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ON THE PICARD GROUP FOR NON-COMPLETE ALGEBRAIC VARIETIES

by

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Abstract. — In this paper we show some relations between the topology of a complex algebraic variety and its algebraic or analytic Picard group. Some of our results involve the subgroup of the Picard group whose elements have a trivial Chern class and the Néron-Severi group, quotient of the Picard group by this subgroup. We are also led to give results concerning their relations with the topology of the complex algebraic variety.

***Résumé* (Sur le groupe de Picard des variétés algébriques non complètes).** — Dans cet article, nous montrons quelques relations entre la topologie d'une variété algébrique complexe et son groupe de Picard algébrique ou analytique. Certains de nos résultats concernent le sous-groupe du groupe de Picard dont les éléments ont une classe de Chern triviale et le groupe de Néron-Severi, quotient du groupe de Picard par ce sous-groupe. Nous obtenons aussi des résultats sur leurs relations avec la topologie de la variété algébrique complexe.

1. Statements

Let X be a complex algebraic variety, *i.e.* a (sc. separated) integral (*i.e.* irreducible and reduced) scheme of finite type over $\operatorname{Spec}\mathbb{C}$. Then we have a corresponding complex space X^{an}. The notion of the Picard group exists in the category of complex algebraic varieties and in the category of complex spaces, since both algebraic varieties and complex spaces are locally ringed spaces. Recall that, for a locally ringed space, the Picard group is the group of isomorphism classes of invertible sheaves. For algebraic varieties it coincides with the Cartier divisor class group [**H**] II 6.15.

If X is complete, *i.e.* X^{an} is compact, both Picard groups are isomorphic to each other by the GAGA principle: $\operatorname{Pic} X \simeq \operatorname{Pic}_{(\mathrm{an})} X^{\mathrm{an}}$. If X is projective, this is a classical result of Serre [**S**]; for the general case see [**G2**] XII Th. 4.4. This is no longer true in general if X is not complete. This fact will be an easy consequence of Corollary

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1.3 below. A more interesting example is due to Serre, *cf.* [**H**] Appendix B 2.0.1; we thank the referee for drawing our attention to it: there are non-singular surfaces X_1 and X_2 such that $X_1^{\mathrm{an}} \simeq X_2^{\mathrm{an}}$ and $\operatorname{Pic} X_1 \not\simeq \operatorname{Pic} X_2$. So $\operatorname{Pic}_{(\mathrm{an})} X_1^{\mathrm{an}} \simeq \operatorname{Pic}_{(\mathrm{an})} X_2^{\mathrm{an}}$, X_1 is not isomorphic to X_2, and we cannot have $\operatorname{Pic} X_j \simeq \operatorname{Pic}_{(\mathrm{an})} X_j^{\mathrm{an}}$, $j = 1, 2$.

We will concentrate here upon the case where X is non-singular. Remember that we have a canonical mixed Hodge structure on the cohomology groups of X^{an} [**D1**].

As usual, if $(\mathbf{H}, F, W)$ is a mixed Hodge structure on $\mathbf{H}$, F is the Hodge filtration $\cdots \supset F^n\mathbf{H}_{\mathbb{C}} \supset F^{n+1}\mathbf{H}_{\mathbb{C}} \supset \ldots$ on $\mathbf{H}_{\mathbb{C}} := \mathbf{H} \otimes \mathbb{C}$ and W is the weight filtration on $\mathbf{H}_{\mathbb{Q}} := \mathbf{H} \otimes \mathbb{Q}$

$$\cdots \subset W_k\mathbf{H}_{\mathbb{Q}} \subset W_{k+1}\mathbf{H}_{\mathbb{Q}} \subset \ldots$$

We write

$$\operatorname{Gr}_\ell^W \mathbf{H}_{\mathbb{Q}} = W_\ell\mathbf{H}_{\mathbb{Q}}/W_{\ell-1}\mathbf{H}_{\mathbb{Q}} \text{ and } \operatorname{Gr}_F^i \mathbf{H}_{\mathbb{C}} = F^i\mathbf{H}_{\mathbb{C}}/F^{i+1}\mathbf{H}_{\mathbb{C}}.$$

Recall also that the Hodge filtration induces a filtration on each $\operatorname{Gr}_\ell^W \mathbf{H}_{\mathbb{C}}$.

In contrast to the approach of A. Grothendieck [**G1**] we apply transcendental methods which lead to results involving transversality conditions.

First let us study the question whether $\operatorname{Pic} X$ is trivial:

1.1. Theorem. — *Let X be a non-singular complex algebraic variety, assume that $\operatorname{Gr}_1^W H^1(X^{\mathrm{an}}; \mathbb{Q}) = 0$, $\operatorname{Gr}_F^1 \operatorname{Gr}_2^W H^2(X^{\mathrm{an}}; \mathbb{C}) = 0$ and $H^2(X^{\mathrm{an}}; \mathbb{Z})$ is torsion free. Then $\operatorname{Pic} X = 0$, i.e. every divisor on X is a principal divisor.*

Note that there is no difference between Weil and Cartier divisors here because X is supposed to be non-singular (see [**H**] Chap. II 6.11.1 A). Of course, this theorem shows that it is sufficient to suppose that $H^1(X^{\mathrm{an}}; \mathbb{Q}) = 0$ and $H^2(X^{\mathrm{an}}; \mathbb{Z}) = 0$ to obtain $\operatorname{Pic} X = 0$. In particular, it is not possible to distinguish X from the affine space $\mathbb{A}_n = \mathbb{A}_n(\mathbb{C}) := \operatorname{Spec} \mathbb{C}[x_1, \ldots, x_n]$ by the Picard group if X^{an} is contractible.

Conversely, we have:

1.2. Theorem. — *Let X be a non-singular complex algebraic variety and suppose $\operatorname{Pic} X = 0$. Then*

$$\operatorname{Gr}_1^W H^1(X^{\mathrm{an}}; \mathbb{Q}) = 0 \textit{ and } H^2(X^{\mathrm{an}}; \mathbb{Z}) \textit{ is torsion free.}$$

Then we also get the following easy consequence of both theorems:

1.3. Corollary. — *Let X be a non-complete non-singular irreducible complex curve, g the genus of its non-singular compactification $\overline{X}$. Then $g = 0$ if and only if the algebraic Picard group $\operatorname{Pic} X$ of X is trivial.*

Note, however, that the analytic Picard group $\operatorname{Pic}_{(\mathrm{an})} X^{\mathrm{an}}$ is always trivial in the case of a non-complete irreducible complex curve.

Now let us turn to the Picard group in the case where it is non-trivial. In general, the structure of the Picard group can be quite complicated but we have a comparison theorem:

1.4. Theorem. — *Let $f : Y \to X$ be a morphism between non-singular complex algebraic varieties. Suppose that the induced map*

$$H^k(X^{\mathrm{an}}; \mathbb{Z}) \longrightarrow H^k(Y^{\mathrm{an}}; \mathbb{Z})$$

is bijective for $k = 1, 2$. Then the natural map $\operatorname{Pic} X \to \operatorname{Pic} Y$ is bijective.

As a consequence, there is a theorem of Zariski-Lefschetz type, using a corresponding topological theorem [**HL**]. Let us state it in a slightly more general form, admitting singularities.

Now, X might be singular. Let $\operatorname{Cl} X$ be the Weil divisor class group of X and $\operatorname{Sing} X$ the singular locus of X.

1.5. Theorem. — *Let $\operatorname{Sing} X$ be of codimension $\geqslant 2$ in X and let $\overline{X}$ be a compactification of X to a projective variety embedded in $\mathbb{P}_m = \mathbb{P}_m(\mathbb{C})$. Let us fix a stratification of $\overline{X}$ such that X and $X \smallsetminus \operatorname{Sing} X$ are unions of strata. Let Z be a complete intersection in $\mathbb{P}_m$ which is non-singular along $\overline{X}$ and intersects all strata of $\overline{X}$ transversally in $\mathbb{P}_m$, and let $Y := X \cap Z$. Suppose $\dim Y \geqslant 3$. Then $\operatorname{Cl} X \simeq \operatorname{Cl} Y$. If X is affine, we have $\operatorname{Pic}_{(\mathrm{an})} X^{\mathrm{an}} \simeq \operatorname{Pic}_{(\mathrm{an})} Y^{\mathrm{an}}$, too.*

Note that Cl may be replaced by Pic if X is non-singular ([**H**] Chap. II 6.16).

