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# AN EXPLICIT CYCLE REPRESENTING THE FULTON-JOHNSON CLASS, I

*by*

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***Abstract.* —** For a singular hypersurface $X$ in a complex manifold we prove, under certain conditions, an explicit formula for the Fulton-Johnson classes in terms of obstruction theory. In this setting, our formula is similar to the expression for the Schwartz-MacPherson classes provided by Brasselet and Schwartz. We use, on the one hand, a generalization of the virtual (or GSV) index of a vector field to the case when the ambient space has non-isolated singularities, and on the other hand a Proportionality Theorem for this index, similar to the one due to Brasselet and Schwartz.

***Résumé* (Une description explicite de la classe de Fulton Johnson, I). —** Pour une hypersurface singulière $X$ d'une variété complexe, et dans certaines conditions, nous montrons une formule explicite pour les classes de Fulton-Johnson en termes de théorie d'obstruction. Dans ce contexte notre formule est similaire à l'expression des classes de Schwartz-MacPherson donnée par Brasselet et Schwartz. Nous utilisons, d'une part, une généralisation de l'indice virtuel (ou GSV-indice) d'un champs de vecteurs au cas où l'espace ambiant a des singularités non-isolées et, d'autre part, un Théorème de Proportionnalité pour cet indice, similaire à celui dû à Brasselet et Schwartz.

## 1. Introduction

There are several different ways to generalize the Chern classes of complex manifolds to the case of singular varieties. Among them are the Schwartz-MacPherson classes [**5, 16, 20**] and the Fulton-Johnson classes [**8, 9**]. Each one of them is defined in a relevant context and has its own interest and advantages. The construction in [**5, 20**] provides a geometric interpretation of the Schwartz-MacPherson classes via

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obstruction theory. This approach is very useful for understanding what these classes measure.

The motivation for this work is to give such a geometric interpretation of the Fulton-Johnson classes, in the spirit of [**5, 20**]. Here we prove that if $X \subset M$ is a singular complex analytic hypersurface of dimension $n$, defined by a holomorphic function on a manifold $M$, then the Fulton-Johnson classes can be regarded as "weighted" Schwartz-MacPherson classes.

In order to explain our result more precisely, let us consider a complex analytic manifold $M$ of dimension $m$, and a compact singular analytic subvariety $X \subset M$. Let us endow $M$ with a Whitney stratification adapted to $X$ [**24**], and consider a triangulation $(K)$ of $M$ compatible with the stratification. We denote by $(D)$ a cellular decomposition of $M$ dual to $(K)$. Let us notice that if the $2q$-cell $d_\alpha$ of $(D)$ meets $X$, it is dual of a $2(m-q)$-simplex $\sigma_\alpha$ of $(K)$ in $X$.

We recall that in her definition of Chern classes, M.H. Schwartz considers particular stratified $r$-frames $v^r$ tangent to $M$, called radial frames. They have no singularity on the $(2q-1)$-skeleton of $(D)$, with $q = m - r + 1$, and isolated singularities on the $2q$-cells $d_\alpha$, at their barycenter $\{\widehat{\sigma}_\alpha\} = d_\alpha \cap \sigma_\alpha$. Let us denote by $I(v^r, \widehat{\sigma}_\alpha)$ the index of the $r$-frame $v^r$ at $\widehat{\sigma}_\alpha$.

The result of [**5**] tells us that the Schwartz-MacPherson class $c_{r-1}(X)$ of $X$ of degree $(r-1)$ is represented in $H_{2(r-1)}(X)$ by the cycle

$$\sum_{\substack{\sigma_\alpha \subset X, \\ \dim \sigma_\alpha = 2(r-1)}} I(v^r, \widehat{\sigma}_\alpha) \cdot \sigma_\alpha$$

In this article we prove:

***Theorem 1.1***. — *Let us assume that $X \subset M$ is a hypersurface, defined by $X = f^{-1}(0)$, where $f : M \to \mathbb{D}$ is a holomorphic function into an open disc around $0$ in $\mathbb{C}$. For each point $a \in X$ let $F_a$ denote a local Milnor fiber, and let $\chi(F_a)$ be its Euler-Poincaré characteristic. Then the Fulton-Johnson class $c^{FJ}_{r-1}(X)$ of $X$ of degree $(r-1)$ is represented in $H_{2(r-1)}(X)$ by the cycle*

$$\sum_{\substack{\sigma_\alpha \subset X, \\ \dim \sigma_\alpha = 2(r-1)}} \chi(F_{\widehat{\sigma}_\alpha}) I(v^r, \widehat{\sigma}_\alpha) \cdot \sigma_\alpha \tag{1.1}$$

On the other hand, the question of understanding the difference between the Schwartz-MacPherson and the Fulton-Johnson classes has been addressed by several authors, and this led to the concept of *Milnor classes*, defined by $\mu_*(X) = (-1)^{n+1}\big(c_*(X) - c^{FJ}_*(X)\big)$, $n = \dim X$, see for instance [**1, 3, 19, 25**]. Let us define the local Milnor number of $X$ at the point $a \in X$ by $\mu(X,a) = (-1)^{n+1}(1-\chi(F_a))$; it coincides with the usual Milnor number of [**17**] when $a$ is an isolated singularity of $X$. It is non zero only on the singular set $\Sigma$ of $X$. We have the following immediate consequence of Theorem 1.1:

***Corollary 1.2***. — *Under the assumptions of Theorem 1.1, the Milnor class $\mu_{r-1}(X)$ in $H_{2(r-1)}(X)$ is represented by the cycle*

$$\sum_{\substack{\sigma_\alpha \subset \Sigma \\ \dim \sigma_\alpha = 2(r-1)}} \mu(X, \widehat{\sigma}_\alpha)\, I(v^r, \widehat{\sigma}_\alpha) \cdot \sigma_\alpha \tag{1.2}$$

One of the key ingredients we use for proving the Theorem 1.1 is a *Proportionality Theorem* for the index of vector fields and frames on singular varieties, similar to the one given in [**5**]. In order to establish it we were led to defining *the local virtual index* at an isolated zero of a smooth vector field on a complex hypersurface with (possibly) non-isolated singularities. This is a generalization of the indices defined previously in [**4, 12, 15**]. We call it "local" virtual index to distinguish it from the "global" virtual index at a whole component of the singular set, as studied in [**4**]

We notice that for hypersurfaces with isolated singularities one also has the homological index of [**11**], which coincides with the index in [**12**]. It would be interesting to know whether our generalized virtual (or GSV) index coincides with the generalized homological index in [**10**] when the ambient space has non-isolated singularities.

Our formulae can also be obtained in another way, using the MacPherson morphism $c_*$ (see [**16**]) together with the Verdier specialization map of constructible functions [**23**], since one knows (see for instance [**19**]) that the Fulton-Johnson and the Milnor classes are image by the morphism $c_*$ of certain constructible functions. The advantage of our construction here is to provide a geometric and explicit point of view, which can be used to study the general case. This is being done in [**6**].

## 2. The local virtual index of a vector field

Let $(X,0)$ be a hypersurface germ in an open set $\mathcal{U} \subset \mathbb{C}^{n+1}$, defined by a holomorphic function $f : (\mathcal{U},0) \to (\mathbb{C},0)$. Let us endow $\mathcal{U}$ with a Whitney stratification $\{V_i\}$ compatible with $X$ and let us consider the subspace $E$ of the tangent bundle $T\mathcal{U}$ of $\mathcal{U}$ consisting of the union of the tangent bundles of all the strata.:

$$E = \bigcup_{V_i} TV_i \tag{2.3}$$

A section of $T\mathcal{U}$ whose image is in $E$ is called a *stratified vector field* on $\mathcal{U}$.

