics in $\mathbf{CP}^2$ with non-abelian fundamental group, *St. Petersburg Math. J.* **11** (2000), no. 5, p. 809–826.

[11] P. Deligne – Le groupe fondamental du complément d'une courbe plane n'ayant que des points doubles ordinaires est abélien, d'après W. Fulton, in *Sém. Bourbaki, 1979/80*, Lect. Notes in Math., vol. 842, Springer, Berlin, 1981, p. 1–10.

[12] A. Dimca – *Singularities and the topology of hypersurfaces*, Universitext, Springer-Verlag, New York, 1992.

[13] I. Dolgachev & A. Libgober – On the fundamental group of the complement to a discriminant variety, in *Algebraic geometry, Chicago, Ill. 1980*, Lect. Notes in Math., vol. 862, Springer, Berlin, 1981, p. 1–25.

[14] H. Esnault – Fibre de Milnor d'un cône sur une courbe plane singulière, *Invent. Math.* **68** (1982), no. 3, p. 477–496.

[15] W. Fulton – On the fundamental group of the complement of a node curve, *Ann. of Math. (2)* **111** (1980), no. 2, p. 407–409.

[16] H.A. Hamm & Lê D.T. – Un théorème de Zariski du type de Lefschetz, *Ann. scient. Éc. Norm. Sup.* 4e *série* **6** (1973), p. 317–355.

[17] J. Harris – On the Severi problem, *Invent. Math.* **84** (1986), no. 3, p. 445–461.

[18] E. van Kampen – On the fundamental group of an algebraic curve, *Amer. J. Math.* **55** (1933), p. 255–260.

[19] M. Kato & Y. Matsumoto – On the connectivity of the Milnor fiber of a holomorphic function at a critical point, in *Manifolds-Tokyo, 1973*, Univ. Tokyo Press, Tokyo, 1975, p. 131–136.

[20] A. Libgober – Alexander polynomial of plane algebraic curves and cyclic multiple planes, *Duke Math. J.* **49** (1982), no. 4, p. 833–851.

[21] ______, Alexander invariants of plane algebraic curves, in *Singularities, Part 2 (Arcata, Calif., 1981)*, Proc. Sympos. Pure Math., vol. 40, American Mathematical Society, Providence, RI, 1983, p. 135–143.

[22] ______, Fundamental groups of the complements to plane singular curves, in *Algebraic geometry (Bowdoin, 1985)*, Proc. Symp. Pure Math., vol. 46, American Mathematical Society, Providence, RI, 1987, p. 29–45.

[23] ______, Characteristic varieties of algebraic curves, in *Applications of algebraic geometry to coding theory, physics and computation (Eilat, 2001)*, NATO Sci. Ser. II Math. Phys. Chem., vol. 36, Kluwer Acad. Publ., Dordrecht, 2001, p. 215–254.

[24] F. Loeser & M. Vaquié – Le polynôme d'Alexander d'une courbe plane projective, *Topology* **29** (1990), no. 2, p. 163–173.

[25] J. Milnor – *Singular points of complex hypersurfaces*, Annals of Mathematics Studies, vol. 61, Princeton University Press, Princeton, NJ, 1968.

[26] B. Moishezon & M. Teicher – Braid group technique in complex geometry, I. Line arrangements in $\mathbf{CP}^2$, in *Braids (Santa Cruz, CA, 1986)*, American Mathematical Society, Providence, RI, 1988, p. 425–555.

[27] A. Némethi – The Milnor fiber and the zeta function of the singularities of type $f = P(h,g)$, *Compositio Math.* **79** (1991), no. 1, p. 63–97.

[28] M.V. Nori – Zariski's conjecture and related problems, *Ann. scient. Éc. Norm. Sup.* 4e *série* **16** (1983), no. 2, p. 305–344.

[29] K. Oguiso & M. Zaidenberg – On fundamental groups of elliptically connected surfaces, in *Complex analysis in modern mathematics*, FAZIS, Moscow, 2001, Russian, p. 231–237.

[30] M. Oka – The monodromy of a curve with ordinary double points, *Invent. Math.* **27** (1974), p. 157–164.

[31] ———, Some plane curves whose complements have non-abelian fundamental groups, *Math. Ann.* **218** (1975), no. 1, p. 55–65.

[32] ———, On the fundamental group of the complement of certain plane curves, *J. Math. Soc. Japan* **30** (1978), no. 4, p. 579–597.

[33] ———, Symmetric plane curves with nodes and cusps, *J. Math. Soc. Japan* **44** (1992), no. 3, p. 376–414.

[34] ———, Two transforms of plane curves and their fundamental groups, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **3** (1996), p. 399–443.

[35] ———, *Non-degenerate complete intersection singularity*, Hermann, Paris, 1997.

[36] ———, Alexander polynomial of sextics, *J. Knot Theory Ramifications* **12** (2003), no. 5, p. 619–636.

[37] M. Oka & D. Pho – Classification of sextics of torus type, arXiv: `math.AG/0201035`, to appear in Tokyo J. Math., 2002.

[38] ———, Fundamental group of sextic of torus type, in *Trends in Singularities* (A. Libgober & M. Tibar, eds.), Birkhäuser, Basel, 2002, p. 151–180.

[39] M. Oka & K. Sakamoto – Product theorem of the fundamental group of a reducible curve, *J. Math. Soc. Japan* **30** (1978), no. 4, p. 599–602.

[40] S.Y. Orevkov – The fundamental group of the complement of a plane algebraic curve, *Mat. Sb. (N.S.)* **137** (1988), no. 2, p. 260–270, 272.

[41] ———, The commutant of the fundamental group of the complement of a plane algebraic curve, *Uspekhi Mat. Nauk* **45** (1990), no. 1, p. 183–184.

[42] D.T. Pho – Classification of singularities on torus curves of type $(2,3)$, *Kodai Math. J.* **24** (2001), no. 2, p. 259–284.

[43] R. Randell – Milnor fibers and Alexander polynomials of plane curves, in *Singularities, Part 2 (Arcata, Calif., 1981)*, American Mathematical Society, Providence, RI, 1983, p. 415–419.

[44] I. Shimada – Fundamental groups of complements to singular plane curves, *Amer. J. Math.* **119** (1997), no. 1, p. 127–157.

[45] E.H. Spanier. – *Algebraic topology*, McGraw-Hill Book Co., New York, 1966.

[46] A.I. Suciu – Translated tori in the characteristic varieties of complex hyperplane arrangements, *Topology Appl.* **118** (2002), no. 1-2, p. 209–223, Arrangements in Boston, a Conference on Hyperplane Arrangements, 1999.

[47] M. Teicher – On the quotient of the braid group by commutators of transversal half-twists and its group actions, *Topology Appl.* **78** (1997), no. 1-2, p. 153–186, Special issue on braid groups and related topics, Jerusalem, 1995.

[48] J.A. Wolf – Differentiable fibre spaces and mappings compatible with Riemannian metrics, *Michigan Math. J.* **11** (1964), p. 65–70.

[49] O. Zariski – On the poincaré group of a rational plane curves, *Amer. J. Math.* **58** (1929), p. 607–619.

[50] ———, On the problem of existence of algebraic functions of two variables possessing a given branch curve, *Amer. J. Math.* **51** (1929), p. 305–328.

[51] ———, On the Poincaré group of a projective hypersurface, *Ann. of Math.* **38** (1937), p. 131–141.

M. Oka, Department of Mathematics, Tokyo Metropolitan University, 1-1 Mimami-Ohsawa, Hachioji-shi, Tokyo 192-0397, Japan • *E-mail :* `oka@comp.metro-u.ac.jp`