1.6. Corollary. — *Suppose that X is a non-singular affine variety of dimension $\geqslant 3$ in $\mathbb{P}_m$. Then there is a linear subspace L of $\mathbb{P}_m$ such that $Y = X \cap L$ is non-singular, $\dim Y = 3$ and $\operatorname{Pic} X \simeq \operatorname{Pic} Y$, $\operatorname{Pic}_{(\mathrm{an})} X^{\mathrm{an}} \simeq \operatorname{Pic}_{(\mathrm{an})} Y^{\mathrm{an}}$.*

1.7. Corollary. — *Let Y be a non-singular closed subvariety of the affine space $\mathbb{A}_m$, $\dim Y \geqslant 3$. Assume that the closure $\overline{Y}$ in $\mathbb{P}_m$ is a non-singular complete intersection which is transversal to $\mathbb{P}_m \smallsetminus \mathbb{A}_m$. Then $\operatorname{Pic} Y = 1$, $\operatorname{Pic}_{(\mathrm{an})} Y^{\mathrm{an}} = 1$.*

In fact, this last corollary is a simultaneous consequence of Theorem 1.1 and 1.5, which justifies to treat both theorems here at the same time.

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2. Proofs of Theorems 1.1 and 1.2

Let X be a smooth complex algebraic variety of dimension n. Recall that we can attach to each invertible sheaf on X its first Chern class. This gives a homomorphism $\alpha : \operatorname{Pic} X \to H^2(X^{\mathrm{an}}; \mathbb{Z})$. Let $\operatorname{Pic}^0 X$ be the kernel.

Since X is separated there is a compactification $\overline{X}$ by Nagata [**N**]. Since X is smooth we can obtain by Hironaka [**Hi**] that $\overline{X}$ is smooth and that $\overline{X} \smallsetminus X$ is a divisor with normal crossings $D = D_1 \cup \cdots \cup D_r$, where the components $D_1, \ldots, D_r$ are smooth.

Recall that, for all k, $W_{k-1}H^k(X^{\mathrm{an}};\mathbb{Q})=0$, because X is non-singular, see [**D1**] 3.2.15.

2.1. Lemma. — *The canonical mapping* $H^1(\overline{X}^{\mathrm{an}};\mathbb{Q})\to H^1(X^{\mathrm{an}};\mathbb{Q})$ *is injective, the image is* $W_1H^1(X^{\mathrm{an}};\mathbb{Q})\simeq \mathrm{Gr}_1^W H^1(X^{\mathrm{an}};\mathbb{Q})$.

Proof. — Let us look at the exact sequence

$$H^1(\overline{X}^{\mathrm{an}},X^{\mathrm{an}};\mathbb{Q})\longrightarrow H^1(\overline{X}^{\mathrm{an}};\mathbb{Q})\longrightarrow H^1(X^{\mathrm{an}};\mathbb{Q})$$

By Lefschetz duality $H^1(\overline{X}^{\mathrm{an}},X^{\mathrm{an}};\mathbb{Q})$ is dual to the vector space $H^{2n-1}(D^{\mathrm{an}};\mathbb{Q})$ which vanishes because $\dim D=n-1$. This proves the injectivity.

On the other hand, the image of $H^1(\overline{X}^{\mathrm{an}};\mathbb{Q})\to H^1(X^{\mathrm{an}};\mathbb{Q})$ is $W_1H^1(X^{\mathrm{an}};\mathbb{Q})\simeq \mathrm{Gr}_1^W H^1(X^{\mathrm{an}};\mathbb{Q})$ by [**D1**] p. 39, Cor. 3.2.17. □

2.2. Proposition. — *The following conditions are equivalent:*

a) $\mathrm{Pic}\,X$ *is a finitely generated group,*
b) $\mathrm{Gr}_1^W H^1(X^{\mathrm{an}};\mathbb{Q})=0$,
c) $\mathrm{Pic}^0 X=0$.

Proof. — Let us first consider the case where X is complete. Since X is also supposed to be smooth, the mixed Hodge structure on $H^1(X^{\mathrm{an}};\mathbb{Q})$ is pure of weight 1, so

$$H^1(X^{\mathrm{an}};\mathbb{Q})=\mathrm{Gr}_1^W H^1(X^{\mathrm{an}};\mathbb{Q}).$$

Therefore b) is equivalent to the condition $b^1(X^{\mathrm{an}})=0$, where b^1 denotes the first Betti number. Now the latter can be expressed by the Hodge numbers: $b^1(X^{\mathrm{an}})=h^{01}(X^{\mathrm{an}})+h^{10}(X^{\mathrm{an}})=2h^{01}(X^{\mathrm{an}})$. Note that X^{an} need not be a Kähler manifold, since X might not be projective. Anyhow X is algebraic, and we have $h^{pq}(X^{\mathrm{an}})=\dim_{\mathbb{C}} H^q(X^{\mathrm{an}},\Omega^p_{X^{\mathrm{an}}})$ because of the definition of the Hodge filtration in general, see [**D1**] (2.2.3) et (2.3.7).

So b) is equivalent to the condition that $H^1(X^{\mathrm{an}},\mathcal{O}_{X^{\mathrm{an}}})=0$.

Now the exponential sequence leads to the following exact sequence:

$$H^1(X^{\mathrm{an}};\mathbb{Z})\longrightarrow H^1(X^{\mathrm{an}},\mathcal{O}_{X^{\mathrm{an}}})\longrightarrow \mathrm{Pic}\,X\xrightarrow{\alpha} H^2(X^{\mathrm{an}};\mathbb{Z}).$$

Here we use the fact that $\mathrm{Pic}\,X\simeq \mathrm{Pic}_{(\mathrm{an})}X^{\mathrm{an}}\simeq H^1(X^{\mathrm{an}},\mathcal{O}^*_{X^{\mathrm{an}}})$ by GAGA, because X is complete. Now $H^1(X^{\mathrm{an}};\mathbb{Z})$ and $H^2(X^{\mathrm{an}};\mathbb{Z})$ are finitely generated abelian groups, *i.e.* Noetherian $\mathbb{Z}$-modules. In particular, $\mathrm{Pic}\,X/\mathrm{Pic}^0 X$ is finitely generated.

a) $\Rightarrow$ b): By the exact sequence above, if $\mathrm{Pic}\,X$ is finitely generated, the cohomology group $H^1(X^{\mathrm{an}},\mathcal{O}_{X^{\mathrm{an}}})$ is also a finitely generated group. But, since we consider a complex vector space, it is a finitely generated group if and only if it is trivial.

b) $\Rightarrow$ c): follows from the surjectivity of $H^1(X^{\mathrm{an}},\mathcal{O}_{X^{\mathrm{an}}})\to \mathrm{Pic}^0 X$.

c) $\Rightarrow$ a) As said before, $\mathrm{Pic}\,X/\mathrm{Pic}^0 X$ is finitely generated.

This finishes the special case where X is complete.

