Séminaires & Congrès
10, 2005, p. 373–389

ON SEMI-STABLE, SINGULAR CUBIC SURFACES

by

Nguyen Chanh Tu

Abstract. — This paper deals with semi-stable and stable singular cubic surfaces from the point of view of the geometric invariant theory. We are interested in properties of the subsets $i\mathcal{A}_1 j\mathcal{A}_2$ corresponding to all semi-stable, singular cubic surfaces with exactly i singular points of type A_1 and j singular points of type A_2. We consider semi-stable cubic surfaces as "csurfaces" of 6-point schemes in almost general position with some conditions of configurations. This is a generalization of the blowing-up of $\mathbb{P}^2$ at 6 points in general position. From relevant configurations of 6-point schemes, we can determine number of star points, the configuration of singular points, of lines and tritangent planes with multiplicities on semi-stable, singular cubic surfaces.

***Résumé* (Sur les surfaces cubiques semi-stables).** — Cet article concerne les surfaces cubiques semi-stables et stables du point de vue de la théorie géométrique des invariants. Nous nous sommes intéressé aux propriétés des sous-ensembles $i\mathcal{A}_1 j\mathcal{A}_2$ correspondant à toutes les surfaces cubiques singulières semi-stables avec exactement i points singuliers de type A_1 et j points singuliers de type A_2. Nous considérons les surfaces cubiques semi-stables comme « c-surfaces » d'ensembles de 6 points en position presque générale avec certaines conditions de configurations. Ceci est une généralisation de l'éclatement de $\mathbb{P}^2$ en 6 points en position générale. À partir de configurations adaptées d'ensembles de 6 points, nous pouvons déterminer le nombre de points « étoile », la configuration des points singuliers, des droites et des plans « tritangents » avec multiplicités sur les surfaces singulières cubiques semi-stables.

1. Introduction

Consider $\mathbb{P}^{19}$ as a parametrizing space of cubic surfaces in $\mathbb{P}^3_k$, where k is an algebraically closed field with characteristic 0. We have the action of PGL(4) on $\mathbb{P}^{19}$. The locus $\Delta \subset \mathbb{P}^{19}$ of singular cubic surfaces is a closed subset of codimension 1. Some

2000 ***Mathematics Subject Classification.*** — Primary 14C05, 14J17, 14J10; Secondary 14C20, 14J25.
Key words and phrases. — Varieties and morphism, special surfaces, singularity.

The author is supported by Japan Society for the Promotion of Science (JSPS) Postdoctoral Fellowship and by Ministry of Education and Training of Vietnam.

classifications of singular cubic surfaces can be found in [**4**] or [**5**]. We are interested in singular cubic surfaces which correspond to semi-stable and stable points under the action of PGL(4) on $\mathbb{P}^{19}$ in the sense of the geometric invariant theory. One reason we are interested in these kinds of singularities is that the quotient space of semi-stable points over PGL(4) exists and it is a compactification of the moduli space of non-singular cubic surfaces.

It is well-known that the blowing-up of $\mathbb{P}^2$ at 6 points in general position is isomorphic to a non-singular cubic surface. Conversely, any non-singular cubic surface can be obtained in that way. A question arises naturally: is there a similar correspondence between a semi-stable, singular cubic surface and a 6-point scheme in some relevant configuration of its points? Showing such a correspondence is one of main goals of this paper. Namely, let X be a semi-stable cubic surface. Then there exists a 6-*point scheme* $\mathcal{P}$ such that the linear system $\mathcal{L}_{\mathcal{P}}$ of cubic forms in four variables through $\mathcal{P}$ has dimension 4; furthermore, for any basis of $\mathcal{L}_{\mathcal{P}}$, the closure of the image of the rational map from $\mathbb{P}^2$ to $\mathbb{P}^3$ defined by the basis is a surface which is isomorphic to X. In this case, we have a morphism $Y \longrightarrow X$, where Y is the blowing-up of $\mathbb{P}^2$ at $\mathcal{P}$. In general, this is a blowing-down and not an isomorphism. A close study of such 6-point schemes enables us to determine the number of lines, the number of singularities of X and their configuration as well.

This also gives a way to compute the multiplicity of lines and tritangent planes on semi-stable, singular cubic surfaces. This investigation shows a clear picture on the configuration of lines and tritangent planes of semi-stable, singular cubic surfaces. Moreover, we will give definitions of *star point* and *proper star point* which are generalizations of the concept of Eckardt point on non-singular cubic surfaces. We will determine the number of (proper) star points on a general one of any class of semi-stable cubic surfaces and study some properties.

2. Stable and semi-stable, singular cubic surfaces

We denote by $i\mathcal{A}_1 j\mathcal{A}_2$ the subset of $\mathbb{P}^{19}$ corresponding to irreducible cubic surfaces with exactly i singular points of type A_1 and j singular points of type A_2. We refer to [**1**] and [**2**] or to [**4**] for general definitions of types of singularities. We will see later that these subsets correspond to all semi-stable, singular cubic surfaces with respect to the action of PGL(4) on $\mathbb{P}^{19}$.

Remark 2.1

(i) In the case of cubic surfaces, the singularities of types A_1 and A_2 are characterized as follows. A point P on a cubic surface with only isolated singularities is called a *singular point of type* A_1 (respectively A_2) if the tangent cone at P is an irreducible quadric surface (respectively if the tangent cone at P consists of two distinct planes whose intersection line does not lie on the surface).

(ii) We have $2i+3j \leqslant 9$, $i \leqslant 4$ and $(i,j) \neq (3,1)$, see [**4**], p. 255 or [**11**], pp. 49-50. We use $j\mathcal{A}_2$ and $i\mathcal{A}_1$ instead of $0\mathcal{A}_1 j\mathcal{A}_2$ and $i\mathcal{A}_1 0\mathcal{A}_2$, respectively.

(iii) By the definition, a semi-stable, singular cubic surface can be given by a polynomial in the following form:

$$F = x_3 f_2(x_0,x_1,x_2) + f_3(x_0,x_1,x_2),$$

where f_i for $i=1,2$ is a homogeneous polynomial of degree i. Then the type of singularity of the surface is characterized by $\operatorname{rank}(f_2)$ and the configuration of points in $V_{\mathbb{P}^2}(f_2,f_3)$.

Some interesting properties of subsets $i\mathcal{A}_1 jA_2$ are shown in the following.

Proposition 2.2. — *The subsets $i\mathcal{A}_1 jA_2$ are irreducible of codimension $i+2j$ in $\mathbb{P}^{19}$ and have a relation as shown in the Figure 1, where $A \longrightarrow B$ means that $\overline{A} \subset \overline{B}$ and subsets are in the same column iff they have the same codimension.*

$$\begin{tikzcd}
\mathcal{A}_1 & 2\mathcal{A}_1 \arrow[l] & 3\mathcal{A}_1 \arrow[l] & 4\mathcal{A}_1 \arrow[l] & & \\
& & \mathcal{A}_1\mathcal{A}_2 \arrow[ul] \arrow[dl] & 2\mathcal{A}_1\mathcal{A}_2 \arrow[l] \arrow[ul] & \mathcal{A}_1 2\mathcal{A}_2 \arrow[l] \arrow[dl] & \\
& \mathcal{A}_2 \arrow[uul] & & 2\mathcal{A}_2 \arrow[ul] & & 3\mathcal{A}_2 \arrow[ul]
\end{tikzcd}$$

FIGURE 1

Proof. — This follows from [**3**], Prop. 2.1. and [**3**], Fig. 1, p. 435. □

Proposition 2.3. — *On the action of* $\mathrm{PGL}(4)$ *on* $\mathbb{P}^{19}$*, we have:*

(i) *The subset of stable points consists of points in $\mathbb{P}^{19}-\Delta$ and those of types $i\mathcal{A}_1$ for $1 \leqslant i \leqslant 4$.*

(ii) *The subset of semi-stable points consists of points in $\mathbb{P}^{19}-\Delta$ and all those of types $i\mathcal{A}_1 jA_2$.*

Proof. — This result was mentioned, for instance, in [**10**], p. 80 or [**9**], p. 51. A detailed proof could be found in [**11**], 3.2.14. □

3. Semi-stable as csurfaces of 6-point schemes in almost general position

As in the case of non-singular cubic surfaces, we show that each semi-stable, singular cubic surface corresponds to a relevant 6-point scheme in almost general position. Moreover we prove that the corresponding semi-stable cubic surfaces are isomorphic if their 6-points schemes are different by quadratic transformations.

