Séminaires & Congrès
10, 2005, p. 429–442

GENERALIZED GINZBURG-CHERN CLASSES

by

Shoji Yokura

Abstract. — For a morphism $f : X \to Y$ with Y being nonsingular, the Ginzburg-Chern class of a constructible function α on the source variety X is defined to be the Chern-Schwartz-MacPherson class of the constructible function α followed by capping with the pull-back of the Segre class of the target variety Y. In this paper we give some generalizations of the Ginzburg-Chern class even when the target variety Y is singular and discuss some properties of them.

***Résumé* (Classes de Ginzburg-Chern généralisées).** — Pour un morphisme algébrique $f : X \to Y$ où la variété Y est non singulière, la classe de Ginzburg-Chern de la fonction constructible α sur la variété source X est définie comme la classe de Chern-Schwartz-MacPherson de la fonction constructible α suivi du cap-produit par l'image réciproque de la classe de Segre de la variété but Y. Dans cet article nous donnons quelques généralisations de la classe de Ginzburg-Chern y compris lorsque la variété but Y est singulière et nous en discutons quelques propriétés.

1. Introduction

In [**G1**] Ginzburg introduced a certain homomorphism from the abelian group of Lagrangian cycles to the Borel-Moore homology group

$$c^{\mathrm{biv}} : L(X_1 \times X_2) \longrightarrow H_*(X_1 \times X_2),$$

which he called a bivariant Chern class. The construction or definition of the homomorphism c^{biv} given in [**G1**] is not direct, but in his survey article [**G2**] he gives an explicit description of it. It assigns to a Lagrangian cycle associated to a subvariety $Y \subset X_1 \times X_2$ the *relative Chern-Mather class of the fibers of the projection*

2000 ***Mathematics Subject Classification.*** — 14C17, 14F99, 55N35.
Key words and phrases. — Bivariant theory, Chern-Schwartz-MacPherson class, Constructible function, Riemann-Roch formula.

Partially supported by Grant-in-Aid for Scientific Research (No.15540086), the Ministry of Education, Science and Culture, Japan.

$p_Y : Y \to X_2$. The projection p_Y is the restriction of the projection $p_2 : X_1 \times X_2 \to X_2$ to the subvariety Y. Let $\nu : \widehat{Y} \to Y$ be the Nash blow-up and $\widehat{TY}$ the tautological Nash tangent bundle over $\widehat{Y}$. Then the above relative Chern-Mather class is defined by

$$c^{\mathrm{biv}}(\Lambda_Y) := i_{Y*}\nu_*\Big(c(\widehat{TY} - \nu^* p_Y^* TX_2) \cap [\widehat{Y}]\Big)$$

where $i_Y : Y \to X_1 \times X_2$ is the inclusion. Then it follows from the projection formula and from $p_Y = p_2 \circ i_Y$ that

$$\begin{aligned} c^{\mathrm{biv}}(\Lambda_Y) &= i_{Y*}\Big(\frac{1}{p_Y^* c(TX_2)} \cap c^M(Y)\Big) \\ &= p_2^* s(TX_2) \cap i_{Y*} c^M(Y). \end{aligned}$$

Here $s(TX_2)$ denotes the Segre class of the tangent bundle TX_2.

Since the Chern-Schwartz-MacPherson class ([**BS**], [**M**], [**Sw1**], [**Sw2**] etc.) is a linear combination of Chern-Mather classes, the above homomorphism c^{biv} can be defined for any morphism $\pi : X \to Y$ from a possibly singular variety X to a smooth variety Y and for any constructible function on the target variety X. Namely we can define the following homomorphism

$$\pi^* s(TY) \cap c_* : F(X) \longrightarrow H_*(X; \mathbf{Z})$$

where $c_* : F(X) \to H_*(X; \mathbf{Z})$ is the usual Chern-Schwartz-MacPherson class transformation. This "twisted" Chern-Schwartz-MacPherson class shall be called the *Ginzburg-Chern class.*

On the other hand, in [**Y3**] we showed that the bivariant Chern class ([**Br**], [**FM**]) for any morphism with nonsingular target variety necessarily has to be the Ginzburg-Chern class. To be more precise, if there exists a bivariant Chern class $\gamma : \mathbf{F} \to \mathbf{H}$ from the Fulton-MacPherson bivariant theory of constructible functions to the Fulton-MacPherson bivariant homology theory, then for any morphism $f : X \to Y$ with Y being nonsingular and any bivariant constructible function $\alpha \in \mathbf{F}(X \to Y)$ the following holds

$$\gamma_f(\alpha) = f^* s(TY) \cap c_*(\alpha),$$

where $\gamma_f : \mathbf{F}(X \xrightarrow{f} Y) \to \mathbf{H}(X \xrightarrow{f} Y)$.

Quickly speaking, this theorem follows from the simple observation that for $\alpha \in \mathbf{F}(X \to Y) \subset F(X)$ we have

$$c_*(\alpha) = \gamma_f(\alpha) \bullet c_*(Y),$$

where $\bullet$ on the right-hand-side is the bivariant product. Thus a naïve solution for $\gamma_f(\alpha)$ is the following "quotient"

$$\gamma_f(\alpha) = \text{“}\frac{c_*(\alpha)}{c_*(Y)}\text{”}.$$

It turns out that in the case when the target variety Y is nonsingular this "quotient" is well-defined and it is nothing but

$$“\frac{c_*(\alpha)}{c_*(Y)}” = \frac{c_*(\alpha)}{f^*c(TY)} = f^*s(TY) \cap c_*(\alpha).$$

From now on the Ginzburg-Chern class of α shall be denoted by $\gamma^{\mathrm{Gin}}(\alpha)$ or $\gamma_f^{\mathrm{Gin}}(\alpha)$ emphasizing the morphism f.

