Séminaires & Congrès
10, 2005, p. 443–460

ISOLATED CRITICAL POINTS AND ADIABATIC LIMITS OF CHERN FORMS

by

Atsuko Yamada Yoshikawa & Ken-Ichi Yoshikawa

Dedicated to Professor Tatsuo Suwa on his 60th birthday

Abstract. — In this note, we compute the adiabatic limit of Chern forms for holomorphic fibrations over complex curves. We assume that the projection of the fibration has only isolated critical points.

***Résumé* (Points critiques isolés et limites adiabatiques des formes de Chern).** — Dans cet article, nous calculons la limite adiabatique des formes de Chern pour les fibrations holomorphes sur des coubes complexes. Nous supposons que le projection de la fibration n'a que des points critiques isolés.

1. Introduction

Let X be a complex manifold of dimension $n+1$ and S a Riemann surface. Let $f : X \to S$ be a proper surjective holomorphic map. The critical locus of f is the analytic subset of X defined by

$$\Sigma_f = \{p \in X\,;\, df_p = 0\}.$$

In this note, we always assume that Σ_f is discrete.

Let g^{TX} be a Hermitian metric on the holomorphic tangent bundle TX. Let g^{TS} be a Hermitian metric on TS. Define the family of Hermitian metrics on TX by

$$g^{TX}_\varepsilon = g^{TX} + \frac{1}{\varepsilon^2} f^* g^{TS} \qquad (\varepsilon > 0).$$

Let $\nabla^{TX, g^{TX}_\varepsilon}$ be the holomorphic Hermitian connection of (TX, g^{TX}_ε), whose curvature form is denoted by $R^{TX, g^{TX}_\varepsilon}$. Then $R^{TX, g^{TX}_\varepsilon}$ is a $(1,1)$-form on X with values in $\mathrm{End}(TX)$. Let $c_i(TX, g^{TX}_\varepsilon)$ be the i-th Chern form of (TX, g^{TX}_ε).

2000 ***Mathematics Subject Classification.*** — 58K05, 58K20, 57R20, 57R70, 58A25.
Key words and phrases. — Isolated critical point, adiabatic limit, Chern form, Milnor number.

Research partially supported by the Grants-in-Aid for Scientific Research (B)(2): 14740035, JSPS.

Let $P(c) = P(c_1, \dots, c_{n+1}) \in \mathbf{C}[c_1, \dots, c_{n+1}]$ be a polynomial in the variables $c_1, \dots, c_{n+1}$. The purpose of this note is to study the family of differential forms $P(TX, g_\varepsilon^{TX}) := P(c(TX, g_\varepsilon^{TX}))$ as $\varepsilon \to 0$, called the *adiabatic limit*, under certain assumptions on the metrics g^{TX}, g^{TS} (see Assumption 2.1).

The study of this problem was initiated by Bismut and Bost in [**3**, Sect. 6 (a)]; they treated the case where $\dim X = 2$, the map f has only non-degenerate critical points, and $P(c)$ is the Todd polynomial. They applied their formula for the adiabatic limit to compute the holonomy of the determinant line bundles on S ([**3**, Sect. 6 (b), (c)]). Then Bismut treated in [**2**, Sect. 1 (e)] the case where $\dim X$ is arbitrary, the critical locus of the map f is locally defined by the equation $f(z_0, z_1, z') = z_0 z_1$, and $P(c)$ is arbitrary; he used his result to study the boundary behavior of Quillen metrics.

The goal of this note is to establish the convergence of the adiabatic limit $\lim_{\varepsilon\to 0} P(TX, g_\varepsilon^{TX})$ in the sense of currents on X and to compute the explicit formula for it. In particular, we extend [**3**, Sect. 6 (a)] to the case where f has only isolated critical points. Our result (Theorem 2.2) is compatible with [**15**].

2. Statement of the Result

Let $f : X \to S$ be a proper surjective holomorphic map between complex manifolds. Throughout this note, we assume the following:

(i) The critical locus Σ_f is a discrete subset of X.
(ii) $\dim X = n + 1$ and $\dim S = 1$.

Let g^{TX} and g^{TS} be Hermitian metrics on TX and TS, respectively. We define the family of Hermitian metrics $\{g_\varepsilon^{TX}\}_{\varepsilon>0}$ by

$$g_\varepsilon^{TX} := g^{TX} + \varepsilon^{-2} f^* g^{TS}.$$

The unit disc $\{s \in \mathbf{C};\ |s| < 1\}$ and the unit punctured disc $\{s \in \mathbf{C};\ 0 < |s| < 1\}$ are denoted by Δ and $\Delta^* = \Delta \setminus \{0\}$, respectively.

2.1. Assumptions on metrics.

— Let $\Gamma_f \subset X \times S$ be the graph of f:

$$\Gamma_f = \{(x, t) \in X \times S;\ f(x) = t\}.$$

Let $\mathrm{pr}_1 : \Gamma_f \to X$ and $\mathrm{pr}_2 : \Gamma_f \to S$ be the natural projections. Let $(U_p, (z_0, \dots, z_n))$ be a coordinate neighborhood of $p \in \Sigma_f$ in X centered at p. Let $(D_{f(p)}, t)$ be a coordinate neighborhood of $f(p)$ in S centered at $f(p)$. Assume that

(i) $U_p \cap U_q = \varnothing$ for $p, q \in \Sigma_f$ with $p \neq q$;
(ii) $(U_p, p) \cong (\Delta^{n+1}, 0)$;
(iii) $(f(U_p), f(p)) \subset (D_{f(p)}, 0)$.

Then $\Gamma_f|_{U_p}$ is a submanifold of $U_p \times D_{f(p)}$. Let $\iota : \Gamma_f|_{U_p} \hookrightarrow U_p \times D_{f(p)}$ be the inclusion. We have the commutative diagram:

$$\begin{array}{ccc} (\Gamma_f|_{U_p}, (p, f(p))) & \xrightarrow{\ \iota\ } & (U_p \times D_{f(p)}, (0,0)) \\ {\scriptstyle \mathrm{pr}_1}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{pr}_2} \\ (U_p, p) & \xrightarrow{\quad f \quad} & (D_{f(p)}, 0). \end{array}$$

Assumption 2.1. — Let $\delta \geqslant 0$ be a constant. Assume that the Hermitian metrics g^{TX} and g^{TS} are expressed as follows on each U_p $(p \in \Sigma_f)$:

$$\mathrm{pr}_1^* g^{TX}|_{(\Gamma_f|_{U_p})} = \left\{ \sum_i dz_i \otimes d\overline{z}_i + \delta \cdot dt \otimes d\overline{t} \right\} \Bigg|_{(\Gamma_f|_{U_p})}, \tag{1}$$

$$g^{TS}|_{D_{f(p)}} = dt \otimes d\overline{t}. \tag{2}$$

We are mainly interested in the case $\delta = 0$ because $g^{TX}|_{U_p}$ is the restriction of the Euclidean metric on $\mathbf{C}^{n+1}$ in this case.

2.2. Chern forms. — Let $M_{n+1}(\mathbf{C})$ be the set of all complex $(n+1) \times (n+1)$ matrices. For $A \in M_{n+1}(\mathbf{C})$, set $c(A) = \det(I_{n+1} + A) = 1 + c_1(A) + \cdots + c_{n+1}(A)$, where $c_i(A)$ is homogeneous of degree i. For a polynomial $P(c) = P(c_1, \ldots, c_{n+1}) \in \mathbf{C}[c_1, \ldots, c_{n+1}]$, set $P(A) = P(c_1(A), \ldots, c_{n+1}(A))$.

Denote by $A_X^{p,q}$ (resp. A_X^r) the vector space of smooth (p,q)-forms (resp. r-forms) on X. For a complex vector bundle F on X, the set of smooth (p,q)-forms on X with values in F is denoted by $A_X^{p,q}(F)$. For $\Phi \in A_X^*$, Φ^{top} denotes the bidegree $(\dim X, \dim X)$-part of Φ. Hence $\Phi^{\mathrm{top}} \in A_X^{n+1,n+1}$.

Let (E, h^E) be a holomorphic Hermitian vector bundle on X. Let ∇^{E,h^E} be the holomorphic Hermitian connection. Namely, the $(0,1)$-part of ∇^{E,h^E} is given by the $\overline{\partial}$-operator and ∇^{E,h^E} is compatible with the metric h^E (*cf.* [**10**, Chap. 1, Sect. 4]). Let $R^{E,h^E} = (\nabla^{E,h^E})^2 \in A_X^{1,1}(\mathrm{End}(E))$ be the curvature form of ∇^{E,h^E}. Set

$$c(E, h^E) = \sum_{i=0}^{\mathrm{rank}(E)} c_i(E, h^E) := c\left(\frac{i}{2\pi} R^{E,h^E} \right) \in \bigoplus_{p \geqslant 0} A_X^{p,p}.$$

2.3. The convergence of adiabatic limits. — Let

$$Tf := \ker\{ f_* : TX|_{X \smallsetminus \Sigma_f} \longrightarrow f^*TS \}$$

be the relative holomorphic tangent bundle of the map $f : X \to S$. Then Tf is a holomorphic subbundle of $TX|_{X \smallsetminus \Sigma_f}$.

Let $g^{Tf} = g^{TX}|_{Tf} = (g_\varepsilon^{TX})|_{Tf}$ be the Hermitian metric on Tf induced from g_ε^{TX}. Then g^{Tf} is independent of $\varepsilon > 0$. Let $R^{Tf,g^{Tf}}$ be the curvature of (Tf, g^{Tf}). The i-th Chern form $c_i(Tf, g^{Tf})$ lies in $A_{X \smallsetminus \Sigma_f}^{i,i}$ for $i = 1, \ldots, n$.

For $p \in \Sigma_f$, let $\mu(f,p) \in \mathbf{N}$ be the *Milnor number* of f at p, *i.e.*,

$$\mu(f,p) := \dim_{\mathbf{C}} \mathbf{C}\{z_0,\dots,z_n\} \Big/ \left(\frac{\partial f}{\partial z_0}(z),\dots,\frac{\partial f}{\partial z_n}(z)\right),$$

where $(\frac{\partial f}{\partial z_0},\dots,\frac{\partial f}{\partial z_n}) \subset \mathbf{C}\{z_0,\dots,z_n\}$ is the ideal generated by the germs $\frac{\partial f}{\partial z_0},\dots,\frac{\partial f}{\partial z_n}$.