Let $v$ be a stratified vector field on $(X,0)$ with an isolated singularity (zero) at $0 \in X$. We want to define an index of $v$ at $0 \in X$ which coincides with the $GSV$-index of [**12**] (or the virtual index in [**4**]) when 0 is also an isolated singularity of $X$. For this, let us consider a (sufficiently small) ball $B_\varepsilon$ around $0 \in \mathcal{U}$ and denote by $\mathcal{T}$ the Milnor tube $f^{-1}(D_\delta) \cap B_\varepsilon$, where $D_\delta$ is a (sufficiently small) disc around $0 \in \mathbb{C}$. We let $\partial\mathcal{T}$ be the "boundary" $f^{-1}(C_\delta) \cap B_\varepsilon$ of $\mathcal{T}$, $C_\delta = \partial D_\delta$.

Let $r$ be the radial vector field in $\mathbb{C}$ whose solutions are straight lines converging to 0. It can be lifted to a vector field $\widetilde{r}$ in $\mathcal{T}$, whose solutions are arcs that start in

$\partial\mathcal{T}$ and finish in $X$; since the corresponding trajectories in $\mathbb{C}$ are transversal to all the circles $(C_\eta)$ around $0 \in \mathbb{C}$ of radius $\eta \in ]0, \delta[$, it follows that the solutions of $\widetilde{r}$ are transversal to all the tubes $f^{-1}(C_\eta)$. This vector field $\widetilde{r}$ defines a $C^\infty$ retraction $\xi$ of $\mathcal{T}$ into $X$, with $X$ as fixed point set. The restriction of $\xi$ to any fixed Milnor fibre $F = f^{-1}(t_0) \cap B_\varepsilon,\ t_0 \in C_\delta$, provides a continuous map $\pi : F \to X$, which is surjective and it is $C^\infty$ over the regular part of $X$. We call such map $\xi$, or also $\pi$, a *degenerating map* for $X$ (this was called a "collapsing map" in [**14**]). Since the singular set $\Sigma$ of $X$ is a Zariski closed subset of $X$, we notice that we can choose the lifting $\widetilde{r}$ so that $\pi^{-1}(X_{\text{reg}})$ is an open dense subset of $F$, where $X_{\text{reg}}$ is the regular part $X_{\text{reg}} = X \setminus \Sigma$.

We want to use $\pi$ to lift the stratified vector field $v$ on $X$ to a vector field on $F$. Firstly, let us consider the case where $X$ has an isolated singularity at 0. The map $\pi$ is a diffeomorphism restricted to a neighbourhood $N \subset F$ of $F \cap \partial B_\varepsilon$. Then $v$ can be lifted to a non-singular vector field on $N$ and extended to the interior of $F$ with finitely many singularities, by elementary obstruction theory. By definition [**12**], the total Poincaré-Hopf index of this vector field on $F$ is the GSV-index of $v$ on $X$.

We want to generalize this construction to the case when the singularity of $X$ at 0 is not necessarily isolated. Let us consider $(X, 0)$ as above, a possibly non-isolated germ. We fix a Milnor fibre $F = f^{-1}(t_o) \cap B_\varepsilon$ for some $t_o \in C_\delta$. Given a point $x \in F$, we let $\gamma_x$ be the solution of $\widetilde{r}$ that starts at $x$. The end-point of $\gamma_x$ is the point $\pi(x) \in X$. We parametrize this arc $\gamma_x$ by the interval $[0, 1]$, with $\gamma_x(0) = x$ and $\gamma_x(1) = \pi(x)$. We assume that this interval $[0, 1]$ is the straight arc in $\mathbb{C}$ going from $t_o$ to 0, so that for each $t \in [0, 1[$, the point $\gamma_x(t)$ is in a unique Milnor fibre $F_t = f^{-1}(t) \cap B_\varepsilon$. The family of tangent spaces to $F_t$ at the points $\gamma_x(t)$ defines a 1-parameter family of $n$-dimensional subspaces of $\mathbb{C}^{n+1}$, $\{TF_t\}_{\gamma_x(t)}$. By [**18**] we may assume that the Whitney stratification $\{V_i\}$ satisfies the strict Thom $w_f$-condition. This implies that for each trajectory $\gamma_x(t)$ the corresponding family $\{TF_t\}_{\gamma_x(t)}$ has a well defined limit space $\Lambda_{\pi(x)}$, *i.e.* it converges to an $n$-plane $\Lambda_{\pi(x)} \subset T_{\pi(x)}(\mathcal{U})$ when $t \to 1$. Hence one has an identification $T_xF \cong \Lambda_{\pi(x)}$ which defines an isomorphism of vector spaces. Moreover, since $w_f$ implies the Thom $a_f$-condition one has that the limit space $\Lambda_{\pi(x)}$ contains the space $T_{\pi(x)}V_i$ tangent to the stratum that contains $\pi(x)$. Therefore the vector $v(\pi(x))$ can be lifted to a vector $\widetilde{v}(x) \in T_xF$. This vector field $\widetilde{v}$ is non-singular over the inverse image of $X_{\text{reg}}$, which is open and dense in $F$. Also $\widetilde{v}$ is non-zero on a neighbourhood of $F \cap \partial B_\varepsilon$, since $v$ is assumed to have an isolated singularity at 0. Furthermore, by the $w_f$-condition the vector field $\widetilde{v}$ is continuous, so it has a well defined Poincaré-Hopf index in $F$. The $w_f$-condition also implies that the angle between $v(\pi(x))$ and $\widetilde{v}(x)$ is small. That is, given any $\alpha > 0$ small, we can choose $\delta$ sufficiently small with respect to $\alpha$ so that the angle between $v(\pi(x))$ and $\widetilde{v}(x)$ is less than $\alpha$. This implies that if we replace $\widetilde{v}$ by some other lifting of $v$, the induced vector fields on $F$ are homotopic. Since $f$ induces a locally trivial fibration over the punctured disc $D_\delta \smallsetminus 0$, then the homotopy class of $\widetilde{v}$ does not depend on the choice of the Milnor fibre. So we obtain:

***Proposition 2.1.*** — *The Poincaré-Hopf index of $\widetilde{v}$ in $F$ depends only on $X \subset \mathcal{U}$ and the vector field $v$. It is independent of the choices of the Milnor fibre $F$ as well as the liftings involved in its definition. We call this integer the local virtual index of $v$ on $X$ at $0$, and we denote it by $\mathcal{I}_{\mathrm{v}}(v,0,X)$.*

In other words, the index $\mathcal{I}_{\mathrm{v}}(v,0,X)$ is the obstruction $\mathrm{Obs}(\widetilde{v}, T^*F, \pi^{-1}(B_\varepsilon))$ to the extension of the lifting $\widetilde{v}$ as a section of $TF$ without singularity on $\pi^{-1}(B_\varepsilon(0))$.

Let us consider now the case where $w$ is a stratified vector field transversal to the boundary $S_\varepsilon = \partial(B_\varepsilon)$ of every small ball $B_\varepsilon$, *pointing outwards*; it has a unique singular point (inside $B_\varepsilon$) at 0. The Poincaré-Hopf index of $w$ at the point 0, denoted by $I(w,0)$, is equal to 1, computed either in $M$ or in the stratum $V_i(0)$ of $X$ containing 0 (if the dimension of $V_i(0)$ is more than 0). The lifting $\widetilde{w}$ is a section of $TF$ on $\pi^{-1}(S_\varepsilon) = F \cap S_\varepsilon$, pointing outwards $\pi^{-1}(B_\varepsilon) = F \cap B_\varepsilon$.

Let us denote by $T^*F$ the fiber bundle over $F$ which is $TF$ minus the zero section. The obstruction to the extension of $\widetilde{w}$ as a section of $T^*F$ inside $\pi^{-1}(B_\varepsilon)$ is equal to the Euler-Poincaré characteristic of the Milnor fiber, *i.e.*

$$\mathrm{Obs}(\widetilde{w}, T^*F, \pi^{-1}(B_\varepsilon)) = \chi(F). \tag{2.4}$$

We obtain:

***Proposition 2.2.*** — *If $w$ is a stratified vector field pointing outwards the ball $B_\varepsilon$ along its boundary $S_\varepsilon = \partial(B_\varepsilon)$, then its local virtual index equals the Euler-Poincaré characteristic of the Milnor fiber:*

$$\mathcal{I}_{\mathrm{v}}(w,0,X) = \chi(F) = 1 + (-1)^n \mu(X,0).$$

In the sequel, for any vector bundle $\xi$ over a space $B$, we will denote by $\xi^*$ the bundle over $B$ which is $\xi$ minus its zero section.

