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ON SOME CLASSES OF WEAKLY KODAIRA SINGULARITIES

by

Tadashi Tomaru

Abstract. — In this paper, we prove some relations between surface singularities and pencils of compact complex algebraic curves. Let (X,o) be a complex normal surface singularity. Let $p_f(X,o)$ be the arithmetic genus of the fundamental cycle associated to (X,o). If there is a pencil of curves of genus $p_f(X,o)$ (*i.e.*, $\Phi\colon S\to\Delta$, where Φ is a proper holomorphic map between a non-singular complex surface and a small open disc in $\mathbb{C}^1$ around the origin $\{0\}$ and the fiber $S_t=\Phi^{-1}(t)$ is a smooth compact algebraic curve of genus $p_f(X,o)$ for any $t\neq 0$) and a resolution $(\widetilde{X},E)\to(X,o)$ such that $(S,\operatorname{supp}(S_o))\supset(\widetilde{X},E)$, then we call (X,o) *a weakly Kodaira singularity*. Any Kodaira singularity in the sense of Karras is a weakly Kodaira singularity. In this paper we show some sufficient conditions for surface singularities of some classes to be weakly Kodaira singularities.

***Résumé* (Sur certaines classes de singularités faiblement Kodaira).** — Dans cet article, nous montrons certaines relations entre les singularités de surfaces et les pinceaux de courbes algébriques complexes compactes. Soit (X,o) une singularité de surface complexe normale. Soit $p_f(X,o)$ le genre arithmétique du cycle fondamental associé à (X,o). S'il existe un pinceau de courbes de genre $p_f(X,o)$ (*i.e.*, s'il existe une application holomorphe propre $\Phi\colon S\to\Delta$, entre une surface complexe non-singulière et un petit disque ouvert dans $\mathbb{C}^1$ autour de l'origine $\{0\}$ tels que la fibre $S_t=\Phi^{-1}(t)$ soit une courbe algébrique lisse compacte de genre $p_f(X,o)$ pour tout $t\neq 0$) et une résolution $(\widetilde{X},E)\to(X,o)$ telle que $(S,\operatorname{supp}(S_o))\supset(\widetilde{X},E)$, alors on dit que (X,o) est *une singularité faiblement Kodaira*. Toute singularité Kodaira dans le sens de Karras est une singularité faiblement Kodaira. Dans cet article, nous montrons certaines conditions suffisantes pour que les singularités de surface de certaines classes soient des singularités faiblement Kodaira.

1. Introduction

After Kulikov's work ([**4**]) on Arnold's uni- and bi-modal singularities, U. Karras ([**3**]) introduced the notion of Kodaira singularities, which was defined by pencils

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of curves (*i.e.*, one parameter families of compact complex algebraic curves). Also, J. Stevens [**8**] studied a subclass of Kodaira singularities (called *Kulikov singularities*). They applied them to deformation theory of singularities. In this paper, we also consider normal surface singularities associated to pencils of curves (*i.e.*, weakly Kodaira singularities).

In [**13**], the author introduced an invariant for normal surface singularities, which is associated to pencils of curves, and proved some results. We explain the definition. Let S be a complex surface and Δ a small open disk in the complex line $\mathbb{C}^1$ around the origin. A holomorphic mapping $\Phi\colon S \to \Delta$ is called *a pencil of curves of genus g* if Φ is proper and surjective and the fiber $S_t = \Phi^{-1}(t)$ is a smooth compact complex curve of genus g for any t with $t \neq 0$. Let (X, o) be a normal surface singularity. We consider the following property:

(1.1) There exists a good resolution $\pi\colon (\widetilde{X}, E) \to (X, o)$ and a pencil of curves $\Phi\colon S \to \Delta$ such that $(S, \operatorname{supp}(S_o)) \supset (\widetilde{X}, E)$ (*i.e.*, $S \supset \widetilde{X}$ and $\operatorname{supp}(S_o) \supset E$).

Definition 1.1

(i) Let us define

$$p_e(X, o) := \min\{\text{the genus of a pencil of curves satisfying (1.1)}\},$$

and call it *the pencil genus of* (X, o).

(ii) Let h be an element of $\mathfrak{m}_{X,o}$ such that the divisor of $\operatorname{red}(h \circ \pi)_{\widetilde{X}}$ is simple normal crossing. Consider pencils of curves $\Phi\colon S \to \Delta$ satisfying (1.1) and $h \circ \pi = \Phi$. Let us define

$$p_e(X, o, h) := \min\{\text{genus of such a pencil of curves}\},$$

and call it $p_e(X, o, h)$ *the pencil genus of a pair of* (X, o) *and* h.

For $\widetilde{X}$ and h as above, the author constructed a pencil of curves of genus $p_e(X, o, h)$ that satisfy (1.1) and $h \circ \pi = \Phi$ ([**13**], Theorem 2.2). The surface S of Definition 1.1 is constructed by glueing $\widetilde{X}$ and suitable resolution spaces of some cyclic quotient singularities. In [**13**], he also proved some results for $p_e(X, o)$ and $p_e(X, o, h)$. For example, Kodaira and Kulikov singularities are characterized by using them. Moreover, the author [**13**] proved an estimate of (1.2) on $p_e(X, o)$. Let (X, o) be a normal surface singularity and $\sigma\colon (\widetilde{X}, E) \to (X, o)$ a resolution and Z_E the fundamental cycle on E. Since the arithmetic genus $p_a(Z_E)$ of Z_E is independent of the choice of a resolution, $p_a(Z_E)$ is an invariant of (X, o) ([**14**]). Then we define it as $p_f(X, o)$ and call it the *fundamental genus* of (X, o). Also, $p_f(X, o)$ is a topological invariant of (X, o) and it is useful for a rough classification of normal surface singularities. In [**13**], the author proved that

$$p_f(X, o) \leqslant p_e(X, o) \leqslant p_a(\mathbb{M}_X) + \operatorname{mult}(X, o) - 1, \tag{1.2}$$

where $\operatorname{mult}(X,o)$ is the multiplicity of (X,o) and $\mathbb{M}_X$ is the maximal ideal cycle on the minimal resolution of (X,o). From Karras's result [**3**], if (X,o) is a Kodaira singularity, we have $p_e(X,o) = p_f(X,o)$. Therefore we give the following definition.

Definition 1.2. — If $p_f(X,o) = p_e(X,o) = g$, then we call (X,o) *a weakly Kodaira singularity of genus* g.

Though any Kodaira singularity is a weakly Kodaira singularity, the converse is not necessarily true. For rational double points, every A_n-singularity is a Kodaira singularity and every D_n-singularity ($n \geqslant 4$) is a weakly Kodaira singularity but not a Kodaira singularity. Since rational double points of E_6, E_7 and E_8 have $p_e(X,o) = 1$ ([**13**]), they are not weakly Kodaira singularities.

In this paper, we give some conditions to be weakly Kodaira singularities for normal surface singularities. In section 2, we consider normal surface singularities obtained through some procedures for pencils of curves, and prove a sufficient condition for them to be weakly Kodaira singularities. From this results, we can see that the class of weakly Kodaira singularities is fairly bigger than the class of Kodaira singularities. Also we prove some results on elliptic (*i.e.*, $p_f(X,o) = 1$) weakly Kodaira singularities. In section 3, we prove a sufficient condition for some cyclic coverings of normal surface singularities to be weakly Kodaira singularities. As a corollary, we obtain a class of weakly Kodaira hypersurface singularities which contains rational double points of D_n-type.

Notation and terminology. — Let M be a complex surface and $E = \bigcup_{j=1}^{r} E_j \subset M$ a 1-dimensional compact analytic subspace, where $E_1, \dots, E_r$ are all irreducible components of E. Suppose that $E = \sum_{j=1}^{r} E_j$ is a simple normal crossing divisor on M with $E_i^2 \leqslant 0$. For (M,E), the *weighted dual graph* (=w.d.graph) Γ_E of E is a graph such that each vertex of Γ_E represents an irreducible component E_j weighted by E_j^2 and $g(E_j)$ (=genus), while each edge connecting to E_i and E_j, $i \neq j$, corresponds to the point $E_i \cap E_j$. For example, if $E_i^2 = -b_i$ and $g(E_i) = g_i > 0$ (resp. $g_i = 0$), then E_i corresponds to a vertex which is figured as follows:

$$\underset{[g_i]}{-b_i} \quad \text{(resp. } -b_i \text{), and} \quad \bigcirc \quad \text{means} \quad -2 \, .$$

Moreover, if $D = \sum_{i=1}^{r} d_i E_i$ is a cycle on E, then we denote by $\operatorname{Coeff}_{E_i} D$ the coefficient d_i. If E_i is a $\mathbb{P}^1$ (*i.e.*, non-singular rational curve) with $E_i^2 = -1$, then we call it a (-1)-curve. If E_i is a (-1)-curve in E which intersects with only one component of E, we call it a (-1)-*edge curve* of E. For a resolution $\pi\colon (\widetilde{X}, E) \to (X,o)$ and an element $h \in \mathcal{O}_{X,o}$, let $(h\circ\pi)_{\widetilde{X}}$ be the divisor defined by $h \circ \pi$ on $\widetilde{X}$. Also let $E(h\circ\pi)$ (resp. $\Delta(h\circ\pi)$) be the exceptional part (resp. the non-exceptional part) of $(h\circ\pi)_{\widetilde{X}}$. Namely, we have $E(h\circ\pi) = \sum_{i=1}^{r} v_{E_i}(h\circ\pi)E_i$ and $\Delta(h\circ\pi) = \sum_{j=1}^{s} v_{C_j}(h\circ\pi)C_j$ if $\operatorname{supp}(\Delta(h\circ\pi)) = \bigcup_{j=1}^{s} C_j$, and so $(h\circ\pi)_{\widetilde{X}} = E(h\circ\pi) + \Delta(h\circ\pi)$. For any real number $a \in \mathbb{R}$, we denote by $\{a\}$ the least number greater than, or equal to a.

