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Twisted (2+1) κ-AdS Algebra, Drinfel'd Doubles and Non-Commutative Spacetimes*

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Abstract. We construct the full quantum algebra, the corresponding Poisson–Lie structure and the associated quantum spacetime for a family of quantum deformations of the isometry algebras of the (2+1)-dimensional anti-de Sitter (AdS), de Sitter (dS) and Minkowski spaces. These deformations correspond to a Drinfel'd double structure on the isometry algebras that are motivated by their role in (2+1)-gravity. The construction includes the cosmological constant Λ as a deformation parameter, which allows one to treat these cases in a common framework and to obtain a twisted version of both space- and time-like κ-AdS and dS quantum algebras; their flat limit $\Lambda \to 0$ leads to a twisted quantum Poincaré algebra. The resulting non-commutative spacetime is a nonlinear Λ-deformation of the κ-Minkowski one plus an additional contribution generated by the twist. For the AdS case, we relate this quantum deformation to two copies of the standard (Drinfel'd–Jimbo) quantum deformation of the Lorentz group in three dimensions, which allows one to determine the impact of the twist.

Key words: (2+1)-gravity; deformation; non-commutative spacetime; anti-de Sitter; cosmological constant; quantum groups; Poisson–Lie groups; contraction

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1 Introduction

Spacetime geometry is widely expected to exhibit discrete features at the Planck scale [36, 53], which gives rise to an interplay between algebra and geometry. A schematic approach to these phenomena is provided by non-commutative models, in which spacetime coordinates are replaced by (non-commuting) operators whose uncertainty relations encode the discrete nature of spacetime geometry.

Non-commutative models based on quantum groups [27, 54] are well-established in this context and serve as a framework which enables one to construct non-commutative spacetimes together with an action of the corresponding generalisations of the classical spacetime kinematical groups [5]. Most of these models are based on q-deformed function algebras and the associated universal enveloping algebras, where the deformation parameter q is related to the Planck scale.

Since the introduction of the so-called κ-Poincaré algebras and their associated κ-Minkowski spacetimes [4, 9, 11, 12, 13, 19, 22, 23, 28, 29, 31, 32, 45, 46, 48, 49, 50, 51, 55, 57, 64], different

*This paper is a contribution to the Special Issue on Deformations of Space-Time and its Symmetries. The full collection is available at http://www.emis.de/journals/SIGMA/space-time.html quantum deformations of kinematical groups of spacetimes have been proposed, and a large number of possible deformations has been outlined. The resulting κ-Poincaré models played an important role in the development of the so-called "doubly special relativity" theories [1, 2, 3, 8, 24, 35, 44, 47, 52], where the deformation parameter κ is interpreted as a second fundamental constant (related to the Plank length) in addition to the speed of light.

However, most of the work in this context focused on deformations of Minkowski space and the Poincaré group, while quantum anti-de Sitter (AdS) space and de Sitter (dS) space and the associated isometry groups have received less attention (see [10, 12, 56] and references therein). This is regrettable since it does not allow one to investigate effects due to a non-trivial cosmological constant or, more generally, non-trivial curvature in these models. Indeed, the interplay between curvature and Planck scale effects should be taken into account when the possible implications in astrophysics and cosmology of the novel properties of spacetime at the Planck scale are considered [2, 56].

Since quantum deformations are highly non-unique, another essential open problem is the question regarding which of the models under consideration is suitable for the description of effects at the Planck scale. This question is relevant since even quantum deformations related by a twist give rise to different composition laws for multi-particle systems. However, in (3+1) dimensions, arguments about the physical interpretation of models generated by different quantum deformations are largely heuristic and phenomenological.

In contrast, the (2+1)-dimensional scenario offers some insight into this question, as quantum group structures arise naturally in the quantisation of (2+1)-gravity. They can be viewed as the quantum counterparts of certain Poisson–Lie symmetries which describe the Poisson structure on the phase space of the theory. This restricts the possible quantum deformations that are compatible with classical (2+1)-gravity [59]. It was shown in [16] that a criterium that ensures compatibility with (2+1)-gravity is the requirement that the deformation arise from an underlying Drinfel'd double structure on the isometry group of spacetime. Further evidence for the relevance of Drinfel'd doubles is their role in state sum or spin foam models of spacetimes, which are related to the Turaev–Viro invariant [43, 62].