As one sees, for the definition of the Ginzburg-Chern class the nonsingularity of the target variety Y is clearly essential. In this paper, we put aside the bivariant-theoretic aspect of the Ginzburg-Chern class ([**Y4**], [**Y5**], [**Y6**]) and, using Nash blow-ups and also resolutions of singularities we introduce reasonably modified versions of the Ginzburg-Chern class, even when the target variety is arbitrarily singular. We discuss some properties of them and in particular we obtain some results concerning the convolution product of them.

Acknowledgements. — The author is greatful for the hospitality and financial support he received from the Erwin Schrödinger International Institute for Mathematical Physics (ESI) in Vienna, Austria, where most of this work was done in July and August 2002. In particular the author would like to thank Professor Peter Michor, Professor Franz Kamber and the staff of the ESI. The author thanks Professor Jean-Paul Brasselet and Professor Tatsuo Suwa, the organizers of "Singularités franco-japonaises" held at the CIRM, Luminy, September 9–12, 2002, for giving him the opportunity to give a talk on the present work at the conference and for the financial support. The author also thanks the referee for his/her valuable comments and suggestions.

2. Generalized Ginzburg-Chern classes

The Ginzburg-Chern class is a unique natural transformation satisfying a certain normalization in the following sense:

Theorem 2.1 **([Y2**, Theorem (2.1)]). — *For the category of Y-varieties, i.e., morphisms $\pi : X \to Y$, with Y being a nonsingular variety, $\gamma_\pi^{\mathrm{Gin}} : F(X) \to H_*(X;\mathbf{Z})$ is the unique natural transformation from the constructible functions to the homology theory such that for a smooth variety X we have*

$$\gamma_\pi^{\mathrm{Gin}}(1\!\!1_X) = c(T_\pi) \cap [X], \tag{2.2}$$

where $T_\pi := TX - \pi^*TY$ is the relative virtual tangent bundle. Namely, for any commutative diagram

$$\begin{tikzcd} X_1 \arrow[rr, "f"] \arrow[dr, "\pi_1"'] & & X_2 \arrow[dl, "\pi_2"] \\ & Y & \end{tikzcd}$$

where f is proper, we have the following commutaive diagram

$$\begin{tikzcd} F(X_1) \arrow[r, "f_*"] \arrow[d, "\gamma_{\pi_1}^{\mathrm{Gin}}"'] & F(X_2) \arrow[d, "\gamma_{\pi_2}^{\mathrm{Gin}}"] \\ H_*(X_1) \arrow[r, "f_*"'] & H_*(X_2). \end{tikzcd}$$

A natural question or problem on the Ginzburg-Chern class is whether or not one can extend it to the case when the target variety Y is singular and we want to see if a theorem similar to the above one holds.

Suppose that Y is singular and we consider the Nash blow-up $\nu : \widehat{Y} \to Y$ and the following fiber square

$$\begin{tikzcd} \widehat{X} \arrow[r, "\widehat{\nu}"] \arrow[d, "\widehat{\pi}"'] & X \arrow[d, "\pi"] \\ \widehat{Y} \arrow[r, "\nu"'] & Y. \end{tikzcd}$$

Then we define the homomorphism

$$\widehat{\gamma}_{\pi}^{\mathrm{Gin}} : F(X) \longrightarrow H_*(X; \mathbf{Z})$$

by

$$\widehat{\gamma}_{\pi}^{\mathrm{Gin}} := \widehat{\nu}_* \Big(\widehat{\pi}^* s(\widehat{TY}) \cap c_*(\widehat{\nu}^* \alpha) \Big). \tag{2.3}$$

This class shall be called *a Nash-type Ginzburg-Chern class*, abusing words. Then we have the following theorem:

Theorem 2.4. — *Let Y be a possibly singular variety. Then, for any commutative diagram*

$$\begin{tikzcd} X_1 \arrow[rr, "f"] \arrow[dr, "\pi_1"'] & & X_2 \arrow[dl, "\pi_2"] \\ & Y & \end{tikzcd}$$

where f is proper, we have the following commutative diagram

$$\begin{tikzcd} F(X_1) \arrow[r, "f_*"] \arrow[d, "\widehat{\gamma}_{\pi_1}^{\mathrm{Gin}}"'] & F(X_2) \arrow[d, "\widehat{\gamma}_{\pi_2}^{\mathrm{Gin}}"] \\ H_*(X_1) \arrow[r, "f_*"'] & H_*(X_2). \end{tikzcd}$$

Proof. — First we recall the following fact ([**Er**, Proposition 3.5], [**FM**, Axiom (A_{23})]): for any fiber square

$$\begin{tikzcd} W' \arrow[r, "g'"] \arrow[d, "h'"'] & W \arrow[d, "h"] \\ Z' \arrow[r, "g"] & Z, \end{tikzcd}$$

the following diagram commutes

$$\begin{tikzcd} F(W) \arrow[r, "g'^*"] \arrow[d, "h'_*"'] & F(W') \arrow[d, "h_*"] \\ F(Z) \arrow[r, "g^*"] & F(Z'). \end{tikzcd}$$

Now we have the following commutative diagrams:

$$\begin{tikzcd} \widehat{X}_1 \arrow[rrr, "\widehat{\nu}_1"] \arrow[dr, "\widehat{f}"] \arrow[dd, "\widehat{\pi}_1"'] & & & X_1 \arrow[dl, "f"'] \arrow[dd, "\pi_1"] \\ & \widehat{X}_2 \arrow[r, "\widehat{\nu}_2"] \arrow[dl, "\widehat{\pi}_2"] & X_2 \arrow[dr, "\pi_2"'] & \\ \widehat{Y} \arrow[rrr, "\nu"] & & & Y. \end{tikzcd}$$