Now let us turn to the general case. Since X and $\overline{X}$ are smooth we can replace the Picard group by the Weil divisor class group, so we have an exact sequence of the form

$$\mathbb{Z}^r \longrightarrow \operatorname{Pic}\overline{X} \longrightarrow \operatorname{Pic}X \longrightarrow 0$$

see [**H**] II Prop. 6.5, p. 133 in the case $r = 1$.

a) $\Rightarrow$ b): Since $\operatorname{Pic}X$ is finitely generated, the same holds for $\operatorname{Pic}\overline{X}$. By the first case, $H^1(\overline{X}^{\mathrm{an}}, \mathbb{Q}) = 0$. Now Lemma 2.1 yields $\mathrm{Gr}_1^W H^1(X^{\mathrm{an}}; \mathbb{Q}) = 0$.

b) $\Rightarrow$ c): By Lemma 2.1, $H^1(\overline{X}^{\mathrm{an}}; \mathbb{Q}) \simeq \mathrm{Gr}_1^W H^1(X^{\mathrm{an}}; \mathbb{Q}) = 0$, so

$$\operatorname{Pic}^0 \overline{X} = 0$$

by the first case applied to $\overline{X}$ which is complete. Now, let us consider the commutative diagram with exact rows:

$$\begin{tikzcd}
\mathbb{Z}^r \arrow[r] \arrow[d, "\wr"'] & \operatorname{Pic}\overline{X} \arrow[r] \arrow[d, "\overline{\alpha}"'] & \operatorname{Pic}X \arrow[r] \arrow[d, "\alpha"'] & 0 \arrow[d] \\
H^2(\overline{X}^{\mathrm{an}}, X^{\mathrm{an}}; \mathbb{Z}) \arrow[r] & H^2(\overline{X}^{\mathrm{an}}; \mathbb{Z}) \arrow[r] & H^2(X^{\mathrm{an}}; \mathbb{Z}) \arrow[r] & H^3(\overline{X}^{\mathrm{an}}, X^{\mathrm{an}}; \mathbb{Z})
\end{tikzcd}$$

Here we were allowed to put the right hand vertical arrow by a diagram chase. Since $\operatorname{Pic}^0 \overline{X} = 0$, we know that $\overline{\alpha}$ is injective. The five lemma shows therefore that α is also injective. This means that $\operatorname{Pic}^0 X = 0$.

c) $\Rightarrow$ a): This follows from the fact that $\operatorname{Pic}X/\operatorname{Pic}^0 X$ is finitely generated. $\square$

Proof of Theorem 1.1. — By Proposition 2.2 we have that $\operatorname{Pic}^0 X = 0$, so the natural mapping $\operatorname{Pic}X \to H^2(X^{\mathrm{an}}; \mathbb{Z})$ is injective.

If X is complete, we obtain an exact sequence

$$0 \longrightarrow \operatorname{Pic}X \longrightarrow H^2(X^{\mathrm{an}}; \mathbb{Z}) \longrightarrow H^2(X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}})$$

We can factorize the last map through $H^2(X^{\mathrm{an}}; \mathbb{C})$. The image P of $\operatorname{Pic}X$ in $H^2(X^{\mathrm{an}}; \mathbb{C})$ is obviously contained in the kernel of the map

$$H^2(X^{\mathrm{an}}; \mathbb{C}) \longrightarrow H^2(X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}}) \simeq \mathrm{Gr}_F^0 H^2(X^{\mathrm{an}}; \mathbb{C}).$$

This kernel is

$$U := F^1 H^2(X^{\mathrm{an}}; \mathbb{C})$$

because $H^2(X^{\mathrm{an}}; \mathbb{C}) = F^0(H^2(X^{\mathrm{an}}; \mathbb{C}))$ and

$$\mathrm{Gr}_F^0(H^2(X^{\mathrm{an}}; \mathbb{C})) = F^0(H^2(X^{\mathrm{an}}; \mathbb{C}))/F^1(H^2(X^{\mathrm{an}}; \mathbb{C})).$$

Since $\operatorname{Pic}X$ injects in $H^2(X^{\mathrm{an}}; \mathbb{Z})$, P is also invariant under conjugation, so P is contained in $U \cap \overline{U}$.

We observe that, by definition of the Hodge structure (see [**D1**] (B) of (2.2.1)) we have

$$U \cap \overline{U} \simeq \mathrm{Gr}_F^1 H^2(X^{\mathrm{an}}; \mathbb{C})$$

because $H^2(X^{\mathrm{an}};\mathbb{C}) = F^2(H^2(X^{\mathrm{an}};\mathbb{C})) \oplus \overline{F}^1(H^2(X^{\mathrm{an}};\mathbb{C}))$ which induces

$$F^1(H^2(X^{\mathrm{an}};\mathbb{C})) = F^2(H^2(X^{\mathrm{an}};\mathbb{C})) \oplus (F^1(H^2(X^{\mathrm{an}};\mathbb{C}) \cap \overline{F}^1(H^2(X^{\mathrm{an}};\mathbb{C})))$$

and gives

$$\mathrm{Gr}^1_F H^2(X^{\mathrm{an}};\mathbb{C}) \simeq (F^1(H^2(X^{\mathrm{an}};\mathbb{C}) \cap \overline{F}^1(H^2(X^{\mathrm{an}};\mathbb{C}))).$$

Since X is complete, $H^2(X^{\mathrm{an}};\mathbb{C}) = \mathrm{Gr}^W_2 H^2(X^{\mathrm{an}};\mathbb{C})$, so $P = 0$, which means that $\operatorname{Pic} X$ is a torsion group, but by hypothesis $H^2(X^{\mathrm{an}};\mathbb{Z})$ has no torsion, so $\operatorname{Pic} X = 0$.

If X is not necessarily complete, we get that $\operatorname{Pic} X$ is mapped to

$$V \cap \overline{V},$$

with $V := F^1 W_2 H^2(X^{\mathrm{an}};\mathbb{C})$, because of the commutative diagram with surjective upper row

$$\begin{array}{ccc}
\operatorname{Pic} \overline{X} & \longrightarrow & \operatorname{Pic} X \\
\big\downarrow & & \big\downarrow \\
U \cap \overline{U} & \longrightarrow & W_2 H^2(X^{\mathrm{an}};\mathbb{C})
\end{array}$$

where $U := F^1 H^2(\overline{X}^{\mathrm{an}};\mathbb{C})$.

As above we have by definition of the Hodge structure (see [**D1**] (B) of (2.2.1))

$$V \cap \overline{V} \simeq \mathrm{Gr}^1_F W_2 H^2(X^{\mathrm{an}};\mathbb{C}).$$

In fact, since X is non-singular, we have

$$W_2 H^2(X^{\mathrm{an}};\mathbb{C}) = \mathrm{Gr}^W_2 H^2(X^{\mathrm{an}};\mathbb{C}).$$

Since $\mathrm{Gr}^1_F \mathrm{Gr}^W_2 H^2(X^{\mathrm{an}};\mathbb{C}) = 0$ by hypothesis, we get that

$$\operatorname{Pic} X \subset \operatorname{Tor} H^2(X^{\mathrm{an}};\mathbb{Z}) = 0. \qquad \square$$

Proof of Theorem 1.2. — By Proposition 2.2 we get that

$$\mathrm{Gr}^W_1 H^1(X^{\mathrm{an}};\mathbb{Q}) = 0.$$

Let us look at the commutative diagram

$$\begin{array}{ccccccc}
\mathbb{Z}^r & \longrightarrow & \operatorname{Pic} \overline{X} & \longrightarrow & \operatorname{Pic} X & \longrightarrow & 0 \\
\wr\big\downarrow & & \overline{\alpha}\big\downarrow & & \alpha\big\downarrow & & \big\downarrow \\
H^2(\overline{X}^{\mathrm{an}}, X^{\mathrm{an}};\mathbb{Z}) & \longrightarrow & H^2(\overline{X}^{\mathrm{an}};\mathbb{Z}) & \longrightarrow & H^2(X^{\mathrm{an}};\mathbb{Z}) & \longrightarrow & H^3(\overline{X}^{\mathrm{an}}, X^{\mathrm{an}};\mathbb{Z})
\end{array}$$

Let z be a torsion element of $H^2(X^{\mathrm{an}};\mathbb{Z})$, so $nz = 0$ for some integer $n > 0$. The image of z in $H^3(\overline{X}^{\mathrm{an}}, X^{\mathrm{an}};\mathbb{Z})$ is a torsion element, too, but $H^3(\overline{X}^{\mathrm{an}}, X^{\mathrm{an}};\mathbb{Z})$ is without torsion, by the universal coefficient formula, because otherwise $H_2(\overline{X}^{\mathrm{an}}, X^{\mathrm{an}};\mathbb{Z}) \simeq H^{2n-2}(D^{\mathrm{an}};\mathbb{Z}) \simeq \mathbb{Z}^r$ would have torsion, which is a contradiction. So z is the image of some element $y \in H^2(\overline{X}^{\mathrm{an}};\mathbb{Z})$. Then ny is mapped to $nz = 0$, so ny is the image of some $x \in H^2(\overline{X}^{\mathrm{an}}, X^{\mathrm{an}};\mathbb{Z})$, hence ny has an inverse image in $\operatorname{Pic} \overline{X}$. So in the exponential sequence of $\overline{X}$, ny is mapped to 0 in $H^2(\overline{X}, \mathcal{O}_{\overline{X}})$. Therefore y is mapped to 0, too, because $H^2(\overline{X}, \mathcal{O}_{\overline{X}})$ is without torsion, which means that y has an inverse image by $\overline{\alpha}$ in $\operatorname{Pic}\overline{X}$, which implies that $z = 0$, since $\operatorname{Pic} X = 0$. Therefore, $H^2(X^{\mathrm{an}}; \mathbb{Z})$ has no torsion. □