A classification of the Drinfel'd double structures on the isometry groups of the classical homogeneous spacetimes can therefore shed light on the question of which quantum deformations are relevant to the quantisation of (2+1)-gravity. A further desirable property is that the possible deformations form a family that admit the cosmological constant as a deformation parameter and hence permits one to include curvature effects in these models.

Based on these considerations, it was shown in [16] that the isometry algebra of the (2+1)-dimensional AdS space admits *two* classical r-matrices arising from a Drinfel'd double structure that are compatible with (2+1)-gravity and its pairing [63], that determines its Poisson structure. The quantum deformation associated with the first one was already presented in [15], and the dependence of the associated non-commutative spacetime on the cosmological constant was recently studied in [17]. The second classical r-matrix turns out to be a twist of the quantum κ-AdS algebra [12] that, to our knowledge, has not been previously considered in the literature, except for partial results. In particular, its Poincaré counterpart was constructed in [28] and the corresponding pure κ-AdS spacetime (without the twist) was considered in [10] and further studied in [56].

Moreover, it was shown in [16] that both r-matrices have counterparts for the dS and Poincaré algebras that are compatible with (2+1)-gravity and give rise to quantum deformations of the isometry groups of these spacetimes. This allows one to implement the cosmological constant as a deformation parameter and to investigate the quantum effects arising from curvature.

The aim of the present article is to construct the full quantum algebra for this twisted κ-deformation of the AdS, dS and Poincaré algebras and to analyse the fundamental properties of their associated non-commutative spacetimes. We will construct these algebras simultaneously by considering the associated Lie algebras as a family AdS_ω, in which the cosmological constant plays the role of a real Lie algebra (graded contraction) parameter ω. The isometry algebras of AdS, dS and Minkowski spaces correspond, respectively, to positive, negative and zero values of ω, which coincides with the *sectional curvature* of the underlying classical spacetime.

The structure of the paper is as follows. In the next section we review the basic geometrical properties of (2+1)-dimensional AdS, dS and Minkowski spaces, their isometry groups and the associated Lie algebras [38, 40] and discuss how the latter can be grouped into a family AdS_ω for which the cosmological constant plays the role of a deformation parameter ω. Section 3 exhibits the underlying Drinfel'd double structure that generates the twisted κ-AdS_ω via its canonical classical r-matrix. We show that this Drinfel'd double structure leads to a *space-like* κ-deformation [11, 12] and that the passage to the usual *time-like* one (i.e., the proper κ-deformation [51]) involves a change of basis with complex coefficients. In Section 4 we compute the corresponding *first-order* structure of the full quantum deformation which is given by its Lie bialgebra. This Lie bialgebra depends on three *deformation* parameters (η, z, ϑ): the cosmological constant $\Lambda = -\omega = -\eta^2$, the usual quantum (Plank scale) deformation parameter $\kappa = 1/z$, and the additional twist parameter ϑ. Furthermore, we construct and analyse the *first-order* non-commutative quantum spacetime for AdS_ω, which does not depend on the cosmological constant and hence is common to the three cases under consideration.

The construction of the full quantum twisted κ-AdS_ω algebra $U_{\kappa,\vartheta}(\mathrm{AdS}_\omega)$ is undertaken in Section 5. At this point, it is important to emphasise that the compatibility condition of the deformation with the underlying Drinfel'd double structure is satisfied only for very specific values of the deformation parameters $z = 1/\kappa$ and ϑ, and this, in turn, depends on whether one chooses the space-like or time-like deformation from Section 3. We construct the (flat) Poincaré limit $\omega \to 0$ of the quantum algebra and show that this leads to one of the known twisted κ-Poincaré algebras given in [28].

In Section 6, we construct the full Poisson–Lie group associated to this twisted deformation by making use of a simultaneous parametrisation of the (2+1) AdS, dS and Poincaré groups in terms of local coordinates. The Poisson subalgebra generated by the local spacetime coordinates $\{x_0, x_1, x_2\}$ provides the classical counterpart of the non-commutative twisted κ-AdS_ω spacetime associated to this deformation. As expected, this Poisson AdS_ω spacetime is a nonlinear algebra when the cosmological constant Λ is non-zero, and in the limit $\Lambda \to 0$ reduces to linear non-commutative Minkowski spacetime that is consistent with the results in [28]. Due to the nonlinear nature of this new AdS_ω spacetime, its quantisation is far from being trivial. Nevertheless, in Section 7 it is shown that by using the Lie algebra isomorphism between $\mathfrak{so}(2,2)$ and $\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{sl}(2,\mathbb{R})$, the twisted κ-AdS Poisson–Lie group can be reconstructed and fully quantised. A final section with comments and a discussion of open problems closes the paper.