The *Dirac δ-current* supported at $p \in \Sigma_f$ is the $(n+1,n+1)$-current δ_p on X defined by

$$\int_X \varphi\, \delta_p := \varphi(p), \qquad \forall\, \varphi \in C_0^\infty(X).$$

For a formal power series of one variable $\varphi(t) \in \mathbf{C}[[t]]$, let $\varphi(t)|_{t^m}$ be the coefficient of the term t^m in $\varphi(t)$, *i.e.*, $\varphi(t)|_{t^m} = \frac{1}{m!}\left(\frac{d}{dt}\right)^m|_{t=0}\,\varphi(t)$.

Main Theorem 2.2. — *With the same notation as above, assume that Σ_f is a discrete subset of X and that the metrics g^{TX}, g^{TS} verify Assumption* 2.1. *Then the following hold:*

(1) *The differential form $P(Tf \oplus f^*TS,\, g^{Tf} \oplus f^*g^{TS})^{\mathrm{top}} \in A^{n+1,n+1}_{X\smallsetminus\Sigma_f}$ extends trivially to a smooth $(n+1,n+1)$-form on X.*

(2) *The adiabatic limit $\lim_{\varepsilon\to 0} P(TX, g^{TX}_\varepsilon)^{\mathrm{top}}$ converges to a $(n+1,n+1)$-current on X. Moreover, the following identity holds:*

$$\lim_{\varepsilon\to 0} P(TX, g^{TX}_\varepsilon)^{\mathrm{top}} = P(Tf \oplus f^*TS,\, g^{Tf} \oplus f^*g^{TS})^{\mathrm{top}} + P(-t,\dots,(-t)^{n+1})|_{t^{n+1}} \cdot \sum_{p\in\Sigma_f} \mu(f,p)\,\delta_p, \tag{2.1}$$

In particular, the following equation of currents on U_p holds:

$$\lim_{\varepsilon\to 0} P(TX, g^{TX}_\varepsilon)^{\mathrm{top}}|_{U_p} = P(-t,\dots,(-t)^{n+1})|_{t^{n+1}} \cdot \mu(f,p)\,\delta_p. \tag{2.2}$$

Corollary 2.3 **([8], [4**, Example 14.1.5**], [7**, Chap. VI, 3**], [9**, Cor. 2.4**])**

Let X be a compact complex manifold of dimension $n+1$ and S a compact Riemann surface. Let $f : X \to S$ be a proper surjective holomorphic map with general fiber F. Let $\chi_{\mathrm{EP}}(X)$, $\chi_{\mathrm{EP}}(F)$, $\chi_{\mathrm{EP}}(S)$ be the topological Euler-Poincaré numbers of X, F, S, respectively. If Σ_f is a finite set, then the following identity holds:

$$\chi_{\mathrm{EP}}(X) = \chi_{\mathrm{EP}}(F)\chi_{\mathrm{EP}}(S) + (-1)^{n+1} \sum_{p\in\Sigma_f} \mu(f,p).$$

Proof of Corollary 2.3. — Consider the polynomial $P(A) = c_{n+1}(A) = \det(A)$. Then the corresponding genus is the Euler characteristic. Since

$$c_{n+1}(Tf \oplus f^*TS, g^{Tf} \oplus f^*g^{TS}) = c_n(Tf, g^{Tf}) \wedge f^*c_1(TS, g^{TS}) \in A^{n+1,n+1}_X$$

by Theorem 2.2 (1), the result follows from (2.1) and the projection formula:

$$\begin{aligned}\int_X c_{n+1}(Tf \oplus f^*TS, g^{Tf} \oplus f^*g^{TS}) &= \int_F c_n(Tf, g^{Tf})|_F \int_S c_1(TS, g^{TS}) \\ &= \chi_{\mathrm{EP}}(F)\chi_{\mathrm{EP}}(S).\end{aligned}$$

□

Example 2.4. — Let A be an Abelian variety of dimension g and E an elliptic curve. Let $X \subset A \times E$ be a smooth hypersurface such that the restriction of the projection $\mathrm{pr}_2|_X : X \to E$ has only isolated critical points. Set $f = \mathrm{pr}_2|_X$.

Let g^{TA} and g^{TE} be the *flat* Kähler metrics on TA and TE, respectively. For $\varepsilon > 0$, set

$$g_\varepsilon^{TX} = g^{TA} \oplus \left(1 + \frac{1}{\varepsilon^2}\right) g^{TE}\Big|_X .$$

Then, for all $x \in X$, there is a neighborhood U_x in $A \times E$ such that the metrics $g^{TX} := g_\infty^{TX}$ and g^{TE} verify Assumption 2.1 on U_x. The first term of the R.H.S. of (2.1) vanishes identically on X by Propositions 4.1 and 4.2 below. Hence it follows from (2.1) that

$$\lim_{\varepsilon \to 0} P(TX, g_\varepsilon^{TX})^{\mathrm{top}} = P(-t, \dots, (-t)^g)|_{t^g} \cdot \sum_{p \in \Sigma_f} \mu(f,p)\, \delta_p. \tag{2.3}$$

In particular, the support of the adiabatic limit $\lim_{\varepsilon \to 0} P(TX, g_\varepsilon^{TX})^{\mathrm{top}}$ concentrates on the critical locus Σ_f in this example.

Remark 2.5. — We can verify (2.3) as an identity of cohomology classes as follows. Let N be the normal bundle of X in $A \times E$. Then we have the exact sequence of holomorphic vector bundles on X:

$$0 \longrightarrow TX \longrightarrow T(A \times E)|_X = \mathbf{C}^{g+1} \longrightarrow N \longrightarrow 0,$$

from which we obtain $c(X) = c(N)^{-1} = (1 + c_1(N))^{-1}$. Hence $c_i(X) = (-c_1(N))^i$ for $i = 1, \dots, g$ and

$$P(c(X)) = P(-t, \dots, (-t)^g)|_{t^g} \cdot c_1(N)^g = (-1)^g P(-t, \dots, (-t)^g)|_{t^g} \cdot c_g(X).$$

Since $\chi_{\mathrm{EP}}(E) = 0$, this yields that

$$\begin{aligned}
\int_X P(c(X)) &= (-1)^g P(-t, \dots, (-t)^g)|_{t^g} \cdot \chi_{\mathrm{EP}}(X) \\
&= (-1)^g P(-t, \dots, (-t)^g)|_{t^g} \cdot \Big\{ \chi_{\mathrm{EP}}(F) \chi_{\mathrm{EP}}(E) + (-1)^g \sum_{p \in \Sigma_f} \mu(f,p) \Big\} \\
&= P(-t, \dots, (-t)^g)|_{t^g} \cdot \sum_{p \in \Sigma_f} \mu(f,p).
\end{aligned}$$

3. An analytic characterization of the Milnor number

Set $U := \Delta^{n+1}$. We denote by $z = (z_0, \dots, z_n)$ the system of coordinates of U. Let $f : (U, 0) \to (\mathbf{C}, 0)$ be a holomorphic function on U such that

$$\Sigma_f = \{0\}.$$

The Milnor number $\mu(f,0)$ is denoted by $\mu(f)$, for short. Set $\|df\|^2 = \sum_{i=0}^n |\frac{\partial f}{\partial z_i}|^2$. We prove the following result in this section, which shall be used in the proof of the Main Theorem 2.2 in Section 5.

Theorem 3.1. — *The following equation of currents on U holds:*

$$\lim_{\varepsilon\to 0}\left\{\frac{i}{2\pi}\partial\overline{\partial}\log(\|df\|^2+\varepsilon^2)\right\}^{n+1} = \mu(f)\,\delta_0.$$

Following [**2**, Sect. 1 (c)], we regard ε as a complex parameter and replace ε^2 by $|\varepsilon|^2$ in what follows. Hence $\varepsilon\in\Delta$.

3.1. Proof of Theorem 3.1. — Define the holomorphic map $\nu : (U\times\Delta)\smallsetminus\{(0,0)\}\to\mathbf{P}^{n+1}$ by

$$\nu(z,\varepsilon) = \left(\frac{\partial f}{\partial z_0}(z):\cdots:\frac{\partial f}{\partial z_n}(z):\varepsilon\right).$$

Then ν extends to a meromorphic map from $U\times\Delta$ into $\mathbf{P}^{n+1}$ with indeterminacy locus $\{(0,0)\}$. Let $\pi : (\widetilde{U\times\Delta}, E)\to(U\times\Delta,(0,0))$ be the resolution of the indeterminacy of ν. Hence $E=\pi^{-1}(0,0)$. Then there exists a holomorphic map $\widetilde{\nu}:\widetilde{U\times\Delta}\to\mathbf{P}^{n+1}$ such that $\widetilde{\nu}|_{(\widetilde{U\times\Delta})\smallsetminus E} = \nu\circ\pi$. Let $\widetilde{U\times\{0\}}\subset\widetilde{U\times\Delta}$ be the proper transform of the divisor $U\times\{0\}\subset U\times\Delta$.

Set

$$H=\{(z:\varepsilon)\in\mathbf{P}^{n+1};\ \varepsilon=0\}\subset\mathbf{P}^{n+1},$$

where $(z:\varepsilon)=(z_0:\cdots:z_n:\varepsilon)$ are the homogeneous coordinates of $\mathbf{P}^{n+1}$. Then $H\cong\mathbf{P}^n$. Since $\nu(U\times\{0\}\smallsetminus\{(0,0)\})\subset H$ and hence $\widetilde{\nu}(\widetilde{U\times\{0\}}\smallsetminus E)\subset H$, we get

$$\widetilde{\nu}(\widetilde{U\times\{0\}})\subset H. \tag{3.1}$$

Let $p:U\times\Delta\to\Delta$ be the natural projection. Set $\widetilde{p}=p\circ\pi$. Then $\widetilde{p}:\widetilde{U\times\Delta}\to\Delta$ is a holomorphic map such that

$$\widetilde{p}^{-1}(\varepsilon)=\begin{cases} U\times\{\varepsilon\} & (\varepsilon\neq 0)\\ \widetilde{U\times\{0\}}+\widetilde{E} & (\varepsilon=0).\end{cases} \tag{3.2}$$

Here $\widetilde{E}$ is a (possibly non-reduced) divisor on $\widetilde{U\times\Delta}$ such that $\mathrm{Supp}(\widetilde{E})\subset E$.