Proof of Corollary 1.3. — Of course, since X is a non-complete curve, $H^2(X^{\mathrm{an}}, \mathbb{Z}) = 0$. By Lemma 2.1, we have

$$\mathrm{Gr}_1^W H^1(X^{\mathrm{an}}; \mathbb{Q}) \simeq H^1(\overline{X^{\mathrm{an}}}; \mathbb{Q}) = 0 \qquad (\text{resp.} \neq 0)$$

if $g = 0$ (resp. $g > 0$). This implies our statement. □

More complicated examples can be constructed using the theory of toric varieties. Using the results of [**DK**] one can calculate the mixed Hodge numbers for X if $\overline{X}$ is a non-degenerate complete intersection, in particular one can calculate the dimension of

$$\mathrm{Gr}_1^W H^1(X^{\mathrm{an}}; \mathbb{Q}) \simeq H^1(\overline{X}^{\mathrm{an}}; \mathbb{Q}),$$

see Lemma 2.1.

An easier example is given by the Cartesian product of two non-complete nonsingular curves: $X := X_1 \times X_2$, where $\overline{X}_j$ has the genus g_j. Here to get $\operatorname{Pic} X = 0$, we have to decide whether $b^1(\overline{X}_1 \times \overline{X}_2) = 0$. By the Künneth formula, $b^1(\overline{X}_1 \times \overline{X}_2) = 2(g_1 + g_2)$. So to get $\operatorname{Pic} X = 0$, it is necessary that $g_1 = g_2 = 0$. The condition is obviously sufficient, too. Recall that the Künneth formula respects the mixed Hodge structures [**D2**] 8.2.10.

As for the analytic Picard group that we denote here by $\mathrm{Pic}_{(\mathrm{an})}$ to avoid possible confusions, we have the following obvious lemma, using the exponential sequence. Note that in contrast to algebraic varieties the integral cohomology of a complex space may not be finitely generated.

2.3. Lemma. — *Let $\mathfrak{X}$ be a complex space such that the cohomology groups $H^1(\mathfrak{X}; \mathbb{Z})$ and $H^2(\mathfrak{X}; \mathbb{Z})$ are finitely generated. Then the following conditions are equivalent:*

a) $H^1(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) = 0$,
b) $\mathrm{Pic}^0_{(\mathrm{an})}\ \mathfrak{X} = 0$,
c) $\mathrm{Pic}_{(\mathrm{an})}\ \mathfrak{X}$ *is finitely generated.*

For the triviality of the analytic Picard group we have the following criterion:

2.4. Lemma. — *Let $\mathfrak{X}$ be a complex space such that $H^1(\mathfrak{X}; \mathbb{Z})$ is finitely generated. The following conditions are equivalent:*

a) $H^1(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) = 0$ *and the mapping* $H^2(\mathfrak{X}; \mathbb{Z}) \to H^2(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ *is injective,*
b) $\mathrm{Pic}_{(\mathrm{an})}\ \mathfrak{X} = 0$.

The case of Stein spaces is quite easy:

2.5. Lemma. — *If $\mathfrak{X}$ is a Stein space the map $\mathcal{L} \mapsto c_1(\mathcal{L})$ induces an isomorphism* $\mathrm{Pic}_{(\mathrm{an})}\ \mathfrak{X} \to H^2(\mathfrak{X}; \mathbb{Z})$.

Proof. — Use the exponential sequence. □

In particular, X^{an} is Stein if X is a non-complete curve. Therefore we have $\operatorname{Pic}_{(\mathrm{an})} X^{\mathrm{an}} = 0$, independently of the genus g, in contrast to $\operatorname{Pic} X$ (see Corollary 1.3).

Note that in general $\operatorname{Pic}_{(\mathrm{an})} X^{\mathrm{an}}$ need not be simpler than $\operatorname{Pic} X$.

For instance, if $X = X_1 \times \overline{X}_2$, where X_1 and X_2 are curves chosen as above, we have an exact sequence

$$\mathbb{Z}^{2g_1+2g_2+c-1} \longrightarrow \Gamma(X_1^{\mathrm{an}}, \mathcal{O}_{X_1^{\mathrm{an}}})^{g_2} \longrightarrow \operatorname{Pic}_{(\mathrm{an})} X^{\mathrm{an}} \longrightarrow \mathbb{Z}^{(2g_1+c-1)2g_2+1}$$

where $c := \#(\overline{X}_1 \smallsetminus X_1)$, so $\operatorname{Pic}_{(\mathrm{an})} X^{\mathrm{an}}$ has to be very large. Note that

$$H^1(X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}}) = \Gamma(X_1^{\mathrm{an}}, R^1 p_*^{\mathrm{an}} \mathcal{O}_{X^{\mathrm{an}}}) = \Gamma(X_1^{\mathrm{an}}, \mathcal{O}_{X_1^{\mathrm{an}}})^{g_2},$$

by using Grauert's continuity theorem (see Theorem 4.12 p. 134 of [**BS**]), $p : X \to X_1$ being the projection. So $\dim_{\mathbb{C}} H^1(X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}}) = \infty$ if $g_2 > 0$.

As for Corollary 1.7, note that by the topological Lefschetz-Zariski theorem we have $H^k(Y^{\mathrm{an}}; \mathbb{Z}) \simeq H^k(\mathbb{C}^m; \mathbb{Z}) = 0$, $k = 1, 2$. Therefore the Corollary follows from Theorem 1.1 resp. Lemma 2.5.

3. The Néron-Severi group

Let X be a non-singular complex algebraic variety. Let $\operatorname{Pic}^0 X$ be the kernel of the morphism

$$\alpha : \operatorname{Pic} X \longrightarrow H^2(X^{\mathrm{an}}; \mathbb{Z}).$$

We denote the Néron-Severi group of X by $\mathrm{NS}(X) := \operatorname{Pic} X / \operatorname{Pic}^0 X$. It is isomorphic to the image of the Chern class homomorphism α, hence it is a finitely generated abelian group. The following result should be compared with [**J**] Theorem 5.13.

3.1. Theorem. — *The Chern class homomorphism α induces an isomorphism of the Néron-Severi group $\mathrm{NS}(X)$ of X with $\gamma^{-1}(V)$, where*

$$\gamma : H^2(X^{\mathrm{an}}; \mathbb{Z}) \longrightarrow H^2(X^{\mathrm{an}}; \mathbb{C})$$

is the canonical mapping, $V := F^1 W_2 H^2(X^{\mathrm{an}}; \mathbb{C})$.

Proof: a) Let us treat first the case where X is compact. Then the exponential sequence leads to the exact sequence:

$$\operatorname{Pic} X \longrightarrow H^2(X^{\mathrm{an}}; \mathbb{Z}) \longrightarrow H^2(X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}})$$

The image of α coincides with the kernel of the map

$$H^2(X^{\mathrm{an}}; \mathbb{Z}) \longrightarrow H^2(X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}})$$

which can be factorized through $H^2(X^{\mathrm{an}}; \mathbb{C})$. Now

$$H^2(X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}}) \simeq \mathrm{Gr}_F^0 H^2(X^{\mathrm{an}}; \mathbb{C}) \simeq H^2(X^{\mathrm{an}}; \mathbb{C}) / F^1 H^2(X^{\mathrm{an}}; \mathbb{C}),$$

so the kernel in question coincides with $\gamma^{-1}(F^1H^2(X^{\mathrm{an}};\mathbb{C}))$ which is our assertion in this case.