2. Weakly Kodaira singularities obtained by Kulikov process for pencils of curves

In this section we consider a procedure to obtain normal surface singularities from pencils of curves (originally introduced by Kulikov [**4**]). We give conditions for such singularities to be weakly Kodaira singularities. Also we prove a formula of the geometric genus when such singularities are elliptic.

Let E be the exceptional set of a resolution of a normal surface singularity or $\mathrm{supp}(S_o)$ for a pencil of curves $\Phi\colon S \to \Delta$. Let $F = \bigcup_{i=1}^r F_i$ and A be two 1-dimensional analytic subsets of E such that $F_i \not\subset A$ for $i = 1, \dots, r$. Let us consider the following three conditions:

(i) $F_i \simeq \mathbb{P}^1$ and $A \cdot F_1 = F_1 \cdot F_2 = \cdots = F_{r-1} \cdot F_r = 1$,
(ii) F intersects A only at $F_1 \cap A$,
(iii) $\bigcup_{i=2}^r F_i$ does not contain any (-1) curve.

If F satisfies (i) and (ii), then we call it a $\mathbb{P}^1$*-chain (of length r) started from A*. If $b_i = -F_i^2$ for any i, then we call it *a $\mathbb{P}^1$-chain of type $(b_1, \dots, b_r)$ started from A*. If a $\mathbb{P}^1$-chain F satisfies (iii), then we call it a *minimal $\mathbb{P}^1$-chain started from A*.

Let $\overline{\Phi}\colon \overline{S} \to \Delta$ be a pencil of curves and let $\overline{S}_o = \overline{\Phi}^{-1}(o) = \sum_{j=1}^r a_j A_j$ be the singular fiber. If $\gcd(a_1, \dots, a_r) > 1$ (resp. $= 1$), then we say that the pencil is multiple (resp. non-multiple).

Definition 2.1

(i) Let $\overline{\Phi}\colon \overline{S} \to \Delta$ be a non-multiple pencil of curves without any (-1)-edge curve. Let $S^{(0)} = \overline{S} \overset{\sigma_1}{\longleftarrow} S^{(1)}$ be blow-ups at non-singular points $P_1^{(1)}, \dots, P_{t_1}^{(1)}$ of $\mathrm{red}(S_o^{(0)})$. As next step, let $P_1^{(2)}, \dots, P_{t_2}^{(2)} \in \bigcup_{j=1}^{t_1} \sigma_1^{-1}(P_j^{(1)})$ be non-singular points of $\mathrm{red}(S_o^{(1)})$ and let $S^{(1)} \overset{\sigma_2}{\longleftarrow} S^{(2)}$ be blow-ups at these points. After continuing this process m times, we get $S^{(0)} = \overline{S} \overset{\sigma_1}{\longleftarrow} S^{(1)} \overset{\sigma_2}{\longleftarrow} \cdots \overset{\sigma_m}{\longleftarrow} S^{(m)} = S$ and put $\sigma = \sigma_1 \circ \cdots \circ \sigma_m$. Hence we get a new pencil $\Phi = \overline{\Phi} \circ \sigma\colon S \to \Delta$ and call this procedure Kulikov process of type I started from $P_1, \dots, P_k$ (or I-process started from $P_1, \dots, P_k$).

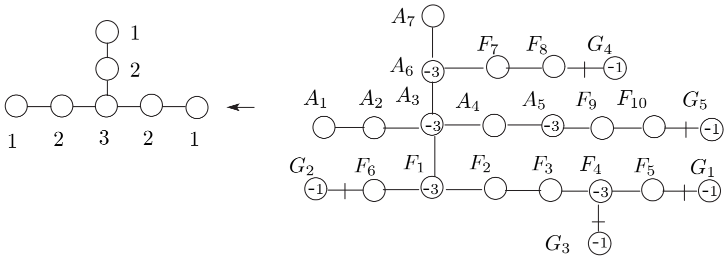
(ii) In I-process of (i), if a component $\overline{A}_{k_j}$ of $\mathrm{supp}(\overline{S}_o)$ contains $P_j^{(1)}$ $(j = 1, \dots, t_1)$ and $A_{k_j} = \sigma_*^{-1}(\overline{A}_{k_j})$ (*i.e.*, the strict transform of $\overline{A}_{k_j}$ by σ), then we call A_{k_j} *a root component* of this I-process. Let $B_1, \dots, B_{t_1}$ be connected components of $B := \mathrm{supp}(S_o) \smallsetminus \mathrm{supp}(\sigma_*^{-1}(\overline{S}_o)$. Each B_j $(j = 1, \dots, t_1)$ is constructed from all components which are infinitesimally near to $P_j^{(1)}$. We call such B_j *a branch* of $\mathrm{supp}(S_o)$ by this I-process.

(iii) For each branch B_j $(j = 1, \dots, t_1)$, we denote a partial order between all irreducible components of B_j and the root component. First we denote $A_{k_j} = \sigma_*^{-1}(\overline{A}_{k_j}) \succ F_{j_1}^{(1)} := (\sigma_2 \circ \cdots \circ \sigma_m)_*^{-1}(\sigma_1^{-1}(P_{j_1}^{(1)}))$ where $P_{j_1}^{(1)} \in \overline{A}_{k_j}$. Second, we denote $F_{j_1}^{(1)} \succ F_{j_2}^{(2)} := (\sigma_3 \circ \cdots \circ \sigma_m)_*^{-1}(\sigma_2^{-1}(P_{j_2}^{(2)}))$ if $P_{j_2}^{(2)} \in \sigma_1^{-1}(P_{j_1}^{(1)})$. We continue this for $\sigma_3, \dots, \sigma_{m-1}$ and σ_m.

(iv) For any component $F_j^{(i)}$ of a branch B_j, let $\ell(F_j^{(i)})$ be the number of blow-ups to produce $F_j^{(i)}$ from the root component A_j, and we call it *the length of* $F_j^{(i)}$. Also we define $\ell(A_k) = 0$ for any component A_k of the strict transform of $\operatorname{supp}(S_o)$ through σ. Further, let $c_R(F_j^{(i)}) = \operatorname{Coeff}_{A_{k_j}} S_o$ (*i.e.*, coefficient of the root of $F_j^{(i)}$) if A_{k_j} is the root of $F_j^{(i)}$.

We explain these terminologies and the situation through the following example:

(2.1)



where $F_1, \ldots, F_{10}, G_1, \ldots, G_5$ are produced through I-process. There are three branches whose root components are A_3, A_5 and A_6. The order between them are given as follows: $A_3 \succ F_1 \succ F_2 \succ F_3 \succ F_4 \succ F_5 \succ G_1, F_1 \succ F_6 \succ G_2, F_4 \succ G_3, A_6 \succ F_7 \succ F_8 \succ G_4$ and $A_5 \succ F_9 \succ F_{10} \succ G_5$. Also we have $\ell(F_1) = 1, \ell(F_8) = 2, \ell(G_1) = 6$ and $\ell(G_3) = 5$.

Definition 2.2. — Let $\overline{\Phi}\colon \overline{S} \to \Delta$ be a non-multiple pencil of curves and $Q_1, \ldots, Q_\ell$ non-singular points in $\overline{S}_o$. Namely, they are contained in reduced components (*i.e.*, the coefficient of S_o on the component equals one) and non-singular points of $\operatorname{supp}(\overline{S}_o)$. For each point Q_j ($j = 1, \ldots, \ell$), let's blow-up s_j times at same point Q_j, where $s_j \geqslant 2$ for any i. Let $\overline{S} \xleftarrow{\psi} S$ be a birational map obtained by these blow-ups. If $Q_j \in \overline{A}_{j_1}$, then any connected component of $\operatorname{supp}(S_o) \smallsetminus \operatorname{supp}(\psi_*^{-1}(\overline{S}_o))$ is a $\mathbb{P}^1$-chain of type $(1, 2, \ldots, 2)$ started from $A_{j_1} = \sigma_*^{-1}(\overline{A}_{j_1})$. We call this *Kulikov process of type* II started from $Q_1, \ldots, Q_k$ (or *II-process* started from $Q_1, \ldots, Q_\ell$).

Definition 2.3. — Let $\overline{\overline{\Phi}}\colon \overline{\overline{S}} \to \Delta$ be a non-multiple pencil of curves without any (-1)-edge curve. Let $P_1, \ldots, P_k$ (resp. $Q_1, \ldots, Q_\ell$) be non-singular points of S_o (resp. $\operatorname{red}(S_o)$), and assume they are different $k + \ell$ points. Let $\overline{\overline{S}} \xleftarrow{\overline{\overline{\sigma}}} \overline{S}$ be a birational map given by I-processes started from $P_1, \ldots, P_k$, and let $\overline{S} \xleftarrow{\overline{\sigma}} S$ be a birational map given by II-processes started from $Q_1, \ldots, Q_\ell$. We put $\sigma = \overline{\overline{\sigma}} \circ \overline{\sigma}$. Let A be the union of all components of the strict transform of $\operatorname{supp}(\overline{\overline{S}}_o)$ by σ, and let F be the union of all components in branches by the I-process except for (-1)-edge curves. Let $\widetilde{X}$ be a small neighborhood of $A \cup F$ and let (X, o) be a normal surface singularity obtained by contracting $A \cup F$ in $\widetilde{X}$. We call such (X, o) *a singularity obtained from Kulikov-process.*

In Definition 2.3, let G be the union of all (-1) edge curves by I-process and H the union of all exceptional components by II-process. Then there is a decomposition $\mathrm{supp}(S_o) = A \cup F \cup G \cup H$ and $B = F \cup G$.