Let

$$\omega_{\mathbf{P}^{n+1}} = \frac{i}{2\pi}\partial\overline{\partial}\log(\|z\|^2+|\varepsilon|^2)$$

be the Fubini-Study form on $\mathbf{P}^{n+1}$. Then we have the identity on $U\times\Delta\smallsetminus\{(0,0)\}$:

$$\nu^*\omega_{\mathbf{P}^{n+1}} = \frac{i}{2\pi}\partial\overline{\partial}\log(\|df\|^2+|\varepsilon|^2).$$

Proposition 3.2. — *The following equation of currents on U holds:*

$$\lim_{\varepsilon\to 0}\left\{\frac{i}{2\pi}\partial\overline{\partial}\log(\|df\|^2+|\varepsilon|^2)\right\}^{n+1}=\left(\int_{\widetilde{E}}\widetilde{\nu}^*\omega^{n+1}_{\mathbf{P}^{n+1}}\right)\delta_0.$$

Proof. — Let $\varphi\in C^\infty_0(U)$. Since $\pi:\widetilde{U\times\Delta}\smallsetminus\widetilde{p}^{-1}(0)\to U\times\Delta\smallsetminus p^{-1}(0)$ is an isomorphism and since $\widetilde{\nu}=\nu\circ\pi$ on $\widetilde{U\times\Delta}\smallsetminus\widetilde{p}^{-1}(0)$, we have for all $\varepsilon\in\Delta^*=\Delta\smallsetminus\{0\}$:

$$\int_{U\times\{\varepsilon\}}\varphi\cdot\nu^*\omega^{n+1}_{\mathbf{P}^{n+1}}=\int_{\widetilde{p}^{-1}(\varepsilon)}\pi^*\varphi\cdot\pi^*\nu^*\omega^{n+1}_{\mathbf{P}^{n+1}}=\int_{\widetilde{p}^{-1}(\varepsilon)}\pi^*\varphi\cdot\widetilde{\nu}^*\omega^{n+1}_{\mathbf{P}^{n+1}}.$$

Since $\pi^*\varphi\cdot\widetilde{\nu}^*\omega^{n+1}_{\mathbf{P}^{n+1}}\in A^{n+1,n+1}_{\widetilde{U\times\Delta}}$, we obtain from [**1**, Th.1] that

$$\lim_{\varepsilon\to 0}\int_{\widetilde{p}^{-1}(\varepsilon)}\pi^*\varphi\cdot\widetilde{\nu}^*\omega^{n+1}_{\mathbf{P}^{n+1}}=\int_{\widetilde{p}^{-1}(0)}\pi^*\varphi\cdot\widetilde{\nu}^*\omega^{n+1}_{\mathbf{P}^{n+1}},$$

which, together with (3.2), yields that

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{U\times\{\varepsilon\}}\varphi\cdot\nu^*\omega^{n+1}_{\mathbf{P}^{n+1}}&=\int_{\widetilde{U\times\{0\}}}\pi^*\varphi\cdot\widetilde{\nu}^*\omega^{n+1}_{\mathbf{P}^{n+1}}+\varphi(0)\int_{\widetilde{E}}\widetilde{\nu}^*\omega^{n+1}_{\mathbf{P}^{n+1}}\\&=\varphi(0)\int_{\widetilde{E}}\widetilde{\nu}^*\omega^{n+1}_{\mathbf{P}^{n+1}}.\end{aligned}$$

Here the second term of the R.H.S. of the first equality follows from $(\pi^*\varphi)|_E=\varphi(0)$ and the second equality from (3.1) because $(\omega_{\mathbf{P}^{n+1}}|_H)^{n+1}\equiv 0$. ☐

To prove that $\int_{\widetilde{E}}\widetilde{\nu}^*\omega^{n+1}_{\mathbf{P}^{n+1}}=\mu(f)$, we need the following:

Proposition 3.3. — *Let $\chi(z)\in C^\infty_0(U)$ and assume that $\chi(z)=1$ when $\|z\|\leqslant\frac{3}{4}$. For $\varepsilon\in\Delta^*=\Delta\smallsetminus\{0\}$, set*

$$a(\varepsilon):=\int_{U\times\{\varepsilon\}}\chi(z)\log\left(\frac{|\varepsilon|^2}{\|df(z)\|^2+|\varepsilon|^2}\right)\nu^*\omega^{n+1}_{\mathbf{P}^{n+1}},$$

$$b(\varepsilon):=\int_{U\times\{\varepsilon\}}\chi(z)\log(\|df(z)\|^2+|\varepsilon|^2)\,\nu^*\omega^{n+1}_{\mathbf{P}^{n+1}}.$$

Then there exist $\psi_1(\varepsilon),\psi_2(\varepsilon)\in C^0(\Delta)$ such that for all $\varepsilon\in\Delta^*=\Delta\smallsetminus\{0\}$,

$$a(\varepsilon)=\psi_1(\varepsilon),\qquad b(\varepsilon)=\mu(f)\log|\varepsilon|^2+\psi_2(\varepsilon).$$

The proof of Proposition 3.3 is technical and shall be given in Section 3.2. However, it is easy to verify the proposition when f has a *non-degenerate* critical point at 0 (see Lemma 3.11 below).

Proof of Theorem 3.1. — By Proposition 3.3, we have

$$\log|\varepsilon|^2\int_{U\times\{\varepsilon\}}\chi(z)\,\nu^*\omega^{n+1}_{\mathbf{P}^{n+1}}=a(\varepsilon)+b(\varepsilon)=\mu(f)\log|\varepsilon|^2+\psi_1(\varepsilon)+\psi_2(\varepsilon).$$

Hence, as $\varepsilon\to 0$,

$$\int_{U\times\{\varepsilon\}}\chi(z)\,\nu^*\omega^{n+1}_{\mathbf{P}^{n+1}}=\mu(f)+O\left(\frac{1}{\log|\varepsilon|}\right).$$

Comparing this with Proposition 3.2 and using $\chi(0)=1$, we get

$$\mu(f)=\lim_{\varepsilon\to 0}\int_{U\times\{\varepsilon\}}\chi(z)\,\nu^*\omega_{\mathbf{P}^{n+1}}^{n+1}=\chi(0)\int_{\widetilde{E}}\widetilde{\nu}^*\omega_{\mathbf{P}^{n+1}}^{n+1}=\int_{\widetilde{E}}\widetilde{\nu}^*\omega_{\mathbf{P}^{n+1}}^{n+1},$$

which, together with Proposition 3.2, yields the theorem. □

3.2. Proof of Proposition 3.3 via the Picard-Lefschetz principle. — In the rest of Section 3, we prove Proposition 3.3. Our approach is as follows:

(I) We take a morsification $F(z,w)$ of $f(z)$ and extend the meromorphic map ν to a meromorphic map $\mathcal{N}$ from $U\times\Delta^2$ into $\mathbf{P}^{n+1}$.

(II) Replacing df by d_zF and ν by $\mathcal{N}$ in the definitions of $a(\varepsilon)$ and $b(\varepsilon)$, we obtain their extensions $A(\varepsilon,w), B(\varepsilon,w)\in C^\infty(\Delta^*\times\Delta)$ such that $A(\varepsilon,0)=a(\varepsilon)$ and $B(\varepsilon,0)=b(\varepsilon)$.

(III) Proposition 3.3 is deduced from the regularities of $A(\varepsilon,w)$ and $B(\varepsilon,w)$; we prove that $A(\varepsilon,w)\in C^1(\Delta^2)$ and $B(\varepsilon,w)-\mu(f)\log|\varepsilon|^2\in C^\infty(\Delta^2)$.

To distinguish between the target $\mathbf{C}$ of $f(z)$ and the parameter space Δ^2, we denote by (ε,w) the coordinates of Δ^2.

3.2.1. Preliminaries

a) A holomorphic function $F(z,w)\in\mathcal{O}(U\times\Delta)$ satisfying the following properties (i) and (ii) is called a *morsification* of $f(z)$:

(i) $F(z,0)=f(z)$;

(ii) $F|_{U\times\{w\}}\in\mathcal{O}(U)$ has only non-degenerate critical points when $w\neq 0$.

There always exists a morsification of $f(z)$ if we replace U by a smaller open subset of $0\in\mathbf{C}^{n+1}$ (*cf.* [**13**, Loo, Cor. 4.10 and 4.11 and Prop. 4.12]).

Let $F(z,w)$ be a morsification of $f(z)$. Assume that for every $w\in\Delta$,

$$\Sigma_{F(\cdot,w)}\subset\left\{z\in U;\ \|z\|\leqslant\frac{1}{2}\right\}. \tag{3.3}$$

This can be satisfied if we replace the disc $\Delta=\{w\in\mathbf{C};\ |w|<1\}$ by a smaller one.

Associated to the morsification $F(z,w)$, we deform the meromorphic map ν as follows: Define the meromorphic map $\mathcal{N}:U\times\Delta^2\to\mathbf{P}^{n+1}$ by

$$\mathcal{N}(z,\varepsilon,w)=\left(\frac{\partial F}{\partial z_0}(z,w):\cdots:\frac{\partial F}{\partial z_n}(z,w):\varepsilon\right).$$

Then we have $\mathcal{N}|_{U\times\Delta^*\times\{0\}}=\nu|_{U\times\Delta^*}$. Outside the indeterminacy locus of $\mathcal{N}$,

$$\mathcal{N}^*\omega_{\mathbf{P}^{n+1}}(z,\varepsilon,w)=\frac{i}{2\pi}\partial\overline{\partial}\log(\|d_zF(z,w)\|^2+|\varepsilon|^2),$$

where $d_zF=(\frac{\partial F}{\partial z_0},\dots,\frac{\partial F}{\partial z_n})$. The indeterminacy locus of $\mathcal{N}$ is given by the set $\{(z,0,w)\in U\times\Delta^2;\ d_zF(z,w)=0\}=\bigcup_{w\in\Delta}(\Sigma_{F(\cdot,w)},0,w)$.