Definition. — A 6-*point scheme* is a closed subscheme in $\mathbb{P}^2$ of dimension zero and of length 6. Any 6-point scheme $\mathcal{P}$ defines a formal cycle $c(\mathcal{P}) = \sum n_i P_i$ for $\sum n_i = 6$; the set of the points P_i is called *the support* of $\mathcal{P}$ and denoted by $\operatorname{Supp}(\mathcal{P})$. If the linear system of all cubic forms passing through a 6-point scheme $\mathcal{P}$ has (linear) dimension 4, then $\mathcal{P}$ is called a 6-point scheme *in almost general position*.

Let Hilb_n denote the Hilbert scheme of zero-dimensional closed subschemes of length n in $\mathbb{P}^2$. We denote by $\mathcal{H}^a$ the subscheme of Hilb_6 consisting of all 6-point schemes in almost general position.

Let $\mathcal{P} \in \mathcal{H}^a$ and let l be any line in $\mathbb{P}^2$ such that $l \cap \mathcal{P} \neq \varnothing$. Then the length of $l \cap \mathcal{P}$ is not greater than 4.

Definition. — Let $\mathcal{P} \in \mathcal{H}^a$. We say that $\mathcal{P}$ is a 6-point scheme with *no 4 points on a line* if there does not exist any line l in $\mathbb{P}^2$ such that the length of $l \cap \mathcal{P}$ is equal to 4. Denote by $\mathcal{H}^o$ the subset of 6-point schemes with no 4 points on a line.

Lemma 3.1. — *Let $\mathcal{P} \in \mathcal{H}^o$. Let $\mathcal{L}_{\mathcal{P}}$ be the linear system of cubic forms passing through $\mathcal{P}$.*

(i) *The base locus of $\mathcal{L}_{\mathcal{P}}$ is the support of $\mathcal{P}$.*

(ii) *Let $\{f_1, \ldots, f_4\}$ be a basis of $\mathcal{L}_{\mathcal{P}}$. Consider the morphism*

$$\begin{aligned} \psi : \mathbb{P}^2 - \operatorname{Supp}(\mathcal{P}) &\longrightarrow \mathbb{P}^3 \\ P \qquad &\longmapsto (f_1(P) : f_2(P) : f_3(P) : f_4(P)). \end{aligned}$$

Let X be the closure of the image of ψ. Then X is a cubic surface.

(iii) *If $\{g_1, \ldots, g_4\}$ is another basis of $\mathcal{L}_{\mathcal{P}}$ and X' is the cubic surface obtained as in (ii), then X and X' are isomorphic.*

Proof

(i) Let $P \in \mathbb{P}^2 - \operatorname{Supp}(\mathcal{P})$. Since $\mathcal{P}$ does not have 4 points on a line, there exists a cubic form in $\mathcal{L}_{\mathcal{P}}$ which does not contain P. This implies that the base locus of $\mathcal{L}_{\mathcal{P}}$ is the support of $\mathcal{P}$.

(ii) Let Q_1, Q_2 be two general points in $\mathbb{P}^2 - \operatorname{Supp}(\mathcal{P})$. The linear subspaces consisting of cubic forms through $\mathcal{P} \cup \{Q_1\}$ and $\mathcal{P} \cup \{Q_1, Q_2\}$ respectively have dimension 3 and 2. This implies that there exists a cubic form in $\mathcal{L}_{\mathcal{P}}$ which contains Q_1 but does not contain Q_2 and conversely. This means that ψ is injective over an open subset of $\mathbb{P}^2$. Moreover, any two general cubic forms in $\mathcal{L}_{\mathcal{P}}$ have 3 other points in common which do not belong to $\mathcal{P}$. This implies that X is a cubic surface.

(iii) Let $A = (a_{ij})_{4\times 4}$ be the base change matrix from $\{f_1, \ldots, f_4\}$ to $\{g_1, \ldots, g_4\}$. Then A defines a projective transformation which transforms X to X'. □

Definition. — A *csurface* is an algebraic variety Y such that there exists a cubic surface $X \subset \mathbb{P}^3$ such that $X \cong Y$. From the lemma, we see that each $\mathcal{P} \in \mathcal{H}^o$ determines uniquely (up to isomorphisms) a csurface, which is called *the csurface* *of* $\mathcal{P}$. If $\mathcal{P}$ consists of 6 points in general position, then the csurface of $\mathcal{P}$ is the blowing-up of $\mathbb{P}^2$ at $\mathcal{P}$.

Definition. — Let $P_0 = (1:0:0)$, $P_1 = (0:1:0)$ and $P_2 = (0:0:1)$. Let $\varphi : \mathbb{P}^2 - - \to \mathbb{P}^2$ be the quadratic transformation with respect to P_0, P_1 and P_2 (see [**8**], V.4.2.3). Let C be the cubic curve given by

$$F = \sum a_{ijk}\, x_0^i x_1^j x_2^k \text{ for } i+j+k = 3 \text{ and } 0 \leqslant i,j,k \leqslant 2. \tag{1}$$

The cubic curve defined by $F_\varphi := \sum a_{ijk}\, y_0^{2-i} y_1^{2-j} y_2^{2-k}$ in $\mathbb{P}^2$ is called *the image of* C *by* φ and is denoted by C_φ.

Lemma 3.2. — *Let* $\mathcal{P} \in \mathcal{H}^o$. *Suppose that* $\mathrm{Supp}(\mathcal{P})$ *contains 3 distinct points* P_1, P_2 *and* P_3. *Suppose further that there exists a cubic form in* $\mathcal{L}_\mathcal{P}$ *which is non-singular at any* P_i *for* $i = 1,2,3$. *Let* φ *be the quadratic transformation with respect to* P_1, P_2 *and* P_3. *Then the set* $\varphi(\mathcal{L}_\mathcal{P}) := \{F\varphi \mid F \in \mathcal{L}_\mathcal{P}\}$ *is a 4-dimensional linear space whose base locus is of dimension* 0.

Proof. — Choose coordinates such that $P_1 = (1:0:0)$, $P_2 = (0:1:0)$ and $P_3 = (0:0:1)$. Suppose that the base locus of $\varphi(\mathcal{L}_\mathcal{P})$ contains an irreducible component Y of positive dimension. Since φ is one-to-one in $\mathbb{P}^2 - V(x_0x_1x_2)$, the variety Y is contained in $V(y_0y_1y_2)$. Assume that Y contains the line $d_{12} = V(y_0)$. This means that for any $F \in \mathcal{L}_\mathcal{P}$, we have $F_\varphi = y_0 g_2(y_0, y_1, y_2)$ where g_2 is a homogeneous polynomial of degree 2 and vanishes at $Q_3 = (0:0:1)$. Then $F = (F_\varphi)_{\varphi^{-1}}$ is singular at $P_1 = (1:0:0)$. A contradiction! $\square$

Definition. — Let $\mathcal{P} \in \mathcal{H}^o$ satisfy the conditions as in the previous lemma. Let I be the ideal generated by all cubic forms in $\varphi(\mathcal{L}_\mathcal{P})$. The scheme defined by this ideal is called *the image of* $\mathcal{P}$ and denoted by $\varphi(\mathcal{P})$.

Proposition 3.3. — *Every semi-stable cubic surface is isomorphic to the csurface of some* 6*-point scheme in almost general position with no 4 points on a line.*

Proof. — Let X be a semi-stable cubic surface. If X is a non-singular cubic surface then it is isomorphic to the blowing-up of a 6-point scheme in general position. We consider the case that X is singular.