## 3. Proportionality Theorems

Let us consider again a stratified vector field $v$ defined on the ball $B_\varepsilon \subset \mathcal{U}$, with a unique singularity at 0. We assume further that $v$ is constructed by the radial extension process of M. H. Schwartz [**20**]. This means, essentially, that if $V_j$ is any stratum containing $V_i(0)$ in its closure, then the vector field $v$ is transversal to the boundary of every tubular neighbourhood of $V_i(0)$ in $X$, pointing outwards. The Poincaré-Hopf index of $v$, computed in $V_i(0)$ and denoted $I(v,0)$, can be any integer, and the fact that $v$ is constructed by radial extension implies that $I(v,0)$ equals the Poincaré-Hopf index of $v$ computed in $\mathcal{U}$. We shall call $v$ a *vector field constructed by radial extension*, or simply *a radial vector field* if this does not lead to confusion, as in Theorem 3.1 below. If the stratum $V_i(0)$ has dimension 0, this implies that $v$ is actually radial and its local virtual index equals $\chi(F)$, by the proposition above. In the next section we will show that, more generally, we have:

***Theorem 3.1*** **(Proportionality Theorem for vector fields)**. — *Let $v$ be a radial vector field. Then the local virtual index of $v$ in $X$, $\mathcal{I}_{\mathrm{v}}(v,0,X)$, is proportional to the Poincaré-Hopf index $I(v,0)$ of $v$ in the ambient space $B_\varepsilon$:*

$$\mathcal{I}_{\mathrm{v}}(v,0,X) = \chi(F)\cdot I(v,0).$$

Let us recall some basic facts about the notion of radial frames, as defined by M.H. Schwartz [**21**], in order to generalize the notion of radial vector fields. A radial $r$-frame is a set $v^r = (v_1, v_2, \dots, v_r)$ of $r$ stratified vector fields constructed by the M.H. Schwartz method of radial extension.

Let us consider a Whitney stratification of $\mathcal{U}$ compatible with $X$ and a triangulation $(K)$ of $\mathcal{U}$ compatible with the stratification. Let us consider a cell decomposition $(D)$ of $\mathcal{U}$ dual of $(K)$. Each cell of $(D)$ meets the strata transversally. The union of cells which meet $X$ is a tubular neighbourhood of $X$ in $\mathcal{U}$. A $k$-cell $d_\alpha$ meeting $X$ is dual of a $(2(n+1)-k)$-dimensional $K$-simplex $\sigma_\alpha$ in $X$. Let us denote by $T^r\mathcal{U}$ the fiber bundle associated to $T\mathcal{U}$ whose fiber at $x\in\mathcal{U}$ is the set of (complex) $r$-frames in $T_x\mathcal{U}$. A section of $T^r\mathcal{U}$ on a subset $A$ of $\mathcal{U}$ is an $r$-frame tangent to $\mathcal{U}$ over $A$.

The general obstruction theory (see [**22**]) tells us that the obstruction dimension to the construction of an $r$-frame tangent to $\mathcal{U}$ is equal to $2q = 2((n+1)-r+1)$. In the same way, on $X_{\mathrm{reg}}$, the obstruction dimension is $2p = 2(n-r+1)$ and on $V_i^{2s}$ it is equal to $2e = 2(s-r+1)$. This implies that we can construct a stratified $r$-frame $v^r$ with isolated singularities on the $2q$-cells $d_\alpha^{2q}$ of a cellular decomposition $(D)$ of $\mathcal{U}$, with index $I(v^r, T^r\mathcal{U}, d_\alpha^{2q})$ in the barycenter $\{\widehat{\sigma}_\alpha\} = d_\alpha^{2q}\cap\sigma_\alpha$.

Since the $r$-frame is stratified, we can also consider the index $I(v^r|_{V_i}, T^rV_i, d_\alpha^{2q}\cap V_i)$ of its restriction to the stratum $V_i$ containing $\widehat{\sigma}_\alpha$. The main property of the radial frames [**21**] is that these two indices are equal:

$$I(v^r, T^r\mathcal{U}, d_\alpha^{2q}) = I(v^r|_{V_i}, T^rV_i, d_\alpha^{2q}\cap V_i).$$

We denote this common index by $I(v^r,\widehat{\sigma}_\alpha)$. The method above for lifting a vector field from $X$ to a local Milnor fiber works for frames and we have:

***Theorem 3.2*** **(Proportionality Theorem for frames)**. — *Let $v^r$ be a radial $r$-frame with isolated singularities on the $2q$-cells $d_\alpha^{2q}$ with index $I(v^r,\widehat{\sigma}_\alpha)$ in the barycenter $\{\widehat{\sigma}_\alpha\} = d_\alpha^{2q}\cap\sigma_\alpha$. Then the obstruction to the extension of $\widetilde{v}^r$ as a section of $T^rF$ on $\widetilde{\beta}^{2p} = \pi^{-1}(d_\alpha^{2q}\cap X)$ is*

$$\mathrm{Obs}(\widetilde{v}^r, T^rF, \widetilde{\beta}^{2p}) = \chi(F_{\widehat{\sigma}_\alpha})\cdot I(v^r,\widehat{\sigma}_\alpha).$$

## 4. Proof of the Proportionality Theorems

The proofs of Theorems 3.1 and 3.2 are analogous to the proof of Théorème 11.1 in [**5**]. We first give some topological properties of the Milnor fiber. Then we prove independence and proportionality properties for the obstructions in question. We will prove Theorem 3.1 in section 4.4 and Theorem 3.2 in section 4.5.

**4.1. Topological properties of the Milnor fiber.** — We will denote by $\{V_i\}$ the strata of a stratification of $X \cap B_\varepsilon$, restriction of a Whitney stratification of $\mathcal{U}$ to $X$, and we denote by $\{W_j\}$ a Whitney stratification of $F$ such that:

(i) $\pi : F \to X \cap B_\varepsilon$ is a stratified map,

(ii) for every $j$, the restriction of $\pi$ to $W_j$ is a map of constant rank from $W_j$ to a stratum $V_i$ of $X$.

Such stratifications exist by [**13**]. We notice that each $\pi^{-1}(V_i)$ is union of strata $\{W_j\}$.

In the case of isolated singularities, the construction of "polyèdres d'effondrement" by Lê [**14**] allows us to prove that there are triangulations of $\mathcal{U}$ and $F$ compatible with the previous stratifications, and such that $\pi$ is a simplicial map. For non necessarily isolated singularities, let us consider a triangulation $(K)$ of $X$ compatible with the stratification $\{V_i\}$; as the restriction of $\pi$ to each stratum $\{W_j\}$ of $F$ has constant rank, the intersection of the inverse image of a simplex of $(K)$ with the strata $W_j$ can be decomposed into cells $\widetilde{\sigma}_\beta$ satisfying the following proposition:

***Proposition 4.1.*** — *There is a simplicial triangulation $(K)$ of $\mathcal{U}$ compatible with the stratification $\{V_i\}$ and a cellular decomposition $(\widetilde{K})$ of $F$ compatible with the stratification $\{W_j\}$, such that for each cell $\widetilde{\sigma}_\beta$ of $(\widetilde{K})$, there is a simplex $\sigma_\alpha$ of $(K)$ such that $\pi(\widetilde{\sigma}_\beta) = \sigma_\alpha$ and the restriction of $\pi$ to each open cell $\widetilde{\sigma}_\beta$ has constant rank.*

Let us denote by $(\Delta)$ a barycentric subdivision of $(K)$ and by $(D)$ the cell decomposition dual of $(K)$ defined by $(\Delta)$. The intersection of a $(D)$-cell $d^\ell_\alpha$ with $X$ is a $(\Delta)$-subcomplex of dimension $\ell - 2$, denoted by $\delta^{\ell-2}_\alpha$. Using [**5**] one can construct a cell decomposition $(\widetilde{D})$ of $F$ dual of $(\widetilde{K})$ satisfying the following property:

***Proposition 4.2* ([5], Proposition 3).** — *Let us consider a $(K)$-simplex $\sigma_\alpha$, its dual cell $d^\ell_\alpha$ and $\delta^{\ell-2}_\alpha = d^\ell_\alpha \cap X$. Let us denote by $\{\widetilde{\sigma}_\beta\}_{\beta \in B_\alpha}$ the set of $(\widetilde{K})$-cells such that $\pi(\widetilde{\sigma}_\beta) = \sigma_\alpha$ and $\dim \pi(\widetilde{\sigma}_\beta) = \dim(\sigma_\alpha)$. Let us denote by $\widetilde{d}_\beta$ the dual cell of $\widetilde{\sigma}_\beta$ in $(\widetilde{D})$. One has:*

$$\textit{Closure of } \pi^{-1}(\delta^{\ell-2}_\alpha) = \textit{Closure of } \bigcup_{\beta \in B_\alpha} \widetilde{d}_\beta$$

We can suppose that the barycenter $\widehat{\sigma}_\alpha$ of the cell $d^{2n+2}_\alpha$ in the cellular decomposition $(D)$ corresponds to the point 0 in $\mathcal{U}$, open subset in $\mathbb{C}^{n+1}$. Let us denote by $2s$ the dimension of $V_i$, by $b^{2s}$ a small euclidean ball centered at 0 in $V_i$ and by $D^{2n+2-2s}$

a small disc, transverse to $b^{2s}$. The tube $b^{2n+2} = b^{2s} \times D^{2n+2-2s}$ is homeomorphic to a $(2n+2)$-ball, neighbourhood of 0 in the dual cell $d_\alpha^{2n+2}$. The intersection

$$\beta^{2n} = b^{2n+2} \cap X$$

is not always homeomorphic to a ball, but it is contractible to 0. One defines

$$\partial\beta^{2n} = \partial b^{2n+2} \cap X.$$

Let us denote

$$\widetilde{\beta}^{2n} = \pi^{-1}(\beta^{2n}) \quad \text{and} \quad \partial\widetilde{\beta}^{2n} = \pi^{-1}(\partial\beta^{2n})$$

in the Milnor fiber $F$.

***Proposition 4.3***. — *Let $x \in V_i^{2s}$. Then $\dim \pi^{-1}(x) \leqslant 2(n-s-1)$ for all $x \in b^{2s}$. More precisely:*

$$\dim \pi^{-1}(x) = \begin{cases} 0 & \text{if } s = n \\ 2d \leqslant 2(n-s-1) & \text{if } s \leqslant n-1. \end{cases}$$

*Proof.* — Using the stratifications of $F$ and $X \cap B_\varepsilon$, we see that $\pi^{-1}(V_i)$ is a union of strata of $F$ such that on each of them the restriction of $\pi$ has constant rank. The strata of $\pi^{-1}(V_i)$ of maximal dimension have dimension $\dim(V_i) + 2d$. Moreover, as $\pi^{-1}(V_i)$ is an analytic subspace of $F$ contained in the closure of $\pi^{-1}(X_{\text{reg}})$, one has

$$\dim \pi^{-1}(V_i) = \dim(V_i) + 2d < \dim \pi^{-1}(X_{\text{reg}}) = 2n$$

and the result follows. □

One obtains that $\dim \pi^{-1}(b^{2s}) \leqslant 2(n-s-1) + 2s = 2(n-1)$. On the other hand, Proposition 4.1 implies that $\dim \widetilde{\beta}^{2n} \leqslant 2n$. As $\beta^{2n} \cap X_{\text{reg}}$ is not empty, one gets $\dim \widetilde{\beta}^{2n} = 2n$.

**4.2. The obstruction depends only on the index.** — In this section, we show that $\mathrm{Obs}(\widetilde{v}, T^*F, \pi^{-1}(B_\varepsilon))$ depends only on the Poincaré-Hopf index $I(v,0)$ of $v$ at 0 as a section of $TV_i$ and not on the vector field $v$ itself. Moreover, if $I(v,0) = 0$, then $\mathrm{Obs}(\widetilde{v}, T^*F, \pi^{-1}(B_\varepsilon)) = 0$.

A non-zero section $v$ of $Tb^{2s}$ over $\partial b^{2s}$ determines a cycle $\gamma$ of $T^*b^{2s}$ whose index $I(\gamma)$ is, by definition, the class of $\gamma$ in $H_{2s-1}(T^*b^{2s}) \cong \mathbb{Z}$. One can extend $v$ as a section of $Tb^{2s}$ inside $b^{2s}$ with an isolated zero at 0, by a homothety centered at 0, along the rays of $b^{2s}$. This section can now be extended by the radial extension process [**20**] as a section of $E$ (see (2.3)) over $b^{2n+2}$. One obtains a section of $E$, still denoted by $v$, without zero over $b^{2n+2} \smallsetminus \{0\}$, in particular over $\partial b^{2n+2}$. Let us consider the restriction of $v$ on $\partial\beta^{2n} = \partial b^{2n+2} \cap X$, one denotes by $\widetilde{v}$ the section of $T^*F$ over $\partial\widetilde{\beta}^{2n} = \pi^{-1}(\partial\beta^{2n})$ defined by a lifting of $v$.

Since working in the ball $B_\varepsilon$ is equivalent to working in the tube $b^{2n+2}$, one has

$$\mathrm{Obs}(\widetilde{v}, T^*F, \pi^{-1}(B_\varepsilon)) = \mathrm{Obs}(\widetilde{v}, T^*F, \widetilde{\beta}^{2n}).$$

***Proposition 4.4.*** — *Let $v_0$ and $v_1$ be two sections of $T^*b^{2s}$ over $\partial b^{2s}$. They define two cycles $\gamma_0$ and $\gamma_1$ of $T^*b^{2s}$. Let $\widetilde{v}_0$ and $\widetilde{v}_1$ be liftings of $v_0$ and $v_1$ respectively, over $\partial\widetilde{\beta}^{2n}$.*

(a) *If $I(\gamma_0) = I(\gamma_1)$, then* $\mathrm{Obs}(\widetilde{v}_0, T^*F, \widetilde{\beta}^{2n}) = \mathrm{Obs}(\widetilde{v}_1, T^*F, \widetilde{\beta}^{2n})$*,*

(b) *If $I(\gamma_0) = 0$, then,* $\mathrm{Obs}(\widetilde{v}_0, T^*F, \widetilde{\beta}^{2n}) = 0$.

*Proof*

a) If $I(\gamma_0) = I(\gamma_1)$, then $v_0$ and $v_1$ are homotopic over $\partial b^{2s}$. The same holds for their extensions over $b^{2s}$ and $b^{2n+2}$. The liftings $\widetilde{v}_0$ and $\widetilde{v}_1$ over $\partial\widetilde{\beta}^{2n}$ are homotopic as sections of $TF$, so the obstructions $\mathrm{Obs}(\widetilde{v}_0, T^*F, \widetilde{\beta}^{2n})$ and $\mathrm{Obs}(\widetilde{v}_1, T^*F, \widetilde{\beta}^{2n})$ are equal.

b) If $I(\gamma_0) = 0$, then by a) one can take for $v_0$ the restriction to $\partial b^{2s}$ of a vector field $v_1$ without singularities in $b^{2s}$. The lifting of $v_1$ in $F$ is a section of $TF$ without singularities over $\widetilde{\beta}^{2n}$. One has $\mathrm{Obs}(\widetilde{v}_1, T^*F, \widetilde{\beta}^{2n}) = 0$ and the result follows by a). □

**4.3. The obstruction is proportional to** $I(\gamma)$.— In this section, we prove the proportionality itself, *i.e.* we show that there is a constant $C$ such that $\mathrm{Obs}(\widetilde{v}, T^*F, \pi^{-1}(B_\varepsilon)) = C \cdot I(v, 0)$.