2 The AdS_ω Lie algebra

In this section we describe, in a unified setting, the realisation of the (2+1)-dimensional AdS, dS and Minkowski spaces as symmetric homogeneous spaces, their isometry groups, their Lie algebras and the associated left- and right-invariant vector fields. We start by considering their Lie algebras, which form a family of three six-dimensional real Lie algebras which are related by a real contraction parameter ω and will be denoted by $\mathrm{AdS}_\omega \equiv \mathfrak{so}_\omega(2,2)$ in the following.

In terms of a basis $\{P_0, P_i, J, K_i\}$, $i = 1, 2$, consisting of the infinitesimal generators of, respectively, a time translation, spatial translations, a spatial rotation and boosts, the Lie brackets of AdS_ω take the form

$$
\begin{aligned}
&[J, P_i] = \epsilon_{ij} P_j, \qquad &&[J, K_i] = \epsilon_{ij} K_j, \qquad &&[J, P_0] = 0, \qquad &&[P_i, K_j] = -\delta_{ij} P_0,\\
&[P_0, K_i] = -P_i, \qquad &&[K_1, K_2] = -J, \qquad &&[P_0, P_i] = \omega K_i, \qquad &&[P_1, P_2] = -\omega J,
\end{aligned}
\tag{2.1}
$$

where $i, j = 1, 2$ and ϵ_{ij} is a skew-symmetric tensor normalised such that $\epsilon_{12} = 1$. For positive, zero and negative values of ω, this Lie bracket defines a Lie algebra isomorphic to $\mathfrak{so}(2,2)$, $\mathfrak{iso}(2,1) = \mathfrak{so}(2,1) \ltimes \mathbb{R}^3$ and $\mathfrak{so}(3,1)$, respectively. Note that when $\omega \neq 0$, this parameter can always be transformed to $\omega = \pm 1$ by a rescaling of the basis. Moreover, the case $\omega = 0$ can be understood as an Inönü–Wigner contraction [41]: $\mathfrak{so}(2,2) \to \mathfrak{iso}(2,1) \leftarrow \mathfrak{so}(3,1)$.

The two quadratic Casimir invariants of the Lie algebra AdS_ω are given by

$$\mathcal{C} = P_0^2 - \mathbf{P}^2 + \omega\big(J^2 - \mathbf{K}^2\big), \qquad \mathcal{W} = -JP_0 + K_1P_2 - K_2P_1, \tag{2.2}$$

where here and in the following we denote $\mathbf{P} = (P_1, P_2)$, $\mathbf{P}^2 = P_1^2 + P_2^2$ and similarly for any other vector object with two components. Recall that $\mathcal{C}$ corresponds to the Killing–Cartan form and it is related to the energy of a point particle, while $\mathcal{W}$ is the Pauli–Lubanski vector.

It is well-known that parity Π, time-reversal Θ and their composition $\Pi\Theta$ defined in [5]

$$\begin{aligned}
\Pi: &\quad (P_0, \mathbf{P}, J, \mathbf{K}) \to (P_0, -\mathbf{P}, J, -\mathbf{K}),\\
\Theta: &\quad (P_0, \mathbf{P}, J, \mathbf{K}) \to (-P_0, \mathbf{P}, J, -\mathbf{K}),\\
\Pi\Theta: &\quad (P_0, \mathbf{P}, J, \mathbf{K}) \to (-P_0, -\mathbf{P}, J, \mathbf{K}),
\end{aligned} \tag{2.3}$$

are involutive automorphisms of AdS_ω which, together with the identity, define an Abelian group isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. From this viewpoint, ω is a graded contraction parameter related to the $\mathbb{Z}_2$-grading for $\Pi\Theta$ [39] and gives rise to the following Cartan decomposition:

$$\mathrm{AdS}_\omega = \mathfrak{so}_\omega(2,2) = \mathfrak{h} \oplus \mathfrak{p}, \qquad \mathfrak{h} = \mathrm{span}\{J, \mathbf{K}\} = \mathfrak{so}(2,1), \qquad \mathfrak{p} = \mathrm{span}\{P_0, \mathbf{P}\}.$$