Suppose that X does not have any A_2 singularity. By choosing coordinates, we may assume X to be defined by

$$F = x_3 f_2(x_0, x_1, x_2) + f_3(x_0, x_1, x_2),$$

where f_i for $i = 2,3$ is a homogeneous polynomial of degree i and f_2 is irreducible. The scheme $\mathcal{P} = V_{\mathbb{P}^2}(f_2, f_3)$ defines an element in $\mathcal{H}^o$. The 6-point scheme $\mathcal{P}$ is contained in an irreducible conic curve defined by f_2 and the cycle $c(\mathcal{P})$ corresponds to a partition $(2^{i-1}1^k)$ of 6. Let $\mathcal{L}_\mathcal{P}$ be the linear space of cubic forms passing through $\mathcal{P}$. Since $\mathcal{P}$ does not contain any triple point, we see that the cubic forms

x_0f_2, x_1f_2 and x_2f_2 are elements of $\mathcal{L}_{\mathcal{P}}$. Moreover, we see that $\{x_0f_2, x_1f_2, x_2f_2, -f_3\}$ is a basis of $\mathcal{L}_{\mathcal{P}}$.

Consider the morphism $\psi : \mathbb{P}^2 - \mathrm{Supp}(\mathcal{P}) \longrightarrow \mathbb{P}^3$ determined by this basis. Then we see that $F(x_0f_2, x_1f_2, x_2f_2, -f_3) = -f_3f_2^3 + f_3f_2^3 = 0$. This means that X is isomorphic to the csurface of $\mathcal{P}$.

Suppose that X contains at least one A_2 singularity. By choosing coordinates, we may assume X to be defined by

$$F = x_3f_2(x_0, x_1, x_2) + f_3(x_0, x_1, x_2),$$

where f_i for $i = 2, 3$ is a homogeneous polynomial of degree i and f_2 is reducible. The scheme $\mathcal{P} = V_{\mathbb{P}^2}(f_2, f_3)$ defines an element in $\mathcal{H}^o$ which corresponds to a partition $(3^{j-1}2^i1^k)$ of 6, where $j \geqslant 1$. Let $\mathcal{L}_{\mathcal{P}}$ be the linear space of cubic forms passing through $\mathcal{P}$. Note that, if $\mathcal{P}$ has a multiple point then the direction at the multiple point is contained in the reducible conic defined by f_2. This implies that the cubic forms x_0f_2, x_1f_2 and x_2f_2 are elements of $\mathcal{L}_{\mathcal{P}}$. Moreover, we have $\{x_0f_2, x_1f_2, x_2f_2, -f_3\}$ is a basis of $\mathcal{L}_{\mathcal{P}}$. As above, we see that X is isomorphic to the csurface of $\mathcal{P}$. □

Remark 3.4. — In [**11**], Prop. 2.1.3, we see that the blowing-up of $\mathbb{P}^2$ at a given $\mathcal{P}$ in general position is isomorphic to the blowing-up of $\mathbb{P}^2$ at $\varphi(\mathcal{P})$. We now show that a similar property holds for all semi-stable cases.

Let $\mathcal{P} \in \mathcal{H}^o$ such that the csurface of $\mathcal{P}$ is isomorphic to a semi-stable, singular cubic surface and the support of $\mathcal{P}$ contains at least 3 distinct points. Let P_1, P_2, P_3 be some 3 distinct points contained in $\mathcal{P}$. Choose coordinates such that $P_1 = (1:0:0)$, $P_2 = (0:1:0)$ and $P_3 = (0:0:1)$. Let φ be the quadratic transformation with respect to P_1, P_2 and P_3. As in the proof of the previous proposition, there exists a basis of $\mathcal{L}_{\mathcal{P}}$ of the form $\{x_0f_2, x_1f_2, x_2f_2, -f_3\}$ where $f_2, f_3 \in k[x_0, x_1, x_2]$ are homogeneous polynomials such that the csurface of $\mathcal{P}$ is isomorphic to the surface $X = V(x_3f_2 + f_3)$.

On the other hand, we see that $\{(x_0f_2)_\varphi, (x_1f_2)_\varphi, (x_2f_2)_\varphi, -(f_3)_\varphi\}$ is a basis of the linear space $\varphi(\mathcal{L}_{\mathcal{P}})$. Consider the morphism:

$$\begin{aligned} \mathbb{P}^2 - \mathrm{Supp}(\varphi(\mathcal{P})) &\longrightarrow \mathbb{P}^3 \\ (y_0 : y_1 : y_2) &\longmapsto ((x_0f_2)_\varphi : (x_1f_2)_\varphi : (x_2f_2)_\varphi : (-f_3)_\varphi) \end{aligned}$$

defined by this basis. The closure of the image of this morphism is a surface Y. We will see that the surface Y is isomorphic to X. For this, let $f_2 = a_1x_0x_1 + a_2x_0x_2 + a_3x_1x_2$. Then f_2 defines a conic curve containing P_1, P_2, P_3. We have:

$$\begin{aligned} (x_0f_2)_\varphi &= y_1y_2(a_1y_2 + a_2y_1 + a_3y_0), \\ (x_1f_2)_\varphi &= y_0y_2(a_1y_2 + a_2y_1 + a_3y_0), \\ (x_2f_2)_\varphi &= y_0y_1(a_1y_2 + a_2y_1 + a_3y_0). \end{aligned}$$

Let $h_1 = a_1y_2 + a_2y_1 + a_3y_0$ and $F = x_3f_2 + f_3$. We have

$$\begin{aligned}F((x_0f_2)_\varphi, (x_1f_2)_\varphi, (x_2f_2)_\varphi, (-f_3)_\varphi) \\ &= (-f_3)_\varphi f_2(y_1y_2h_1, y_0y_2h_1, y_0y_1h_1) + f_3(y_1y_2h_1, y_0y_2h_1, y_0y_1h_1) \\ &= (-f_3)_\varphi h_1^2 f_2(y_1y_2, y_0y_2, y_0y_1) + h_1^3 f_3(y_1y_2, y_0y_2, y_0y_1).\end{aligned}$$

Note that

$$\begin{aligned}f_2(y_1y_2, y_0y_2, y_0y_1) &= a_1y_0y_1y_2^2 + a_2y_0y_1^2y_2 + a_3y_0^2y_1y_2 \\ &= y_0y_1y_2(a_1y_2 + a_2y_1 + a_3y_0) = y_0y_1y_2h_1,\end{aligned}$$

and $f_3(y_1y_2, y_0y_2, y_0y_1) = y_0y_1y_2(f_3)_\varphi$. So we have

$$F((x_0f_2)_\varphi, (x_1f_2)_\varphi, (x_2f_2)_\varphi, (-f_3)_\varphi) = 0.$$

Since F is irreducible, the surface Y is defined by the polynomial F. This implies that $\varphi(\mathcal{P})$ is a 6-point scheme in almost general position. Therefore, we have proved the following proposition.

Proposition 3.5. — *Let $\mathcal{P} \in \mathcal{H}^o$. Suppose that the csurface of $\mathcal{P}$ is isomorphic to a semi-stable cubic surface and the support of $\mathcal{P}$ contains at least 3 distinct points. Let φ be the quadratic transformation with respect to some 3 distinct points of $\mathcal{P}$. Then the subscheme $\varphi(\mathcal{P})$ is a* 6*-point scheme in almost general position and the csurface of $\varphi(\mathcal{P})$ is isomorphic to the csurface of $\mathcal{P}$.*

From the above propositions, we can easily describe the configuration of a 6-point scheme of any semi-stable cubic surface.