***Proposition 4.5.*** — *Let $v$ be the radial vector field previously defined, $\gamma$ the cycle in $T^*b^{2s}$ defined by the restriction of $v$ to $\partial b^{2s}$ and $\widetilde{v}$ a lifting of $v$ over $\partial\widetilde{\beta}^{2n}$. Then there is a constant $C$ such that*

$$\mathrm{Obs}(\widetilde{v}, T^*F, \widetilde{\beta}^{2n}) = C \cdot I(\gamma).$$

*Proof.* — Proposition 4.4 shows that $\mathrm{Obs}(\widetilde{v}, T^*F, \widetilde{\beta}^{2n})$ does not depend on the cycle $\gamma$ defined by a section $v$ of $T^*b^{2s}$ over $\partial b^{2s}$ and whose index is $I(\gamma)$. Let us consider two cycles in $T^*b^{2s}$ defined in the following way:

i) The cycle $\gamma$ is defined by a smooth map

$$\psi_1 : \partial b^{2s} \longrightarrow T^*b^{2s},$$

such that $\psi_1(\xi) = v(\xi)$ for the unitary vector field $v$ tangent to $b^{2s}$ along the boundary $\partial b^{2s}$, defining a smooth section of $T\partial b^{2s}$, *i.e.* $\gamma = \psi_1(\partial b^{2s})$.

ii) The cycle $\gamma_0$ is defined by the smooth map

$$\psi_0 : \partial b^{2s} \longrightarrow T^*b^{2s} \tag{4.5}$$

such that $\psi_0(\xi)$ is the unitary vector in $T_0b^{2s}$ parallel to $v(\xi)$ and with origin 0. Then, $\gamma_0 = \psi_0(\partial b^{2s})$ is a cycle in the fiber $T_0b^{2s}$ and $\psi_0$ is a map with rank $2s-1$ nearly everywhere and it preserves the orientations.

In the case of the radial vector field $w$ pointing outwards $b^{2s}$ along the boundary $\partial b^{2s}$, the cycle $\gamma_0$ is a cycle of index 1 in $H_{2s-1}(T_0^*b^{2s})$. We denote it by $[c_0]$.

*4.3.1. Homotopy between $\psi_0$ and $\psi_1$ over $\partial b^{2s}$.* — Let us construct a homotopy $\psi$ between $\psi_0$ and $\psi_1$ in $T^*b^{2s}$. In order to do that, one extends on $b^{2s}$ the vector field $v$ previously defined on $\partial b^{2s}$, by a homothety of center 0. One denotes by $v'$ the extension; it has an isolated singularity at the point 0. One defines a map

$$J : ]0,1] \times b^{2s} \longrightarrow Tb^{2s}$$

such that $J(\rho,\xi)$ is the unitary vector parallel to $v'(\xi)$ at the point $\rho\xi$; we will denote it by $v_\rho(\rho\xi)$.

The map $\psi$ is the restriction of $J$ to $\partial b^{2s}$, it is a diffeomorphism over its image. Let us define $\psi_\rho$ by $\psi_\rho(\xi) = \psi(\rho,\xi)$. If $\rho$ goes to 0, then the limit of $\psi_\rho$ coincides with the map $\psi_0$ defined in (4.5). Let us denote by $\mathcal{S}$ the unit sphere of the fiber $T_0 b^{2s}$. As $\psi_0$ and $\psi_1$ are homotopic, $\psi_0$ is a ($C^2$-differentiable) map $\psi_0 : \partial b^{2s} \cong S^{2s-1} \to \mathcal{S} \cong S^{2s-1}$ with topological degree $I(\gamma)$.

One has, at the level of chains and cycles in $H_{2s-1}(T^*b^{2s})$:

$$\partial \operatorname{Im}\psi = \operatorname{Im}\psi_1 - I(\gamma)\cdot[c_0]. \tag{4.6}$$

The proof of Proposition 4.5 consists of showing that one has still a formula of type (4.6) at the level of the radial extension of $v$, still denoted by $v$, over $\partial\beta^{2n}$ (formula (iii) of Lemma 4.6) and at the level of the lifting of $v$ in $F$, over $\partial\widetilde{\beta}^{2n}$ (formula (4.7)). We will conclude the proof of Proposition 4.5 using the Transgression Lemma (Lemma 4.7).

*4.3.2. Construction of the homotopy $\Psi$ over $\partial\beta^{2n}$.* — Let us denote by $\beta^{2n-2s} = D^{2n+2-2s} \cap X$, the intersection of $X$ with the transversal disc to $b^{2s}$ in $\mathcal{U}$, and by $\theta$ the piecewise differentiable homeomorphism

$$\theta : b^{2s} \times \beta^{2n-2s} \longrightarrow \beta^{2n}$$

such that $\theta(\xi,\zeta)$ is the point of $\beta^{2n}$ whose barycentric coordinates, relative to the vertices of $(\Delta) \cap (\partial b^{2n+2} \setminus V_i)$, are equal to those of $\zeta$ and the others, corresponding to the vertices of $\partial b^{2s}$, are proportional to those of $\xi$. On the one hand, for $\xi$ fixed, $\theta(\xi,\zeta)$ is on a ray of $D_\xi^{2n+2-2s}$, on the other hand, $\zeta$ and $\theta(\xi,\zeta)$ are in the same stratum.

The boundary $\partial\beta^{2n}$ is

$$\partial\beta^{2n} = \theta\left((\partial b^{2s} \times \beta^{2n-2s}) \cup (b^{2s} \times \partial\beta^{2n-2s})\right).$$

Let us define a map

$$\Psi : ]0,1] \times \partial\beta^{2n} \longrightarrow E^*$$

such that $\Psi(\rho,y) = \Psi(\rho,\theta(\xi,\zeta))$ is the vector at the point $y_\rho = \theta(\rho\xi,\zeta)$ obtained by radial extension, at this point, of $v_\rho(\rho\xi)$.

Let us denote, for $\rho \in ]0,1]$, $\Psi_\rho(y) = \Psi(\rho,y)$. Then $\Psi_1(y)$ is the original vector field $v$ defined on $\partial\beta^{2n}$. One defines $\Psi_0$ as the limit of $\Psi_\rho$ for $\rho$ going to zero.

We define the cycle $\Gamma$ in $E^*$ in the following way: one considers the radial extension, along $\beta^{2n-2s}$, of the radial vector field $w$ constructed on $b^{2s}$. It defines a chain of $E(\beta^{2n})$, canonically oriented by $b^{2s}$ and $\beta^{2n-2s}$ and whose oriented boundary is $\Gamma$. One has $\Gamma \cap E_0^* = \Gamma \cap T_0^* V_i \cong \mathcal{S} \cong \partial b^{2s}$. In fact $\Gamma$ can be written $\Gamma' \cup \Gamma''$ where $\Gamma'$ is the union of radial extensions, along $\beta^{2n-2s}$ of vectors of $\mathcal{S}$ and $\Gamma''$ is the union of radial extensions, in $\partial\beta^{2n-2s}$ of vectors of $b^{2s}$.

***Lemma 4.6***

i) *For* $\rho > 0$, $\Psi_\rho$ *is a piecewise differentiable homeomorphism from* $\partial\beta^{2n}$ *onto its image,*

ii) $\Psi_0 : \partial\beta^{2n} \to \Gamma$ *is a piecewise differentiable homeomorphism, with topological degree* $I(\gamma)$*,*

iii) $\partial \operatorname{Im} \Psi = \operatorname{Im} \Psi_1 - I(\gamma) \cdot \Gamma$.