Lemma 3.4. — *Set* $V:=\{z\in U;\ \|z\|>\frac{3}{4}\}$. *Then* $d_zF(z,w)$ *is nowhere vanishing on* $V\times\Delta^2$. *Moreover,* $\mathcal{N}^*\omega_{\mathbf{P}^{n+1}}^{n+1}\wedge\frac{d\varepsilon}{\varepsilon}\in A_{V\times\Delta^2}^{n+2,n+1}$ *and* $\mathcal{N}^*\omega_{\mathbf{P}^{n+1}}^{n+1}\wedge\frac{d\overline{\varepsilon}}{\overline{\varepsilon}}\in A_{V\times\Delta^2}^{n+1,n+2}$.

Proof. — For $i = 0, \dots, n$, set $\mathcal{V}_i = \{(z, \varepsilon, w) \in V \times \Delta^2;\ \frac{\partial F}{\partial z_i}(z, w) \neq 0\}$ and $f_i = \frac{\partial F}{\partial z_i}$. Then every $\mathcal{V}_i$ is an open subset of $V \times \Delta^2$. On $\mathcal{V}_0$, the differential forms

$$\omega_0 := f_0^{-1} d\left(\frac{f_1}{f_0}\right) \wedge \cdots \wedge d\left(\frac{f_n}{f_0}\right), \qquad \omega_1 := f_0^{-n-1} df_1 \wedge \cdots \wedge df_n \wedge df_0$$

are holomorphic. Let $(\zeta_1, \dots, \zeta_{n+1})$ be the inhomogeneous coordinates of $\mathbf{P}^{n+1}$, where $\zeta_i = z_i/z_0$ for $i = 1, \dots, n$ and $\zeta_{n+1} = \varepsilon/z_0$. Then one can verify that

$$\mathcal{N}^*(d\zeta_1 \wedge \cdots \wedge d\zeta_{n+1})|_{\mathcal{V}_0} = \omega_0 \wedge d\varepsilon - \varepsilon\, \omega_1.$$

Hence there exists a smooth function $g(z, \varepsilon, w)$ on $\mathcal{V}_0$ such that

$$\begin{aligned} \mathcal{N}^* \omega_{\mathbf{P}^{n+1}}^{n+1}|_{\mathcal{V}_0 \times \Delta} &= g\,(\omega_0 \wedge d\varepsilon - \varepsilon\,\omega_1) \wedge \overline{(\omega_0 \wedge d\varepsilon - \varepsilon\,\omega_1)} \\ &= g\,\{(-1)^n \omega_0 \wedge \overline{\omega}_0 \wedge d\varepsilon \wedge d\overline{\varepsilon} - \omega_1 \wedge \overline{\omega}_0 \wedge \varepsilon\, d\overline{\varepsilon} \\ &\qquad - (-1)^n \omega_0 \wedge \overline{\omega}_1 \wedge \overline{\varepsilon}\, d\varepsilon + |\varepsilon|^2 \omega_1 \wedge \overline{\omega}_1\}. \end{aligned}$$

By this formula, we get $\mathcal{N}^* \omega_{\mathbf{P}^{n+1}}^{n+1} \wedge \frac{d\varepsilon}{\varepsilon} \in A_{\mathcal{V}_0}^{n+2,n+1}$ and $\mathcal{N}^* \omega_{\mathbf{P}^{n+1}}^{n+1} \wedge \frac{d\overline{\varepsilon}}{\overline{\varepsilon}} \in A_{\mathcal{V}_0}^{n+1,n+2}$. Similarly, we can verify that $\mathcal{N}^* \omega_{\mathbf{P}^{n+1}}^{n+1} \wedge \frac{d\varepsilon}{\varepsilon} \in A_{\mathcal{V}_i}^{n+2,n+1}$ and $\mathcal{N}^* \omega_{\mathbf{P}^{n+1}}^{n+1} \wedge \frac{d\overline{\varepsilon}}{\overline{\varepsilon}} \in A_{\mathcal{V}_i}^{n+1,n+2}$ for $i = 1, \dots, n$. Since $V \times \Delta^2 = \bigcup_{i=0}^n \mathcal{V}_i$ by (3.3), this implies the result. $\square$

b) Let $\Omega \subset \Delta^2$ be a domain. Define the subspace $A_{U \times \Omega, vc}^* \subset A_{U \times \Omega}^*$ by

$$A_{U \times \Omega, vc}^* := \{\omega \in A_{U \times \Omega}^*;\ \mathrm{Supp}(\omega) \subset K \times \Omega \text{ for some compact subset } K \subset U\}.$$

We define the linear map $\int_U : A_{U \times \Omega, vc}^* \to A_\Omega^{*-2n-2}$ as follows: For $\theta(\varepsilon, w) \in A_\Omega^*$ and $\omega(z, \varepsilon, w) = a(z, \varepsilon, w)\, dz^I \wedge d\overline{z}^J \wedge \theta(\varepsilon, w) \in A_{U \times \Omega, vc}^{*+|I|+|J|}$,

$$\Big(\int_U \omega\Big)(\varepsilon, w) := \begin{cases} (\int_U a(z, \varepsilon, w)\, dz_0 \dots dz_n d\overline{z}_0 \dots d\overline{z}_n)\theta(\varepsilon, w) & (I = J = \{0, \dots, n\}), \\ 0 & (\text{otherwise}), \end{cases}$$

where $dz^I = dz_{i_1} \wedge \cdots \wedge dz_{i_p}$ and $|I| = p$ for $I = \{i_1 < \cdots < i_p\}$. Then we extend linearly the map $\int_U$ to $A_{U \times \Omega, vc}^*$. One can verify that for all $\omega \in A_{U \times \Omega, vc}^*$,

$$d_{\Delta^2}\Big(\int_U \omega\Big) = \int_U d_{U \times \Delta^2}\, \omega, \quad \partial_{\Delta^2} \overline{\partial}_{\Delta^2}\Big(\int_U \omega\Big) = \int_U \partial_{U \times \Delta^2} \overline{\partial}_{U \times \Delta^2}\, \omega. \tag{3.4}$$

c) Identify $\mathbf{C}^2$ with $\mathbf{R}^4$. Then we may regard $\Omega \subset\subset \mathbf{R}^4$. For $p \geqslant 1$, $L^p(\Omega)$ (resp. $L_{\mathrm{loc}}^p(\Omega)$) denotes the vector space of (resp. locally) L^p-integrable functions on Ω. When $p = \infty$, $L^\infty(\Omega)$ (resp. $L_{\mathrm{loc}}^\infty(\Omega)$) denotes the vector space of (resp. locally) bounded functions on Ω. For a multi-index $k = (k_1, \dots, k_4)$, $k_1, \dots, k_4 \geqslant 0$ and for a function $f \in L_{\mathrm{loc}}^p(\Omega)$, set $|k| = k_1 + \cdots + k_4$ and $D^k f(x) = \partial_{x_1}^{k_1} \cdots \partial_{x_4}^{k_4} f(x)$, where $D^k f$ is the derivative of f of order $|k|$ in the sense of distributions on Ω. Obviously, $D^k f \notin L_{\mathrm{loc}}^p(\Omega)$ in general. For a real number $1 \leqslant p < \infty$ and an integer $l \geqslant 1$, we define the Sobolev spaces $W^{l,p}(\Omega) \subset W_{\mathrm{loc}}^{l,p}(\Omega)$ by

$$\begin{aligned} W^{l,p}(\Omega) &:= \{f \in L^p(\Omega);\ D^k f \in L^p(\Omega) \text{ if } |k| \leqslant l\}, \\ W_{\mathrm{loc}}^{l,p}(\Omega) &:= \{f \in L_{\mathrm{loc}}^p(\Omega);\ D^k f \in L_{\mathrm{loc}}^p(\Omega) \text{ if } |k| \leqslant l\}. \end{aligned}$$

We refer to [**5**, Chap. 1-9] and [**6**, Chap. 3] for distributions, currents, Sobolev spaces, and the regularity theory of the Laplace operator.

3.2.2. Some lemmas. — Recall that $V = \{z \in U;\ \|z\| > \frac{3}{4}\}$ and that $\chi \in C_0^\infty(U)$ is a function such that $\chi \equiv 1$ on $U \setminus V = \{z \in U;\ \|z\| \leqslant \frac{3}{4}\}$ (*cf.* Proposition 3.3). Hence $\mathrm{Supp}(d\chi) \subset \overline{V}$. By (3.3), we have the following for all $w \in \Delta$:

$$\mathrm{Supp}(d\chi) \cap \Sigma_{F(\cdot,w)} \subset \overline{V} \cap \left\{z \in U;\ \|z\| \leqslant \frac{1}{2}\right\} = \varnothing. \tag{3.5}$$

Definition 3.5. — For $(\varepsilon, w) \in \Delta^* \times \Delta$, set

$$\begin{aligned} A(\varepsilon,w) &:= \int_U \chi(z) \log\left(\frac{|\varepsilon|^2}{\|d_zF(z,w)\|^2 + |\varepsilon|^2}\right) \mathcal{N}^*\omega_{\mathbf{P}^{n+1}}^{n+1}, \\ B(\varepsilon,w) &:= \int_U \chi(z) \log(\|d_zF(z,w)\|^2 + |\varepsilon|^2)\, \mathcal{N}^*\omega_{\mathbf{P}^{n+1}}^{n+1}. \end{aligned}$$

Then $A(\varepsilon,w)$ and $B(\varepsilon,w)$ are smooth functions on $\Delta^* \times \Delta$ such that $A(\varepsilon,0) = a(\varepsilon)$ and $B(\varepsilon,0) = b(\varepsilon)$. To establish (III), we study the regularities of $\partial_\Delta\overline{\partial}_\Delta A$ and $\partial_\Delta\overline{\partial}_\Delta B$. For this purpose, we introduce the following $(1,1)$-forms on $\Delta^* \times \Delta$:

Write $\partial = \partial_{U\times\Delta^2}$ and $\overline{\partial} = \overline{\partial}_{U\times\Delta^2}$ in what follows.