***Example 1* (6-point schemes for $i\mathcal{A}_1$).** — By (3.3), we see that the points $P_1, \ldots, P_6$ (not necessarily distinct) of a 6-point scheme $\mathcal{P}$ corresponding to $i\mathcal{A}_1$ lie on an irreducible conic. There are at least 3 distinct points in the support of $\mathcal{P}$, say $\{P_1, P_2, P_3\}$, and suppose that this set contains multiple points of $\mathcal{P}$ if it has. Applying the quadratic transformation with respect to $\{P_1, P_2, P_3\}$ we obtain configurations as mentioned in [**7**], pp. 641-646. In Figure 2 we see some 6-point schemes for $2\mathcal{A}_1$.

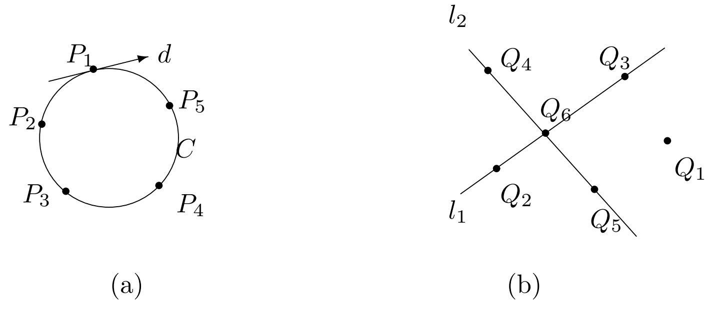

FIGURE 2. 6-point schemes giving points in $2\mathcal{A}_1$

Example 2. — Similarly, we give in Figure 3 some configurations of 6-point schemes for $\mathcal{A}_1\mathcal{A}_2$.

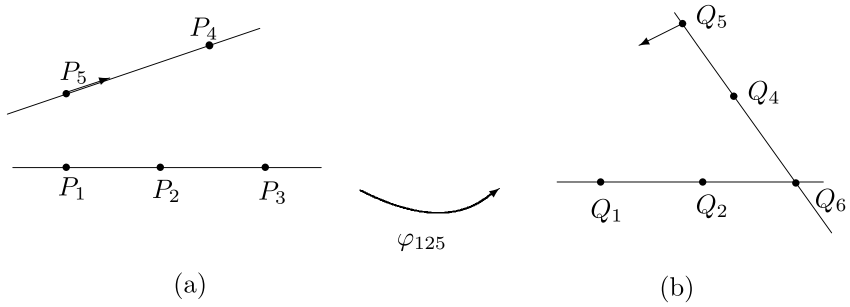

FIGURE 3. 6-point schemes corresponding to elements in $\mathcal{A}_1\mathcal{A}_2$

4. Configurations of singular points, star points, lines and tritangent planes with multiplicities

Definition. — Let X be a semi-stable cubic surface. A *tritangent plane* of X is a plane such that the hyperplane intersection factors into 3 lines (not necessarily distinct). A point $P \in X$ is called a *star point* if it is contained in all lines of the hyperplane intersection of some tritangent plane. In that case, the lines of the hyperplane intersection is called a *star triple.*

It is well-known that a non-singular cubic surface has exactly 27 lines and 45 tritangent planes with a special configuration. The numbers of distinct lines and tritangent planes of a semi-stable, singular cubic surface decrease. But with multiplicities, these numbers are the same for all semi-stable cubic surfaces.

A description of configurations of lines and tritangent planes with multiplicities on cubic surfaces could be found in [**6**]. In [**6**], the author has classified 23 classes of cubic surfaces with normal forms. The explicit equations of the lines on any cubic surface were carried out from the normal form. Moreover, when reducing from the non-singular class to a singular class of cubic surfaces (with only isolated singularities), the 27 lines and 45 tritangent planes on a non-singular cubic surface reduce to the lines and tritangent planes on the corresponding singular cubic surface. The *multiplicity* of a line l (tritangent plane T) of a singular cubic surface (with only isolated singularity) is nothing but the number of lines (tritangent planes) which reduce to l (respectively T). See [**6**], Articles 35-201 for details.

We now see that the correspondence between semi-stable cubic surfaces and 6-point schemes, as considered in the previous section, enables us to describe configurations of lines and tritangent planes, to determine easily not only the multiplicities of lines and tritangent planes but star points on a generic singular cubic surface with respect to any $i\mathcal{A}_1 j\mathcal{A}_2$.

First of all, we recall how to determine the lines and tritangent planes on a semi-stable cubic surface from a 6-point scheme of it. Let X be a semi-stable cubic surface and let $\mathcal{P}$ be a 6-point scheme of X. Let $\mathcal{L}_{\mathcal{P}}$ be the linear space of cubic forms in $\mathbb{P}^2$ containing $\mathcal{P}$. Then $\mathcal{L}_{\mathcal{P}}$ has linear dimension 4. Let $\{f_1, \ldots, f_4\}$ be a basis of $\mathcal{L}_{\mathcal{P}}$. Consider the morphism

$$\begin{aligned} \psi : \mathbb{P}^2 - \mathrm{Supp}(\mathcal{P}) &\longrightarrow \mathbb{P}^3 \\ P \quad\quad &\longmapsto (f_1(P) : f_2(P) : f_3(P) : f_4(P)). \end{aligned}$$

The surface X is the closure of the image of ψ. Note that cubic forms of $\mathcal{L}_{\mathcal{P}}$ are in 1-1 correspondence with hyperplanes in $\mathbb{P}^3$. We denote by S_{ij} the two-dimensional linear subspace consisting of all cubic forms factoring into the linear form defining $l_{ij} = \overline{P_iP_j}$ and quadratic forms passing through $\mathcal{P} - \{P_i, P_j\}$. This subspace determines uniquely a line on X which is denoted by $\widetilde{l}_{ij}$. The line $\widetilde{l}_{ij}$ is the closure of the image of $l_{ij} - \{P_i, P_j\}$. There are 15 lines of this kind. Similarly, we denote by S_{P_i} the two-dimensional linear subspace consisting of cubic forms singular at P_i. This determines uniquely a line on X which we denote by $\widetilde{P}_i$. There are 6 lines of this kind. Let S_{C_i} denote the two-dimensional linear subspace consisting of all cubic forms factoring into the quadratic form defining the conic C_i through $\{P_1, \ldots, P_6\} - \{P_i\}$ and linear forms vanishing at P_i. This subspace determines uniquely a line on X, which is denoted by $\widetilde{C}_i$. The line $\widetilde{C}_i$ is nothing but the closure of the image of $C_i - \{P_1, P_2, P_3\}$. There are 6 lines of this kind.

Any tritangent plane of X has the form either $(\widetilde{P}_i, \widetilde{C}_j, \widetilde{l}_{ij})$ for $1 \leqslant i \neq j \leqslant 6$ or $(\widetilde{l}_{ij}, \widetilde{l}_{mn}, \widetilde{l}_{kh})$ for $\{i, j, m, n, h, k\} = \{1, \ldots, 6\}$.

If $\mathcal{P}$ consists of 6 points in general position, then the above 27 two-dimensional linear subspaces are all distinct. In general, some of the 27 two-dimensional linear subspaces may coincide. The coincidence of them determines the multiplicities of lines and tritangent planes on semi-stable, singular cubic surfaces. Formulating this idea, we have:

Proposition 4.1. — *Let X be a semi-stable cubic surface and l be a line on X.*

(i) *Suppose that l contains exactly one singular point.*
  (a) *If the singular point is A_1, then l is of multiplicity* 2.(1)
  (b) *If the singular point is A_2, then l is of multiplicity* 3.

(ii) *Suppose that l contains* 2 *singular points.*
  (a) *If both of singularities are A_1, then l is of multiplicity* 4.
  (b) *If both of singularities are A_2, then l is of multiplicity* 9.
  (c) *If two singularities are of different types, then l is of multiplicity* 6.

(iii) *If l does not contain any singular point, then l is of multiplicity* 1.

(1)This result was also mentioned in [**13**], p. 39 for the cases of real and complex fields.