*Proof.* — The only point to be proved is (ii). We show that the topological degrees of $\Psi_0$ and $\psi_0$ are the same. Let $\zeta \in \Gamma \cap E_0^*$ such that $\psi_0^{-1}(\zeta)$ consists of $I(\gamma)$ points $\xi_j \in \partial b^{2s}$, and at each of them $\psi_0$ is differentiable of rank $2s-1$. From the definition of the local radial extension (see [**5**] Proposition 7.4) one obtains that $\Psi_0$ is still an homeomorphism in the neighbourhood of each point $\xi_j$, considered as in $\partial\beta^{2n}$, and that $\Psi_0$ respects the orientations of $\partial\beta^{2n}$ and $\Gamma$. One has $\Psi_0^{-1}(\zeta) = \psi_0^{-1}(\zeta)$, proving the Lemma. ☐

*4.3.3. Lifting of the homotopy over* $\partial\widetilde{\beta}^{2n}$. — Let us define the map

$$\widetilde{\Psi} : ]0,1] \times \partial\widetilde{\beta}^{2n} \longrightarrow TF|_{\widetilde{\beta}^{2n}}$$

such that $\widetilde{\Psi}(\rho, \widetilde{y})$ is the lifting at $\widetilde{y}$ of $\Psi(\rho, \pi(\widetilde{y}))$, for $\pi(\widetilde{y}) \in \partial\beta^{2n}$. We define $\widetilde{\Psi}_\rho(\widetilde{y}) = \widetilde{\Psi}(\rho, \widetilde{y})$.

If $\rho = 1$, then $\widetilde{\Psi}_1$ is the lifting of the radial extension of $v$, along $\partial\beta^{2n}$, *i.e.* $\widetilde{v}$.

If $\rho = 0$, then the image of the map $\widetilde{\Psi}_0$ is the lifting of $\Gamma$, denoted by $\widetilde{\Gamma}$. It can be oriented with the orientation induced by the one of $\Gamma|_{X_{\mathrm{reg}}}$, and we claim that it is a $(2n-1)$-cycle. In fact, the dimension of $\widetilde{\Gamma}|_{\pi^{-1}(X_{\mathrm{reg}})}$ is the same as the dimension of $\Gamma|_{X_{\mathrm{reg}}}$, *i.e.* $2n-1$. If $V_j^{2h}$ is a stratum whose dimension $2h$ is bigger than or equal to $2s$, then $\widetilde{\Gamma}|_{\pi^{-1}(V_j)} = \pi^{-1}(\Gamma|_{V_j})$. Now, for transversality reasons, the dimension of $\Gamma|_{V_j} = \psi_0(\partial\beta^{2n} \cap V_j^{2h})$ is $2h-1$. By Lemma 4.3, one has, for $x \in V_j^{2h}$, $\dim \pi^{-1}(x) \leqslant 2(n-h-1)$. One obtains $\dim \widetilde{\Gamma}|_{\pi^{-1}(V_j)} \leqslant 2n-3$, that proves the claim.

One has

$$\partial \operatorname{Im} \widetilde{\Psi} = \operatorname{Im} \widetilde{\Psi}_1 - I(\gamma) \cdot \widetilde{\Gamma} \quad \text{and} \quad \operatorname{Im} \widetilde{\Psi}_1 = \widetilde{v}(\partial\widetilde{\beta}^{2n}). \tag{4.7}$$

*4.3.4. End of the Proof of Proposition 4.5.* — Let us recall the Transgression Lemma ([**7**], see also [**5**] and [**21**]):

***Lemma 4.7***. — *Let* $p : TF \to F$ *be the projection of the tangent bundle to* $F$*. There are differential forms* $\Omega^{2n}$ *and* $\Pi^{2n-1}$ *on* $TF$*, and* $\Omega_0^{2n}$ *on* $F$*, such that:*

(i) $\Pi^{2n-1}$ *induces on each fiber* $T_{\widetilde{y}}F$ *the fundamental form of* $H^{2n-1}(T_{\widetilde{y}}F)$,
(ii) $\Omega^{2n} = p^*(\Omega_0^{2n}) = -d\Pi^{2n-1}$.

*Proof.* — The differential forms are the transgression differential forms, induced from the classical Chern transgression differential forms [**7**] on the universal bundle over the Grassmanian, as classifying space. The induced transgression forms verify (i) and (ii). □

Let us denote by $\widetilde{y}_i$ the singularities of $\widetilde{v}$ inside $\widetilde{\beta}^{2n}$ with Poincaré-Hopf index $I(\widetilde{v}, \widetilde{y}_i)$. Let us denote by $\widetilde{\gamma}_i$ the cycle defined in $T^*_{\widetilde{y}_i}F$ in the same way as in (4.5). By Lemma 4.7(i), one has

$$\int_{\widetilde{\gamma}_i} \Pi^{2n-1} = I(\widetilde{v}, \widetilde{y}_i).$$

Let us apply the Stokes formula in $TF$ to the differential forms $-\Pi^{2n-1}$ and $\Omega^{2n}$ and to the variety defined by $\widetilde{v}(\widetilde{\beta}^{2n})$. One has

$$\int_{\widetilde{v}(\widetilde{\beta}^{2n})} \Omega^{2n} = -\int_{\widetilde{v}(\widetilde{\beta}^{2n})} d\Pi^{2n-1} = -\int_{\partial\widetilde{v}(\widetilde{\beta}^{2n})} \Pi^{2n-1}$$

Observing that

$$\partial\widetilde{v}(\widetilde{\beta}^{2n}) = \widetilde{v}(\partial\widetilde{\beta}^{2n}) \cup_i \widetilde{\gamma}_i$$

one obtains:

$$\mathrm{Obs}(\widetilde{v}, T^*F, \widetilde{\beta}^{2n}) = \int_{\widetilde{v}(\partial\widetilde{\beta}^{2n})} \Pi^{2n-1} + \int_{\widetilde{\beta}^{2n}} \Omega_0^{2n}. \tag{4.8}$$

By integration of the form $\Pi^{2n-1}$ on $\partial \operatorname{Im} \widetilde{\Psi}$ and using (4.7), one has

$$\begin{aligned}\int_{\operatorname{Im}\widetilde{\Psi}_1} \Pi^{2n-1} - I(\gamma)\cdot\int_{\widetilde{\Gamma}} \Pi^{2n-1} &= \int_{\partial\operatorname{Im}\widetilde{\Psi}} \Pi^{2n-1} = \int_{\operatorname{Im}\widetilde{\Psi}} d\Pi^{2n-1}\\ &= -\int_{p(\operatorname{Im}\widetilde{\Psi})} \Omega_0^{2n} = -\int_{\widetilde{\beta}^{2n}} \Omega_0^{2n}.\end{aligned}$$

Then, using (4.8), one has

$$\mathrm{Obs}(\widetilde{v}, T^*F, \widetilde{\beta}^{2n}) = \int_{\operatorname{Im}\widetilde{\Psi}_1} \Pi^{2n-1} + \int_{\widetilde{\beta}^{2n}} \Omega_0^{2n} = I(\gamma). \int_{\widetilde{\Gamma}} \Pi^{2n-1}$$

and Proposition 4.5 follows with $C = \int_{\widetilde{\Gamma}} \Pi^{2n-1}$.

One the other hand, if $I(\gamma) = 0$, the result is obvious. □

**4.4. Proof of Theorem 3.1.** — The proof of Theorem 3.1 now goes as follows: firstly, we showed in 4.2 that the obstruction $\mathrm{Obs}(\widetilde{v}, T^*F, \pi^{-1}(B_\varepsilon))$ depends only on the index $I(v, 0)$ of $v$ at 0 as a section of $TV_i$ and not on the vector field $v$ itself. Moreover, if $I(v, 0) = 0$, then $\mathrm{Obs}(\widetilde{v}, T^*F, \pi^{-1}(B_\varepsilon)) = 0$. Then we proved Proposition 4.5, which is the proportionality itself, *i.e.* we showed that there is a constant $C$ such that $\mathrm{Obs}(\widetilde{v}, T^*F, \pi^{-1}(B_\varepsilon)) = C \cdot I(v, 0)$. Using 2.4 one obtains that if $w$ is a radial vector field of index +1, then $C = \chi(F)$. This proves the theorem.

**4.5. Proof of Theorem 3.2.** — The previous argument is also valid in the case of $r$-frames. Since an important part of the proof is similar to the case of vector fields, we give only the main indications for the proof.