Now let's prepare some notations to compute the fundamental cycle Z_E. For any component F_j of F and any (-1)-edge curve G_k with $F_j \succ G_k$, let $\ell(F_j, G_k) = \ell(G_k) - \ell(F_j)$ and call it the *length between F_j and G_k*. Also we denote a non-negative integer $\varepsilon(F_j)$ as follow:

$$\varepsilon(F_j) := \min_{j,k}\{\ell(F_j, G_k) \mid F_j \succ G_k\}.$$

Furthermore we define positive integers $\{\lambda(F_j)\}$ inductively as follows:

$$\lambda(F_j) := \begin{cases} \min\{c_R(F_j), \varepsilon(F_j)\}, & \text{if } F_j \cdot A \neq 0 \\ \min\{\lambda(F_i), \varepsilon(F_j)\}, & \text{if } F_i \cdot F_j \neq 0 \text{ and } F_i \succ F_j. \end{cases}$$

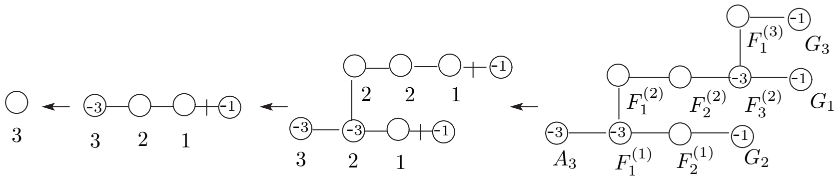
Then we have $\lambda(F_k) \geqslant \lambda(F_j)$ if $F_k \succeq F_j$. In the example of (2.1), we have $\varepsilon(F_1) = 2$, $\varepsilon(F_2) = 3$, $\varepsilon(F_3) = 2$, $\varepsilon(F_4) = 1$, $\varepsilon(F_5) = 1$, $\varepsilon(F_6) = 1$ and $\lambda(F_1) = \lambda(F_2) = \lambda(F_3) = 2$, $\lambda(F_4) = \lambda(F_5) = \lambda(F_6) = 1$.

Lemma 2.4. — *Under the condition of Definition 2.3, suppose $\ell(G_j) \geqslant c_R(G_j)$ for any (-1) edge curve G_i. Then the fundamental cycle Z_E is equal to $\sigma_*^{-1}(\overline{S}_o) + \sum_{F_j \subset F} \lambda(F_j) F_j$.*

Proof. — For any branch by a I-process, we consider a following canonical reconstruction of B_j. Let A_{k_j} be a root component of B_j. Let $G_{i_1}, \ldots, G_{i_s}$ be all (-1)-edge curves in B_j, and let's assume that $\ell_1 = \ell(G_{i_1}) \leqslant \ell_2 = \ell(G_{i_2}) \leqslant \cdots \leqslant \ell_s = \ell(G_{i_s})$. First let $\overline{S} \xleftarrow{\sigma_1} S^1$ be ℓ_1 successive blow-ups which make a $\mathbb{P}^1$-chain from A_{k_j} to G_{i_1}, and we put it $\{A_{k_j}, F_1^{(1)}, \ldots, F_{\ell_1-1}^{(1)}, G_{i_1}\}$. Let $E^{(1)}$ be the union of $\bigcup_{i=1}^{\ell_1-1} F_i^{(1)}$ and the strict transform of $\mathrm{supp}(S_o)$ by σ_1. From $\ell_1 \geqslant c_R(G_{i_1})$, we can easily check that the coefficients on $F_1^{(1)}, \ldots, F_{\ell_1-1}^{(1)}$ of the fundamental cycle $Z_{E^{(1)}}$ are given by $\lambda(F_1^{(1)}), \ldots, \lambda(F_{\ell_1-1}^{(1)})$ respectively. Second, let $S^1 \xleftarrow{\sigma_2} S^2$ be ℓ_2 blow-ups which produce a $\mathbb{P}^1$-chain from $F_{j_1}^{(1)}$ to G_{i_2} and put it $\{F_{j_1}^{(1)}, F_1^{(2)}, \ldots, F_{\ell_2-1}^{(2)}, G_{i_2}\}$. Let $E^{(2)}$ be the union of $\bigcup_{i=1}^{\ell_2-1} F_i^{(2)}$ and the strict transform of $E^{(1)}$ by σ_2 . From the assumption $\ell_2 \geqslant \ell_1$, we have $\ell_2 - j_1 \geqslant \mathrm{Coeff}_{F_{j_1}^{(1)}} Z_{E^{(1)}}$. Then the coefficients of $F_1^{(2)}, \ldots, F_{\ell_2-1}^{(2)}$ of the fundamental cycle $Z_{E^{(2)}}$ are given by $\lambda(F_1^{(2)}), \ldots, \lambda(F_{\ell_2-1}^{(2)})$ respectively. Continuing this procedure s times, we can reconstruct the branch B_j and so we have $\mathrm{Coeff}_{F_i} Z_E = \lambda(F_i)$ for any F_i in B_j. $\square$

The following figure shows the canonical reconstruction of a branch starting from A_3 in (2.1):

(2.2)



Let $(\widetilde{X}, E) \to (X, o)$ be a resolution of a normal surface singularity of $p_f(X, o) \geqslant 1$. Let consider a cycle D_o such that $0 < D_o \leqslant Z_E$ and $p_a(D_o) = p_f(X, o)$ and $p_a(D) < p_f(X, o)$ for any cycle D with $D < D_o$. Such D_o is always exists and we call it *the minimal cycle on* E and write it $Z_{\min}(E)$ ([**11**], Definition 1.2 and Proposition 1.3). A resolution is called *a good resolution* if the exceptional set is a simple normal crossing divisor in the resolution space.

Theorem 2.5. — *Let $\overline{\Phi}\colon \overline{S} \to \Delta$ be a non-multiple pencil of curves of genus $g \geqslant 1$ without any (-1)-edge curve. Let (X, o) be a normal surface singularity obtained from Kulikov-process $\overline{S} \xleftarrow{\sigma} S$ and $(\widetilde{X}, E) \subset (S, \mathrm{supp}(S_o))$ the associated good resolution, where $E = A \cup F$ and $\mathrm{supp}(S_o) = E \cup G \cup H$ as in 2.3. Also we put $F' = \bigcup_{\ell(F_j) < c_R(F_j)} F_j$ and $E' = A \cup F'$.*

(i) *Assume $\ell(G_i) \geqslant c_R(G_i)$ for any (-1) edge curve G_i. Then (X, o) is a weakly Kodaira singularity of genus g. Furthermore, assume $\overline{S}$ is minimal (i.e., $\overline{S}$ doesn't contain any (-1) curve). Then $Z_{E'} = Z_{\min}(E) = \sigma_*^{-1}(\overline{S}_o) + \sum_{F_i \subset F'} (c_R(F_i) - \ell(F_i))F_i$.*

(ii) *Conversely, if (X, o) is a weakly Kodaira singularity of genus g and S is minimal, then $\ell(G_i) \geqslant c_R(G_i)$ for any (-1) edge curve G_i.*

(iii) *Suppose that $\overline{S}$ is minimal. Then (X, o) is a weakly Kodaira singularity satisfying the minimality condition $Z_E = Z_{\min}(E)$ if and only if $\ell(G_i) = c_R(G_i)$ for any (-1) edge curve G_i.*

Proof

(i) From Lemma 2.4, we can easily see that

$$Z_{E'} = \sigma_*^{-1}(\overline{S}_o) + \sum_{F_j \subset F'} \lambda(F_j) F_j = \sigma_*^{-1}(\overline{S}_o) + \sum_{F_j \subset F'} \{c_R(F_j) - \ell(F_j)\} F_j.$$

From this we can easily check that $p_a(Z_{E'}) = g$. Since $E' \subset E$, we have

$$g = p_a(Z_{E'}) \leqslant p_a(Z_E) = p_f(X, o) \leqslant p_e(X, o) \leqslant g.$$

Then (X, o) is a weakly Kodaira singularity.

Now let's assume that $\overline{S}$ is minimal. From the above, we have

$$Z_{E'} = \sigma_*^{-1}(\overline{S}_o) + \sum_{F_i \subset F'} (c_R(F_i) - \ell(F_i)) F_i$$

and $p_a(Z_{E'}) = g$. It is easy to see the following:

$$Z_{E'} \cdot F_i = 0 \quad \text{for any } F_i \subset F' \text{ with } F_i^2 = -2. \tag{2.3}$$

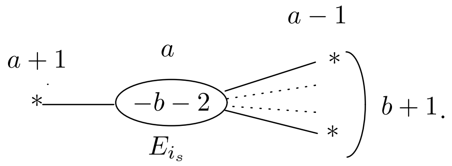
From now on we prove that $Z_{E'} = Z_{\min}(E)$. Assume that $Z_{\min}(E) < Z_{E'}$. There is a computation sequence (see p. 273 in [**11**]) from $Z_{\min}(E)$ to $Z_{E'}$ as follows:

$$Z_o := Z_{\min}(E), Z_1 = Z_o + E_{i_1}, \ldots, Z_s = Z_{E'} = Z_{s-1} + E_{i_s},$$

where $Z_{j-1} \cdot E_{i_j} > 0$ for $j = 1, \ldots, s$. Then we have

$$p_a(Z_o) = \cdots = p_a(Z_s) = g.$$

Then $(Z_{E'} - E_{i_s}) \cdot E_{i_s} = 1$ (Lemma 1.4 in [**11**]) and so $Z_{E'} \cdot E_{i_s} < 0$. Since $\overline{S}$ is minimal, we can easily check that E_{i_s} is a component of F with $E_{i_s}^2 \leqslant -3$ or the root of a branch from (2.3). In the former case, a part of $Z_{E'}$ near by E_{i_s} is written as follows:



Then we have $(Z_{E'} - E_{i_s}) \cdot E_{i_s} = 2$. In the later case we have $(Z_{E'} - E_{i_s}) \cdot E_{i_s} = -\sigma(E_{i_s})^2 > 1$ similarly. They contradict the above. Therefore we proved that $Z_{E'} = Z_{\min}(E)$.