Now let us turn to the general case. We take the notations of the proof of Proposition 2.2. Let us consider the commutative diagram with exact rows:

$$\begin{tikzcd}
\mathbb{Z}^r \arrow[r] \arrow[d, "\wr"'] & \operatorname{Pic}\overline{X} \arrow[r] \arrow[d, "\overline{\alpha}"'] & \operatorname{Pic}X \arrow[r] \arrow[d, "\alpha"'] & 0 \arrow[d] \\
H^2(\overline{X}^{\mathrm{an}}, X^{\mathrm{an}};\mathbb{Z}) \arrow[r] & H^2(\overline{X}^{\mathrm{an}};\mathbb{Z}) \arrow[r] & H^2(X^{\mathrm{an}};\mathbb{Z}) \arrow[r] & H^3(\overline{X}^{\mathrm{an}}, X^{\mathrm{an}};\mathbb{Z})
\end{tikzcd}$$

Here we were allowed to put the right hand vertical arrow by a diagram chase. We can apply our preliminary result to $\overline{X}$ and deduce that the image of $\overline{\alpha}$ is $\overline{\gamma}^{-1}(U)$ where $\overline{\gamma}: H^2(\overline{X}^{\mathrm{an}};\mathbb{Z}) \to H^2(\overline{X}^{\mathrm{an}};\mathbb{C})$ is the canonical mapping and $U := F^1H^2(\overline{X}^{\mathrm{an}};\mathbb{C})$. If we tensorize the diagram above with $\mathbb{C}$ we get a commutative diagram with exact rows:

$$\begin{tikzcd}
\mathbb{C}^r \arrow[r] \arrow[d, "\wr"'] & \operatorname{Pic}\overline{X}\otimes_{\mathbb{Z}}\mathbb{C} \arrow[r] \arrow[d] & \operatorname{Pic}X\otimes_{\mathbb{Z}}\mathbb{C} \arrow[r] \arrow[d] & 0 \arrow[d] \\
H^2(\overline{X}^{\mathrm{an}}, X^{\mathrm{an}};\mathbb{C}) \arrow[r] & H^2(\overline{X}^{\mathrm{an}};\mathbb{C}) \arrow[r] & H^2(X^{\mathrm{an}};\mathbb{C}) \arrow[r] & H^3(\overline{X}^{\mathrm{an}}, X^{\mathrm{an}};\mathbb{C})
\end{tikzcd}$$

Note that $H^2(\overline{X}^{\mathrm{an}}, X^{\mathrm{an}};\mathbb{Q})$ is dual to $H^{2n-2}(D^{\mathrm{an}};\mathbb{Q}) \simeq \oplus_{j=1}^r H^{2n-2}(D_j^{\mathrm{an}};\mathbb{Q})$, so, by the Theorem 1.7.1 of [**F**], the mixed Hodge structure on $H^2(\overline{X}^{\mathrm{an}}, X^{\mathrm{an}};\mathbb{Q})$ is pure of weight 1, the corresponding Hodge numbers being trivial except maybe h^{11}. So

$$H^2(\overline{X}^{\mathrm{an}}, X^{\mathrm{an}};\mathbb{C}) = F^1W_2H^2(\overline{X}^{\mathrm{an}}, X^{\mathrm{an}};\mathbb{C}).$$

By the first case we have that the image of

$$\operatorname{Pic}\overline{X}\otimes_{\mathbb{Z}}\mathbb{C} \longrightarrow H^2(\overline{X}^{\mathrm{an}};\mathbb{C})$$

is contained in $F^1H^2(\overline{X}^{\mathrm{an}};\mathbb{C}) = F^1W_2H^2(\overline{X}^{\mathrm{an}};\mathbb{C})$. Since $\overline{X}$ is smooth, we have $W_kH^3(\overline{X}^{\mathrm{an}}, X^{\mathrm{an}};\mathbb{C}) = 0$ for $k < 3$. So we get an induced diagram with exact rows

$$\begin{tikzcd}
\mathbb{C}^r \arrow[r] \arrow[d, "\wr"'] & \operatorname{Pic}\overline{X}\otimes_{\mathbb{Z}}\mathbb{C} \arrow[r] \arrow[d] & \operatorname{Pic}X\otimes_{\mathbb{Z}}\mathbb{C} \arrow[r] \arrow[d] & 0 \\
F^1W_2H^2(\overline{X}^{\mathrm{an}}, X^{\mathrm{an}};\mathbb{C}) \arrow[r] & F^1W_2H^2(\overline{X}^{\mathrm{an}};\mathbb{C}) \arrow[r] & F^1W_2H^2(X^{\mathrm{an}};\mathbb{C}) \arrow[r] & 0
\end{tikzcd}$$

In particular, this shows that the image of α is contained in $\gamma^{-1}(V)$.

Let us prove that every element of $\gamma^{-1}(V)$ is contained in the image of α. Consider the commutative diagram

$$\begin{tikzcd}
H^2(X^{\mathrm{an}};\mathbb{Z}) \arrow[r] \arrow[d] & H^3(\overline{X}^{\mathrm{an}}, X^{\mathrm{an}};\mathbb{Z}) \arrow[d] \\
H^2(X^{\mathrm{an}};\mathbb{C}) \arrow[r] & H^3(\overline{X}^{\mathrm{an}}, X^{\mathrm{an}};\mathbb{C})
\end{tikzcd}$$

Now $H^3(\overline{X}^{\mathrm{an}}, X^{\mathrm{an}};\mathbb{Z})$ is without torsion. Otherwise, by the universal coefficient formula, we would have torsion in

$$H_2(\overline{X}^{\mathrm{an}}, X^{\mathrm{an}};\mathbb{Z}) \simeq H^{2n-2}(D^{\mathrm{an}};\mathbb{Z}) \simeq \bigoplus_{i=1}^{r} H^{2n-2}(D_i^{\mathrm{an}};\mathbb{Z}) \simeq \mathbb{Z}^r,$$

which is a contradiction.

This shows that the right vertical arrow is injective.

Let $z \in \gamma^{-1}(V)$. Then z is mapped to an element $z_{\mathbb{C}}$ of $V = F^1W_2H^2(X^{\mathrm{an}};\mathbb{C})$, so it is mapped to 0 in $H^3(\overline{X}^{\mathrm{an}}, X^{\mathrm{an}};\mathbb{C})$, since $W_2H^3(\overline{X}^{\mathrm{an}}, X^{\mathrm{an}};\mathbb{C}) = 0$. By the preceding remark, this implies that z is mapped to 0 in $H^3(\overline{X}^{\mathrm{an}}, X^{\mathrm{an}};\mathbb{Z})$. So z has a preimage y in $H^2(\overline{X}^{\mathrm{an}};\mathbb{Z})$. Let $y_{\mathbb{C}}$ be the image of y in $H^2(\overline{X}^{\mathrm{an}};\mathbb{C})$. On the other hand, $z_{\mathbb{C}}$ has a preimage y' in U. So the element $y'-y_{\mathbb{C}}$ is mapped to 0 in $H^2(X^{\mathrm{an}};\mathbb{C})$, which implies that it is the image of some element $x \in H^2(\overline{X}^{\mathrm{an}}, X^{\mathrm{an}};\mathbb{C})$. But as we saw this space coincides with $F^1W_2H^2(\overline{X}^{\mathrm{an}}, X^{\mathrm{an}};\mathbb{C})$, so $y' - y_{\mathbb{C}} \in U$, hence $y_{\mathbb{C}} \in U$, which means

$$y \in \gamma^{-1}(U).$$

By our preliminary result, we can find an element in $\operatorname{Pic}\overline{X}$, whose image in $\operatorname{Pic} X$ is mapped to z. So z is in the image of α.

3.2. Corollary. — *Let X be a non-singular complex algebraic variety.*

a) $\operatorname{rk}\operatorname{NS} X \leqslant \dim_{\mathbb{C}} \operatorname{Gr}_F^1\operatorname{Gr}_2^W H^2(X^{\mathrm{an}};\mathbb{C})$,

b) α *induces an isomorphism* $\operatorname{Tor}(\operatorname{NS} X) \simeq \operatorname{Tor} H^2(X^{\mathrm{an}};\mathbb{Z})$.