Proof

(1) Suppose that X contains only A_1 singularities. By choosing coordinates, we assume that X is given by

$$F = x_3 f_2(x_0, x_1, x_2) + f_3(x_0, x_1, x_2),$$

where f_i for $i = 1, 2$ is a homogeneous polynomial of degree i and f_2 is irreducible. Let $\mathcal{P}$ be the 6-point scheme $V_{\mathbb{P}^2}(f_2, f_3)$. Let $c(\mathcal{P}) = \sum_{i=1}^{6} P_i$ where the points P_i for $1 \leqslant i \leqslant 6$ are unnecessarily different. We know that X is the closure of the image of the morphism from $\mathbb{P}^2 - \mathrm{Supp}(\mathcal{P})$ to $\mathbb{P}^3$ determined by the basis $\{x_0 f_2, x_1 f_2, x_2 f_2, -f_3\}$ of $\mathcal{L}_{\mathcal{P}}$. Let C be the conic curve in $\mathbb{P}^2 = V(x_3)$ defined by f_2. It is clear that the image of any point on $C - \mathrm{Supp}(\mathcal{P})$ is the point $S = (0 : 0 : 0 : 1)$, which is an A_1 singularity. Let P_i be a point in the support of $\mathcal{P}$. Each cubic form in S_{P_i} factors into f_2 and a linear form vanishing at P_i. This implies that the line $\widetilde{P}_i$ contains the singular point S. Moreover, we prove that $\widetilde{P}_i$ is the line containing S and P_i. For this, suppose that $P_i = (1 : 0 : 0 : 0)$. Any line d containing P_i is given by $V_{\mathbb{P}^2}(a_1 x_1 + a_2 x_2)$. We see that $d \cup \widetilde{P}_i = V(F, a_1 x_1 + a_2 x_2)$. The line connecting S and P_i is given by $x_1 = x_2 = 0$. This implies that $\widetilde{P}_i$ is the line containing S and P_i.

Let l be a line on X containing at least one A_1 singularity; we may assume l to be one of the $\widetilde{P}_i$'s. If l contains exactly one A_1 singularity, then the corresponding point P_i is a single point of $V_{\mathbb{P}^2}(f_2, f_3)$. It is easy to check that the linear subspaces S_{P_i} and S_{C_i} are the same. Moreover, they are different from other linear subspaces of the forms S_{P_i} and S_{ij}. Therefore, the multiplicity of l is 2. If l contains two A_1 singularities, then the corresponding point P_i is a double point of $V_{\mathbb{P}^2}(f_2, f_3)$. So we may assume that in the cycle $c(\mathcal{P}) = \sum_{i=1}^{6} P_i$, the point P_1 coincides with P_2. This implies that the linear subspaces $S_{P_1}, S_{P_2}, S_{C_1}$ and S_{C_2} are the same; in fact, the line l is of multiplicity 4.

Consider that l does not contain any singular point. If X has exactly one A_1 singularity, then there exist exactly 6 lines of multiplicity 2. Note that X has exactly 21 lines. This implies that the other 15 lines are of multiplicity 1. So l is of multiplicity 1. If X has exactly two A_1 singularities, then there exist exactly 8 lines with multiplicity 2; there exists one line with multiplicity 4. Note that X has exactly 16 lines. This implies that the other 7 lines of X are of multiplicity 1. So l is of multiplicity 1 in this case. If X has exactly three A_1 singularities, then there exist exactly 6 lines with multiplicity 2, there exist exactly 3 lines with multiplicity 4. In this case, the surface X has exactly 12 lines. This implies that the other 3 lines are of multiplicity 1. This means that l is of multiplicity 1. Finally, if X has exactly four A_1 singularities, then there exist exactly 6 lines with multiplicity 4. Since X has exactly 9 lines, the other 3 lines are of multiplicity 1. So l is of multiplicity 1.

(2) Suppose that X contains at least one A_2 singularity. The reader can perform the result using a similar argument as used in (1). $\square$

As illustrations, the rest of this section is used to work out some cases of $i\mathcal{A}_1 j\mathcal{A}_2$. We will describe the configurations of lines, tritangent planes with multiplicities, determine the number of star points and describe how to recognize singular points of semi-stable cubic surfaces from the corresponding 6-point schemes. From now on, unless stating differently, when we write the formal cycle $c(\mathcal{P})$ of a given 6-point scheme $\mathcal{P}$, we always mean that the points in the cycle are mutually distinct.

$\mathcal{A}_1$. Let $x \in \mathcal{A}_1$. We know that the corresponding cubic surface X_x is isomorphic to the csurface of a 6-point scheme $\mathcal{P} \in \mathcal{H}^o$ such that $c(\mathcal{P}) = \sum_{i=1}^{6} P_i$ where the 6 mutually distinct points lie on an irreducible conic curve C (see Figure 4).

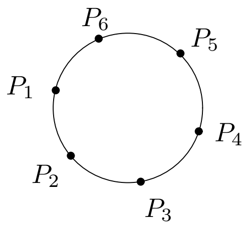

FIGURE 4. 6-point schemes corresponding to points in $\mathcal{A}_1$

By (4.1), we see that the image of $C - \mathrm{Supp}(\mathcal{P})$ (via any morphism from $\mathbb{P}^2 - \mathrm{Supp}(\mathcal{P})$ to $\mathbb{P}^3$ determined by a basis of $\mathcal{L}_{\mathcal{P}}$) is the singular point; the lines $\widetilde{P}_i = \widetilde{C}_i$ for $1 \leqslant i \leqslant 6$ are the 6 lines through the singular point. Other lines of X_x are $\widetilde{l}_{ij}$ for $1 \leqslant i < j \leqslant 6$. The 21 lines of X_x with multiplicities correspond to the partition $(2^6, 1^{15})$ of 27. Moreover, we see that the tritangent planes $(\widetilde{P}_i, \widetilde{C}_j, \widetilde{l}_{ij})$, $(\widetilde{P}_j, \widetilde{C}_i, \widetilde{l}_{ij})$ and $(\widetilde{P}_i, \widetilde{P}_j, \widetilde{l}_{ij})$ for $1 \leqslant i < j \leqslant 6$ are the same. This means that every tritangent plane $(\widetilde{P}_i, \widetilde{P}_j, \widetilde{l}_{ij})$ for $1 \leqslant i < j \leqslant 6$ is of multiplicity 2. The corresponding cubic surface X_x has 30 distinct tritangent planes which correspond to the partition $(2^{15}, 1^{15})$ of 45.

$\mathcal{A}_2$. Let $x \in \mathcal{A}_2$. The corresponding cubic surface X_x is isomorphic to the csurface of a 6-point scheme $\mathcal{P} \in \mathcal{H}^o$ such that $c(\mathcal{P}) = \sum_{i=1}^{6} P_i$ where 3 points P_1, P_2, P_3 lie on a line l_1; three points P_4, P_5, P_6 lie on another line l_2; the intersection point of l_1 and l_2 does not belong to $\mathcal{P}$ (see Figure 5).

Let $\mathcal{L}_{\mathcal{P}}$ be the linear space of cubic forms passing through $\mathcal{P}$. Consider any morphism from $\mathbb{P}^2 - \mathrm{Supp}(\mathcal{P})$ to $\mathbb{P}^3$ determined by a basis of $\mathcal{L}_{\mathcal{P}}$. By (4.1), the image of $(l_1 \cup l_2) - \mathrm{Supp}(\mathcal{P})$ is the singular point. The 6 lines $\widetilde{P}_i$ for $1 \leqslant i \leqslant 6$ contain the singular point and they are of multiplicity 3. The other 9 lines of X_x are $\widetilde{l}_{ij}$ for $i \in \{1,2,3\}$ and $j \in \{4,5,6\}$. These lines are of multiplicity 1. The 15 lines of X_x with multiplicities correspond to the partition $(3^6, 1^9)$ of 27.