(ii) Assume that there is a (-1) edge G_i with $\ell(G_i) < c_R(G_i)$. Let $S \xleftarrow{\sigma'} \widetilde{S}$ be an iteration of blow-ups at some points on those (-1) edge curves such that $\overline{S} \xleftarrow{\sigma\circ\sigma'} \widetilde{S}$ is a I-process and $\ell(K_i) \geqslant c_R(K_i)$ for any (-1) edge curve K_i in $\widetilde{S}_o$. Let $\widetilde{F}$ be the union of all components in branches which are not (-1) edge curves in $\widetilde{S}_o$. Also let A' be the strict transform of A by σ' and put $\widetilde{E} = A' \cup \widetilde{F}$. We put $E'' = A' \cup (\bigcup_{\ell(\widetilde{F}_j)<c_R(\widetilde{F}_j)} \widetilde{F}_j)$. From Lemma 2.4 and Theorem 2.5, we have $Z_{E''} = Z_{\min}(\widetilde{E}) = (\sigma\circ\sigma')_*^{-1}(\overline{S}_o) + \sum_{\ell(\widetilde{F}_j)<c_R(\widetilde{F}_j)} \lambda(\widetilde{F}_j)\cdot \widetilde{F}_j$ and $p_a(Z_{E''}) = g$. If we put $D_1 = (\sigma')_*^{-1}(Z_E)$, then $p_a(D_1) = p_a(Z_E)$ since σ' is isomorphic near by $\operatorname{supp}(D_1)$. Let $D_2 = \min\{D_1, Z_{E''}\} = \sum_{E''_j \subseteq E''} \min\{\mathrm{Coeff}_{E''_j} D_1, \mathrm{Coeff}_{E''_j} Z_{E''}\} E''_j$. Then $D_2 < Z_{E''} = Z_{\min}(\widetilde{E})$ and $p_a(D_1) = p_a(Z_E) = p_f(X,o) = g$ and $p_a(Z_{E''}) = g$. Hence $p_a(D_2) = g$ and $D_2 < Z_{\min}(\widetilde{E})$, and so yields a contradiction.

(iii) is obvious from (i), (ii). $\square$

Definition 2.6 **([15], Definition 3.3 and 3.10)**. — Let $\pi\colon (\widetilde{X}, E) \to (X, o)$ be the minimal good resolution of an elliptic singularity. If $Z_E \cdot Z_{\min}(E) < 0$, we say that the *elliptic sequence* is $\{Z_E\}$ and the *length of elliptic sequence* is equal to one. Suppose $Z_E \cdot Z_{\min}(E) = 0$. Let $B(1)(\subsetneqq E)$ be the maximal connected subvariety of E such that $B(1) \supset \operatorname{supp}(Z_{\min}(E))$ and $Z_E \cdot E_i = 0$ for any $E_i \subset B(1)$. Suppose $Z_{B(1)} \cdot Z_{\min}(E) = 0$. Let $B(2)(\subsetneqq B(1))$ be the maximal connected subvariety of $B(1)$ such that $B(2) \supset \mathrm{supp}(Z_{\min}(E))$ and $Z_{B(1)} \cdot E_i = 0$ for any $E_i \subset B(2)$. Continuing this process, we finally obtain $B(m)$ with $Z_{B(m)} \cdot Z_{\min}(E) < 0$. We call $\{Z_{B(0)} = Z_E, Z_{B(1)}, \ldots, Z_{B(m)}\}$ the *elliptic sequence* and *length of elliptic sequence* is $m+1$. Further, if (X,o) is a numerically Gorenstein singularity and $p_g(X,o)$ equals the length of elliptic sequence, then we call (X,o) a *maximally elliptic singularity.*

If $p_f(X,o) = 1$, then we call (X,o) an elliptic singularity. The following result generalizes results by Karras [**2**] and Stevens [**7**] on the geometric genus $p_g(X,o)$ $(= \dim_{\mathbb{C}} H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}))$ for elliptic Kulikov singularities. They proved this result under the condition of $\mathrm{Coeff}_{E_j} S_o = 1$ for any root component E_j.

Proposition 2.7. — *Let $\overline{\overline{\Phi}}\colon \overline{\overline{S}} \to \Delta$ be a minimal non-multiple pencil of genus* 1*. Let (X,o) be a normal surface singularity obtained by a Kulikov process for $\overline{\overline{S}}$ as in 2.3. Then we have the following.*

(i) $p_g(X,o) = \min\Big\{\Big[\dfrac{\ell(G_j)}{c_R(G_j)}\Big] \mid G_j$ *is any* (-1) *edge curve*$\Big\}$, *where* $[a] = \max\{n \in \mathbb{Z} \mid n \leqslant a\}$ *for any* $a \in \mathbb{R}$. *Further, if* (X,o) *is an elliptic singularity, then* $p_g(X,o)$ *equals the length of the elliptic sequence.*

(ii) *Suppose that* $\ell(G_j) \geqslant c_R(G_j)$ *for any* (-1) *edge curve* G_j. *Then the following four conditions are equivalent.*

(a) *There is a constant integer* k *such that* $\ell(G_j) = k \cdot c_R(G_i)$ *for any* (-1) *edge curve* G_j.

(b) (X,o) *is a numerically Gorenstein singularity.*

(c) (X,o) *is a Gorenstein singularity.*

(d) (X,o) *is a maximally elliptic singularity.*

(iii) (X,o) *is a minimally elliptic singularity (i.e.,* $p_g(X,o) = 1$ *and* (X,o) *is a Gorenstein singularity) if and only if* $\ell(G_j) = c_R(G_j)$ *for any* (-1) *edge curve* G_j.

Proof. — Let $\overline{\overline{S}} \xleftarrow{\sigma} S$ be a pencil of curves obtained from a Kulikov process as in 2.3 and also $(\widetilde{X}, E)$ a good resolution space with $(S, \mathrm{supp}(S_o)) \supset (\widetilde{X}, E)$.

(i) From Theorem 2.5, (X,o) is a weakly Kodaira singularity of genus 1 if and only if $\ell(G_j) \geqslant c_R(G_j)$ for any (-1) edge G_j. Hence, (X,o) is a rational singularity (*i.e.*, $\Leftrightarrow p_f(X,o) = 0 \Leftrightarrow p_g(X,o) = 0$: [**14**]) if and only if $\ell(G_{i_o}) < c_R(G_{i_o})$ for a (-1) edge G_{i_o}. Then we may assume that (X,o) is an elliptic singularity (*i.e.*, $p_f(X,o) = 1$) to prove (i). Let us consider the elliptic sequence for $(\widetilde{X}, E)$. If $Z_E \cdot Z_{\min}(E) = 0$, then we can easily check that $B(k) \subsetneqq A \cup (\bigcup_{\varepsilon(F_i) > k c_R(F_i)} F_i)$ and $B(k) \not\subset A \cup (\cup_{\varepsilon(F_i) > (k+1) c_R(F_i)} F_i)$ for $k = 1, 2, \ldots$ Then the length of the elliptic sequence is equal to

$$L := \min\Big\{\Big[\frac{\ell(G_j)}{c_R(G_j)}\Big] \mid G_j \text{ is a } (-1) \text{ edge curve }\Big\}.$$

We have $p_g(X,o) \leqslant L$ by Theorem 3.9 in [**15**]. On the other hand, there exists a nowhere zero holomorphic 2-form ω on $\overline{\overline{S}}$ since $\overline{\overline{S}}$ is the total space of an elliptic pencil. Let $\overline{\overline{E_i}}$ be an irreducible component of $\mathrm{supp}(\overline{\overline{S_o}})$ and P a non-singular point of $\overline{\overline{E_i}}$. Let make a branch $\bigcup_{j=1}^{s} F_{i_j}$ started from E_i through a I-process started from P, where $E_i = \sigma_*^{-1}(\overline{\overline{E_i}})$. Let (x, y) be a local coordinate near P such that $\overline{\overline{E_i}} = \{y = 0\}$ and ω is represented by $dx \wedge dy$. Let us consider the blow-up $\sigma_1(u, v) = (uv, v) = (u', u'v') = (x, y)$ at P. Then we have $\sigma_1^*(dx \wedge dy) = vdu \wedge dv = v'du' \wedge dv'$. Then $\sigma_1^*(\omega)$ has a zero of order 1 along a (-1) curve $\sigma_1^{-1}(P)$. Continuing this argument we can say that $\widetilde{\omega} = \sigma^*(\omega)$ is a holomorphic 2-form on S which has a zero of order $\ell(F_i)$ along a component F_i in a branch. Further, $f := \Phi \circ \sigma$ has a zero of order $\mathrm{Coeff}_{E_i} S_o$ ($= c_R(F_i)$ for any i) along any component of the branch started from E_i. Then we can see that $f^{-1} \cdot \widetilde{\omega}, f^{-2} \cdot \widetilde{\omega}, \ldots, f^{-L} \cdot \widetilde{\omega}$ are 2-forms which are meromorphic on S and also holomorphic on $S \smallsetminus \bigcup_{j=1}^{s} F_{i_j}$. They make a basis of a $\mathbb{C}$-vector space $H^o(\widetilde{X} \smallsetminus E, \mathcal{O}(K_{\widetilde{X}}))/H^o(\widetilde{X}, \mathcal{O}(K_{\widetilde{X}}))$ ($\simeq H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}})$) by Laufer's result ([**5**], Theorem 3.4). Then $p_g(X, o) \geqslant L$ and completes the proof of (i).