Proof. — a) We saw from the proof of Theorem 1.1 that the image of $\operatorname{Pic} X$ is contained in $V \cap \overline{V}$ and also in $H^2(X^{\mathrm{an}};\mathbb{R})$. Since X is smooth, we have that $W_1H^2(X^{\mathrm{an}};\mathbb{Q}) = 0$, so $W_2H^2(X^{\mathrm{an}};\mathbb{Q}) = \operatorname{Gr}_2^W H^2(X^{\mathrm{an}};\mathbb{Q})$ has a pure Hodge structure of weight 2. In the proof of Theorem 1.1 we found that $V \cap \overline{V} \simeq \operatorname{Gr}_F^1 W_2H^2(X^{\mathrm{an}};\mathbb{C})$, so $V \cap \overline{V} \simeq \operatorname{Gr}_F^1\operatorname{Gr}_2^W H^2(X^{\mathrm{an}};\mathbb{C})$.

Now let $\dim_{\mathbb{C}} V \cap \overline{V} = k := \dim_{\mathbb{C}} \operatorname{Gr}_F^1\operatorname{Gr}_2^W H^2(X^{\mathrm{an}};\mathbb{C})$. This implies that

$$\dim_{\mathbb{R}} V \cap \overline{V} \cap H^2(X^{\mathrm{an}};\mathbb{R}) = k.$$

(This is due to some general fact of linear algebra: Let W be a k-dimensional complex linear subspace of $\mathbb{C}^n$ such that $W = \overline{W}$. Then $\dim_{\mathbb{R}} W \cap \mathbb{R}^n = k$. Let $f : W \to W$ be the conjugation map $z \mapsto \overline{z}$. Since $f^2 = \mathrm{id}_W$ the minimal polynomial divides $X^2 - 1$, so W is the direct sum of the eigenspaces W_1 and W_{-1} corresponding to the eigenvalues ± 1. Of course, $W_1 = W \cap \mathbb{R}^n$, $W_{-1} = W \cap i\mathbb{R}^n$, so the multiplication by i defines a (real) isomorphism of W_1 onto W_{-1}, so $\dim_{\mathbb{R}} W_1 = k$.)

Note that the image of $H^2(X^{\mathrm{an}};\mathbb{Z})$ in $H^2(X^{\mathrm{an}};\mathbb{C})$ is discrete, the same holds for the image of $\operatorname{Pic} X$ which is a discrete subgroup of a real vector space of real dimension k. Since $\operatorname{NS} X \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\operatorname{Pic} X \otimes_{\mathbb{Z}} \mathbb{Q}$ have the same image in $H^2(X^{\mathrm{an}};\mathbb{C})$, $\operatorname{NS} X \otimes_{\mathbb{Z}} \mathbb{Q}$ is also embedded in a k dimensional real vector space, so we obtain that $\operatorname{rk}\operatorname{NS} X \leqslant k$.

b) This is an obvious consequence because the torsion of $H^2(X^{\mathrm{an}};\mathbb{Z})$ is mapped to 0 in $H^2(X^{\mathrm{an}};\mathbb{C})$, so it belongs to $\gamma^{-1}(V)$. □

3.3. Corollary. — *Assume that $f: Y \to X$ is a morphism of smooth algebraic varieties such that f induces an injective (resp. bijective) mapping $H^2(X^{\mathrm{an}};\mathbb{Z}) \to H^2(Y^{\mathrm{an}};\mathbb{Z})$. Then the natural mapping $\mathrm{NS}\, X \to \mathrm{NS}\, Y$ is injective (resp. bijective).*

Of course, the injectivity is obvious.

For any analytic space $\mathfrak{X}$, define $\mathrm{NS}^{\mathrm{an}}\,\mathfrak{X}$ as the quotient of $\mathrm{Pic}_{(\mathrm{an})}\,\mathfrak{X}$ by $\mathrm{Pic}^0_{(\mathrm{an})}\,\mathfrak{X}$. Using Theorem 3.1 we get

3.4. Proposition. — *Let X be a smooth algebraic variety. Then the mappings $\mathrm{NS}\, X \to \mathrm{NS}^{\mathrm{an}}\, X^{\mathrm{an}} \to H^2(X^{\mathrm{an}};\mathbb{Z})$ are injective, they induce isomorphisms $\mathrm{Tor}(\mathrm{NS}\, X) \simeq \mathrm{Tor}(\mathrm{NS}^{\mathrm{an}}\, X^{\mathrm{an}}) \simeq \mathrm{Tor}\, H^2(X^{\mathrm{an}};\mathbb{Z})$.*

Proof. — We know that the mappings $\mathrm{NS}\, X \to H^2(X^{\mathrm{an}};\mathbb{Z})$ is injective, as well as

$$\mathrm{NS}^{\mathrm{an}}\, X^{\mathrm{an}} \longrightarrow H^2(X^{\mathrm{an}};\mathbb{Z})$$

since, by definition, $\mathrm{Pic}^0_{(\mathrm{an})}\, X^{\mathrm{an}}$ is the kernel of the map of $\mathrm{Pic}_{(\mathrm{an})}\, X^{\mathrm{an}}$ into $H^2(X^{\mathrm{an}};\mathbb{Z})$, so $\mathrm{NS}\, X \to \mathrm{NS}^{\mathrm{an}}\, X^{\mathrm{an}}$ is also injective. In particular we get injective mappings $\mathrm{Tor}(\mathrm{NS}\, X) \to \mathrm{Tor}(\mathrm{NS}^{\mathrm{an}}\, X^{\mathrm{an}}) \to \mathrm{Tor}\, H^2(X^{\mathrm{an}};\mathbb{Z})$. Since $\mathrm{Tor}(\mathrm{NS}\, X) \to \mathrm{Tor}\, H^2(X^{\mathrm{an}};\mathbb{Z})$ is bijective by Corollary 3.2, the mapping $\mathrm{Tor}(\mathrm{NS}^{\mathrm{an}}\, X^{\mathrm{an}}) \to \mathrm{Tor}\, H^2(X^{\mathrm{an}};\mathbb{Z})$ is surjective, hence bijective, and so is

$$\mathrm{Tor}(\mathrm{NS}\, X) \longrightarrow \mathrm{Tor}(\mathrm{NS}^{\mathrm{an}}\, X^{\mathrm{an}}).$$ □

4. The group $\mathrm{Pic}^0 X$

Again, let X be a non-singular complex algebraic variety. Now let us consider $\mathrm{Pic}^0 X$. It seems quite difficult to say too much about this group if X is not compact:

4.1. Proposition. — *We have:*

a) $\mathrm{Pic}^0 X \simeq \mathbb{C}^k/H$, *where $k = \dim \mathrm{Gr}^0_F \mathrm{Gr}^W_1 H^1(X^{\mathrm{an}};\mathbb{C})$ and H is a free abelian subgroup of rank $\leqslant s := \mathrm{rk}\, H^1(X^{\mathrm{an}};\mathbb{Z})$.*

b) *If $\mathrm{Gr}^W_2 H^1(X^{\mathrm{an}};\mathbb{Q}) = 0$ the image H of $H^1(X^{\mathrm{an}};\mathbb{Z}) \to \mathrm{Gr}^0_F H^1(X^{\mathrm{an}};\mathbb{C})$ is a lattice, and $\mathrm{Pic}^0 X \simeq \mathrm{Gr}^0_F H^1(X^{\mathrm{an}};\mathbb{C})/H$ has the structure of an abelian variety, so we can speak of the Picard variety of X.*

Proof. — a) In the case where X is complete, this is a well-known consequence of the exponential sequence, which gives the following exact sequence

$$0 \longrightarrow H^1(X^{\mathrm{an}};\mathbb{Z}) \longrightarrow H^1(X^{\mathrm{an}};\mathcal{O}_{X^{\mathrm{an}}}) \longrightarrow \mathrm{Pic}^0 X \longrightarrow 0.$$

Therefore we obtain

$$\mathrm{Pic}^0 X = H^1(X^{\mathrm{an}};\mathcal{O}_{X^{\mathrm{an}}})/H^1(X^{\mathrm{an}};\mathbb{Z}).$$