Note that the linear subspaces S_{P_i}, S_{C_i} and S_{jk} for $\{i,j,k\} = \{1,2,3\}$ or $\{i,j,k\} = \{4,5,6\}$ are the same. This implies that the tritangent plane $(\widetilde{P}_1, \widetilde{P}_2, \widetilde{P}_3)$ has multiplicity 6 since it coincides with $(\widetilde{P}_1, \widetilde{C}_2, \widetilde{l}_{12})$, $(\widetilde{C}_1, \widetilde{P}_2, \widetilde{l}_{12})$, $(\widetilde{P}_1, \widetilde{l}_{13}, \widetilde{C}_3)$, $(\widetilde{C}_1, \widetilde{l}_{13}, \widetilde{P}_3)$,

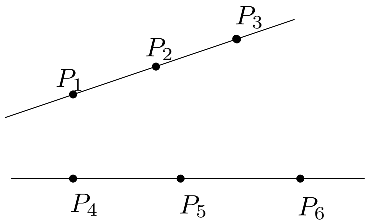

FIGURE 5. 6-point schemes corresponding to points in $\mathcal{A}_2$

$(\widetilde{l}_{23}, \widetilde{P}_2, \widetilde{C}_3)$ and $(\widetilde{l}_{23}, \widetilde{C}_2, \widetilde{P}_3)$. Similarly, the tritangent plane $(\widetilde{P}_4, \widetilde{P}_5, \widetilde{P}_6)$ has multiplicity 6. Every tritangent plane $(\widetilde{P}_i, \widetilde{P}_j, \widetilde{l}_{ij})$ for $1 \leqslant i \leqslant 3$ and $4 \leqslant j \leqslant 6$ has multiplicity 3 since it coincides with $(\widetilde{P}_i, \widetilde{C}_j, \widetilde{l}_{ij}), (\widetilde{C}_i, \widetilde{P}_j, \widetilde{l}_{ij})$ and $(\widetilde{l}_{mn}, \widetilde{l}_{kh}, \widetilde{l}_{ij})$ where $\{m, n\} = \{1, 2, 3\} - \{i\}$, $\{k, h\} = \{4, 5, 6\} - \{j\}$. Every tritangent plane $(\widetilde{l}_{ij}, \widetilde{l}_{mk}, \widetilde{l}_{nh})$ for $\{i, m, n\} = \{1, 2, 3\}$, $\{j, k, h\} = \{4, 5, 6\}$ has multiplicity 1. So X_x has 17 distinct tritangent planes. The 17 tritangent planes with their multiplicities correspond to the partition $(6^2, 3^9, 1^6)$ of 45.

Moreover, we see that the singular point is a star point of X, since it is contained in all lines of the tritangent plane $(\widetilde{P}_1, \widetilde{P}_2, \widetilde{P}_3)$.

$\mathcal{A}_1\mathcal{A}_2$. Let $x \in \mathcal{A}_1\mathcal{A}_2$. The corresponding cubic surface X_x is isomorphic to the csurface of a 6-point scheme $\mathcal{P} \in \mathcal{H}^o$ where $c(\mathcal{P}) = P_1 + P_2 + P_3 + P_4 + 2P_5$ such that P_4 and $2P_5$ are contained in a line l_1; three points P_1, P_2, P_3 are contained in another line l_2; the intersection point of l_1 and l_2 does not belong to $\mathcal{P}$ (Figure 3. (a)).

View X as a point in the closure of $\mathbb{P}^{19} - \Delta$. Consider $\mathcal{P}$ as a specialization position of some family of 6-point schemes in general position. Suppose further that the family has 6 sections of points. We may assume that the double point $2P_5$ is contained in the two sections corresponding to the points P_5 and P_6. Consider any morphism from $\mathbb{P}^2 - \mathrm{Supp}(\mathcal{P})$ to $\mathbb{P}^3$ determined by a basis of $\mathcal{L}_{\mathcal{P}}$. By (4.1), we see that the image of $(l_1 \cup l_2) - \mathrm{Supp}(\mathcal{P})$ is the A_2 singularity; the line $\widetilde{P}_5$ is of multiplicity 6 and is the line containing 2 singularities; the lines $\widetilde{P}_i$ for $1 \leqslant i \leqslant 4$ contain the A_2 singularity and they are of multiplicity 3. Moreover, we see that the lines $\widetilde{l}_{i5}$ for $1 \leqslant i \leqslant 3$ are of multiplicity 2. So they contain the A_1 singularity. The other lines of X_x are $\widetilde{l}_{4i}$ for $1 \leqslant i \leqslant 3$ which are of multiplicity 1. The 11 lines of X_x with their multiplicities correspond to the partition $(6^1, 3^4, 2^3, 1^3)$ of 27.

As in the case of $\mathcal{A}_2$, we see that the tritangent planes $(\widetilde{P}_1, \widetilde{P}_2, \widetilde{P}_3)$ and $(2\widetilde{P}_5, \widetilde{P}_4)$ are of multiplicity 6; every tritangent plane $(\widetilde{P}_4, \widetilde{P}_i, \widetilde{l}_{4i})$ for $1 \leqslant i \leqslant 3$ has multiplicity 3. Every tritangent plane $(\widetilde{P}_5, \widetilde{P}_i, \widetilde{l}_{5i})$ for $1 \leqslant i \leqslant 3$ has multiplicity 6 since it coincides with $(\widetilde{P}_5, \widetilde{C}_i, \widetilde{l}_{5i}), (\widetilde{C}_5, \widetilde{P}_i, \widetilde{l}_{5i}), (\widetilde{l}_{46}, \widetilde{l}_{kh}, \widetilde{l}_{5i}), (\widetilde{P}_6, \widetilde{C}_i, \widetilde{l}_{6i}), (\widetilde{C}_6, \widetilde{P}_i, \widetilde{l}_{6i})$ and $(\widetilde{l}_{45}, \widetilde{l}_{kh}, \widetilde{l}_{6i})$ for $\{k, h\} = \{1, 2, 3\} - \{i\}$. Finally, every tritangent plane $(\widetilde{l}_{i5}, \widetilde{l}_{j5}, \widetilde{l}_{k4})$ for $\{i, j, k\} = \{1, 2, 3\}$ has multiplicity 2 since it coincides with $(\widetilde{l}_{i5}, \widetilde{l}_{j6}, \widetilde{l}_{k4})$ and $(\widetilde{l}_{i6}, \widetilde{l}_{j5}, \widetilde{l}_{k4})$. So X

has 11 distinct tritangent planes. With multiplicities, the tritangent planes of X_x correspond to the partition $(6^5, 3^3, 2^3)$ of 45.

The A_2 singularity is a star point of X_x, since it is the intersection of all lines of the tritangent plane $(2\widetilde{P}_5, \widetilde{P}_4)$.

Remark 4.2. — If we consider the above 6-point scheme, it is not clear how to obtain the A_1 singularity. Consider the quadratic transformation φ_{125} with respect to P_1, P_2, P_5. Let $\mathcal{Q} = \varphi(\mathcal{P})$ be the image of $\mathcal{P}$. We see that $c(\mathcal{Q}) = 2Q_5 + Q_1 + Q_2 + Q_4 + Q_6$, where Q_1, Q_2, Q_6 lie on the line d_1; three points Q_4, Q_5, Q_6 lie on another line d_2 (Figure 3. (b)). The csurface of $\mathcal{Q}$ is isomorphic to X_x. Consider any morphism from $\mathbb{P}^2 - \mathrm{Supp}(\mathcal{Q})$ to $\mathbb{P}^3$ determined by a basis of $\mathcal{L}_{\mathcal{Q}}$. In this case, the image of $\big(l_1 - \{Q_1, Q_2, Q_6\}\big)$ is the A_1 singularity; the image of $\big(l_2 - \{Q_4, Q_5, Q_6\}\big)$ is the A_2 singularity; the line $\widetilde{Q}_6$ is the line containing two singularities.

Similarly, the reader easily performs all remaining cases. We list the results in Table 1 with some remarks as follows.

(i) If a semi-stable cubic surface X contains two A_1 singularities, denote by l the line connecting the two singularities, then there exists exactly another line d intersecting l such that $(2l, d)$ is a tritangent plane. Therefore the intersection point of l and d is a star point.