(ii) If we assume (a), then we can easily see that the length of the elliptic sequence is equal to k. Then (X, o) is a Gorenstein singularity because $f^{-k} \cdot \widetilde{\omega}$ is a nowhere zero holomorphic 2-form on $\widetilde{X} \smallsetminus E$ and so $(a) \Rightarrow (c)$. Since $(c) \Rightarrow (b)$ is obvious, we prove $(b) \Rightarrow (a)$. Now assume that (a) doesn't hold. Let $Z_{B(0)} = Z_E, Z_{B(1)}, \ldots, Z_{B(m)}$ be the elliptic sequence, where $B(0) = E \supsetneqq B(1) \supsetneqq \cdots \supsetneqq B(m)$. Let $E' = \bigcup_{\ell(E_i) < C_R(E_i)} E_i$ be the subset of E and so $Z_{E'} = Z_{\min}(E)$ from (i) of Theorem 2.5. Then we can easily check that $E' \subsetneqq B(m)$ since (a) doesn't hold. Hence we have $Z_{\min}(E) = Z_{E'} \lneqq Z_{B(m)}$. However we have $Z_{E'} = Z_{B(m)}$ from Theorem 3.7 in [**15**]. This is a contradiction. Therefore we have $(a) \Leftrightarrow (b) \Leftrightarrow (c)$. Further, we have $(b) \Rightarrow (d)$ from (i) and the definition of maximally elliptic singularities, and also $(d) \Rightarrow (c)$ from Theorem 3.11 in [**15**]. □

Remark 2.8. — Let (X, o) be a weakly Kodaira singularity obtained as in 2.7 and assume conditions (a)-(d). Then, from Némethi and Tomari's results ([**6**], [**9**]), we can get the value of $\mathrm{mult}(X, o)$ and embedding dimension of (X, o). Moreover, Némethi [**6**] proved that if (X, o) is a Gorenstein elliptic singularity and $H^1(A, \mathbb{Z}) = 0$ (A is the exceptional set of the minimal resolution), then (X, o) is a maximal elliptic singularity. Then, when (X, o) is a Gorenstein singularity and any component of E is a smooth rational curve, the formula of p_g of Proposition 2.7 (i) is also obtained from his result.

3. Weakly Kodaira singularities given by cyclic coverings of normal surface singularities

Let (Y, o) be a normal surface singularity and $h \in \mathfrak{m}_{Y,o} \subset \mathcal{O}_{Y,o}$. If h defines a reduced curve on Y, then h is called *a reduced element*. Let I be the defining ideal of (Y, o) and so $I \subset \mathbb{C}\{y_1, \ldots, y_N\}$, where N is the embedding dimension of (Y, o). Let J be the ideal generated by $z^n - h$ and I in $\mathbb{C}\{y_1, \ldots, y_N, z\}$. Let (X, o) be the surface singularity defined by J. Then h is a reduced element if and only if (X, o) is normal (Theorem 3.2 in [**10**]).

In this section, we prove some sufficient conditions for normal surface singularities given by cyclic coverings to be weakly Kodaira singularities.

***Definition 3.1* ([2], [3])**. — Let $\Phi\colon S \to \Delta$ be a non-multiple pencil of curves of genus g and let $S \xleftarrow{\sigma} S'$ be finite number of blow-ups at finite non-singular points of S_o. Let $\widetilde{X}$ be an open neighborhood of the proper transform E of $\mathrm{supp}(S_o)$ in S'. By contracting E in $\widetilde{X}$, we obtain a normal surface singularity (X, o). Then $\varphi\colon (\widetilde{X}, E) \to (X, o)$ is a resolution of (X, o). If a normal surface singularity is obtained in this way, then it is called *a Kodaira singularity of genus g*. Also, if σ is just one blow-up at every center in the above construction, then (X, o) is called *a Kulikov singularity of genus g* ([**7**], [**8**]). Moreover, if $h \in \mathfrak{m}_{X,o}$ satisfies $h \circ \varphi = \Phi \circ \sigma|_{\widetilde{X}}$, then we call h (or $h \circ \varphi$) a *projection function* of a Kodaira singularity (X, o).

Theorem 3.2. — *Let (Y, o) be a Kulikov singularity of genus g_o and $h \in \mathfrak{m}_{Y,o}$ its projection function and f a reduced element of $\mathfrak{m}_{Y,o}$ with $f \neq h$. Let $\sigma\colon (\widetilde{Y}, E) \to (Y, o)$ be a good resolution such that* $\mathrm{supp}(\Delta(f \circ \sigma)) \cap \mathrm{supp}(\Delta(h \circ \sigma)) = \varnothing$ *on $\widetilde{Y}$. Suppose that n divides $v_{E_j}(f \circ \sigma)$ for any E_j with $Z_E \cdot E_j < 0$. Let (X, o) be the n-th cyclic covering defined by $z^n = fh$ over (Y, o) for $n \geqslant 2$. Let $\gamma = -Z_E \cdot E(f \circ \sigma)$ and $g_1 = ng_o + (n-1)(\gamma - 2)/2$. Then we have the following.*

(i) *There is a pencil $\Phi\colon S \to \Delta$ of genus $p_e(X, o, h \circ \psi) = g_1$ and a good resolution $\pi\colon (\widetilde{X}, \widetilde{E}) \to (X, o)$ such that $(\widetilde{X}, \widetilde{E}) \subset (S, \mathrm{supp}(S_o))$ and $\Phi|_{\widetilde{X}} = h \circ \psi \circ \pi$ and any connected component of* $\mathrm{supp}(S_o) \smallsetminus \widetilde{E}$ *is a minimal $\mathbb{P}^1$-chain, where $\psi\colon (X, o) \to (Y, o)$ is the covering map.*

(ii) *Let $\mathbb{Z}_X$ (resp. $\mathbb{M}_X$) be the fundamental (resp. maximal ideal) cycle on the minimal resolution of (X, o). Let $\widehat{\Phi}\colon \widehat{S} \to \Delta$ be any pencil of curves satisfying the condition of* (i)*. Then $\mathbb{M}_X = \mathbb{Z}_X$ if and only if $\widehat{\Phi}$ is a non-multiple pencil.*

Further, if $\mathbb{M}_X = \mathbb{Z}_X$, then (X, o) is a weakly Kodaira singularity of genus g_1 and $\mathbb{Z}_X^2 = n\mathbb{Z}_Y^2$.

Proof

(i) For σ and ψ, let's consider the following diagram:

$$\begin{tikzcd}
(X, o) \arrow[d, "\psi"'] & X' \arrow[l, "\phi_1"'] \arrow[d, "\psi'"'] & X'' \arrow[l, "\phi_2"'] & (\widetilde{X}, \widetilde{E}) \arrow[l, "\phi_3"'] \arrow[dll, "\delta"'] \\
(Y, 0) & (\widetilde{Y}, E) \arrow[l, "\sigma"'] \subset (\overline{S}, \mathrm{supp}(\overline{S}_o)), & &
\end{tikzcd} \tag{3.1}$$

where $X' = X \times_Y \widetilde{Y}$ and ϕ_2 is the normalization of X' (so X' has only cyclic quotient singularities) and ϕ_3 is the minimal resolution of X''. Then $\phi := \phi_1 \circ \phi_2 \circ \phi_3\colon (\widetilde{X}, \widetilde{E}) \to (X, o)$ is a good resolution such that $(f \circ \psi \circ \phi)_{\widetilde{X}}$ and $(h \circ \psi \circ \phi)_{\widetilde{X}}$ are simple normal crossing. Also $\overline{\Phi}\colon \overline{S} \to \Delta$ is an associated pencil of curves to (Y, o) such that $(\widetilde{Y}, E) \subset (\overline{S}, \operatorname{supp}(\overline{S}_o))$ and $\overline{\Phi}|_{\widetilde{Y}} = h \circ \sigma$. Let $\Phi\colon S \to \Delta$ be a pencil of curves constructed from $\widetilde{X}$ and $h \circ \psi \circ \phi$ as in Theorem 2.2 in [**13**]. Hence, the genus of Φ is $p_e(X, o, h \circ \psi)$ and $\Phi|_{\widetilde{X}} = h \circ \psi \circ \phi$. Then we need to show that $p_e(X, o, h \circ \psi)$ is g_1 to prove (i). Since $0 \sim (f \circ \sigma)_{\widetilde{Y}} = E(f \circ \sigma) + \Delta(f \circ \sigma)$, we have $Z_E \cdot \Delta(f \circ \sigma) = -Z_E \cdot E(f \circ \sigma)$. If we put $\ell = -\mathbb{Z}_Y^2$, there are just ℓ irreducible curves $C_1, \dots, C_\ell$ satisfying $C_j \cap E \neq \varnothing$. Let $E_{i_1}, \dots, E_{i_\ell}$ be (not necessarily different) irreducible components of E with $E_{i_j} \cap C_j \neq \varnothing$ respectively for $j = 1, \dots, \ell$. If we put $\alpha_j = v_{E_{i_j}}(f \circ \sigma)$, then α_j is divided by n from the assumption. Since $v_{E_{i_j}}(h \circ \sigma) = v_{C_j}(h \circ \sigma) = 1$, X' is locally represented by $z^n = uv^{\alpha_j + 1}$ over an open neighborhood U_j of $E_{i_j} \cap C_j$ in $\widetilde{Y}$, where $E_{i_j} = \{v = o\}$ and $C_j = \{u = 0\}$. By Lemma 2.5 in [**12**], the normalization of $z^n = uv^{\alpha_j+1}$ is isomorphic to A_k-singularity. Then the following figure shows the exceptional set of $\delta^{-1}(U_j)$:

$$
\begin{array}{cccccc}
n & n & n & \cdots & n & n \\
* & \text{---}\bigcirc\text{---} & \bigcirc\text{---} & \cdots & \text{---}\bigcirc\text{---} & * \\
\widetilde{E}_{i_j} & F_{j,1} & F_{j,2} & \cdots & F_{j,n-1} & \widetilde{C}_j
\end{array}
\qquad (3.2)
$$

The integers at the top of components indicate the coefficients of the divisor $(h \circ \sigma \circ \delta)_{\widetilde{Y}}$ from Lemma 3.1 in [**12**]. Since $\overline{S}$ (resp. S) is constructed from $\widetilde{Y}$ (resp. $\widetilde{X}$) by glueing some open neighborhoods of (-1) curves, we can say that $\overline{S}_t \cap \widetilde{Y}$ (resp. $S_t \cap \widetilde{X}$) is an open Riemann surface $\overline{S}_t \smallsetminus \bigcup_{j=1}^{\ell} D_j$ (resp. $S_t \smallsetminus \bigcup_{j=1}^{\ell} \bigcup_{k=1}^{n} D_{j,k}$), where each D_j and $D_{j,k}$ are isomorphic to a closed disc in $\mathbb{C}$ and the boundary $\partial D_{j,k}$ corresponds ∂D_j by δ. If $|t|$ is sufficiently small, then S_t intersects $\Delta(f \circ \sigma)$ transversally. From the assumption of $\Delta(f \circ \sigma) \cap \Delta(h \circ \sigma) = \varnothing$, we have $D_j \cap \Delta(f \circ \sigma) = \varnothing$. Hence a holomorphic map $\delta_t\colon = \delta|_{S \cap \widetilde{X}}\colon S_t \cap \widetilde{X} \to \overline{S}_t \cap \widetilde{Y}$ is a branched covering map which has γ branch points at $\delta_t^{-1}(\overline{S}_t \cap \Delta(f \circ \sigma))$ of ramification indices n. It can be extended to a continuous finite covering map $\widetilde{\delta}_t\colon S_t \to \overline{S}_t$ which is unramified outside of $\widetilde{X}$. By the Riemann-Hurwitz formula for finite covering maps between two compact oriented real surfaces, we have $n(2 - 2g_o) - (n-1)\gamma = 2 - 2g_1$. This gives the formula of g_1.