It follows that the three classical (2+1)-dimensional Lorentzian symmetric homogeneous spacetimes with *constant sectional curvature* ω are obtained as quotients $\mathrm{SO}_\omega(2,2)/\mathrm{SO}(2,1)$ where $H = \mathrm{SO}(2,1)$ and $\mathrm{SO}_\omega(2,2)$ are the Lie groups corresponding to $\mathfrak{h}$ and AdS_ω, respectively. In the gravitational setting, the parameter ω is related to the *cosmological constant* Λ via $\omega = -\Lambda$. More explicitly, we have the following description:

$$\begin{array}{lll}
\omega > 0,\ \Lambda < 0: \ \text{AdS space} & \omega = \Lambda = 0: \ \text{Minkowski space} & \omega < 0,\ \Lambda > 0: \ \text{dS space}\\
\mathbf{AdS}^{2+1} = \mathrm{SO}(2,2)/\mathrm{SO}(2,1) & \mathbf{M}^{2+1} = \mathrm{ISO}(2,1)/\mathrm{SO}(2,1) & \mathbf{dS}^{2+1} = \mathrm{SO}(3,1)/\mathrm{SO}(2,1)
\end{array}$$

Here, $\mathrm{SO}(2,1)$ is the Lorentz group in three dimensions whose Lie algebra $\mathfrak{so}(2,1) = \mathrm{span}\{J, \mathbf{K}\}$ is spanned by the generators of boosts and spatial rotations, and the momenta P_0, $\mathbf{P}$ span the tangent space at the origin. The curvature ω can also be written as $\omega = \pm 1/R^2$, where R is radius of the AdS and dS spaces, and the limit $R \to \infty$ corresponds to their contraction to the Minkowski space.

2.1 Vector model: ambient and geodesic parallel coordinates

The action of the isometry groups $\mathrm{SO}_\omega(2,2)$ on their (2+1)-dimensional homogeneous spaces is nonlinear. However, this problem can be circumvented by considering the vector representation of AdS_ω which makes use of an ambient space with an "extra" dimension (called s_3 below). This leads to the *vector representation* of AdS_ω, in which the basis $\{P_0, \mathbf{P}, J, \mathbf{K}\}$ is represented by the following 4×4 real matrices [12]:

$$P_0 = \begin{pmatrix} 0 & -\omega & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad P_1 = \begin{pmatrix} 0 & 0 & \omega & 0\\ 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad P_2 = \begin{pmatrix} 0 & 0 & 0 & \omega\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$J = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad K_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad K_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

The corresponding one-parameter subgroups of $\mathrm{SO}_\omega(2,2)$ are obtained by exponentiation:

$$\mathrm{e}^{x_0 P_0} = \begin{pmatrix} \cos\eta x_0 & -\eta\sin\eta x_0 & 0 & 0 \\ \frac{1}{\eta}\sin\eta x_0 & \cos\eta x_0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \mathrm{e}^{\theta J} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{pmatrix},$$
$$\mathrm{e}^{x_1 P_1} = \begin{pmatrix} \cosh\eta x_1 & 0 & \eta\sinh\eta x_1 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{\eta}\sinh\eta x_1 & 0 & \cosh\eta x_1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \mathrm{e}^{\xi_1 K_1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cosh\xi_1 & \sinh\xi_1 & 0 \\ 0 & \sinh\xi_1 & \cosh\xi_1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$
$$\mathrm{e}^{x_2 P_2} = \begin{pmatrix} \cosh\eta x_2 & 0 & 0 & \eta\sinh\eta x_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{\eta}\sinh\eta x_2 & 0 & 0 & \cosh\eta x_2 \end{pmatrix}, \qquad \mathrm{e}^{\xi_2 K_2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cosh\xi_2 & 0 & \sinh\xi_2 \\ 0 & 0 & 1 & 0 \\ 0 & \sinh\xi_2 & 0 & \cosh\xi_2 \end{pmatrix}, \qquad (2.4)$$

where hereafter the parameter η is related to the curvature by $\omega = \eta^2 = -\Lambda$. This means that η is either a real number ($\eta = 1/R$) for $\mathbf{AdS}^{2+1}$ or a purely imaginary one ($\eta = \mathrm{i}/R$) for $\mathbf{dS}^{2+1}$.