(ii) If a semi-stable cubic surface X contains two A_2 singularities, denote by l the line connecting the two singularities, then there is a tritangent plane such that the hyperplane intersection consists of $\{3l\}$. Therefore, any point on l is a star point.

(iii) The number of star points mentioned in each column of the table holds at the generic point of the corresponding stratum.

5. Proper star points

In this section we study star points of semi-stable cubic surfaces which are specialization positions in some specialization process. Such a star point is called a *proper star point*. We will show that every star point is a proper star point.

Definition. — Let x be a semi-stable point in $\mathbb{P}^{19}$. Suppose that x is a specialization of a given one-dimensional family of semi-stable points, which locally possesses a section of star points. The specialization position of the section of star point on the corresponding cubic surface X_x is called *a proper star point* with respect to the family. It is clear that a proper star point is a star point.

Definition. — Let $\mathcal{H}_1$ be the subvariety of $\mathbb{P}^{19} - \Delta$ parametrizing all non-singular cubic surfaces with at least one star point. In fact, the subset $\mathcal{H}_1$ is irreducible of codimension one in $\mathbb{P}^{19}$ ([**12**], p. 288).

	$\mathbb{P}^{19}-\Delta$	$\mathcal{A}_1$	$2\mathcal{A}_1$	$\mathcal{A}_2$	$3\mathcal{A}_1$	$\mathcal{A}_1\mathcal{A}_2$
Lines with Mult.	27 (1^{27})	21 $(2^6,1^{15})$	16 $(4^1,2^8,1^7)$	15 $(3^6,1^9)$	12 $(4^3,2^6,1^3)$	11 $(6^1,3^4,2^3,1^3)$
Tritangent with Mult.	45 (1^{45})	30 $(2^{15},1^{15})$	20 $(4^4,2^{13},1^3)$	17 $(6^2,3^9,1^6)$	14 $(8^1,4^6,2^6,1^1)$	11 $(6^5,3^3,2^3)$
Star points	0	0	1	1	3	1

	$4\mathcal{A}_1$	$2\mathcal{A}_1\mathcal{A}_2$	$2\mathcal{A}_2$	$\mathcal{A}_12\mathcal{A}_2$	$3\mathcal{A}_2$
Lines with Mult.	9 $(4^6,1^3)$	8 $(6^2,4^1,3^2,2^2,1^1)$	7 $(9^1,3^6)$	5 $(9^1,6^2,3^2)$	3 (9^3)
Tritangent with Mult.	11 $(8^4,2^6,1^1)$	8 $(12^1,6^4,4^1,3^1,2^1)$	6 $(9^3,6^3)$	5 $(18^1,9^1,6^3)$	4 $(27^1,6^3)$
Star points	6	2	∞	∞	∞

TABLE 1. Information about lines, tritangent planes and star points on semi-stable cubic surfaces

Lemma 5.1. — *The subset $2\mathcal{A}_1$ is contained in the closure of $\mathcal{H}_1$. Consequently the star point on the line with multiplicity 4 of any cubic surface corresponding to a point of $2\mathcal{A}_1$ is a proper star point.*

Proof. — Let $x \in 2\mathcal{A}_1$. The corresponding cubic surface X_x is isomorphic to the csurface of a 6-point scheme $\mathcal{Q} = \sum_{i=1}^{6} Q_i$ where 3 points Q_2, Q_3, Q_6 lie on a line l_1; the three points Q_4, Q_5, Q_6 lie on another line l_2; no 3 of the five points $Q_1, \dots, Q_5$ are collinear (Figure 6).

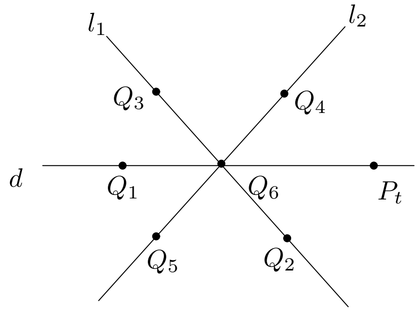

FIGURE 6. 6-point schemes giving points in $2\mathcal{A}_1$

Let P_t be a moving point on the line $d = \overline{Q_1Q_6}$. At a general position of P_t on d, the 6-point scheme $\mathcal{P}_t = \sum_{i=1}^{6} P_i$ where $P_i = Q_i$ for $1 \leqslant i \leqslant 5$ and $P_6 = P_t$, gives a non-singular cubic surface with at least one star point. Except for a finite number of positions, when P_t moves on the line d, we have a family in $\mathcal{H}_1$. This implies that x lies on the closure of $\mathcal{H}_1$. Moreover, we see that the section of star points over the family is defined by the tritangent planes $H_t = (\widetilde{l}_{23}, \widetilde{l}_{45}, \widetilde{l}_{1t})$ where $\widetilde{l}_{ij}$ is the line on the csurface of a 6-point scheme in the family determined by the linear subspace S_{ij}. In the specialization position, the linear subspaces S_{23}, S_{45}, S_{C_1} and S_{Q_6} coincide. This means that $\widetilde{Q}_6$ is the line connecting the 2 singular points and the section of tritangent planes H_t contains the tritangent plane $(2\widetilde{Q}_1, \widetilde{l}_{16})$. So the section of star points contains the star point on the line $\widetilde{Q}_6$ of multiplicity 4. □

Lemma 5.2. — *Any $x \in \mathcal{A}_2$ lies on the closure of $\mathcal{H}_1$. Consequently, the A_2 singularity of the corresponding cubic surface X_x, as a star point, is a proper star point.*

Proof. — Let $\mathcal{R}$ be a 6-point scheme consisting of 6 distinct points $R_1, \ldots, R_6$ such that the 3 points R_1, R_2, R_3 as well as the 3 points R_4, R_5, R_6 are collinear (Figure 7. (b)). We know that the csurface of $\mathcal{R}$ is isomorphic to a cubic surface with exactly one A_2 singularity. Consider the quadratic transformation with respect to R_1, R_4, R_5. Then the image of $\mathcal{R}$ is a 6-point scheme $\mathcal{Q}$ where $c(\mathcal{Q}) = 2Q_1 + \sum_{i=2}^{5} Q_i$, such that three points Q_1, Q_2, Q_3 are collinear; the corresponding direction at double point $2Q_1$ does not contain any Q_i for $i = 4, 5$; the four points Q_1, Q_2, Q_4, Q_5 as well as the four points Q_1, Q_3, Q_4, Q_5 are in general position (Figure 7. (a)).

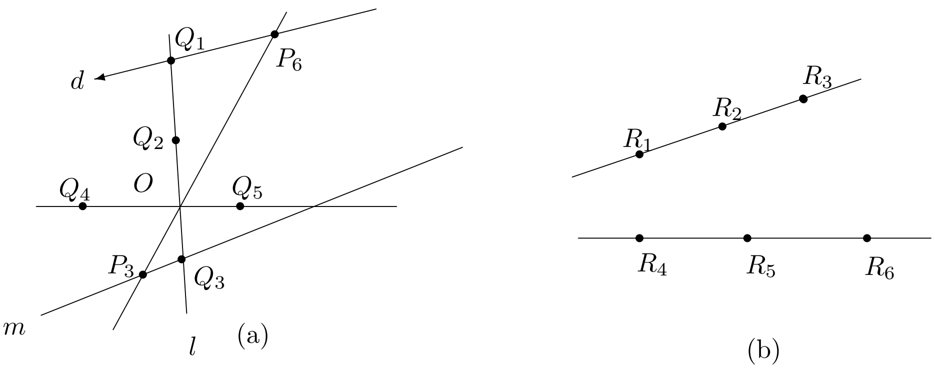

FIGURE 7. 6-point schemes giving points in $\mathcal{A}_2$

Let $x \in \mathcal{A}_2$. The corresponding cubic surface X_x is isomorphic to the csurface of a 6-point scheme $\mathcal{Q}$ where $c(\mathcal{Q}) = 2Q_1 + \sum_{i=2}^{5} Q_i$ described as above.