(ii) If $g_1 = 0$, then any pencil of (i) is non-multiple and (X, o) is a rational singularity and so $\mathbb{M}_X = \mathbb{Z}_X$. Hence we may assume $g_1 \geqslant 1$. Now we prove "only if part". Let $\phi\colon (\widetilde{X}, \widetilde{E}) \to (X, o)$ be the resolution and $\Phi\colon S \to \Delta$ the pencil of curves constructed in (i). Then there is a following diagram:

$$
\begin{tikzcd}
 & & (\widehat{X}, \widehat{E}) \subset (\widehat{S}, \operatorname{supp}(\widehat{S}_o)) \arrow[lld, "\widehat{\phi}"'] \arrow[ld, "\widehat{\delta}"] \\
(X, o) & (\overline{X}, \overline{E}) \arrow[l, "\overline{\phi}"'] & \\
 & & (\widetilde{X}, \widetilde{E}) \subset (S, \operatorname{supp}(S_o)) \arrow[llu, "\phi"] \arrow[lu, "\widetilde{\delta}"']
\end{tikzcd}
\qquad (3.3)
$$

where $\overline{\phi}$ is the minimal resolution and ϕ is the good resolution in (3.1), and also $\widehat{\delta}$ and $\widetilde{\delta}$ are iterations of blow-ups from $\overline{X}$. Let E_{i_j} be a component of $E \subset \widetilde{Y}$ with $E_{i_j} \cap C_j \neq \varnothing$. Then there is a $\mathbb{P}^1$-chain of type $(2, \ldots, 2)$ between $\widetilde{E}_{i_j}$ and $\widetilde{C}_j$ as in (3.2). By considering the $\mathbb{P}^1$-chain of (3.2), we can see that $\widetilde{E}_{i_j}$ isn't contracted to a point by $\widetilde{\delta}$. In fact, if it is true, then $\widetilde{E}_{i_j}$ doesn't intersect other components except for $F_{j,1}$ and so Φ is a rational pencil and so $g_1 = 0$. This contradicts the assumption $g_1 \geqslant 1$. Hence, $\overline{E}_{i_j} :\, = \widetilde{\delta}(\widetilde{E}_{i_j})$ is an irreducible component of $\overline{E}$ and $v_{\overline{E}_{i_j}}(h \circ \psi \circ \overline{\phi}) = n$. If $\widehat{E}_{i_j}$ is the strict transform of $\overline{E}_{i_j}$ by $\widehat{\delta}$, then $v_{\widehat{E}_{i_j}}(h \circ \psi \circ \widehat{\phi}) = n$. On the other hand, from Lemma 3.1 in [**12**] and $\gcd(n, \alpha_j + 1) = 1$, we have $v_{\widetilde{E}_{i_j}}(z \circ \phi) = \dfrac{\alpha_j + 1}{\gcd(n, \alpha_j + 1)} = \alpha_j + 1$ and $v_{\widetilde{E}_{i_j}}(y_i \circ \psi \circ \phi) \geqslant n$ for any generator y_i of $\mathfrak{m}_{Y,o}$ and $v_{\widetilde{E}_{i_j}}(h \circ \sigma \circ \delta) = n$. Then $\operatorname{Coeff}_{\widehat{E}_{i_j}} Z_{\widehat{X}} = \operatorname{Coeff}_{\overline{E}_{i_j}} \mathbb{Z}_X = \operatorname{Coeff}_{\overline{E}_{i_j}} \mathbb{M}_X = \operatorname{Coeff}_{\overline{E}_{i_j}} M_{\widehat{X}} = n$ and $\operatorname{Coeff}_{\widehat{E}_{i_j}} \widehat{S}_o = v_{\widehat{E}_{i_j}}(h \circ \psi \circ \widehat{\phi}) = n$. Since $\operatorname{Coeff}_{\widehat{E}_{i_j}} \widehat{S}_o = \operatorname{Coeff}_{\widehat{E}_{i_j}} Z_{\widehat{X}} = n$, $\widehat{\Phi}$ is a non-multiple pencil.

Now we prove "if part". Since $\widehat{\Phi}$ is non-multiple, we can easily check that $\Phi \colon S \to \Delta$ constructed in (i) is also non-multiple by the construction in Theorem 2.2 in [**13**]. By the construction of S, each $\widetilde{C}_j$ is contained in a (-1) curve $G_j \subset \operatorname{supp}(S_o) \smallsetminus \widetilde{E}$. From (3.2), we can consider the following diagram:

$$\begin{tikzcd} S \arrow[rr, "\varphi"] \arrow[dr, "\Phi"'] & & \check{S} \arrow[dl, "\check{\Phi}"] \\ & \Delta & \end{tikzcd}$$

where φ is the contraction map of $(\bigcup_{j=1}^{r} \bigcup_{k=1}^{n-1} F_{j,k}) \cup (\bigcup_{j=1}^{r} G_j)$. We put $\widetilde{E}' = \widetilde{E} \smallsetminus (\bigcup_{j=1}^{r} \bigcup_{k=1}^{n-1} F_{j,k})$. Since $\operatorname{Coeff}_{\widetilde{E}_{i_j}}(S_o) = \operatorname{Coeff}_{\varphi(\widetilde{E}_{i_j})}(\check{S}_o) = n$ from (3.2), we have

$$Z_{\widetilde{E}} = S_o|_{\widetilde{E}'} + \sum_{j=1}^{r} \sum_{k=1}^{n-1} (n-k) F_{j,k} \tag{3.4}$$

from Theorem 2.5 (i). Let $y_1, \ldots, y_m$ be generators of $\mathfrak{m}_{Y,o}$, where m is the embedding dimension of (Y, o). Then an element $g := \beta_1 y_1 + \cdots + \beta_m y_m \in \mathfrak{m}_{Y,o}$ for general elements $\beta_1, \ldots, \beta_m \in \mathbb{C}$ satisfies $E(g \circ \sigma) = Z_E$ and $\operatorname{supp}(\Delta(g \circ \sigma)) \cap \operatorname{supp}(\Delta(h \circ \sigma)) = \varnothing$ and $\operatorname{supp}(\Delta(g \circ \sigma)) \cap \operatorname{supp}(\Delta(f \circ \sigma)) = \varnothing$. Hence we can easily see that $E(g \circ \sigma \circ \delta)$ is equal to the right hand side of (3.4) from $\operatorname{Coeff}_{\widetilde{E}_{i_j}} E(g \circ \sigma \circ \delta) = n$. Then $E(g \circ \sigma \circ \delta) = Z_{\widetilde{E}}$ and so $M_{\widetilde{E}} = Z_{\widetilde{E}}$. Therefore, we have $\mathbb{Z}_X^2 \geqslant \mathbb{M}_X^2 \geqslant M_{\widetilde{E}}^2 = Z_{\widetilde{E}}^2 = \mathbb{Z}_X^2$ and then $\mathbb{M}_X^2 = \mathbb{Z}_X^2$. Hence $\mathbb{M}_X = \mathbb{Z}_X$ from the result in p. 426 of [**14**]. □

Theorem 3.3. — *Let $\mathfrak{n}$ be the maximal ideal (x, y) of $\mathcal{O}_{\mathbb{C}^2,o} = \mathbb{C}\{x, y\}$ and $h \in \mathfrak{n} \smallsetminus \mathfrak{n}^2$ and $f \in \mathfrak{n}$. Suppose $(X, o) = \{z^n = fh\}$ is a normal surface singularity and n $(\geqslant 2)$ divides* $\operatorname{ord}(f)$ *and $T_o(h) \not\subset T_o(f)$, where $T_o(f)$ is the tangent cone of a curve singularity $(\{f = 0\}, o)$ at $\{o\}$ and so on. Then we have the following.*

(i) $p_e(X,o) = p_e(X,o,h) = p_a(\mathbb{M}_X) = (n-1)(\mathrm{ord}(f)-2)/2$.

(ii) *If* $\mathbb{Z}_X = \mathbb{M}_X$*, then* (X,o) *is a weakly Kodaira singularity of genus* $p_e(X,o,h)$.

(iii) *If* $\mathbb{Z}_X \neq \mathbb{M}_X$*, then there exists a multiple pencil of curves* $\Phi\colon S \to \Delta$ *of genus* $p_e(X,o)$ *and multiplicity* $-n/\mathbb{Z}_X^2$ *and exists a good resolution* $\pi\colon (\widetilde{X},E) \to (X,o)$ *such that* $(\widetilde{X},E) \subset (S, \mathrm{supp}(S_o))$ *and* $\Phi|_{\widetilde{X}} = h \circ \pi$.