The matrix representation of the isometry groups $\mathrm{SO}_\omega(2,2)$ and their Lie algebras AdS_ω can be characterised in terms of the bilinear form represented by the matrix

$$\mathbf{I}_\omega = \mathrm{diag}(+1, \omega, -\omega, -\omega), \qquad (2.5)$$

which identifies them with the isometry groups of the four-dimensional linear space $(\mathbb{R}^4, \mathbf{I}_\omega)$ with *ambient* or *Weierstrass* coordinates (s_3, s_0, s_1, s_2):

$$\mathrm{SO}_\omega(2,2) = \big\{G \in \mathrm{GL}(4,\mathbb{R}) : \ G^T \mathbf{I}_\omega G = \mathbf{I}_\omega\big\},$$
$$\mathrm{AdS}_\omega = \big\{Y \in \mathrm{Mat}(4,\mathbb{R}) : \ Y^T \mathbf{I}_\omega + \mathbf{I}_\omega Y = 0\big\},$$

where A^T denotes the transpose of A. The origin of the ambient space has ambient coordinates $O = (1,0,0,0)$ and is invariant under the Lorentz subgroup $\mathrm{SO}(2,1) \subset \mathrm{SO}_\omega(2,2)$ given by (2.4). The orbit passing through O corresponds to the (2+1)-dimensional homogeneous spacetime which is contained in the pseudosphere

$$\Sigma_\omega \equiv s_3^2 + \omega\big(s_0^2 - \mathbf{s}^2\big) = 1,$$

determined by $\mathbf{I}_\omega$ (2.5). Note that in the limit $\omega \to 0$ ($R \to \infty$), which corresponds to the contraction to Minkowski space, the pseudosphere Σ_ω gives rise to two hyperplanes, which are characterised by the condition $s_3 = \pm 1$ in Cartesian coordinates $(s_0, \mathbf{s})$. From now on, we will identify the Minkowski space with the hyperplane given by $s_3 = +1$.

The metric on the homogeneous spacetime is obtained from the flat ambient metric given by $\mathbf{I}_\omega$ by dividing by the curvature and restricting the resulting metric to the pseudosphere Σ_ω:

$$\begin{aligned} \mathrm{d}\sigma^2 &= \frac{1}{\omega}\left(\mathrm{d}s_3^2 + \omega\big(\mathrm{d}s_0^2 - \mathrm{d}s_1^2 - \mathrm{d}s_2^2\big)\right)\Big|_{\Sigma_\omega} \\ &= \mathrm{d}s_0^2 - \mathrm{d}s_1^2 - \mathrm{d}s_2^2 + \omega\frac{(s_0\mathrm{d}s_0 - s_1\mathrm{d}s_1 - s_2\mathrm{d}s_2)^2}{1 - \omega(s_0^2 - \mathbf{s}^2)}. \end{aligned} \qquad (2.6)$$

Now let us introduce three *intrinsic* spacetime coordinates that will be helpful in the sequel: these are the so-called *geodesic parallel coordinates* (x_0, x_1, x_2) [40], which can be regarded as a generalisation of the flat Cartesian coordinates to non-vanishing curvature. They are defined in terms of the action of the one-parameter subgroups (2.4) for P_0, $\mathbf{P}$ on the origin $O = (1, 0, 0, 0)$:

$$
\begin{aligned}
&(s_3, s_0, \mathbf{s})(x_0, \mathbf{x}) = \exp(x_0 P_0) \exp(x_1 P_1) \exp(x_2 P_2) O, \\
&s_3 = \cos\eta x_0 \cosh\eta x_1 \cosh\eta x_2, \qquad s_1 = \frac{\sinh\eta x_1}{\eta} \cosh\eta x_2, \\
&s_0 = \frac{\sin\eta x_0}{\eta} \cosh\eta x_1 \cosh\eta x_2, \qquad s_2 = \frac{\sinh\eta x_2}{\eta}.
\end{aligned}
\tag{2.7}
$$

The geometrical meaning of the coordinates $(x_0, \mathbf{x})$ that parametrise a generic point Q in the spacetime via (2.7) is as follows. Let l_0 be a time-like geodesic and l_1, l_2 two space-like geodesics such that these three basis geodesics are orthogonal at O. Then x_0 is the geodesic distance from O up to a point Q_1 measured along the time-like geodesic l_0; x_1 is the geodesic distance between Q_1 and another point Q_2 along a space-like geodesic l_1' orthogonal to l_0 through Q_1 and parallel to l_1; and x_2 is the geodesic distance between Q_2 and Q along a space-like geodesic l_2' orthogonal to l_1' through Q_2 and parallel to l_2.