Then by definition we have

$$\begin{aligned} \widehat{\gamma}^{\mathrm{Gin}}_{\pi_2}(f_*\alpha) &= \widehat{\nu}_{2*}\Big(\widehat{\pi}_2^* s(\widehat{TY}) \cap c_*(\widehat{\nu}_2^* f_*\alpha)\Big) \\ &= \widehat{\nu}_{2*}\Big(\widehat{\pi}_2^* s(\widehat{TY}) \cap c_*(\widehat{f}_*\widehat{\nu}_1^*\alpha)\Big) \\ &= \widehat{\nu}_{2*}\Big(\widehat{\pi}_2^* s(\widehat{TY}) \cap \widehat{f}_* c_*(\widehat{\nu}_1^*\alpha)\Big) \\ &= \widehat{\nu}_{2*}\widehat{f}_*\Big(\widehat{f}^*\widehat{\pi}_2^* s(\widehat{TY}) \cap c_*(\widehat{\nu}_1^*\alpha)\Big) \\ &= f_*\widehat{\nu}_{1*}\Big(\widehat{\pi}_1^* s(\widehat{TY}) \cap c_*(\widehat{\nu}_1^*\alpha)\Big) \\ &= f_*\widehat{\gamma}^{\mathrm{Gin}}_{\pi_1}(\alpha). \end{aligned}$$

□

Motivated by the definition of the Nash-type Ginzburg-Chern class, we give another modification of the Ginzburg-Chern class via a resolution of singularities. Let $\rho : \widetilde{Y} \to$

$\widetilde{Y}$ be a resolution of singularities of Y, and consider the following fiber square

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\widetilde{\rho}} & X \\ {\scriptstyle\widetilde{\pi}}\downarrow & & \downarrow{\scriptstyle\pi} \\ \widetilde{Y} & \xrightarrow[\rho]{} & Y. \end{array}$$

Then we define the homomorphism

$$\widetilde{\gamma}^{\mathrm{Gin}}_{\pi} : F(X) \longrightarrow H_*(X; \mathbf{Z})$$

by

$$\widetilde{\gamma}^{\mathrm{Gin}}_{\pi} := \widetilde{\rho}_*\Big(\widetilde{\pi}^* s(T\widetilde{Y}) \cap c_*(\widetilde{\rho}^*\alpha)\Big).$$

This class shall be called *a "resolution" Ginzburg-Chern class*, abusing words. This class of course depends on the choice of resolution of singularities of the target variety Y. Then we can clearly see that the Nash-type Ginzburg-Chern class can be replaced by any resolution Ginzburg-Chern class in the above theorem.

Remark 2.5. — At the moment, we do not know how to define a "canonical" resolution Ginzburg-Chern class independent of the choice of resolution of singularities, which remains to be seen. Of course, a minimal resolution would supply such a thing, if it exists.

Remark 2.6. — Furthermore generalizing the above modifications of the Ginzburg-Chern class, we can see the following. Suppose that $(\overline{Y}, E)$ is a variety $\overline{Y}$, singular or nonsingular, accompanied with a certain vector bundle E and that there is a morphism $\eta : \overline{Y} \to Y$. Here in general there is not any connection between the bundle E and the morphism η. However, to obtain a reasonable and interesting result, we would need some geometric or topological connections between them. Then we consider the fiber square

$$\begin{array}{ccc} \overline{X} & \xrightarrow{\overline{\eta}} & X \\ {\scriptstyle\overline{\pi}}\downarrow & & \downarrow{\scriptstyle\pi} \\ \overline{Y} & \xrightarrow[\eta]{} & Y. \end{array}$$

Then we define the homomorphism

$$\overline{\gamma}^{\mathrm{Gin}}_{\pi} : F(X) \longrightarrow H_*(X; \mathbf{Z})$$

by

$$\overline{\gamma}^{\mathrm{Gin}}_{\pi}(\alpha) := \overline{\eta}_*\Big(\overline{\pi}^* s(E) \cap c_*(\overline{\eta}^*\alpha)\Big).$$

This class is called *a generalized Ginzburg-Chern class associated to the data* {*a morphism* $\eta : \overline{Y} \to Y$*, a vector bundle* E *on* $\overline{Y}$ } .

Note that in Theorem (2.4) the Nash-type Ginzburg-Chern class can be replaced by $\overline{\gamma}_\pi^{\mathrm{Gin}}$.

With this definition, we can consider the following problem: Suppose that $\overline{Y}$ is a smooth compact variety and let G be a (reasonable, *e.g.*, finite) group acting on $\overline{Y}$ and let $\eta : \overline{Y} \to \overline{Y}/G$ be the projection. Then we have for any morphism $\pi : X \to \overline{Y}/G$ and for any constructible function $\alpha \in F(X)$

$$\overline{\gamma}_\pi^{\mathrm{Gin}}(\alpha) = \overline{\eta}_*\Big(\overline{\pi}^* s(T\overline{Y}) \cap c_*(\overline{\eta}^*\alpha)\Big).$$

In particular, for the projection $\overline{\pi} : \overline{Y} \to \overline{Y}/G$ and for $\alpha \in F(\overline{Y})$ we have

$$\overline{\gamma}_\pi^{\mathrm{Gin}}(\alpha) = \overline{\eta}_*\Big(\overline{\pi}^* s(T\overline{Y}) \cap c_*(\overline{\eta}^*\alpha)\Big) = s(T\overline{Y}) \cap c_*(\overline{\pi}_*\overline{\pi}^*\alpha).$$

It remains to see if there is an application of this class to quotient singularities.