Now, $H^1(X^{\mathrm{an}};\mathbb{Z})$ has a pure Hodge structure, because X is supposed complete and non-singular, so in this case

$$H^1(X^{\mathrm{an}};\mathcal{O}_{X^{\mathrm{an}}}) = \mathrm{Gr}^0_F H^1(X^{\mathrm{an}};\mathbb{C}) = \mathrm{Gr}^0_F W_1 H^1(X^{\mathrm{an}};\mathbb{C}).$$

In general, let $\overline{X}$ be chosen as in section 2. From chasing on the following diagram:

$$\begin{tikzcd}
 & & \mathrm{Pic}^0\,\overline{X} \arrow[r] \arrow[d] & \mathrm{Pic}^0\,X \arrow[d] \\
 & \mathbb{Z}^r \arrow[rr] \arrow[d, "\wr"'] & & \mathrm{Pic}\,\overline{X} \arrow[r] \arrow[d, "\overline{\alpha}"'] & \mathrm{Pic}\,X \arrow[d, "\alpha"] \\
0 \arrow[r] & G \arrow[r] & H^2(\overline{X}^{\mathrm{an}}, X^{\mathrm{an}};\mathbb{Z}) \arrow[r] & H^2(\overline{X}^{\mathrm{an}};\mathbb{Z}) \arrow[r] & H^2(X^{\mathrm{an}};\mathbb{Z})
\end{tikzcd}$$

we have an exact sequence

$$G \longrightarrow \mathrm{Pic}^0\,\overline{X} \longrightarrow \mathrm{Pic}^0\,X \longrightarrow 0$$

where G is the kernel of $H^2(\overline{X}^{\mathrm{an}}, X^{\mathrm{an}};\mathbb{Z}) \to H^2(\overline{X}^{\mathrm{an}};\mathbb{Z})$, *i.e.* the image of $H^1(X^{\mathrm{an}};\mathbb{Z}) \to H^2(\overline{X}^{\mathrm{an}}, X^{\mathrm{an}};\mathbb{Z})$. Since G is contained in

$$H^2(\overline{X}^{\mathrm{an}}, X^{\mathrm{an}};\mathbb{Z}) \simeq \mathbb{Z}^r,$$

the group G is free abelian. Now the kernel of the map

$$H^1(X^{\mathrm{an}};\mathbb{Q}) \longrightarrow H^2(\overline{X}^{\mathrm{an}}, X^{\mathrm{an}};\mathbb{Q})$$

is the image of $H^1(\overline{X}^{\mathrm{an}};\mathbb{Q}) \to H^1(X^{\mathrm{an}};\mathbb{Q})$. By Lemma 2.1, this image is $W_1H^1(X^{\mathrm{an}};\mathbb{Q})$. Therefore we obtain

$$G \otimes_{\mathbb{Z}} \mathbb{Q} \simeq H^1(X^{\mathrm{an}};\mathbb{Q})/W_1H^1(X^{\mathrm{an}};\mathbb{Q}) \simeq \mathrm{Gr}^W_2 H^1(X^{\mathrm{an}};\mathbb{Q}).$$

So $\mathrm{rk}\,G = \dim \mathrm{Gr}^W_2 H^1(X^{\mathrm{an}};\mathbb{Q})$.

On the other hand, by the preliminary consideration, applied to $\overline{X}$, and Lemma 2.1, we have

$$\mathrm{Pic}^0\,\overline{X} = \mathrm{Gr}^0_F H^1(\overline{X}^{\mathrm{an}};\mathbb{C})/H^1(\overline{X}^{\mathrm{an}};\mathbb{Z}) = \mathrm{Gr}^0_F W_1 H^1(X^{\mathrm{an}};\mathbb{C})/H^1(\overline{X}^{\mathrm{an}};\mathbb{Z})$$

Now, $\mathrm{rk}\,H^1(\overline{X}^{\mathrm{an}};\mathbb{Z}) = \dim W_1H^1(X^{\mathrm{an}};\mathbb{Q})$, by Lemma 2.1 again, and

$$W_1H^1(X^{\mathrm{an}};\mathbb{Q}) = \mathrm{Gr}^W_1 H^1(X^{\mathrm{an}};\mathbb{Q}).$$

Since G is a free abelian group we can lift the map $G \to \mathrm{Pic}^0\,\overline{X}$ to

$$G \longrightarrow \mathrm{Gr}^0_F W_1 H^1(X^{\mathrm{an}};\mathbb{C})$$

We have

$$\mathrm{Pic}^0\,X \simeq \mathrm{Gr}^0_F W_1 H^1(X^{\mathrm{an}};\mathbb{C})/H = \mathrm{Gr}^0_F \mathrm{Gr}^W_1 H^1(X^{\mathrm{an}};\mathbb{C})/H$$

where H is generated by the images of G and $H^1(\overline{X}^{\mathrm{an}};\mathbb{Z})$, so

$$\mathrm{rk}\,H \leqslant \mathrm{rk}\,G + \mathrm{rk}\,H^1(\overline{X}^{\mathrm{an}};\mathbb{Z})$$

Since

$$\operatorname{rk} G + \operatorname{rk} H^1(\overline{X}^{\mathrm{an}};\mathbb{Z}) = \dim \operatorname{Gr}_2^W H^1(X^{\mathrm{an}};\mathbb{Q}) + \dim \operatorname{Gr}_1^W H^1(X^{\mathrm{an}};\mathbb{Q})$$

and

$$\dim \operatorname{Gr}_2^W H^1(X^{\mathrm{an}};\mathbb{Q}) + \dim \operatorname{Gr}_1^W H^1(X^{\mathrm{an}};\mathbb{Q}) = \operatorname{rk} H^1(X^{\mathrm{an}};\mathbb{Z}),$$

this implies our statement.

b) Since we assume $\operatorname{Gr}_2^W H^1(X^{\mathrm{an}};\mathbb{Q}) = 0$, the group G vanishes. The proof above shows that $\operatorname{Pic}^0 \overline{X} \simeq \operatorname{Pic}^0 X$, so we have an abelian variety. Furthermore

$$\operatorname{Gr}_2^W H^1(X^{\mathrm{an}};\mathbb{Q}) = W_2 H^1(X^{\mathrm{an}};\mathbb{Q})/W_1 H^1(X^{\mathrm{an}};\mathbb{Q})$$

and $W_2 H^1(X^{\mathrm{an}};\mathbb{Q}) = H^1(X^{\mathrm{an}};\mathbb{Q})$. Lemma 2.1 gives

$$W_1 H^1(X^{\mathrm{an}};\mathbb{Q}) = H^1(\overline{X}^{\mathrm{an}};\mathbb{Q})$$

So our assumption implies $H^1(\overline{X}^{\mathrm{an}};\mathbb{Q}) \simeq H^1(X^{\mathrm{an}};\mathbb{Q})$. In particular

$$\operatorname{Gr}_F^0 H^1(X^{\mathrm{an}};\mathbb{C}) = \operatorname{Gr}_F^0 H^1(\overline{X}^{\mathrm{an}};\mathbb{C}) = \operatorname{Gr}_F^0 W_1 H^1(X^{\mathrm{an}};\mathbb{C}).$$

From the exact sequence

$$0 \longrightarrow H^1(\overline{X}^{\mathrm{an}};\mathbb{Z}) \longrightarrow H^1(X^{\mathrm{an}};\mathbb{Z}) \longrightarrow H^2(\overline{X}^{\mathrm{an}}, X^{\mathrm{an}};\mathbb{Z})$$

where the last group is free abelian, we get $H^1(\overline{X}^{\mathrm{an}};\mathbb{Z}) \simeq H^1(X^{\mathrm{an}};\mathbb{Z})$. This gives

$$\operatorname{Pic}^0 X \simeq \operatorname{Gr}_F^0 H^1(X^{\mathrm{an}};\mathbb{C})/H. \qquad \square$$

4.2. Theorem. — *Let $f : Y \to X$ be a morphism between smooth algebraic varieties, and suppose that f induces an isomorphism*

$$H^1(X^{\mathrm{an}};\mathbb{Z}) \longrightarrow H^1(Y^{\mathrm{an}};\mathbb{Z})$$

Then the natural mapping $\operatorname{Pic}^0 X \to \operatorname{Pic}^0 Y$ is bijective.