Let O be the intersection point of $\overline{Q_1Q_2}$ and $\overline{Q_4Q_5}$. Let d be the direction at the double point $2Q_1$. Let m be a fixed line which contains Q_3 and does not contain any other point of $\mathrm{Supp}(\mathcal{Q})$. Let (P_6, P_3) be a pair of moving points where $P_6 \in d$ and $P_3 \in m$ such that $\overline{P_3P_6}$ contains O. Except for a finite number of positions, when moving (P_6, P_3), the csurfaces of 6-point schemes $\mathcal{P} = \sum_{i=1}^{6} P_i$, where $P_i = Q_i$ for $i \in \{1, 2, 4, 5\}$, are isomorphic to non-singular cubic surfaces with at least one star point. This defines a family in $\mathcal{H}_1$. When $(P_6, P_3) = (Q_1, Q_3)$, we get the 6-point scheme $\mathcal{Q}$ whose csurface is isomorphic to X_x. So x lies on the closure of $\mathcal{H}_1$. Moreover, the star section over the family is defined by the tritangent planes $(\widetilde{l}_{12}, \widetilde{l}_{45}, \widetilde{l}_{36})$ where the line $\widetilde{l}_{ij}$ on a surface corresponding to a point of the family is determined by the linear subspace S_{ij}. In the specialization position, the linear subspaces S_{12}, S_{26} and S_{Q_3} coincide; the linear subspaces S_{36}, S_{13} and S_{Q_2} coincide. Note that the 6 lines $\widetilde{Q}_1, \widetilde{Q}_2, \widetilde{Q}_3, \widetilde{l}_{45}, \widetilde{l}_{14}$ and $\widetilde{l}_{15}$ have multiplicity 3 and they contain the A_2 singularity. It is clear that the section of star points gives a specialization to the intersection of $\widetilde{Q}_2, \widetilde{Q}_3$ and $\widetilde{l}_{45}$, which is the A_2 singularity. □

Proposition 5.3. — *Let X be a semi-stable cubic surface. Any star point of X is a proper star point.*

Proof. — Let P be a star point of X. The result is clear if P is the intersection of a star triple whose lines are of multiplicity 1. If P is an A_2 singularity, then the result follows from (5.2). Suppose that X has at least two A_1 singularities. Let d be the line containing two A_1 singularities. Let $(2d, l)$ be the star triple which factors into $2d$ and another line l on X. Suppose that $\{P\} = d \cap l$, then the result follows from (5.1).

Suppose that X has at least two A_2 singularities and P is a point in the line connecting two A_2 singularities. We only consider the case that P is not a singular point of X. Choose coordinates such that X is given by the polynomial (see [**4**], p. 249):

$$F_0 = x_3x_0x_1 + x_1(a_1x_1^2 + a_2x_1x_2 + a_3x_2^2) + x_2^3.$$

The surface X contains two A_2 singularities, namely $S_1 = (0:0:0:1)$ and $S_2 = (1:0:0:0)$. The line $d = V(x_1, x_2)$ contains the two A_2 singularities. Let $P = (\lambda, 0:0:1) \in d$ where $\lambda \neq 0$.

Consider the family given by

$$F_t = x_3(x_0x_1 + t(\lambda+t)x_2^2) + x_1(a_1x_1^2 + a_2x_1x_2 + a_3x_2^2) + x_2^3 - tx_0x_2^2, \tag{2}$$

where $t \in k$. Let $f_2^t = x_0x_1 + t(\lambda+t)x_2^2$ and $f_3^t = x_1(a_1x_1^2 + a_2x_1x_2 + a_3x_2^2) + x_2^3 - tx_0x_2^2$. For $t \notin \{0, -\lambda\}$, the polynomial f_2^t has rank 3. Consider $\mathcal{P}_t = V_{\mathbb{P}^2}(f_2^t, f_3^t)$. We see that the point $(1:0:0)$ is a double point of $V_{\mathbb{P}^2}(f_2^t, f_3^t)$. Other four points of $\mathcal{P}_t$ are determined by $(-t(\lambda+t)b^2 : 1 : b)$ where b is a solution of the following equation:

$$a_1 + a_2x_2 + a_3x_2^2 + x_2^3 + t^2(\lambda+t)x_2^4 = 0. \tag{3}$$

The above equation has a multiple solution with multiplicity 4 for only a finite number of t. It means that (2) defines a family Γ_t of semi-stable cubic surfaces which gives a specialization to the surface X.

Each corresponding cubic surface $X_t := V(F_t)$ of any element in Γ_t contains two A_1 singularities, namely $S_1 = (0:0:0:1)$ and $S_2 = (1:0:0:0)$. We see that $T_t = V(x_1) \cap X_t = 2d \cup l_t$, where $d = V(x_1, x_2)$ and $l_t = V(x_1, t(\lambda+t)x_3 + x_2 - tx_0)$ is a star triple. The surface X_t contains the star point $P_t = (\lambda{+}t : 0 : 0 : 1) = d \cap l_t$. When the family Γ_t gives a specialization to $X \equiv X_0$, the sections of the A_1 singularities contain the A_2 singularities of X. Moreover, the section of star points over Γ_t contains $P = (\lambda : 0 : 0 : 1)$ in X. This completes the proof. □

Acknowledgment. — Some results of this paper were obtained during the Ph.D. study of the author under a fellowship of Utrecht University, The Netherlands and were also contained in Chapter 3 of his thesis [**11**]. The author would like to express deep gratitude to his supervisor, Prof. Dr. F. Oort for careful guidance and endless support. This paper is completed when the author is supported by the JSPS postdoctoral fellowship and then by Ministry of Education and Training of Vietnam.

References

[1] V.I. Arnol'd – Normal forms for functions near degenerate critical points, the Weyl groups of A_k, D_k, E_k and Lagrangian singularities, *Functional Anal. Appl.* **6** (1972), p. 254–272.

[2] ______, Normal forms of functions near degenerate critical points, *Russian Math. Surveys* **29** (1974), no. 2, p. 11–49.

[3] J.W. Bruce – A stratification of the space of cubic sufaces, *Math. Proc. Cambridge Philos. Soc.* (1980), p. 427–441.

[4] J.W. Bruce & C.T.C. Wall – On the classification of cubic surfaces, *J. London Math. Soc. (2)* **19** (1979), p. 245–256.

[5] M. Brundu & A. Logar – Parametrization of the orbits of cubic surfaces, in *Transformation Groups*, vol. III, 1998, p. 1–31.

[6] A. Cayley – A memoir on cubic surfaces, *Phil. Trans. Roy. Soc.* **159** (1869), p. 231–326.

[7] P.A. Griffiths & J. Harris – *Principles of Algebraic Geometry*, A. Wiley-Interscience, New York, 1978.

[8] R. Hartshorne – *Algebraic Geometry*, Graduate Texts in Math., vol. 52, Springer-Verlag, 1980.

[9] D. Mumford – Stability of projective varieties, *Enseign. Math.* **23** (1977), p. 39–110.

[10] D. Mumford, J. Fogarty & F. Kirwan – *Geometric Invariant Theory*, third enlarged ed., Springer-Verlag, 1994.

[11] Nguyen C.T. – Star points on cubic surfaces, Doctoral thesis, Utrecht University, The Netherlands, 2000, `http://www.library.uu.nl/digiarchief/dip/diss/1933178`.

[12] ______, Non-singular cubic surfaces with star points, *Vietnam J. Math.* **29** (2001), no. 3, p. 287–292, preprint no. 1082, Department of Mathematics, Utrecht University (12/1998).

[13] B. Segre – *The Non-Singular Cubic Surfaces*, Clarendon Press, Oxford, 1942.

N.C. Tu, Department of Mathematics, Tokyo Metropolitan University, Minami-Ohsawa 1-1, Hachioji-shi, Tokyo 192-0397, Japan • Department of Mathematics, Hue University, 32 Leloi, Hue, Vietnam • *E-mail :* `nctu2000@yahoo.com`