A non-trivial uniqueness of $\widehat{\gamma}_\pi^{\mathrm{Gin}}$ such as one in Theorem (2.1) is not available, except for the following obvious one: for the category of Y-varieties (*i.e.*, for morphisms $\pi : X \to Y$) $\widehat{\gamma}_\pi^{\mathrm{Gin}} : F(X) \to H_*(X; \mathbf{Z})$ is the unique natural transformation satisfying the condition that for a nonsingular variety X and any morphism $\pi : X \to Y$ the following holds

$$\widehat{\gamma}_\pi^{\mathrm{Gin}}(\mathbb{1}_X) = \widehat{\nu}_*\Big(\widehat{\pi}^* s(\widehat{TY}) \cap c_*(\widehat{Y} \times_Y X)\Big).$$

(Which is simply obtained by replacing α by $\mathbb{1}_X$ in the definition (2.3). But it is a best one for the uniqueness so far.)

Theorem 2.7. — *For a smooth morphism $\pi : X \to Y$ of possibly singular varieties X and Y, we have*

$$\widehat{\gamma}_\pi^{\mathrm{Gin}}(\mathbb{1}_X) = c(T_\pi) \cap \pi^*\nu_*\Big(s(\widehat{TY}) \cap c_*(\widehat{Y})\Big) \tag{2.8}$$

where T_π is the virtual relative tangent bundle of the smooth morphism.

(2.8) could be considered as an analogue of (2.2).

To prove Theorem (2.7), we need the so-called Verdier-Riemann-Roch for Chern class ([**FM**], [**Su**], [**Y1**]):

Theorem 2.9. — *Let $f : X \to Y$ be a smooth morphism of possibly singular varieties X and Y. Then the following diagram commutes:*

$$\begin{array}{ccc} F(Y) & \xrightarrow{c_*} & H_*(Y) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle c(T_f)\cap f^*} \\ F(X) & \xrightarrow[c_*]{} & H_*(X). \end{array}$$

Proof of Theorem (2.7). — By definition we have

$$\widehat{\gamma}_{\pi}^{\mathrm{Gin}}(1\!\!1_X) = \widehat{\nu}_* \Big(\widehat{\pi}^* s(\widehat{TY}) \cap c_*(\widehat{\nu}^* 1\!\!1_X)\Big).$$

The key trick is that $\widehat{\nu}^* 1\!\!1_X = 1\!\!1_{\widehat{X}} = \widehat{\pi}^* 1\!\!1_{\widehat{Y}}$. Therefore we have

$$\widehat{\gamma}_{\pi}^{\mathrm{Gin}}(1\!\!1_X) = \widehat{\nu}_* \Big(\widehat{\pi}^* s(\widehat{TY}) \cap c_*(\widehat{\pi}^* 1\!\!1_{\widehat{Y}})\Big).$$

Since π is smooth, the pullback $\widehat{\pi}$ is also smooth. Hence, using the above Verdier-Riemann-Roch for Chern class, we have

$$\begin{aligned}
\widehat{\gamma}_{\pi}^{\mathrm{Gin}}(1\!\!1_X) &= \widehat{\nu}_* \Big(\widehat{\pi}^* s(\widehat{TY}) \cap \big(c(T_{\widehat{\pi}}) \cap \widehat{\pi}^* c_*(1\!\!1_{\widehat{Y}})\big)\Big) \\
&= \widehat{\nu}_* \Big(\Big(\widehat{\pi}^* s(\widehat{TY}) \cup \widehat{\nu}^* c(T_\pi)\Big) \cap \widehat{\pi}^* c_*(1\!\!1_{\widehat{Y}})\Big) \\
&= \widehat{\nu}_* \Big(\widehat{\nu}^* c(T_\pi) \cap \Big(\widehat{\pi}^* s(\widehat{TY}) \cap \widehat{\pi}^* c_*(1\!\!1_{\widehat{Y}})\Big)\Big) \\
&= \widehat{\nu}_* \Big(\widehat{\nu}^* c(T_\pi) \cap \widehat{\pi}^* \Big(s(\widehat{TY}) \cap c_*(1\!\!1_{\widehat{Y}})\Big)\Big) \\
&= c(T_\pi) \cap \widehat{\nu}_* \widehat{\pi}^* \big(s(\widehat{TY}) \cap c_*(1\!\!1_{\widehat{Y}})\big) \\
&= c(T_\pi) \cap \pi^* \nu_* \Big(s(\widehat{TY}) \cap c_*(\widehat{Y})\Big).
\end{aligned}$$

□

As a corollary of the proof of the theorem, we get the following

Corollary 2.10. — *Let the situation be as in Theorem (2.7). For a resolution Ginzburg-Chern class* $\widetilde{\gamma}_{\pi}^{\mathrm{Gin}}$ *we have the following*

$$\widetilde{\gamma}_{\pi}^{\mathrm{Gin}}(1\!\!1_X) = c(T_\pi) \cap [X].$$

Remark 2.11. — One might expect that the formula (2.8) for a smooth morphism $\pi : X \to Y$ could be used as a condition required for the the uniqueness of the Nash-type Ginzburg-Chern class $\widehat{\gamma}_{\pi}^{\mathrm{Gin}}$, but it is not the case. Namely the following speculation (which certainly implies the uniqueness of $\widehat{\gamma}_{\pi}^{\mathrm{Gin}}$) is not necessarily true: For any construcible function $\alpha \in F(X)$ there would exist some varieties W_i's, morphisms $g_i : W_i \to X$ and integers n_i such that

(i) $\alpha = \sum_i n_i g_{i*} 1\!\!1_{W_i}$, and
(ii) the composite $f \circ g_i : W_i \to Y$ is smooth.