Proof. — Note that f can be extended to $\overline{f} : \overline{Y} \to \overline{X}$ where $\overline{Y}$ and $\overline{X}$ are smooth and compact and the complement of Y resp. X in $\overline{Y}$ resp. $\overline{X}$ is a divisor with normal crossings, using passage to the graph and resolution of singularities (see [**D1**] p. 38 remark before 3.2.12).

By Lemma 2.1, $H^1(\overline{X}^{\mathrm{an}};\mathbb{Q}) \simeq \operatorname{Gr}_1^W H^1(X^{\mathrm{an}};\mathbb{Q})$, similarly for Y

$$H^1(\overline{Y}^{\mathrm{an}};\mathbb{Q}) \simeq \operatorname{Gr}_1^W H^1(Y^{\mathrm{an}};\mathbb{Q}),$$

so we get $H^1(\overline{X}^{\mathrm{an}};\mathbb{Q}) \simeq H^1(\overline{Y}^{\mathrm{an}};\mathbb{Q})$ since the isomorphism

$$H^1(X^{\mathrm{an}};\mathbb{Z}) \longrightarrow H^1(Y^{\mathrm{an}};\mathbb{Z})$$

induces a strictly compatible morphism of the corresponding Hodge structures (see Theorem 2.3.5 of [**D1**]). In particular

$$H^1(\overline{X}, \mathcal{O}_{\overline{X}}) \simeq H^1(\overline{Y}, \mathcal{O}_{\overline{Y}}).$$

Furthermore, we have a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & H^1(\overline{X}^{\mathrm{an}};\mathbb{Z}) \arrow[r] \arrow[d] & H^1(X^{\mathrm{an}};\mathbb{Z}) \arrow[r] \arrow[d, "\wr"'] & H^2(\overline{X}^{\mathrm{an}}, X^{\mathrm{an}};\mathbb{Z}) \arrow[d] \\ 0 \arrow[r] & H^1(\overline{Y}^{\mathrm{an}};\mathbb{Z}) \arrow[r] & H^1(Y^{\mathrm{an}};\mathbb{Z}) \arrow[r] & H^2(\overline{Y}^{\mathrm{an}}, Y^{\mathrm{an}};\mathbb{Z}) \end{tikzcd}$$

So the first vertical is injective and, the ranks being equal, the cokernel is a finite group.

Now all involved groups are free abelian (for the first cohomology groups this is obvious from the universal coefficient formula). Let us look at $H_1 := H^1(\overline{X}^{\mathrm{an}};\mathbb{Z})$ and $H_2 := H^1(\overline{Y}^{\mathrm{an}};\mathbb{Z})$ as subgroups of the free abelian group $G := H^1(Y^{\mathrm{an}};\mathbb{Z})$. Then $H_1 \subset H_2$, and there is a natural number $n \neq 0$ with $nH_2 \subset H_1$. Now H_1 is a saturated subgroup of G, so $nx \in H_1 \Rightarrow x \in H_1$ for $x \in G$. So $H_1 = H_2$, which implies that the first vertical is an isomorphism. Furthermore, we get the commutative diagram

$$\begin{tikzcd} H^1(\overline{X}^{\mathrm{an}};\mathbb{Z}) \arrow[r] \arrow[d, "\wr"'] & H^1(\overline{X},\mathcal{O}_{\overline{X}}) \arrow[r] \arrow[d, "\wr"'] & \operatorname{Pic}^0\overline{X} \arrow[r] \arrow[d] & 0 \\ H^1(\overline{Y}^{\mathrm{an}};\mathbb{Z}) \arrow[r] & H^1(\overline{Y},\mathcal{O}_{\overline{Y}}) \arrow[r] & \operatorname{Pic}^0\overline{Y} \arrow[r] & 0 \end{tikzcd}$$

Therefore $\operatorname{Pic}^0\overline{X} \to \operatorname{Pic}^0\overline{Y}$ is bijective. Finally let us look at the commutative diagram

$$\begin{tikzcd} G_1 \arrow[r] \arrow[d] & \operatorname{Pic}^0\overline{X} \arrow[r] \arrow[d, "\wr"'] & \operatorname{Pic}^0 X \arrow[r] \arrow[d] & 0 \\ G_2 \arrow[r] & \operatorname{Pic}^0\overline{Y} \arrow[r] & \operatorname{Pic}^0 Y \arrow[r] & 0 \end{tikzcd}$$

where G_1 is the kernel of $H^2(\overline{X}^{\mathrm{an}}, X^{\mathrm{an}};\mathbb{Z}) \to H^2(\overline{X}^{\mathrm{an}};\mathbb{Z})$, and G_2 is the kernel of $H^2(\overline{Y}^{\mathrm{an}}, Y^{\mathrm{an}};\mathbb{Z}) \to H^2(\overline{Y}^{\mathrm{an}};\mathbb{Z})$.

Note that we have the commutative diagram

$$\begin{tikzcd} H^1(X^{\mathrm{an}};\mathbb{Z}) \arrow[r] \arrow[d, "\wr"'] & G_1 \arrow[r] \arrow[d] & 0 \\ H^1(Y^{\mathrm{an}};\mathbb{Z}) \arrow[r] & G_2 \arrow[r] & 0 \end{tikzcd}$$

Therefore the right vertical is surjective, and the previous diagram leads to the desired statement. □

5. Proofs of Theorem 1.4 and 1.5

Theorem 1.4 is a consequence of

5.1. Theorem. — *Let $f : Y \to X$ be a morphism between non-singular complex algebraic varieties. Suppose that the induced mapping*

$$H^k(X^{\mathrm{an}};\mathbb{Z}) \longrightarrow H^k(Y^{\mathrm{an}};\mathbb{Z})$$

is bijective for $k = 1$ *and injective for* $k = 2$ *(resp. bijective for* $k = 1, 2$*). Then the natural mapping* $\operatorname{Pic} X \to \operatorname{Pic} Y$ *is injective (resp. bijective).*

Proof. — Look at the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Pic}^0 X & \longrightarrow & \operatorname{Pic} X & \longrightarrow & \operatorname{NS} X & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \operatorname{Pic}^0 Y & \longrightarrow & \operatorname{Pic} Y & \longrightarrow & \operatorname{NS} Y & \longrightarrow & 0 \end{array}$$

and use Corollary 3.3 and Theorem 4.2. □

Proof of Theorem 1.5. — By the Zariski-Lefschetz theorem (see [**HL**] Theorem 4.2.5 for a very general version) we have: $H^k(X^{\mathrm{an}};\mathbb{Z}) \simeq H^k(Y^{\mathrm{an}};\mathbb{Z})$, $k \leqslant 2$. Furthermore, $\operatorname{Sing} Y$ is of codimension $\geqslant 2$ in Y. As for the statement about the Weil divisor class group, we may therefore replace X by $X \smallsetminus \operatorname{Sing} X$ and Y by $Y \smallsetminus \operatorname{Sing} Y$, *cf.* [**H**] II Proposition 6.5. So we may suppose that X and Y are non-singular and work with the Picard group instead of the Weil divisor class group. By Theorem 1.4 we obtain that $\operatorname{Pic} X \simeq \operatorname{Pic} Y$. Suppose now that X is affine. Then Y is affine, too, and, by Lemma 2.5, $\operatorname{Pic}_{(\mathrm{an})} X^{\mathrm{an}} \simeq H^2(X^{\mathrm{an}};\mathbb{Z})$, $\operatorname{Pic}_{(\mathrm{an})} Y^{\mathrm{an}} \simeq H^2(Y^{\mathrm{an}};\mathbb{Z})$, so $\operatorname{Pic}_{(\mathrm{an})} X^{\mathrm{an}} \simeq \operatorname{Pic}_{(\mathrm{an})} Y^{\mathrm{an}}$. □

Proof of Corollary 1.6. — We may find a compactification $\overline{X}$ of X as in Theorem 1.5. If L is a generic linear subspace of codimension $\dim X - 3$ we get the hypothesis of Theorem 1.5 with L instead of Z. □

Proof of Corollary 1.7. — This follows from Theorem 1.5 taking $X = \mathbb{A}_m$ and $Z = \overline{Y}$. Note that $\operatorname{Pic} \mathbb{A}_m = 1$, $\operatorname{Pic}_{(\mathrm{an})} \mathbb{C}^m = 1$. □

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