Recall that time-like geodesics (as l_0) are compact in $\mathbf{AdS}^{2+1}$ and non-compact in $\mathbf{dS}^{2+1}$, while space-like ones (as l_i, l_i'; $i = 1, 2$) are compact in $\mathbf{dS}^{2+1}$ but non-compact in $\mathbf{AdS}^{2+1}$. Thus the trigonometric functions depending on x_0 are circular in $\mathbf{AdS}^{2+1}$ ($\eta = 1/R$) and hyperbolic in $\mathbf{dS}^{2+1}$ ($\eta = \mathrm{i}/R$) and, conversely, those depending on x_i are circular in $\mathbf{dS}^{2+1}$ but hyperbolic in $\mathbf{AdS}^{2+1}$. By inserting the parametrisation (2.7) into the metric (2.6) we obtain the corresponding expression in terms of geodesic parallel coordinates:

$$
\mathrm{d}\sigma^2 = \cosh^2(\eta x_1)\cosh^2(\eta x_2)\mathrm{d}x_0^2 - \cosh^2(\eta x_2)\mathrm{d}x_1^2 - \mathrm{d}x_2^2.
$$

For $\omega \in \{\pm 1, 0\}$ this expression reduces to

$$
\begin{aligned}
&\mathbf{AdS}^{2+1}\ (\omega = 1,\ \eta = 1): && \mathrm{d}\sigma^2 = \cosh^2 x_1 \cosh^2 x_2\, \mathrm{d}x_0^2 - \cosh^2 x_2\, \mathrm{d}x_1^2 - \mathrm{d}x_2^2, \\
&\mathbf{M}^{2+1}\ (\omega = \eta = 0): && \mathrm{d}\sigma^2 = \mathrm{d}s_0^2 - \mathrm{d}s_1^2 - \mathrm{d}s_2^2 = \mathrm{d}x_0^2 - \mathrm{d}x_1^2 - \mathrm{d}x_2^2, \\
&\mathbf{dS}^{2+1}\ (\omega = -1,\ \eta = \mathrm{i}): && \mathrm{d}\sigma^2 = \cos^2 x_1 \cos^2 x_2\, \mathrm{d}x_0^2 - \cos^2 x_2\, \mathrm{d}x_1^2 - \mathrm{d}x_2^2.
\end{aligned}
$$

2.2 Invariant vector fields

Left- and right-invariant vector fields, Y^L and Y^R, of the group $\mathrm{SO}_\omega(2,2)$ can be described in terms of the matrix representation (2.4). For this, one parametrises the group elements $T \in \mathrm{SO}_\omega(2,2)$ in terms of the matrices (2.4) as

$$
T = \exp(x_0 P_0) \exp(x_1 P_1) \exp(x_2 P_2) \exp(\xi_1 K_1) \exp(\xi_2 K_2) \exp(\theta J).
$$

This yields a matrix of the form

$$
T = \begin{pmatrix} s_3 & A_{31} & A_{32} & A_{33} \\ s_0 & B_{01} & B_{02} & B_{03} \\ s_1 & B_{11} & B_{12} & B_{13} \\ s_2 & B_{21} & B_{22} & B_{23} \end{pmatrix},
$$

where the entries $A_{\alpha\beta}$ and $B_{\mu\nu}$ depend on all the group coordinates $(x_0, \mathbf{x}, \boldsymbol{\xi}, \theta)$ and on the parameter η. In the limit $\eta \to 0$, these expressions reduce to the well-known matrix representation of the Poincaré group $\mathrm{ISO}(2,1)$

$$
\lim_{\eta\to 0} T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ x_0 & \Lambda_{01} & \Lambda_{02} & \Lambda_{03} \\ x_1 & \Lambda_{11} & \Lambda_{12} & \Lambda_{13} \\ x_2 & \Lambda_{21} & \Lambda_{22} & \Lambda_{23} \end{pmatrix},
$$