3. Convolutions

The notion of *convolution* (product) is important and ubiquitous in the geometric representation theory. Here we recall the convolution on the Borel-Moore homology theory.

In this paper the homology theory $H_*(X)$ is the Borel-Moore homology group of a locally compact Hausdorff space X, *i.e.*, the ordinary (singular) cohomology group of the pair $(\overline{X}, \infty)$ where $\overline{X} = X \cup \infty$ is the one-point compactification of X.

For any closed subsets X and X' in a smooth manifold M, we have the cup product

$$\cup : H^p(M, M \smallsetminus X) \otimes H^q(M, M \smallsetminus X') \longrightarrow H^{p+q}(M, M \smallsetminus (X \cap X')),$$

which implies, by the Alexander duality isomorphism $H^\bullet(M, M \smallsetminus W) \cong H_{\dim M-\bullet}(W)$, denoted by $\mathcal{A}_W$, the following intersection product

$$\cdot : H_i(X) \otimes H_j(X') \longrightarrow H_{i+j-\dim M}(X \cap X').$$

Note that this product depends on the ambient manifold M. Let M_1, M_2, M_3 be smooth oriented manifolds and let $p_{ij} : M_1 \times M_2 \times M_3 \to M_i \times M_j$ be the canonical projections. Let $Z \subset M_1 \times M_2$ and $Z' \subset M_2 \times M_3$ be closed subsets and we assume that the restricted map

$$p_{13} : p_{12}^{-1}(Z) \cap p_{23}^{-1}(Z') \longrightarrow M_1 \times M_3$$

is proper. Then its image is denoted by the $Z \circ Z'$, *i.e.*, the composite of the two correspondences Z and Z' (see Fulton's book [F]). The restricted map $p_{12}^{-1}(Z) \cap p_{23}^{-1}(Z') \to Z \circ Z'$ is also denoted by p_{13}. With this set-up, the convolution

$$\star : H_i(Z) \otimes H_j(Z') \longrightarrow H_{i+j-\dim M}(Z \circ Z')$$

is defined by

$$x \star y := p_{13*}(p_{12}^* x \cdot p_{23}^* y). \tag{3.1}$$

Since the intersection product $\cdot$ depends on the ambient manifolds as pointed out above, this convolution product depends on the ambient manifolds. To be more precise, it is defined by

$$x \star y := p_{13*}\Big(\mathcal{A}_{p_{12}^{-1}(Z) \cap p_{23}^{-1}(Z')}\big(p_{12}^* \mathcal{A}_Z^{-1}(x) \cup p_{23}^* \mathcal{A}_{Z'}^{-1}(y)\big)\Big).$$

Given varieties X_1, X_2, X_3, we let $p_{ij} : X_1 \times X_2 \times X_3 \to X_i \times X_j$ be the projection. Then we can consider the following convolution product on constructible functions:

$$\star : F(X_1 \times X_2) \otimes F(X_1 \times X_2) \longrightarrow F(X_1 \times X_3)$$

defined by

$$\alpha \star \beta := p_{13*}(p_{12}^* \alpha \cdot p_{23}^* \beta).$$

Even if each X_i is contained in a smooth manifold M_i, this constructible function convolution product does not depend on the ambinet manifolds and has nothing to do with them at all, unlike the above homological convolution product (3.1). For any varieties X, Y, we set

$$\widetilde{\mathbf{F}}(X \times Y) := 1\!\!1_X \times F(Y) = \{1\!\!1_X \times \alpha \mid \alpha \in F(Y)\}.$$

Here we note that for any pair of constructible functions $\omega \in F(W)$ and $\zeta \in F(Z)$, the cross product $\omega \times \zeta \in F(W \times Z)$ is defined by

$$(\omega \times \zeta)(w, z) := \omega(w)\zeta(z).$$

Let $\pi : X \times Y \to X$ be the projection to the first factor. Then this subgroup $\widetilde{\mathbf{F}}(X \times Y)(\subset F(X \times Y))$ is invariant under the constructible function convolution product, *i.e.*, we have the convolution product:

$$\star : \widetilde{\mathbf{F}}(X_1 \times X_2) \otimes \widetilde{\mathbf{F}}(X_2 \times X_3) \longrightarrow \widetilde{\mathbf{F}}(X_1 \times X_3).$$

Now we observe that for a nonsingular variety X we have

$$\gamma_\pi^{\mathrm{Gin}}(\mathbb{1}_X \times \alpha) = [X] \times c_*(\alpha).$$

Which can been seen by the multiplicativity of the Chern-Schwartz-MacPherson class [**K**] (*cf.* [**KY**]). With this observation we have the following theorem

Theorem 3.2 ([**Y2**, Theorem (3.2)]). — *Let $X_1 = M_1, X_2 = M_2$ be nonsingular varieties and let X_3 be a subvariety in a smooth manifold M_3. And we assume that X_2 is compact, which makes the projection $p_{13} : X_1 \times X_2 \times X_3 \to X_1 \times X_3$ proper. Let $p_1 : X \times Y \to X$ denote the projection for any X and Y. Then the Ginzburg-Chern class $\gamma_{p_1}^{\mathrm{Gin}} : \widetilde{\mathbf{F}}(X \times Y) \to H_*(X \times Y)$ is convolutive; i.e., the following diagram commutes:*

$$\begin{array}{ccc}
\widetilde{\mathbf{F}}(X_1 \times X_2) \otimes \widetilde{\mathbf{F}}(X_2 \times X_3) & \xrightarrow{\star} & \widetilde{\mathbf{F}}(X_1 \times X_3) \\
{\scriptstyle \gamma_{p_1}^{\mathrm{Gin}} \otimes \gamma_{p_1}^{\mathrm{Gin}}}\downarrow & & \downarrow{\scriptstyle \gamma_{p_1}^{\mathrm{Gin}}} \\
H_*(X_1 \times X_2) \otimes H_*(X_2 \times X_3) & \xrightarrow[\star]{} & H_*(X_1 \times X_3).
\end{array}$$