Proof

(i) Let $(Y,o) = (\mathbb{C}^2,o)$ and $(\mathbb{C}^2,o) \xleftarrow{\sigma_1} V_1 \xleftarrow{\sigma_2} \cdots \xleftarrow{\sigma_s} V_s = (\widetilde{Y},E)$ be an embedded resolution of the curve singularity $\{fh=0\} \subset (\mathbb{C}^2,o)$, where each σ_i is a blow-up at a point. For $\sigma := \sigma_1 \circ \cdots \circ \sigma_s\colon (\widetilde{Y},E) \to (Y,o)$ and the covering map $\psi\colon (X,o) \to (Y,o)$ given by the projection $(x,y,z) \mapsto (x,y)$, let's consider the diagram (3.1) and put $\phi = \phi_1 \circ \phi_2 \circ \phi_3$. Since fh defines a reduced curve, we have $\mathrm{supp}\,\Delta(f\circ\sigma) \cap \mathrm{supp}\,\Delta(h\circ\sigma) = \varnothing$. Let $E_1 \subset E$ be the strict transform of $\sigma_1^{-1}(o)$ by $\sigma_2 \circ \cdots \circ \sigma_s$. Then $Z_E \cdot E_1 = -1$ and $Z_E \cdot E_i = 0$ if $i \neq 1$. Also we have $v_{E_1}(f \circ \sigma) = \mathrm{ord}(f)$. Hence $(X,o), (Y,o), f$ and h satisfy the condition of Theorem 3.2. Let put $\ell := (\alpha x + \beta y) \circ \sigma$, where α, β are general elements of $\mathbb{C}$. We have $M_{\widetilde{X}} = E(\ell \circ \delta)$ and $M_{\widetilde{X}}^2 = -n$ from Proposition 3.3 in [**12**]. We may assume that $E_1 \cap \mathrm{supp}\,\Delta(\ell \circ \sigma) \neq \varnothing$. As in the proof of (3.2), there is a $\mathbb{P}^1$-chain $\bigcup_{i=1}^{n-1} F_i \subset \widetilde{E}$ such that $\widetilde{E}_1 \cdot F_1 = F_1 \cdot F_2 = \cdots = F_{n-1} \cdot \Delta(\ell \circ \delta) = 1$, where $\widetilde{E}_1 = \delta_*^{-1}(E_1)$. Since $v_{\widetilde{E}_1}(\ell \circ \phi) = n$ and $v_{\Delta(h\circ\phi)}(\ell \circ \phi) = 0$, then we have $v_{F_i}(\ell \circ \phi) = n - i$ for $i = 1, 2, \ldots, n-1$. Let $\Phi\colon S \to \Delta$ be a pencil constructed by glueing $\widetilde{X}$ and a neighborhood of (-1) curve F_n as in Theorem 3.2 such that $\Phi|_{\widetilde{X}} = h \circ \phi$ (and so $\Delta(h \circ \phi) \subset F_n$). Then $\mathrm{Coeff}_{F_i} S_o = n$ for $i = 1, 2, \ldots, n$ and $v_{\widetilde{E}_i}(h \circ \phi) = \mathrm{Coeff}_{\widetilde{E}_i} S_o$ for any component $\widetilde{E}_i \subset \widetilde{E}$. Therefore we have $p_a(M_{\widetilde{X}}) = p_a(\widetilde{E}(\ell \circ \delta)) = p_a(S_o) = p_e(X,o,h) = (n-1)(\mathrm{ord}(f)-2)/2$ from Theorem 3.2 (i). By using Lemma 1.4 in [**11**], we can easily check that $p_a(\mathbb{M}_X) \leqslant p_e(X,o)$. Also we have $p_e(X,o) \leqslant p_a(S_o) = p_a(M_{\widetilde{X}}) \leqslant p_a(\mathbb{M}_X)$ ([**14**]). Then $p_e(X,o) = p_a(\mathbb{M}_X) = p_e(X,o,h) = (n-1)(\mathrm{ord}(f)-2)/2$ and we complete the proof of (i).

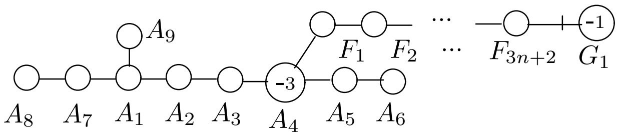
(ii) is obvious from Theorem 3.2 (ii).

(iii) Assume $\mathbb{M}_X \neq \mathbb{Z}_X$. The pencil $\Phi\colon S \to \Delta$ of (i) is multiple from Theorem 3.2 (ii) and its genus is equal to $p_e(X,o)$. Let m be the multiplicity of the pencil. Then m divides n from $\mathrm{Coeff}_{\widetilde{E}_1} S_o = n$ and we have $\mathbb{Z}_X^2 = -n/m$ since $Z_{\widetilde{E}} \cdot \widetilde{E}_1 < 0$ and $Z_{\widetilde{E}} \cdot \widetilde{E}_j = 0$ for any component $\widetilde{E}_j \subset \widetilde{E}$ except for $\widetilde{E}_1$. Hence $m = -n/\mathbb{Z}_X^2$. □

We have already remarked that any D_n-singularity ($n \geqslant 4$) is a weakly Kodaira singularity. We can check this from Theorem 3.3 since it has a defining equation $z^2 = y(x^2 + y^n)$.

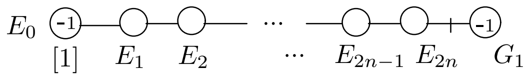
Example 3.4

(i) Let $(X,o) = \{z^3 = y(x^3 + x^2 y^{n+1} + y^{3n+4})\}$ ($n \geqslant 0$). This is a weakly Kodaira elliptic singularity from Theorem 3.3. The w.d.graph of the minimal resolution and the singular fiber of an associated pencil with the projection function y is given as follows:



If we put $D = 6A_1 + 5A_2 + 4A_3 + 3A_4 + 2A_5 + A_6 + 4A_7 + 2A_8 + 3A_9$, then the fundamental cycle equals $D + 3(F_1 + \cdots + F_{3n}) + 2F_{3n+1} + F_{3n+2}$ and the singular fiber of the pencil equals $D + 3(F_1 + \cdots + 3F_{3n+2} + G_1)$. From Proposition 2.7, (X, o) is a maximally elliptic singularity of $p_g(X, o) = n + 1$.

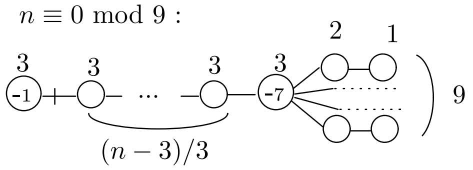
(ii) Let $(X, o) = \{z^2 = y(x^4 + y^{4n+2})\}$. This is an elliptic singularity and it was treated in some papers ([**2**], [**15**]) when $n = 1$. We can check that the minimal resolution is contained in a multiple pencil of multiplicity 2 which is determined by y as follows:



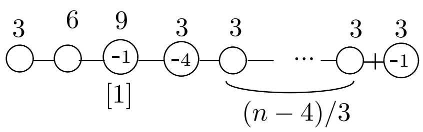
Then $S_o = 2(E_o + \sum_{j=1}^{2n} E_j + G_1)$, $\mathbb{Z}_X = E_o + \sum_{j=1}^{2n} E_j$ and $\mathbb{M}_X = E(\alpha x + \beta y) = 2(E_o + \sum_{j=1}^{2n-1} E_j) + E_{2n}$, where α and β are general elements of $\mathbb{C}$. We put $P := E_o \cap E_1$, and if $R \in E_o$ is a point such that $\mathcal{O}_{E_o}(-R)$ corresponds the normal bundle of E_o in $\widetilde{X}$, then $2P \sim 2R$ but $2P \not\sim 2R$ on E_o. Further, $p_g(X, o) = n + 1$ from Theorem 2.7 (i).

(iii) Let $(X, o) = \{z^3 = y(x^9 + y^n)\}$ $(n \geqslant 9)$. From Theorem 3.3 (i), $p_e(X, o) = 7$ for any $n \geqslant 9$. The author checked the following. If $n \equiv 0, 2, 5$ or $8 \pmod 9$, then (X, o) is a weakly Kodaira singularity with $p_f(X, o) = 7$ and $\mathbb{Z}_X^2 = -3$. For other cases, we have $\mathbb{Z}_X^2 = -1$ and any resolution space of (X, o) is contained in a multiple pencil of genus 7 and multiplicity 3. For example, if $n \equiv 0$ or $1 \pmod 9$, then the associated pencil are given as follows:

$n \equiv 0 \bmod 9$:



$n \equiv 1 \bmod 9$:



The following is a slight modification of a result by Karras [**3**].

Example 3.5. — Let (X, o) be a normal surface singularity. Then (X, o) is a Kodaira (resp. a Kulikov) singularity if and only if there is an element (resp. a reduced element) $h \in \mathfrak{m}_{X,o}$ and a resolution $\pi\colon (\widetilde{X}, E) \to (X, o)$ such that $\mathrm{red}((h \circ \pi)_{\widetilde{X}})$ is a simple normal crossing divisor and $v_{E_i}(h \circ \pi) = 1$ for any component $E_i \subset E$ with $E_i \cdot E(h \circ \pi) < 0$. In this case, a pencil of curves of genus $p_e(X, o, h)$ associated to (X, o) is constructed from $\widetilde{X}$ and h.