Let us consider a complex manifold $M$ of (complex) dimension $(n+1)$, and $0 \leqslant r \leqslant n+1$. We recall that $2q = 2((n+1) - r + 1)$ is the obstruction dimension to the construction of an $r$-frame tangent to $M$. This implies that we can construct a radial $r$-frame $v^r$ with isolated singularities on the $2q$-cells $d_\alpha^{2q}$ of a cellular decomposition $(D)$ of $M$, with index $I(v^r, \widehat{\sigma}_\alpha)$ in the barycenter $\{\widehat{\sigma}_\alpha\} = d_\alpha^{2q} \cap \sigma_\alpha$. One can write the $r$-frame as $v^r = (v^{r-1}, u_r)$, the $(r-1)$-frame $v^{r-1}$ being without singularities on the $(2q)$-skeleton of $(D)$. The singularities of $v^r$ are zeroes of the last vector $u_r$.

In the neighbourhood of 0, the $(r-1)$-frame $v^{r-1}$ generates a sub-bundle $P^{r-1}$ of $TM$, of (complex) rank $(r-1)$. Let us denote by $Q$ the sub-bundle of $TM$ orthogonal to $P^{r-1}$ relatively to an Riemannian metric induced by the one of $\mathbb{C}^{n+1}$. The projection of the vector field $u_r$ on $Q$ parallel to $P^{r-1}$, defines a section of $Q$ over $d_\alpha^{2q}$ with an isolated singularity at 0. The index $I(v^r, \widehat{\sigma}_\alpha)$ is equal to

$$I(v^r, \widehat{\sigma}_\alpha) = I(v^r, T^r M, d_\alpha^{2q}) = I(u_r, Q^*, d_\alpha^{2q}). \tag{4.9}$$

Since the map $\pi$ has constant rank on the strata, the lifting $\widetilde{v}^{r-1}$ defines an $(r-1)$-frame tangent to $F$ over $\widetilde{\beta}^{2p} = \pi^{-1}(d_\alpha^{2q} \cap X)$. In the same way, whenever it is defined, the lifting $\widetilde{u}_r$ is linearly independent of $\widetilde{v}^{r-1}$ and they define an $r$-frame $\widetilde{v}^r = (\widetilde{v}^{r-1}, \widetilde{u}_r)$.

At any point $y$ of $\widetilde{\beta}^{2p}$, the $(r-1)$-frame $\widetilde{v}^{r-1}$ generates a $(r-1)$-subspace $\widetilde{P}^{r-1}(y)$ of $T_yF$. One obtains a trivial fiber sub-bundle of $TF$ of rank $(r-1)$ with basis $\widetilde{\beta}^{2p}$. Let us denote by $\widetilde{Q}(y)$ the vector subspace orthogonal to $\widetilde{P}^{r-1}(y)$ in $T_yF$, with the Riemannian metric induced by the one of $\mathbb{C}^{n+1}$. One obtains a fiber sub-bundle $\widetilde{Q}$ of $TF$ of rank $p$, with basis $\widetilde{\beta}^{2p}$. Let us denote by $\widetilde{Q}^*$ the associated bundle whose fiber is the previous one without the zero section.

One has

$$\mathrm{Obs}(\widetilde{v}^r, T^r F, \widetilde{\beta}^{2p}) = \mathrm{Obs}(\widetilde{u}_r, \widetilde{Q}^*, \widetilde{\beta}^{2p}) \tag{4.10}$$

Now, working with $u_r$ as a section of $Q \subset E$ over $d_\alpha^{2q}$ and with $\widetilde{u}_r$ as a section of $\widetilde{Q}$ over $\widetilde{\beta}^{2p}$, one can use the proof of Theorem 3.1 with the following modifications:

$$\begin{aligned}
b^{2s} &\longrightarrow b^{2e} = d_\alpha^{2q} \cap V_i^{2s}\\
b^{2n+2} = b^{2s} \times D^{2n+2-2s} &\longrightarrow b^{2q} = b^{2e} \times D^{2n+2-2s}\\
\beta^{2n} = b^{2n+2} \cap X \cong b^{2s} \times \beta^{2n-2s} &\longrightarrow \beta^{2p} = b^{2q} \cap X \cong b^{2e} \times \beta^{2n-2s}\\
\widetilde{\beta}^{2n} = \pi^{-1}(\beta^{2n})\,;\, \partial\widetilde{\beta}^{2n} = \pi^{-1}(\partial\beta^{2n}) &\longrightarrow \widetilde{\beta}^{2p} = \pi^{-1}(\beta^{2p})\,;\, \partial\widetilde{\beta}^{2p} = \pi^{-1}(\partial\beta^{2p})\\
\Pi^{2n-1}\,;\, \Omega^{2n-1} &\longrightarrow \Pi^{2p-1}\,;\, \Omega^{2p-1}
\end{aligned}$$

Let us denote by $V_i$ the stratum containing $\sigma_\alpha$. The cell $d_\alpha^{2q}$, dual of $\sigma_\alpha$, is transverse to $X$, *i.e.* to all strata of $X$, in particular to $V_i$. Recalling that we use Whitney stratifications, the intersection $Y := d_\alpha^{2q} \cap X$ is homeomorphic to the cone $c(L_{\widehat{\sigma}_\alpha})$ over the link of $\widehat{\sigma}_\alpha$ and a distinguished neighbourhood $U_{\widehat{\sigma}_\alpha}$ of $\widehat{\sigma}_\alpha$ in $X$ is homeomorphic to $B_i \times c(L_{\widehat{\sigma}_\alpha})$ where $B_i$ is an open ball in $V_i$ whose dimension is the one of $V_i$. One can consider two (local) Milnor fibres of $\widehat{\sigma}_\alpha$. The first one $F_{\widehat{\sigma}_\alpha} = F_{X,\widehat{\sigma}_\alpha}$ is the Milnor fibre of $\widehat{\sigma}_\alpha$ considered as a singularity of $X$, in $M$, the second one $F_{Y,\widehat{\sigma}_\alpha}$ is the Milnor fibre of $\widehat{\sigma}_\alpha$ considered as a singularity of $Y = d_\alpha^{2q} \cap X$, in $d_\alpha^{2q}$.

***Lemma 4.8***. — *The Milnor fibres $F_{X,\widehat{\sigma}_\alpha}$ and $F_{Y,\widehat{\sigma}_\alpha}$ satisfy the following relation:*

$$F_{X,\widehat{\sigma}_\alpha} \cong B_i \times F_{Y,\widehat{\sigma}_\alpha}$$

*and one has*

$$\chi(F_{X,\widehat{\sigma}_\alpha}) = \chi(F_{Y,\widehat{\sigma}_\alpha}). \tag{4.11}$$

Let us return to the proof of Theorem 3.2. Theorem 3.1 implies

$$\mathrm{Obs}(\widetilde{u}_r, \widetilde{Q}^*, \widetilde{\beta}^{2p}) = \chi(F_{Y,\widehat{\sigma}_\alpha}) \cdot I(u_r, Q^*, \widehat{\sigma}_\alpha). \tag{4.12}$$

Combining the equalities (4.9) to (4.12), one obtains the result.

## 5. The Fulton-Johnson classes

Let us consider now a compact complex manifold $M$ of dimension $m = n+1$ and a holomorphic function $f : M \to \mathbb{D}$, where $\mathbb{D}$ is an open disc around 0 in $\mathbb{C}$ and $f$ has a critical value at $0 \in \mathbb{C}$. We set $X = f^{-1}(0)$ and denote by $\Sigma$ the singular set of $X$, which consists of the points in $X$ where the differential $df$ vanishes. We denote by $X_{\mathrm{reg}} = X \setminus \Sigma$ the regular part of $X$. One has an exact sequence of vector bundles:

$$0 \longrightarrow TX_{\mathrm{reg}} \longrightarrow TM|_{X_{\mathrm{reg}}} \longrightarrow L|_{X_{\mathrm{reg}}} \longrightarrow 0,$$

where $L$ is a trivial line bundle, pull back by $f$ of the tangent bundle of $\mathbb{C}$, $TX_{\mathrm{reg}}$ is the tangent bundle of $X_{\mathrm{reg}}$, which is a sub-bundle of the tangent bundle of $M$, $TM$. Thus, $L|_{X_{\mathrm{reg}}}$ is isomorphic to the normal bundle of $X_{\mathrm{reg}}$ in $M$ and, in particular, if $X$ is smooth then its tangent bundle $TX$ is equivalent to $TM|_X - L|_X$ in the K-theory group $KU(X)$. In general, when $X$ is singular, we set

$$\tau(X) = TM|_X - L|_X,$$

and call it *the virtual tangent bundle* of $X$. This is not an actual bundle generally speaking, but it represents an element in $KU(X)$, that we still denote by $\tau(X)$. Thus its total Chern class:

$$c(\tau(X)) = c(TM|_X) \cdot c(L|_X)^{-1}$$

is well defined. The image of $c(\tau(X))$ in $H_*(X)$ under the Poincaré homomorphism coincides with the *Fulton-Johnson* class of $X$, defined in [**8, 9**]. We denote it by

$c_*^{FJ}(X) \in H_{2*}(X)$ and we refer to [**4**] for background on these classes. If $X$ is smooth, these are the Poincaré duals of the Chern classes of the tangent bundle $TX$.