As a matter of fact, it turns out that the commutativity of the diagram in the above theorem holds *even* when X_1 is singular, but that $X_2 = M_2$ has to be nonsingular . Namely we have the following

Theorem 3.3. — *Let X_i be a possibly singular variety in a nonsingular variety M_i $(i = 1, 2, 3)$.*

(i) *If $X_2 = M_2$ is a nonsingular compact variety, then the homomorphism $\widetilde{\gamma} : \widetilde{\mathbf{F}}(X \times Y) \to H_*(X \times Y)$ defined by $\widetilde{\gamma}(\mathbb{1}_X \times \alpha) := [X] \times c_*(\alpha)$ is convolutive; namely for the constructible function convolution and homology convolution defined above the following diagram is commutative:*

$$\begin{array}{ccc}
\widetilde{\mathbf{F}}(X_1 \times X_2) \otimes \widetilde{\mathbf{F}}(X_2 \times X_3) & \xrightarrow{\star} & \widetilde{\mathbf{F}}(X_1 \times X_3) \\
{\scriptstyle \widetilde{\gamma} \otimes \widetilde{\gamma}}\downarrow & & \downarrow{\scriptstyle \widetilde{\gamma}} \\
H_*(X_1 \times X_2) \otimes H_*(X_2 \times X_3) & \xrightarrow[\star]{} & H_*(X_1 \times X_3).
\end{array}$$

(ii) *If X_2 is singular, the above diagram cannot be commutative.*

Before proving this theorem, we first observe the following four lemmas:

Lemma 3.4. — *Let X_i be a possibly singular variety in a nonsingular variety M_i ($i = 1,2,3$). Then the cohomolgy and homology cross products are compatible with the Alexander duality isomorphism; namely we have:*

$$\begin{array}{ccc} H^p(M_i, M_i \smallsetminus X_i) \otimes H^q(M_j, M_j \smallsetminus X_j) & \xrightarrow{\times} & H^{p+q}(M_i \times M_j, M_i \times M_j \smallsetminus X_i \times X_j) \\ \mathcal{A}_{X_i} \otimes \mathcal{A}_{X_j} \Big\downarrow & & \Big\downarrow \mathcal{A}_{X_i \times X_j} \\ H_{\dim M_i - p}(X_i) \otimes H_{\dim M_j - q}(X_j) & \xrightarrow[\times]{} & H_{\dim M_i + \dim M_j - (p+q)}(X_i \times X_j). \end{array}$$

Lemma 3.5. — *Cohomolgy cross product and cup product commute, namely we have*

$$(a \times b) \cup (a' \times b') = (-1)^{\deg a' \deg b} (a \cup a') \times (b \cup b').$$

Here $\deg x$ *denotes the degree of the cohomology class* x.

Note that in our case the sign $(-1)^{\deg a' \deg b}$ is always equal to one because the cohomology classes which we treat are always of even degree, thus we have that $(a \times b) \cup (a' \times b') = (a \cup a') \times (b \cup b')$.

Lemma 3.6. — *With the above situation in Lemma (3.4), let $x_i, y_i \in H_*(X_i)$ be homology classes. Then under the convolution*

$$\star : H_*(X_1 \times X_2) \otimes H_*(X_2 \times X_3) \longrightarrow H_*(X_1 \times X_3)$$

the following holds:

$$(x_1 \times x_2) \star (y_2 \times y_3) = \Big(\int_{X_2} \big(j_2^* \mathcal{A}_{X_2}^{-1}(x_2)\big) \cap y_2 \Big)(x_1 \times y_3).$$

Here $j_2 : X_2 \to M_2$ is the inculsion.

Proof. — Let us denote the cohomology class dual to the fundamental class $[M_i]$ by 1_{M_i}. By definition of the convolution and the above lemmas we have

$$\begin{aligned} &(x_1 \times x_2) \star (y_2 \times y_3) \\ &= p_{13*} \Big(\mathcal{A}_{X_1 \times X_2 \times X_3} \big((\mathcal{A}_{X_1}^{-1}(x_1) \times \mathcal{A}_{X_2}^{-1}(x_2) \times 1_{M_3}) \cup \big(1_{M_1} \times \mathcal{A}_{X_2}^{-1}(y_2) \times \mathcal{A}_{X_3}^{-1}(y_3)\big)\big)\Big) \\ &= p_{13*} \Big(\mathcal{A}_{X_1 \times X_2 \times X_3} \big((\mathcal{A}_{X_1}^{-1}(x_1) \times \big(\mathcal{A}_{X_2}^{-1}(x_2) \cup \mathcal{A}_{X_2}^{-1}(y_2)\big) \times \mathcal{A}_{X_3}^{-1}(y_3))\Big) \\ &= p_{13*} \Big(x_1 \times \mathcal{A}_{X_2} \big(\mathcal{A}_{X_2}^{-1}(x_2) \cup \mathcal{A}_{X_2}^{-1}(y_2)\big) \times y_3 \Big) \\ &= p_{13*} \Big(x_1 \times \big(\big(j_2^* \mathcal{A}_{X_2}^{-1}(x_2)\big) \cap y_2 \big) \times y_3 \Big) \\ &= \Big(\int_{X_2} \big(j_2^* \mathcal{A}_{X_2}^{-1}(x_2)\big) \cap y_2 \Big)(x_1 \times y_3). \end{aligned}$$