Proof. — Suppose (X,o) is a Kodaira singularity. Then there is a pencil $\Phi\colon S\to\Delta$ containing a good resolution $\pi\colon(\widetilde{X},E)\to(X,o)$ and satisfying $\mathrm{Coeff}_{E_i}S_o=1$ for any i with $Z_E\cdot E_i<0$. Let $h\in\mathfrak{m}_{X,o}$ be an element with $\Phi|_{\widetilde{X}}=h\circ\pi$. Then h satisfies conditions to be desired. Now let's consider "if part". As in Theorem 2.2 in [**13**], we can construct a pencil $\Phi\colon S\to\Delta$ extending $h\circ\pi$ with $S_o|_E=E(h\circ\pi)$. If $E_i\cdot\Delta(h\circ\pi)\neq 0$, then $v_{E_i}(h\circ\pi)=1$. Then the pencil above is non-multiple. After suitable contractions of (-1)-curves, we may assume that any connected component of $\mathrm{supp}(S_o)\smallsetminus E$ is a (-1)-curve or a $\mathbb{P}^1$-chain of type $(1,2,\dots,2)$. Therefore, by contraction of all components of $\mathrm{supp}(S_o)\smallsetminus E$, we get a pencil $\Phi'\colon S'\to\Delta$ such that (X,o) is a Kodaira singularity associated to the pencil. The case of Kulikov singularities is obvious from Kodaira's case. □

In [**13**], the author proved that if $(X,o=\{z^n=f(x,y)\}\subset(\mathbb{C}^3,o)$ is a normal surface singularity and $n|\mathrm{ord}(f)$, then (X,o) is a Kodaira singularity. We generalize this in the following.

Let (Y,o) be a Kodaira singularity of genus g_o whose projection function is $h\in\mathfrak{m}_{Y,o}$ and $\overline{\Phi}\colon\overline{S}\to\Delta$ is an associated pencil. Then we have a following diagram:

$$\begin{tikzcd}
(Y,o) \arrow[drr, "h"'] & (\widetilde{Y},E) \arrow[l, "\sigma"'] \subset & \overline{S} \arrow[d, "\overline{\Phi}"] \\
& & \Delta
\end{tikzcd}$$

where $h\circ\sigma=\overline{\Phi}|_{\widetilde{Y}}$ (so $\mathrm{supp}(\overline{S}_o)\supset E$) and $Z_E=E(h\circ\sigma)$.

Theorem 3.6. — *Under the situation above, let $f\in\mathfrak{m}_{Y,o}$ be a reduced element such that $\mathrm{red}(f\circ\sigma)_{\widetilde{Y}}$ is a simple normal crossing divisor and $\mathrm{supp}((f\circ\sigma)_{\widetilde{Y}})\cap\mathrm{supp}((h\circ\sigma)_{\widetilde{Y}})=\varnothing$. Let (X,o) be the n-th cyclic covering over (Y,o) given by $z^n=f$ $(n\geqslant 2)$. If n divides $v_{E_j}(f\circ\sigma)$ for any component E_j with $Z_E\cdot E_j<0$, then (X,o) is a Kodaira singularity of genus $ng_o+(n-1)(c_o-2)/2$ and $\mathbb{Z}_X^2=n\cdot\mathbb{Z}_Y^2$ and whose projection function is f, where $c_o=Z_E\cdot\Delta(f\circ\sigma)$. Further, if (Y,o) is a Kulikov singularity, then (X,o) is so.*

Proof. — Let us consider the same diagram as (3.1). Then $\phi:=\phi_1\circ\phi_2\circ\phi_3$ is a good resolution and we put $\widetilde{h}=h\circ\psi\circ\phi=h\circ\sigma\circ\delta$. Let $\widetilde{E}_i$ be any component of $\widetilde{E}$ with $\widetilde{E}(\widetilde{h})\cdot\widetilde{E}_i<0$. Then there exists a component $\widetilde{C}_{j_i}$ of $\mathrm{supp}(\Delta(\widetilde{h}))$ such that $\widetilde{E}_i\cdot\widetilde{C}_{j_i}\neq 0$. We put $C_{j_i}=\delta(\widetilde{C}_{j_i})$. Hence there is a component E_k of E such that $E_k\cdot C_{j_i}\neq 0$ and so $Z_E\cdot E_k<0$. Let U be a small neighborhood of $E_k\cap C_{j_i}$ in $\widetilde{Y}$ and (u,v) a local coordinate on U. such that $C_{j_i}=\{u=0\}$ and $E_k=\{v=0\}$, and let $f_k=v_{E_k}(f\circ\sigma)$. Then ${\psi'}^{-1}(U)$ is represented by $z^n=uv^{f_k}$ since f is a reduced element, and so $v_{C_{j_i}}(f\circ\sigma)=1$. From the assumption of $n\mid v_{E_k}(f\circ\sigma)$ and Lemma 2.4 in [**12**], X' is resolved by the normalization ϕ_2. Then $E_k=\delta(\widetilde{E}_i)$. Since $v_{E_k}(h\circ\sigma)=1$, we have $v_{\widetilde{E}_i}(h\circ\sigma\circ\delta)=1$ from Lemma 3.1 in [**12**]. Then (X,o) is a Kodaira singularity whose projection function is $h\circ\phi$ from Lemma 3.5. Also if (Y,o)

is a Kulikov singularity, then $v_{C_{j_i}}(h \circ \sigma) = 1$ and so $v_{\widetilde{C}_{j_i}}(\widetilde{h}) = 1$. Hence (X, o) is a Kulikov singularity.

Now let's consider the genus of a pencil associated to (X, o) and consider $\mathbb{Z}_X^2$. Let E_k, C_{j_i} and U be as in above. Since $\mathrm{supp}((f \circ \sigma)_{\widetilde{Y}}) \cap \mathrm{supp}((h \circ \sigma)_{\widetilde{Y}}) = \varnothing$, δ is an unramified covering map. Then $\delta^{-1}(U) = \widetilde{U}_1 \cup \cdots \cup \widetilde{U}_n$ (disjoint union). By suitably glueing of neighborhoods of (-1)-curves onto U and onto $\widetilde{U}_1, \ldots, \widetilde{U}_n$ respectively, we can construct pencils $S \to \Delta$ and $\widetilde{S} \to \Delta$ from $h \circ \sigma$ and $h \circ \sigma \circ \delta$ respectively. Also there is a continuous finite covering map $\widetilde{\delta}\colon \widetilde{S} \to S$ such that $\widetilde{\delta}|_{\widetilde{X}} = \delta$, and so there exist $E_k \cap C_{j_i}$ the following diagram:

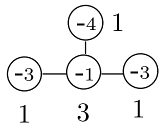
$$\begin{tikzcd} S & \widetilde{S} \arrow[l, "\widetilde{\delta}"'] \\ \widetilde{Y} \arrow[u, phantom, "\cup"] & \widetilde{X} \arrow[l, "\delta"'] \arrow[u, phantom, "\cup"] \end{tikzcd}$$

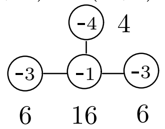
A finite covering map $\widetilde{\delta}_t\colon = \widetilde{\delta}_{\widetilde{S}_t} : \widetilde{S}_t \to S_t$ is ramified at points of

$$\widetilde{\delta}_t^{-1}(S_t \cap \mathrm{supp}(\Delta(f \circ \sigma))).$$

The ramification indices are equal to n for every such point. From the Riemann-Hurwitz formula and $S_t \cdot \Delta(f \circ \sigma) = S_o \cdot \Delta(f \circ \sigma) = Z_E \cdot \Delta(f \circ \sigma) = c_o$, we have the formula for the genus. Since $Z_E = E(h \circ \sigma)$ and $Z_{\widetilde{E}} = \widetilde{E}(h \circ \sigma \circ \delta)$, we have $Z_{\widetilde{E}} = \delta^*(Z_E)$ and so $\mathbb{Z}_X^2 = Z_{\widetilde{E}}^2 = nZ_E^2 = n\mathbb{Z}_Y^2$ (see [**1**], Proposition 8.2). $\square$

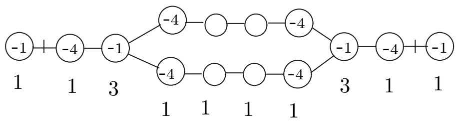
Example 3.7. — Let $(Y, o) = \{x^2 + y^3 + z^8 = 0\}$. It is an elliptic Kulikov singularity of $\mathbb{Z}_Y^2 = -1$ and $h = x^3 + y^2$ is a reduced element of $\mathcal{O}_{Y,o}$. The fundamental cycle Z_E and a cycle $(h \circ \pi)_{\widetilde{Y}}$ on the minimal good resolution $\pi\colon (\widetilde{X}, E) \to (X, o)$ are given as follows:

Z_E : 

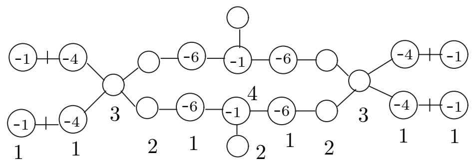
and $E(h \circ \pi)$: 

From Theorem 3.6, if $n = 2$ or 4, then the n-th cyclic covering of (Y, o) defined by $u^n = h$ is a Kulikov singularity of genus 3 and 7 and $\mathbb{Z}_Y^2 = -n$ respectively. Their configurations of singular fibers and resolutions are given as follows:

$$\begin{cases} x^2 + y^3 = z^8 \\ u^2 = x^3 + y^2 \end{cases}$$



$$\begin{cases} x^2 + y^3 = z^8 \\ u^4 = x^3 + y^2 \end{cases}$$



Since $(\mathbb{C}^2, o)$ is a Kulikov singularity, we have the following.

Corollary 3.8 **([12], Theorem 4.1)**. — *Let* $(X, o) = \{z^n = h(x, y)\}$ *be a normal hypersurface singularity with* $n > 1$*, where* $h \in \mathbb{C}\{x, y\}$*. If* $\mathrm{ord}(h)$ *is divided by* n*, then* (X, o) *is a Kulikov singularity associated to a pencil of curves of genus* $(n-1)(\mathrm{ord}(h)-2)/2$ *and* $-\mathbb{Z}_X^2 = n$ *and the projection function is* $\alpha x + \beta y$*, where* α *and* β *are general elements of* $\mathbb{C}$*.*

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T. Tomaru, School of Health Sciences, Gunma University, Maebashi, Gunma 371-8514, Japan
E-mail : ttomaru@health.gunma-u.ac.jp