Our aim now is to prove Theorem 1.1 stated in the introduction. For this, let us denote by $X_t$ the fibers $f^{-1}(t)$, $t \neq 0$. This is a 1-parameter family of $n$-dimensional complex submanifolds of $M$ that degenerate to $X$ when $t = 0$.

Since for $t \neq 0$ each $X_t$ is a smooth complex manifold, its Chern classes $c^i(X_t) \in H^{2i}(X_t)$ are well defined, and since it is compact, by Poincaré duality one can think of these as homology classes in $H_{2n-2i}(X_t)$, denoted by $c_{n-i}(X_t)$. The class in degree 0, corresponding to $c^n(X_t)$, is the Euler-Poincaré characteristic of $X_t$.

We notice that, by the compactness of $X$, given a regular neighbourhood $\mathcal{N}$ of $X$ in $M$, we can find $t$ sufficiently small so that $X_t \subset \mathcal{N}$. Thus, one has a homomorphism,

$$i_* : H_*(X_t) \longrightarrow H_*(\mathcal{N}),$$

induced by the inclusion. One also has:

$$r_* : H_*(\mathcal{N}) \longrightarrow H_*(X),$$

induced by a retraction $r$ from $\mathcal{N}$ into $X$. The composition:

$$\sigma_* = r_* \circ i_* : H_*(X_t) \longrightarrow H_*(X)$$

is the *Verdier specialization map*. Notice that by construction, for each $x \in X$, $\sigma_*$ is induced by the degenerating map $\pi$ of section 2 above, which is now globally defined on all of $X_t$. In other words, the Verdier specialization map is in this case the homomorphism in homology induced by the map $\pi : X_t \to X$ defined (locally) in section 2 above.

For each $X_t$, $t \neq 0$, one has that $[TX_t] = [TM|_{X_t} - L|_{X_t}]$ in $K$-theory. Thus the Chern classes of $X_t$ are those of the virtual bundle $[TM|_{X_t} - L|_{X_t}]$. By [**23**], the homology specialization map $\sigma_*$ carries the Chern classes of $TM|_{X_t}$ and $L|_{X_t}$ into the Chern classes of $TM|_X$ and $L|_X$, respectively. Thus, as noticed in [**19**], one has:

$$c_*^{FJ}(X) = \sigma_* c_*(X_t). \tag{5.13}$$

Let $\widetilde{v}^r$ be, as before, a lifting to $X_t$ via the degenerating map $\pi$, of a frame $v^r$ on the $2p$-skeleton of $X$ with isolated singularities. With the notations of 4.5, the Chern class $c^p(X_t)$ is represented by the obstruction cocycle $\widetilde{\gamma}$ satisfying

$$\langle \widetilde{\gamma}, \widetilde{\beta}^{2p} \rangle = \mathrm{Obs}(\widetilde{v}^r, T^r X_t, \widetilde{\beta}^{2p}) = \sum I(\widetilde{v}^r, y_\lambda)$$

where the points $y_\lambda$ are singular points of $\widetilde{v}^r$ within $\widetilde{\beta}^{2p} = \pi^{-1}(d_\alpha^{2q} \cap X)$.

For each point $a \in X$, the restriction of $f$ to a small neighbourhood of $a$ can be regarded as a holomorphic function from an open set in $\mathbb{C}^{n+1}$ into $\mathbb{C}$. Hence there exists a (local) Milnor fiber $F_a$ of $X$ at $a$. This can be identified with $X_t \cap B_\varepsilon(a)$ for $t \neq 0$ sufficiently near the origin in $\mathbb{C}$ and $B_\varepsilon(a)$ a small ball in $M$ around $a$. We denote by $\chi(F_a)$ the Euler-Poincaré characteristic of $F_a$.

By Theorem 3.2 one has: $\langle\widetilde{\gamma},\widetilde{\beta}^{2p}\rangle = \chi(F_{\widehat{\sigma}_\alpha})\cdot I(v^r,\widehat{\sigma}_\alpha)$. The following lemma will prove Theorem 1.1:

***Lemma 5.1***. — *Let $\widetilde{\gamma}$ be a $(\widetilde{D})$-cocycle representing the Chern class $c^p(X_t)$ and let us denote $k_\alpha = \langle\widetilde{\gamma},\widetilde{\beta}^{2p}\rangle$. Then the cycle*

$$\sum_{\sigma_\alpha^{2r-2}\subset X} k_\alpha \sigma_\alpha^{2r-2}$$

*represents the Fulton-Johnson class $c_{r-1}^{FJ}(X)$.*

*Proof.* — Let us write the cycle $\widetilde{\gamma}$ representing $c^p(X_t)$ as

$$\widetilde{\gamma} = \sum \mu_\beta (\widetilde{d}_\beta^{2p})^*$$

where $(\widetilde{d}_\beta^{2p})^*$ is the elementary $(\widetilde{D})$-cochain whose value is 1 on the cell $\widetilde{d}_\beta^{2p}$ and 0 on all others. In other words, $\mu_\beta = \langle\widetilde{\gamma},\widetilde{d}_\beta^{2p}\rangle$.

Since $X_t$ is smooth, the Chern class $c_{r-1}(X_t)$ is the Poincaré dual of $c^p(X_t)$. This means that if $[X_t]$ denotes the fundamental class of $X_t$ and if $\widetilde{\sigma}_\beta^{2r-2}$ denotes the dual cell of $\widetilde{d}_\beta^{2p}$, then one has (see [**2**]):

$$\widetilde{\gamma}\cap[X_t] = \sum_{\widetilde{\sigma}_\beta^{2r-2}\subset X_t} \mu_\beta \widetilde{\sigma}_\beta^{2r-2}.$$

By (5.13), the Fulton-Johnson class is represented by the cycle $\pi_*(\widetilde{\gamma}\cap[X_t])$. In the image of $\widetilde{\gamma}\cap[X_t]$ by $\pi_*$, the only cells $\widetilde{\sigma}_\beta^{2r-2}$ with non-zero contribution are the cells such that $\pi(\widetilde{\sigma}_\beta)=\sigma_\alpha$ and $\dim\pi(\widetilde{\sigma}_\beta)=\dim(\sigma_\alpha)$. The images of other cells have dimension strictly less than $2r-2$. Thus the cycle $\pi_*(\widetilde{\gamma}\cap[X_t])$ is homologous to

$$\pi_*\Big(\sum_{\widetilde{\sigma}_\beta^{2r-2}\subset X_t} \mu_\beta\widetilde{\sigma}_\beta^{2r-2}\Big) = \sum_{\sigma_\alpha^{2r-2}\subset X} k_\alpha\sigma_\alpha^{2r-2},$$

where $k_\alpha = \sum\mu_\beta = \sum\langle\widetilde{\gamma},\widetilde{d}_\beta^{2p}\rangle$, the sum being extended to all the indices $\beta$ such that $\pi(\widetilde{\sigma}_\beta)=\sigma_\alpha$ and $\dim\pi(\widetilde{\sigma}_\beta)=\dim(\sigma_\alpha)$. By Proposition 4.2 one has $k_\alpha = \langle\widetilde{\gamma},\widetilde{\beta}^{2p}\rangle$, hence the lemma. $\square$

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