Definition 3.6. — For $(\varepsilon, w) \in \Delta^* \times \Delta$, set

$$\begin{aligned} K(\varepsilon,w) := {}& \frac{i}{2\pi}\int_U \log\left(\frac{|\varepsilon|^2}{\|d_zF(z,w)\|^2 + |\varepsilon|^2}\right) \partial\overline{\partial}\chi(z) \wedge \mathcal{N}^*\omega_{\mathbf{P}^{n+1}}^{n+1}(z,\varepsilon,w) \\ &+ \frac{i}{2\pi}\int_U \partial\chi(z) \wedge \overline{\partial}\log\left(\frac{|\varepsilon|^2}{\|d_zF(z,w)\|^2 + |\varepsilon|^2}\right) \wedge \mathcal{N}^*\omega_{\mathbf{P}^{n+1}}^{n+1}(z,\varepsilon,w) \\ &- \frac{i}{2\pi}\int_U \overline{\partial}\chi(z) \wedge \partial\log\left(\frac{|\varepsilon|^2}{\|d_zF(z,w)\|^2 + |\varepsilon|^2}\right) \wedge \mathcal{N}^*\omega_{\mathbf{P}^{n+1}}^{n+1}(z,\varepsilon,w), \end{aligned}$$

$$\begin{aligned} L(\varepsilon,w) := {}& \frac{i}{2\pi}\int_U \log(\|d_zF(z,w)\|^2 + |\varepsilon|^2)\, \partial\overline{\partial}\chi(z) \wedge \mathcal{N}^*\omega_{\mathbf{P}^{n+1}}^{n+1}(z,\varepsilon,w) \\ &+ \frac{i}{2\pi}\int_U \partial\chi(z) \wedge \overline{\partial}\log(\|d_zF(z,w)\|^2 + |\varepsilon|^2) \wedge \mathcal{N}^*\omega_{\mathbf{P}^{n+1}}^{n+1}(z,\varepsilon,w) \\ &- \frac{i}{2\pi}\int_U \overline{\partial}\chi(z) \wedge \partial\log(\|d_zF(z,w)\|^2 + |\varepsilon|^2) \wedge \mathcal{N}^*\omega_{\mathbf{P}^{n+1}}^{n+1}(z,\varepsilon,w). \end{aligned}$$

Then $K(\varepsilon,w)$ and $L(\varepsilon,w)$ are real smooth $(1,1)$-forms on $\Delta^* \times \Delta$ such that

$$\begin{aligned} K(\varepsilon,w) + L(\varepsilon,w) = {}& \left\{\frac{i}{2\pi}\int_U \partial\overline{\partial}\chi(z) \wedge \mathcal{N}^*\omega_{\mathbf{P}^{n+1}}^{n+1}\right\} \log|\varepsilon|^2 \\ &+ \frac{i}{2\pi}\int_U \left\{\partial\chi(z) \wedge \mathcal{N}^*\omega_{\mathbf{P}^{n+1}}^{n+1} \wedge \frac{d\overline{\varepsilon}}{\overline{\varepsilon}} - \overline{\partial}\chi(z) \wedge \mathcal{N}^*\omega_{\mathbf{P}^{n+1}}^{n+1} \wedge \frac{d\varepsilon}{\varepsilon}\right\}. \end{aligned} \tag{3.6}$$

Lemma 3.7. — *On $\varDelta^* \times \varDelta$, the following equations hold:*

$$(1)\quad \frac{i}{2\pi}\partial_{\varDelta^2}\overline{\partial}_{\varDelta^2}A(\varepsilon,w) = K(\varepsilon,w), \qquad (2)\quad \frac{i}{2\pi}\partial_{\varDelta^2}\overline{\partial}_{\varDelta^2}B(\varepsilon,w) = L(\varepsilon,w).$$

Proof

(1) By (3.4), we get

$$\begin{aligned}
\frac{i}{2\pi}\partial_{\varDelta^2}\overline{\partial}_{\varDelta^2}A(\varepsilon,w) &= \frac{i}{2\pi}\int_U \partial\overline{\partial}\left\{\chi(z)\log\left(\frac{|\varepsilon|^2}{\|d_zF(z,w)\|^2+|\varepsilon|^2}\right)\mathcal{N}^*\omega_{\mathbf{P}^{n+1}}^{n+1}\right\}\\
&= \frac{i}{2\pi}\int_U \chi(z)\,\partial\overline{\partial}\left\{\log\left(\frac{|\varepsilon|^2}{\|d_zF(z,w)\|^2+|\varepsilon|^2}\right)\right\}\wedge\mathcal{N}^*\omega_{\mathbf{P}^{n+1}}^{n+1}\\
&\quad+\frac{i}{2\pi}\int_U \log\left(\frac{|\varepsilon|^2}{\|d_zF(z,w)\|^2+|\varepsilon|^2}\right)\partial\overline{\partial}\chi(z)\wedge\mathcal{N}^*\omega_{\mathbf{P}^{n+1}}^{n+1}\\
&\quad+\frac{i}{2\pi}\int_U \partial\chi(z)\wedge\overline{\partial}\log\left(\frac{|\varepsilon|^2}{\|d_zF(z,w)\|^2+|\varepsilon|^2}\right)\wedge\mathcal{N}^*\omega_{\mathbf{P}^{n+1}}^{n+1}\\
&\quad-\frac{i}{2\pi}\int_U \overline{\partial}\chi(z)\wedge\partial\log\left(\frac{|\varepsilon|^2}{\|d_zF(z,w)\|^2+|\varepsilon|^2}\right)\wedge\mathcal{N}^*\omega_{\mathbf{P}^{n+1}}^{n+1}\\
&= \int_U -\chi(z)\,\mathcal{N}^*\omega_{\mathbf{P}^{n+1}}^{n+2} + K(\varepsilon,w) = K(\varepsilon,w),
\end{aligned}$$

where we used the equation $\partial\overline{\partial}\log|\varepsilon|^2 = 0$ on $\varDelta^* = \varDelta \smallsetminus \{0\}$ to get the third equality and the equation $\mathcal{N}^*\omega_{\mathbf{P}^{n+1}}^{n+2} \equiv 0$ to get the last one. This proves (1).

(2) Similarly, we can verify that

$$\frac{i}{2\pi}\partial_{\varDelta^2}\overline{\partial}_{\varDelta^2}B(\varepsilon,w) = \int_U \chi(z)\,\mathcal{N}^*\omega_{\mathbf{P}^{n+1}}^{n+2} + L(\varepsilon,w) = L(\varepsilon,w). \qquad \square$$

Lemma 3.8

(1) *L extends to a smooth $(1,1)$-form on $\varDelta^2$.*
(2) *There exist $\sigma,\tau \in A_{\varDelta^2}^{1,1}$ such that $K = \log|\varepsilon|^2\cdot\sigma + \tau$ on $\varDelta^*\times\varDelta$.*

Proof

(1) Since $\{(z,\varepsilon,w)\in U\times\varDelta^2;\ \varepsilon = d_zF(z,w) = 0\}\cap\mathrm{Supp}(d\chi) = \varnothing$ by (3.5) and since the indeterminacy locus of $\mathcal{N}$ and the singular locus of the function $\log(\|d_zF(z,w)\|^2 + |\varepsilon|^2)$ are given by $\{(z,\varepsilon,w)\in U\times\varDelta^2;\ \varepsilon = d_zF(z,w) = 0\}$,

$$\begin{aligned}
\Phi \;:=\; &\log(\|d_zF(z,w)\|^2+|\varepsilon|^2)\,\partial\overline{\partial}\chi\wedge\mathcal{N}^*\omega_{\mathbf{P}^{n+1}}^{n+1}\\
&+\partial\chi\wedge\overline{\partial}\log(\|d_zF(z,w)\|^2+|\varepsilon|^2)\wedge\mathcal{N}^*\omega_{\mathbf{P}^{n+1}}^{n+1}\\
&-\overline{\partial}\chi\wedge\partial\log(\|d_zF(z,w)\|^2+|\varepsilon|^2)\wedge\mathcal{N}^*\omega_{\mathbf{P}^{n+1}}^{n+1}
\end{aligned}$$

is well defined and is a smooth $(n+2,n+2)$-form on $U\times\varDelta^2$. Since $L = \frac{i}{2\pi}\int_U\Phi$, L is a smooth $(1,1)$-form on $\varDelta^2$. This proves (1).

(2) Similarly, since $\partial\chi\wedge\mathcal{N}^*\omega^{n+1}_{\mathbf{P}^{n+1}}\wedge\overline{\varepsilon}^{-1}d\overline{\varepsilon}$ and $\overline{\partial}\chi\wedge\mathcal{N}^*\omega^{n+1}_{\mathbf{P}^{n+1}}\wedge\varepsilon^{-1}d\varepsilon$ are smooth $(n+2,n+2)$-forms on $U\times\Delta^2$ by Lemma 3.4 and (3.5), we get

$$\int_U\partial\chi\wedge\mathcal{N}^*\omega^{n+1}_{\mathbf{P}^{n+1}}\wedge\frac{d\overline{\varepsilon}}{\overline{\varepsilon}}\in A^{1,1}_{\Delta^2},\qquad\int_U\overline{\partial}\chi\wedge\mathcal{N}^*\omega^{n+1}_{\mathbf{P}^{n+1}}\wedge\frac{d\varepsilon}{\varepsilon}\in A^{1,1}_{\Delta^2}.\tag{3.7}$$

By (3.6), (3.7), and $L\in A^{1,1}_{\Delta^2}$, we get $K(\varepsilon,w)-\frac{i}{2\pi}\{\int_U\partial\overline{\partial}\chi\wedge\mathcal{N}^*\omega^{n+1}_{\mathbf{P}^{n+1}}\}\cdot\log|\varepsilon|^2\in A^{1,1}_{\Delta^2}$. Since $\partial\overline{\partial}\chi\wedge\mathcal{N}^*\omega^{n+1}_{\mathbf{P}^{n+1}}$ is a smooth $(n+2,n+2)$-form on $U\times\Delta^2$ by (3.5), we get $\int_U\partial\overline{\partial}\chi\wedge\mathcal{N}^*\omega^{n+1}_{\mathbf{P}^{n+1}}\in A^{1,1}_{\Delta^2}$. This proves (2). □

Since the coefficients of K and L lie in $L^1_{\mathrm{loc}}(\Delta^2)$ by Lemma 3.8, K and L define real $(1,1)$-currents on Δ^2. By Lemma 3.7, they are d_{Δ^2}-closed on $\Delta^*\times\Delta$ in the ordinary sense.

Lemma 3.9. — *K and L are d_{Δ^2}-closed currents on Δ^2.*

Proof. — Since $d_{\Delta^2}L=0$ on $\Delta^*\times\Delta$ and since L is smooth on Δ^2 by Lemma 3.8 (1), L is a closed $(1,1)$-form on Δ^2. Let us prove that K is a d_{Δ^2}-closed current.