Here we note that the equality $\mathcal{A}_{X_2}\Big(\mathcal{A}_{X_2}^{-1}(x_2) \cup \mathcal{A}_{X_2}^{-1}(y_2)\Big) = \Big(j_2^* \mathcal{A}_{X_2}^{-1}(x_2)\Big) \cap y_2$ follows from, *e.g.*, [**F**, 19.1, (8), p. 371]. ☐

Lemma 3.7 (**[KY]**). — *Let $f : W \to W'$ and $g : Z \to Z'$ be proper morphisms. Then for any constructible functions $\omega \in F(W)$ and $\zeta \in F(Z)$*

$$(f \times g)_*(\omega \times \zeta) = (f_*\omega) \times (g_*\zeta).$$

Namely, the following diagram is commutative:

$$\begin{array}{ccc} F(W) \otimes F(Z) & \xrightarrow{\times} & F(W \times Z) \\ {\scriptstyle f_* \otimes g_*}\big\downarrow & & \big\downarrow{\scriptstyle (f \times g)_*} \\ F(W') \otimes F(Z') & \xrightarrow{\times} & F(W' \times Z'). \end{array}$$

Proof of Theorem (3.3)

(i) Let $1\!\!1_{X_i} \times \alpha_j \in \widetilde{\mathbf{F}}(X_i \times X_j)$ and let $p_2 : X_2 \to pt$ be the morphism to a point pt. Then we have

$$\begin{aligned}(1\!\!1_{X_1} \times \alpha_2)\star(1\!\!1_{X_2} \times \alpha_3) &= p_{13*}(1\!\!1_{X_1} \times \alpha_2 \times \alpha_3) \\ &= (p_{2*}\alpha_2)(1\!\!1_{X_1} \times \alpha_3) \\ &= \Big(\int_{X_2} c_*(\alpha_2)\Big)(1\!\!1_{X_1} \times \alpha_3).\end{aligned}$$

The last equality of course follows from the naturality of the Chern-Schwartz-MacPherson class c_*. The second equality is proved directly or it follows from the above general formula Lemma (3.7).

On the other hand it follows from Lemma (3.6) that

$$([X_1] \times c_*(\alpha_2))\star([X_2] \times c_*(\alpha_3)) = \Big(\int_{X_2} \big(j_2^*\mathcal{A}_{X_2}^{-1}(c_*(\alpha_2))\big) \cap [X_2]\Big)([X_1] \times c_*(\alpha_3)).$$

A crucial assumption in the statement (i) of the theorem is that $X_2 = M_2$ is nonsingular and compact. Thus $\mathcal{A}_{X_2} : H^*(X_2) \to H_{2\dim X_2 - *}(X_2)$ is the Poincaré duality isomorphism and $j_2^* : H^*(X_2) \to H^*(X_2)$ is the identity. Therefore we get

$$\Big(j_2^*\mathcal{A}_{X_2}^{-1}(c_*(\alpha_2))\Big) \cap [X_2] = c_*(\alpha_2),$$

hence in particular we get

$$\int_{X_2} \Big(j_2^*\mathcal{A}_{X_2}^{-1}(c_*(\alpha_2))\Big) \cap [X_2] = \int_{X_2} c_*(\alpha_2).$$

Thus the statement (i) follows.

(ii) Let X_2 be singular, thus $\dim_{\mathbf{C}} X_2 < \dim_{\mathbf{C}} M_2$. Let us take a closer look at the cohomology class $\Big(j_2^*\mathcal{A}_{X_2}^{-1}(c_*(\alpha_2))\Big) \cap [X_2]$, *i.e.*, $\mathcal{A}_{X_2}\Big(\mathcal{A}_{X_2}^{-1}(c_*(\alpha_2)) \cup \mathcal{A}_{X_2}^{-1}([X_2])\Big)$. Let $[c_*(\alpha_2)]_k$ denote the k-dimensional component of $c_*(\alpha_2)$. Let $m = \dim_{\mathbf{C}} M_2$ and $n = \dim_{\mathbf{C}} X_2$. Then it follows from the definition of the Alexander duality isomorphism that

$$\mathcal{A}_{X_2}\Big(\mathcal{A}_{X_2}^{-1}([c_*(\alpha_2)]_k) \cup \mathcal{A}_{X_2}^{-1}([X_2])\Big) \in H_{k-2(m-n)}(X_2). \tag{3.8}$$

Hence to compute $\int_{X_2}\Big(j_2^*\mathcal{A}_{X_2}^{-1}(c_*(\alpha_2))\Big)\cap[X_2]$, *i.e.*, the degree of the 0-dimensional component of $\Big(j_2^*\mathcal{A}_{X_2}^{-1}(c_*(\alpha_2))\Big)\cap[X_2]$, we need that $k-2(m-n)=0$, *i.e.*, $k=2(m-n)>2$. Let $\alpha_2=1\!\!1_p$ with p being a point on X_2, in which case $\int_{X_2}c_*(1\!\!1_p)=1$. On the other hand, the degree of $c_*(1\!\!1_p)$ is zero and hence it follows from (3.8) that

$$\Big(j_2^*\mathcal{A}_{X_2}^{-1}(c_*(\alpha_2))\Big)\cap[X_2]\in H_{-2(m-n)}(X_2).$$

Since $-2(m-n)<0$, $H_{-2(m-n)}(X_2)=0$. This implies that $\Big(j_2^*\mathcal{A}_{X_2}^{-1}(c_*(\alpha_2))\Big)\cap[X_2]=0$, thus $\int_{X_2}\Big(j_2^*\mathcal{A}_{X_2}^{-1}(c_*(\alpha_2))\Big)\cap[X_2]=0$.Therefore the square diagram in the theorem cannot be commutative. Thus the statement (ii) follows. □

So we would like to pose the following

Problem 3.9. — *Generalize Theorem (3.3) to the case when X_1,X_2,X_3 are all singular.*