Let $\xi\in A^1_{\Delta^2}$ and assume that $\mathrm{Supp}(\xi)$ is compact. For $0<r<1$, set $\Delta(r)=\{\varepsilon\in\Delta;\,|\varepsilon|<r\}$. Since $d_{\Delta^2}K=0$ on $\Delta^*\times\Delta$, we obtain from Stokes' formula that

$$\int_{\Delta^2}K\wedge d_{\Delta^2}\xi=\lim_{r\to0}\int_{(\Delta\smallsetminus\Delta(r))\times\Delta}K\wedge d_{\Delta^2}\xi=-\lim_{r\to0}\int_{\partial\Delta(r)\times\Delta}K\wedge\xi.\tag{3.8}$$

Write $K=i\{K_{\varepsilon\overline{\varepsilon}}\,d\varepsilon\wedge d\overline{\varepsilon}+K_{\varepsilon\overline{w}}\,d\varepsilon\wedge d\overline{w}+K_{w\overline{\varepsilon}}\,dw\wedge d\overline{\varepsilon}+K_{w\overline{w}}\,dw\wedge d\overline{w}\}$ and set $|K|^2=|K_{\varepsilon\overline{\varepsilon}}|^2+|K_{\varepsilon\overline{w}}|^2+|K_{w\overline{\varepsilon}}|^2+|K_{w\overline{w}}|^2\in C^\infty(\Delta^*\times\Delta)$. We define the functions $|\xi|^2\in C^\infty_0(\Delta^2)$ and $|\sigma|^2,|\tau|^2\in C^\infty(\Delta^2)$ similarly. Then we have

$$\begin{aligned}\left|\int_{\partial\Delta(r)\times\Delta}K(\varepsilon,w)\wedge\xi(\varepsilon,w)\right|&\leqslant\int_0^{2\pi}\int_\Delta|K(r\,e^{i\theta},w)|\cdot|\xi(r\,e^{i\theta},w)|\,r\,d\theta\,dw\,d\overline{w}\\&\leqslant2\pi^3\Big(\sup_{\mathrm{Supp}(\xi)}|\sigma|\cdot\log r^2+\sup_{\mathrm{Supp}(\xi)}|\tau|\Big)\cdot\sup_{\Delta^2}|\xi|\cdot r\\&\to0\qquad(r\longrightarrow0),\end{aligned}\tag{3.9}$$

where we used Lemma 3.8 (2) to get the second line. Since ξ is an arbitrary test form, the result follows from (3.8), (3.9). □

Lemma 3.10

(1) *There exists a function $\alpha\in C^1(\Delta^2)\cap C^\infty(\Delta^*\times\Delta)$ such that $\frac{i}{2\pi}\partial_{\Delta^2}\overline{\partial}_{\Delta^2}\alpha=K$ in the sense of currents on Δ^2.*

(2) *There exists a function $\beta\in C^\infty(\Delta^2)$ such that $\frac{i}{2\pi}\partial_{\Delta^2}\overline{\partial}_{\Delta^2}\beta=L$.*

Proof

(1) Since K is a real closed $(1,1)$-current on Δ^2 by Lemma 3.9, it follows from the $\partial\overline{\partial}$-Poincaré lemma ([**14**, Proof of Lemma 5.4]) that there exists a *distribution* α on Δ^2 satisfying the equation of currents $\frac{i}{2\pi}\partial_{\Delta^2}\overline{\partial}_{\Delta^2}\alpha=K$ on Δ^2. Write $K=$

$i\{K_{\varepsilon\bar{\varepsilon}}\,d\varepsilon\wedge d\bar{\varepsilon}+K_{\varepsilon\overline{w}}\,d\varepsilon\wedge d\overline{w}+K_{w\bar{\varepsilon}}\,dw\wedge d\bar{\varepsilon}+K_{w\overline{w}}\,dw\wedge d\overline{w}\}$. Then we have the equation of distributions $\Box\alpha = 2\pi(K_{\varepsilon\bar{\varepsilon}}+K_{w\overline{w}})$ on Δ^2, where $\Box = \frac{\partial^2}{\partial\varepsilon\partial\bar{\varepsilon}}+\frac{\partial^2}{\partial w\partial\overline{w}}$ is the Laplacian. Let $\Omega \subset\subset \Delta^2$ be an arbitrary relatively compact domain. Since $K_{\varepsilon\bar{\varepsilon}}+K_{w\overline{w}} \in L^p(\Omega)$ for every $p>1$ by Lemma 3.8 (2), there exists a function $\widetilde{\alpha} \in W^{2,p}(\Omega)$ by [**5**, Th. 9.9] such that $\Box\widetilde{\alpha} = 2\pi(K_{\varepsilon\bar{\varepsilon}}+K_{w\overline{w}})$ on Ω. Then $\Box(\alpha|_\Omega - \widetilde{\alpha}) = 0$ in the sense of distributions on Ω. By [**6**, pp. 379, Lemma], $\alpha|_\Omega - \widetilde{\alpha}$ is a harmonic function on Ω. Hence $\alpha|_\Omega - \widetilde{\alpha} \in C^\omega(\Omega)$. Since $\Omega \subset\subset \Delta^2$ is arbitrary, we get $\alpha \in W^{2,p}_{\mathrm{loc}}(\Delta^2)$ for every $p>1$ and hence $\alpha \in C^1(\Delta^2)$ by the Sobolev embedding theorem $W^{2,p}_{\mathrm{loc}}(\Omega) \subset C^1(\Omega)$ $(p>4)$ (*cf.* [**5**, pp. 158, (7.30)]). Since $K_{\varepsilon\bar{\varepsilon}}+K_{w\overline{w}} \in C^\infty(\Delta^*\times\Delta)$ and $\Box\alpha = K_{\varepsilon\bar{\varepsilon}}+K_{w\overline{w}}$, we get $\alpha \in C^\infty(\Delta^*\times\Delta)$ by [**5**, Th. 6.17].

(2) Since $d_{\Delta^2}L = 0$ and $L \in A^{1,1}_{\Delta^2}$, the result follows from the $\partial\overline{\partial}$-Poincaré lemma. □

Lemma 3.11. — *Set* $C(n) = \int_{\mathbf{C}^{n+1}} \log(\|z\|^2+1)\{\frac{i}{2\pi}\partial\overline{\partial}\log(\|z\|^2+1)\}^{n+1} \in \mathbf{R}$. *Then the following identities hold for all* $\varepsilon \in \mathbf{C}\smallsetminus\{0\}$*:*

$$\text{(1)}\qquad \int_{\mathbf{C}^{n+1}} \log(\|z\|^2+|\varepsilon|^2)\left\{\frac{i}{2\pi}\partial\overline{\partial}\log(\|z\|^2+|\varepsilon|^2)\right\}^{n+1} = \log|\varepsilon|^2 + C(n),$$

$$\text{(2)}\qquad \int_{\mathbf{C}^{n+1}} \log\left(\frac{\|z\|^2+|\varepsilon|^2}{|\varepsilon|^2}\right)\left\{\frac{i}{2\pi}\partial\overline{\partial}\log(\|z\|^2+|\varepsilon|^2)\right\}^{n+1} = C(n).$$

Proof. — By setting $\zeta := \varepsilon^{-1}z$ and using $\int_{\mathbf{C}^{n+1}}\omega^{n+1}_{\mathbf{P}^{n+1}} = 1$, we can verify (1), (2). □

Lemma 3.12

(1) $A \in C^\infty(\Delta^*\times\Delta)$ *extends to a* C^1*-function on* Δ^2.

(2) $B - \mu(f)\log|\varepsilon|^2 \in C^\infty(\Delta^*\times\Delta)$ *extends to a* C^∞*-function on* Δ^2.

Proof. — Let $w \in \Delta^*$. Since $F(\cdot,w) \in \mathcal{O}(U)$ has only non-degenerate critical points, $(\frac{\partial F}{\partial z_0}(\cdot,w),\ldots,\frac{\partial F}{\partial z_n}(\cdot,w))$ is a system of coordinates around $\Sigma_{F(\cdot,w)}$. Hence there is a system of coordinates $(U_p,(u_0^{(p)},\ldots,u_n^{(p)}))$ around each critical point $p \in \Sigma_{F(\cdot,w)}$ such that $U_p\cap U_q = \varnothing$ $(p\neq q)$ and such that $\|d_zF(\cdot,w)\|^2 = \sum_{i=0}^n |u_i^{(p)}|^2$ on U_p.

(1) We have $A|_{\Delta^*\times\{w\}} \in L^\infty_{\mathrm{loc}}(\Delta)$ for every $w\in\Delta^*$ by Lemma 3.11 (2) because

$$\begin{aligned}A(\varepsilon,w) &= \int_{\|z\|<1}\chi(z)\log\left(\frac{|\varepsilon|^2}{\|d_zF(z,w)\|^2+|\varepsilon|^2}\right)\left\{\frac{i}{2\pi}\partial\overline{\partial}\log(\|d_zF(z,w)\|^2+|\varepsilon|^2)\right\}^{n+1}\\ &= \sum_{p\in\Sigma_{F(\cdot,w)}}\int_{U_p}\log\left(\frac{|\varepsilon|^2}{\|u^{(p)}\|^2+|\varepsilon|^2}\right)\left\{\frac{i}{2\pi}\partial\overline{\partial}\log(\|u^{(p)}\|^2+|\varepsilon|^2)\right\}^{n+1} + O(1)\\ &= O(1)\qquad(\varepsilon\longrightarrow 0).\end{aligned}$$

Hence $(A-\alpha)|_{\Delta^*\times\{w\}} \in L^\infty_{\mathrm{loc}}(\Delta)$ because $\alpha|_{\Delta\times\{w\}} \in C^1(\Delta)$ by Lemma 3.10 (1). Since $\partial_{\Delta^2}\overline{\partial}_{\Delta^2}(A-\alpha) = 0$ on $\Delta^*\times\Delta$ by Lemmas 3.7 (1) and 3.10 (1), $(A-\alpha)|_{\Delta^*\times\{w\}}$

is a harmonic function on Δ^*. By Riemann's removable singularities theorem, $(A-\alpha)|_{\Delta^*\times\{w\}}$ extends to a harmonic function on Δ.

Let $r \in (0,1)$ be an arbitrary number. Since $(A-\alpha)|_{\Delta\times\{w\}}$ is harmonic on Δ, we obtain from Poisson's formula ([**5**, Th. 2.6]) that for all $|\varepsilon|<r$ and $w \in \Delta^*$,

$$A(\varepsilon,w)-\alpha(\varepsilon,w)=\frac{1}{2\pi}\int_0^{2\pi}\{A(r\,e^{i\theta},w)-\alpha(r\,e^{i\theta},w)\}\frac{r^2-|\varepsilon|^2}{|r\,e^{i\theta}-\varepsilon|^2}\,d\theta,$$

which implies that $A-\alpha \in C^\infty(\Delta\times\Delta^*)$. This, together with Lemma 3.10 (1), yields that $A-\alpha \in C^\infty(\Delta\times\Delta^*)\cap C^\infty(\Delta^*\times\Delta) = C^\infty(\Delta^2\setminus\{(0,0)\})$. Hence $\partial_{\Delta^2}(A-\alpha)$ is a *holomorphic* 1-form on $\Delta^2\setminus\{(0,0)\}$ because $\overline{\partial}_{\Delta^2}\{\partial_{\Delta^2}(A-\alpha)\}=0$ on $\Delta^*\times\Delta$ by Lemmas 3.7 (1) and 3.10 (1). By Hartogs' principle, $\partial_{\Delta^2}(A-\alpha)$ extends to a holomorphic 1-form on Δ^2. Since $\ker\partial_{\Delta^2}$ consists of anti-holomorphic functions on Δ^2, we get $A-\alpha \in C^\omega(\Delta^2)$. Since $\alpha \in C^1(\Delta^2)$ by Lemma 3.10 (1), this implies that $A \in C^1(\Delta^2)$.

(2) We have $B|_{\Delta\times\{w\}}-\mu(f)\log|\varepsilon|^2 \in L^\infty_{\mathrm{loc}}(\Delta)$ by Lemma 3.11 (1) because

$$\begin{aligned}B(\varepsilon,w)&=\int_{\|z\|<1}\chi(z)\log(\|d_zF(z,w)\|^2+|\varepsilon|^2)\left\{\frac{i}{2\pi}\partial\overline{\partial}\log(\|d_zF(z,w)\|^2+|\varepsilon|^2)\right\}^{n+1}\\&=\sum_{p\in\Sigma_{F(\cdot,w)}}\int_{U_p}\log(\|u^{(p)}\|^2+|\varepsilon|^2)\left\{\frac{i}{2\pi}\partial\overline{\partial}\log(\|u^{(p)}\|^2+|\varepsilon|^2)\right\}^{n+1}+O(1)\\&=\#(\Sigma_{F(\cdot,w)})\log|\varepsilon|^2+O(1)=\mu(f)\log|\varepsilon|^2+O(1)\qquad(\varepsilon\longrightarrow 0).\end{aligned}$$

Here we used [**13**, pp.64 l.1-l.12] to get the last equality. Since we have the equation $\partial_{\Delta^2}\overline{\partial}_{\Delta^2}(B-\mu(f)\log|\varepsilon|^2-\beta)=0$ on $\Delta^*\times\Delta$ by Lemmas 3.7 (2) and 3.10 (2), the same argument as above using Riemann's removable singularities theorem, Poisson's formula, and Hartogs' principle, yields that $B-\mu(f)\log|\varepsilon|^2-\beta \in C^\omega(\Delta^2)$. Since $\beta \in C^\infty(\Delta^2)$ by Lemma 3.10 (2), we get $B-\mu(f)\log|\varepsilon|^2 \in C^\infty(\Delta^2)$. □

Proof of Proposition 3.3. — Since $a(\varepsilon)=A(\varepsilon,0)$ and $b(\varepsilon)=B(\varepsilon,0)$ by the definitions of $a(\varepsilon)$, $b(\varepsilon)$, $A(\varepsilon,w)$, $B(\varepsilon,w)$, the assertion follows from Lemma 3.12. □

Remark 3.13. — Theorem 3.1 seems to be similar to [**11**], [**12**]. However, no higher Milnor numbers appear in Theorem 3.1, since our proof is based on the "Picard-Lefschetz principle" (*cf.* [**15**, Th. 4.1]). Is it possible to derive Theorem 3.1 from [**11**], [**12**]?

4. Explicit formulas for the Chern forms around the critical point

As in Section 3, set $U=\Delta^{n+1}$ and let $f:(U,0)\to(\mathbf{C},0)$ be a holomorphic function such that $\Sigma_f=\{0\}$. We do not assume that f is surjective. The relative tangent bundle $Tf=\ker f_*$ is a holomorphic subbundle of $TU|_{U\setminus\{0\}}=T\mathbf{C}^{n+1}|_{U\setminus\{0\}}$. As in Section 2, let t be the coordinate of $\mathbf{C}$, which is the target of the map f.

Define the Hermitian metrics $g^{T\mathbf{C}^{n+1}}$, $g^{T\mathbf{C}}$, g^{Tf} on TU, $T\varDelta$, Tf, respectively by

$$g^{T\mathbf{C}^{n+1}} := \sum_{i=0}^{n} dz_i \otimes d\overline{z}_i, \qquad g^{T\mathbf{C}} := dt \otimes d\overline{t}, \qquad g^{Tf} := g^{T\mathbf{C}^{n+1}}|_{Tf}.$$

Let

$$\gamma : U \smallsetminus \{0\} \ni z \longrightarrow \left(\frac{\partial f}{\partial z_0}(z) : \cdots : \frac{\partial f}{\partial z_n}(z) \right) \in \mathbf{P}^n$$

be the *Gauss map*. Under the identification $H = \mathbf{P}^n$ as in Section 3.1, we have $\gamma = \nu|_{(U\smallsetminus\{0\})\times\{0\}}$ and $\gamma^*\omega_{\mathbf{P}^n} = \frac{i}{2\pi}\partial\overline{\partial}\log\|df\|^2$.

Proposition 4.1. — *The following equation of closed forms on $U \smallsetminus \{0\}$ holds:*

$$c(Tf, g^{Tf}) = \frac{1}{1+\gamma^*\omega_{\mathbf{P}^n}}. \tag{4.1}$$

In particular, for every polynomial $P(c) \in \mathbf{C}[c_1, \dots, c_n]$ and for every flat Hermitian vector bundle (F, h^F) on U, $P(Tf \oplus F, g^{Tf} \oplus h^F)^{\mathrm{top}}|_{U\smallsetminus\{0\}} = 0$.

Proof. — The equation (4.1) follows from [**15**, Lemma 2.1, (2.7), (2.13)]. Since $c_i(Tf, g^{Tf}) = (-1)^i\gamma^*\omega^i_{\mathbf{P}^n}$ by (4.1) and since the curvature of $(Tf \oplus F, g^{Tf} \oplus h^F)$ is given by $R^{Tf,g^{Tf}}$, we get $c_i(Tf \oplus F, g^{Tf} \oplus h^F) = (-1)^i\gamma^*\omega^i_{\mathbf{P}^n}$ $(i \geqslant 1)$ and hence

$$P(Tf \oplus F, g^{Tf} \oplus h^F)^{\mathrm{top}} = P(-t, \dots, (-t)^{n+1})|_{t^{n+1}} \cdot \gamma^*\omega^{n+1}_{\mathbf{P}^n} = 0. \qquad \square$$

Recall that $\Gamma_f \subset U \times \mathbf{C}$ is the graph of f. We identify U with Γ_f via the obvious projection $\mathrm{pr}_1 : \Gamma_f \to U$. Let $\delta \geqslant 0$. Define the Hermitian metric $g^{T\Gamma_f}$ on TU by

$$g^{T\Gamma_f} := (g^{T\mathbf{C}^{n+1}} \oplus \delta g^{T\mathbf{C}})|_{\Gamma_f}.$$

In this section, we regard ε as a *real* parameter again. For $\varepsilon > 0$, set

$$g^{TU}_\varepsilon := g^{T\Gamma_f} + \frac{1}{\varepsilon^2}f^*g^{T\mathbf{C}} = g^{T\mathbf{C}^{n+1}} + \left(\delta + \frac{1}{\varepsilon^2}\right) f^*g^{T\mathbf{C}}.$$

Proposition 4.2. — *For all $\varepsilon > 0$, the following equation of closed forms on U holds:*

$$c(TU, g^{TU}_\varepsilon) = \frac{1}{1 + \frac{i}{2\pi}\partial\overline{\partial}\log\left(\|df\|^2 + \frac{\varepsilon^2}{1+\varepsilon^2\delta}\right)}.$$

Proof. — Identify U with Γ_f. Let $N = N_{\Gamma_f/(U\times\mathbf{C})}$ be the normal bundle of Γ_f in $U \times \mathbf{C}$. Consider the following short exact sequence of holomorphic vector bundles on Γ_f,

$$0 \longrightarrow T\Gamma_f \longrightarrow T(U\times\mathbf{C})|_{\Gamma_f} \longrightarrow N \longrightarrow 0.$$

Let $g^{T(U\times\mathbf{C})}_\varepsilon$ be the Hermitian metric on $T(U\times\mathbf{C})$ defined by

$$g^{T(U\times\mathbf{C})}_\varepsilon := g^{T\mathbf{C}^{n+1}} \oplus (\delta + \varepsilon^{-2})g^{T\mathbf{C}}.$$

Then $g^{TU}_\varepsilon = g^{T(U\times\mathbf{C})}_\varepsilon|_{\Gamma_f}$. Let g^N_ε be the metric on N induced from $g^{T(U\times\mathbf{C})}_\varepsilon$ by the C^∞-isomorphism $N \cong (T\Gamma_f)^\perp$. Since $(T(U\times\mathbf{C}), g^{T(U\times\mathbf{C})}_\varepsilon)$ is a *flat* Hermitian vector bundle on $U\times\mathbf{C}$, we have $c(TU,g^{TU}_{\varepsilon})\wedge c(N,g^{N}_{\varepsilon})=1$ (*cf.* [**15**, Lemma 2.1, (2.6), (2.7)]). Hence

$$c(TU,g^{TU}_{\varepsilon})=\frac{1}{c(N,g^{N}_{\varepsilon})}=\frac{1}{1+c_1(N,g^{N}_{\varepsilon})}=\frac{1}{1-c_1(N^*,(g^{N}_{\varepsilon})^{-1})},\tag{4.2}$$

where N^* is the conormal bundle of Γ_f in $U\times\mathbf{C}$. Since N^* is generated by the global section $df(z)-dt$, we get

$$\begin{aligned}c_1(N^*,(g^{N}_{\varepsilon})^{-1})&=-\frac{i}{2\pi}\partial\overline{\partial}\log\|df(z)-dt\|^2_{\varepsilon}\\&=-\frac{i}{2\pi}\partial\overline{\partial}\log\{\|df\|^2+(\delta+\varepsilon^{-2})^{-1}\}\\&=-\frac{i}{2\pi}\partial\overline{\partial}\log\left\{\|df\|^2+\frac{\varepsilon^2}{1+\varepsilon^2\delta}\right\},\end{aligned}\tag{4.3}$$

where $\|\cdot\|_{\varepsilon}$ denotes the norm on $N^*\subset T^*(U\times\mathbf{C})$ with respect to the Hermitian metric induced from $g^{T(U\times\mathbf{C})}_{\varepsilon}$. The assertion follows from (4.2) and (4.3). □

5. Proof of the Main Theorem 2.2

5.1. The convergence of the curvature form outside Σ_f. — In this section, we keep the notation and the assumptions of Section 2.