An interesting feature of the above homomorphism $\widetilde{\gamma}:\widetilde{\mathbf{F}}(X\times Y)\to H_*(X\times Y)$ defined by $\widetilde{\gamma}(1\!\!1_X\times\alpha):=[X]\times c_*(\alpha)$ is that it can be described as any resolution Ginzburg-Chern class. Namely we have the following

Proposition 3.10. — *Let $p_1:X\times Y\to X$ denote the projection to the first factor as before. Then we have*

$$\widetilde{\gamma}^{\mathrm{Gin}}_{p_1}(1\!\!1_X\times\alpha)=[X]\times c_*(\alpha).$$

When it comes to the Nash-type Ginzburg-Chern class, we have that

$$\widehat{\gamma}^{\mathrm{Gin}}_{p_1}(1\!\!1_X\times\alpha)=\nu_*\Big(s(\widehat{TX})\cap c_*(\widehat{X})\Big)\times c_*(\alpha).$$

And we can see the following theorem, using Lemma (3.6):

Theorem 3.11. — *Let X_i be a possibly singular variety in a nonsingular variety M_i ($i=1,3$). And let $X_2=M_2$ be a nonsingular compact variety. Then we have the following commutative diagram.*

$$\begin{tikzcd}
\widetilde{\mathbf{F}}(X_1\times X_2)\otimes\widetilde{\mathbf{F}}(X_2\times X_3) \arrow[r, "\star"] \arrow[d, "\widehat{\gamma}^{\mathrm{Gin}}_{p_1}\otimes\widehat{\gamma}^{\mathrm{Gin}}_{p_1}"'] & \widetilde{\mathbf{F}}(X_1\times X_3) \arrow[d, "\gamma^{\mathrm{Gin}}_{p_1}"] \\
H_*(X_1\times X_2)\otimes H_*(X_2\times X_3) \arrow[r, "\star"'] & H_*(X_1\times X_3).
\end{tikzcd}$$

References

[Br] J.-P. Brasselet – Existence des classes de Chern en théorie bivariante, in *Analyse et topologie sur les espaces singuliers (CIRM, 1981)*, Astérisque, vol. 101-102, Société Mathématique de France, 1981, p. 7–22.

[BS] J.-P. Brasselet & M.-H. Schwartz – Sur les classes de Chern d'une ensemble analytique complexe, in *Caractéristique d'Euler-Poincaré (séminaire ENS, 1978-1979)*, Astérisque, vol. 82-83, Société Mathématique de France, 1981, p. 93–148.

[Er] L. Ernström – Topological Radon transforms and the local Euler obstruction, *Duke Math. J.* **76** (1994), p. 1–21.

[F] W. Fulton – *Intersection Theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Folge 3 Band 2, Springer-Verlag, 1984.

[FM] W. Fulton & R. MacPherson – *Categorical frameworks for the study of singular spaces*, Mem. Amer. Math. Soc., vol. 243, American Mathematical Society, 1981.

[G1] V. Ginzburg – $\mathcal{G}$-Modules, Springer's Representations and Bivariant Chern Classes, *Adv. Math.* **61** (1986), p. 1–48.

[G2] ———, Geometric methods in the representation theory of Hecke algebras and quantum groups, in *Representation theories and algebraic geometry (Montreal, PQ 1997)* (A. Broer & A. Daigneault, eds.), Kluwer Acad. Publ., Dordrecht, 1998, p. 127–183.

[K] M. Kwieciński – Formule du produit pour les classes caract'eristiques de Chern-Schwartz-MacPherson et homologie d'intersection, *C. R. Acad. Sci. Paris Sér. I Math.* **314** (1992), p. 625–628.

[KY] M. Kwieciński & S. Yokura – Product formula of the twisted MacPherson class, *Proc. Japan Acad. Ser. A Math. Sci.* **68** (1992), p. 167–171.

[M] R. MacPherson – Chern classes for singular algebraic varieties, *Ann. of Math.* **100** (1974), p. 423–432.

[Su] J. Schürmann – A generalized Verdier-type Riemann-Roch theorem for Chern-Schwartz-MacPherson classes, preprint, 2001.

[Sw1] M.-H. Schwartz – Classes caractéristiques définies par une stratiification d'une variété analytique complexe, *C. R. Acad. Sci. Paris Sér. I Math.* **260** (1965), p. 3262–3264, 3535–3537.

[Sw2] ———, Classes et caractères de Chern des espaces linéaires, *Pub. Int. Univ. Lille* **2** (1980), no. 3.

[Y1] S. Yokura – On a Verdier-type Riemann-Roch for Chern-Schwartz-MacPherson class, *Topology Appl.* **94** (1999), p. 315–327.

[Y3] ———, On the uniqueness problem of bivariant Chern classes, *Doc. Math.* **7** (2002), p. 133–142.

[Y2] ———, Remarks on Ginzburg's bivariant Chern classes, *Proc. Amer. Math. Soc.* **130** (2002), p. 3465–3471.

[Y4] ———, On Ginzburg's bivariant Chern classes, *Trans. Amer. Math. Soc.* **355** (2003), p. 2501–2521.

[Y5] ———, On Ginzburg's bivariant Chern classes, II, *Geom. Dedicata* **101** (2003), p. 185–201.

[Y6] ———, Quasi-bivariant Chern classes obtained by resolutions of singularities, *Rep. Fac. Sci. Kagoshima Univ.* **36** (2003), p. 17–28.

S. Yokura, Department of Mathematics and Computer Science, Faculty of Science, University of Kagoshima, 21-35 Korimoto 1-chome, Kagoshima 890-0065, Japan
E-mail : yokura@sci.kagoshima-u.ac.jp