Let $(Tf)^{\perp}\subset TX$ be the orthogonal complement of Tf in TX with respect to g^{TX}. Then $(Tf)^{\perp}$ is a C^{∞}-vector bundle on $X\setminus\Sigma_f$. Let $g^{(Tf)^{\perp}}$ be the Hermitian metric on $(Tf)^{\perp}$ induced from g^{TX}, *i.e.*, $g^{(Tf)^{\perp}}=g^{TX}|_{(Tf)^{\perp}}$. Under the C^{∞}-identification $f^*TS\cong(Tf)^{\perp}$ via the projection $f_*:TX\to f^*TS$, there exists a positive C^{∞}-function h on $X\setminus\Sigma_f$ such that

$$f^*g^{TS}=h\cdot g^{(Tf)^{\perp}}.$$

Then the C^{∞}-decomposition $TX|_{X\setminus\Sigma_f}\cong Tf\oplus(Tf)^{\perp}$ is orthogonal with respect to the Hermitian metrics

$$g^{TX}_{\varepsilon}=g^{Tf}\oplus(1+\varepsilon^{-2}h)\,g^{(Tf)^{\perp}}$$

for all $\varepsilon>0$. We define the family of positive functions $\{a_{\varepsilon}\}_{\varepsilon>0}$ on $X\setminus\Sigma_f$ by

$$a_{\varepsilon}=1+\varepsilon^{-2}h.$$

Let $A\in A^{1,0}_{X\setminus\Sigma_f}(\mathrm{Hom}(Tf,(Tf)^{\perp}))$ be the second fundamental form of the following exact sequence of holomorphic vector bundles on $X\setminus\Sigma_f$,

$$0\longrightarrow Tf\longrightarrow TX|_{X\setminus\Sigma_f}\longrightarrow f^*TS\longrightarrow 0,$$

with respect to the Hermitian metrics g^{Tf}, g^{TX}, $g^{(Tf)^{\perp}}$ on Tf, TX, $(Tf)^{\perp}$, respectively ([**10**, Chap. 1, Sect. 6]). Notice that A is independent of $\varepsilon>0$.

Proposition 5.1. — *As $\varepsilon \to 0$, the curvature $R^{TX,g_\varepsilon^{TX}}$ converges uniformly on every compact subset of $X \setminus \Sigma_f$ to the following matrix:*

$$R^{TX,g_\varepsilon^{TX}} = \begin{pmatrix} R^{Tf,g^{Tf}} & -(\partial A^* - \partial \log h \wedge A^*) \\ 0 & f^* R^{TS,g^{TS}} \end{pmatrix}.$$

Proof. — We follow [**3**, pp.37 l.1-l.15]. By a straightforward computation (*cf.* [**10**, Chap. I, (6.1)]), the curvature matrix of $(TX, g_\varepsilon^{TX})|_{X\setminus\Sigma_f}$ with respect to the orthogonal decomposition $TX = Tf \oplus (Tf)^\perp$ is given by

$$R^{TX,g_\varepsilon^{TX}} = \begin{pmatrix} R^{Tf,g^{Tf}} - \frac{1}{a_\varepsilon} A^* \wedge A & -(\partial A^* - \partial \log a_\varepsilon \wedge A^*) \\ \frac{1}{a_\varepsilon}(\overline{\partial} A - \overline{\partial} \log a_\varepsilon \wedge A) & R^{f^*TS,g^{(Tf)^\perp}} + \overline{\partial}\partial \log a_\varepsilon - \frac{1}{a_\varepsilon} A \wedge A^* \end{pmatrix}. \tag{5.1}$$

Then the assertion follows from (5.1) because we have the following uniform convergences on every compact subset of $X \setminus \Sigma_f$ as $\varepsilon \to 0$:

$$\frac{1}{a_\varepsilon} = \frac{\varepsilon^2}{\varepsilon^2 + h} \longrightarrow 0, \qquad \partial \log a_\varepsilon = \frac{\partial h}{\varepsilon^2 + h} \longrightarrow \partial \log h, \qquad \overline{\partial}\partial \log a_\varepsilon \longrightarrow \overline{\partial}\partial \log h$$

and also the identity $f^* R^{TS,g^{TS}} = R^{f^*TS,g^{(Tf)^\perp}} + \overline{\partial}\partial \log h$. ☐

5.2. Proof of the Main Theorem 2.2. — Since (f^*TS, f^*g^{TS}) is a flat line bundle on each U_p by Assumption 2.1 (2), the assertion (1) follows from Proposition 4.1. On $X \setminus \bigcup_{p\in\Sigma_f} U_p$, the assertion (2) follows from Proposition 5.1. Since $P(Tf \oplus f^*TS, g^{Tf} \oplus f^*g^{TS})^{\mathrm{top}}$ vanishes on $\bigcup_{p\in\Sigma_f} U_p \setminus \{p\}$ again by Proposition 4.1, it suffices to verify (2.2) on each U_p.

By Proposition 4.2, we have the following identities on U_p for $k = 1, \dots, n+1$:

$$c_k(TU_p, g_\varepsilon^{TU_p}) = (-1)^k \left\{ \frac{i}{2\pi} \partial\overline{\partial} \log \left(\|df\|^2 + \frac{\varepsilon^2}{1 + \varepsilon^2 \delta} \right) \right\}^k,$$

which yields that

$$\begin{aligned} & P(TU_p, g_\varepsilon^{TU_p})^{\mathrm{top}} \\ &= P\left(-\frac{i}{2\pi} \partial\overline{\partial} \log \left(\|df\|^2 + \frac{\varepsilon^2}{1 + \varepsilon^2 \delta} \right), \dots, \left\{ -\frac{i}{2\pi} \partial\overline{\partial} \log \left(\|df\|^2 + \frac{\varepsilon^2}{1 + \varepsilon^2 \delta} \right) \right\}^{n+1} \right)^{\mathrm{top}} \\ &= P(-t, \dots, (-t)^{n+1})|_{t^{n+1}} \cdot \left\{ \frac{i}{2\pi} \partial\overline{\partial} \log \left(\|df\|^2 + \frac{\varepsilon^2}{1 + \varepsilon^2 \delta} \right) \right\}^{n+1} \\ &\to P(-t, \dots, (-t)^{n+1})|_{t^{n+1}} \cdot \mu(f, p)\, \delta_p \qquad (\varepsilon \longrightarrow 0). \end{aligned}$$

Here we used Theorem 3.1 to get the last line. This completes the proof of Theorem 2.2. ☐

References

[1] D. Barlet – Développement asymptotique des fonctions obtenues par intégration dans la fibre, *Invent. Math.* **68** (1982), p. 129–174.

[2] J.-M. Bismut – Quillen metrics and singular fibers in arbitrary relative dimension, *J. Algebraic Geom.* **6** (1997), p. 19–149.

[3] J.-M. Bismut & J.-B. Bost – Fibrés déterminants, métriques de Quillen et dégénérescence des courbes, *Acta Math.* **165** (1990), p. 1–103.

[4] W. Fulton – *Intersection Theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Folge 3 Band 2, Springer-Verlag, 1984.

[5] D. Gilbarg & N.S. Trudinger – *Elliptic Partial Differential Equations of Second Order*, 2nd ed., Springer, Berlin, 1983.

[6] P.A. Griffiths & J. Harris – *Principles of Algebraic Geometry*, A. Wiley-Interscience, New York, 1978.

[7] F.R. Harvey & H.B. Lawson – *A theory of characteristic currents associated with a singular connection*, Astérisque, vol. 213, Société Mathématique de France, 1993.

[8] B. Iversen – Critical points of an algebraic function, *Invent. Math.* **12** (1971), p. 210–224.

[9] T. Izawa & T. Suwa – Multiplicity of functions on singular varieties, *Internat. J. Math.* **14** (2003), p. 541–558.

[10] S. Kobayashi – *Differential Geometry of Complex Vector Bundles*, Iwanami Shoten Publishers and Princeton University Press, 1987.

[11] R. Langevin – Courbure et singularités complexes, *Comment. Math. Helv.* **54** (1979), p. 6–16.

[12] F. Loeser – Formules intégrales pour certains invariants locaux des espaces analytiques complexes, *Comment. Math. Helv.* **59** (1984), p. 204–225.

[13] E. Looigenga – *Isolated Singular Points on Complete Intersections*, Cambridge University Press, 1984.

[14] Y.-T. Siu – Analyticity of sets associated to Lelong numbers, *Invent. Math.* **27** (1974), p. 53–156.

[15] K.-I. Yoshikawa – Smoothing of isolated hypersurface singularities and Quillen metrics, *Asian J. Math.* **2** (1998), p. 325–344.

A.Y. Yoshikawa, Graduate School of Humanities and Sciences, Ochanomizu University, Tokyo 112-8610, Japan • *E-mail :* `ayamada@math.nagoya-u.ac.jp`

K. Yoshikawa, Graduate School of Mathematical Sciences, University of Tokyo, Tokyo 153-8914, Japan • *E-mail :* `yosikawa@ms.u-tokyo.ac.jp`