Table 1. Left- and right-invariant vector fields on the isometry groups of the (2+1)-dimensional (anti-)de Sitter and Minkowski spaces in terms of the sectional curvature $\omega = \eta^2 = -\Lambda$.

$$
\begin{aligned}
Y^L_{P_0} &= \frac{\cosh\xi_1\cosh\xi_2}{\cosh\eta x_1\cosh\eta x_2}(\partial_{x_0}-\eta\sinh\eta x_1\partial_{\xi_1}) + \frac{\sinh\xi_1\cosh\xi_2}{\cosh\eta x_2}\partial_{x_1} + \sinh\xi_2\partial_{x_2} - \eta\tanh\eta x_2\cosh\xi_2\partial_{\xi_2}\\
Y^L_{P_1} &= \left(\frac{\sinh\xi_1\cos\theta+\cosh\xi_1\sinh\xi_2\sin\theta}{\cosh\eta x_1\cosh\eta x_2}\right)(\partial_{x_0}-\eta\sinh\eta x_1\partial_{\xi_1}) + \left(\frac{\cosh\xi_1\cos\theta+\sinh\xi_1\sinh\xi_2\sin\theta}{\cosh\eta x_2}\right)\partial_{x_1}\\
&\quad + \cosh\xi_2\sin\theta\partial_{x_2} - \eta\tanh\eta x_2\left(\tanh\xi_2\cos\theta\partial_{\xi_1} + \sinh\xi_2\sin\theta\partial_{\xi_2} - \frac{\cos\theta}{\cosh\xi_2}\partial_\theta\right)\\
Y^L_{P_2} &= \left(\frac{\cosh\xi_1\sinh\xi_2\cos\theta-\sinh\xi_1\sin\theta}{\cosh\eta x_1\cosh\eta x_2}\right)(\partial_{x_0}-\eta\sinh\eta x_1\partial_{\xi_1}) + \left(\frac{\sinh\xi_1\sinh\xi_2\cos\theta-\cosh\xi_1\sin\theta}{\cosh\eta x_2}\right)\partial_{x_1}\\
&\quad + \cosh\xi_2\cos\theta\partial_{x_2} + \eta\tanh\eta x_2\left(\tanh\xi_2\sin\theta\partial_{\xi_1} - \sinh\xi_2\cos\theta\partial_{\xi_2} - \frac{\sin\theta}{\cosh\xi_2}\partial_\theta\right)\\
Y^L_{K_1} &= \frac{\cos\theta}{\cosh\xi_2}\partial_{\xi_1} + \sin\theta\partial_{\xi_2} + \tanh\xi_2\cos\theta\partial_\theta\\
Y^L_{K_2} &= -\frac{\sin\theta}{\cosh\xi_2}\partial_{\xi_1} + \cos\theta\partial_{\xi_2} - \tanh\xi_2\sin\theta\partial_\theta\\
Y^L_J &= \partial_\theta
\end{aligned}
$$

$$
\begin{aligned}
Y^R_{P_0} &= \partial_{x_0}\\
Y^R_{P_1} &= -\sin\eta x_0\tanh\eta x_1\partial_{x_0} + \cos\eta x_0\partial_{x_1} - \eta\frac{\sin\eta x_0}{\cosh\eta x_1}\partial_{\xi_1}\\
Y^R_{P_2} &= -\frac{\sin\eta x_0\tanh\eta x_2}{\cosh\eta x_1}(\partial_{x_0}-\eta\sinh\eta x_1\partial_{\xi_1}) - \cos\eta x_0\sinh\eta x_1\tanh\eta x_2\partial_{x_1} + \cos\eta x_0\cosh\eta x_1\partial_{x_2}\\
&\quad + \eta\left(\frac{\cos\eta x_0\sinh\eta x_1\sinh\xi_1-\sin\eta x_0\cosh\xi_1}{\cosh\eta x_2}\right)\partial_{\xi_2}\\
&\quad + \eta\left(\frac{\cos\eta x_0\sinh\eta x_1\cosh\xi_1-\sin\eta x_0\sinh\xi_1}{\cosh\eta x_2\cosh\xi_2}\right)(\partial_\theta-\sinh\xi_2\partial_{\xi_1})\\
Y^R_{K_1} &= \frac{\cos\eta x_0\tanh\eta x_1}{\eta}\partial_{x_0} + \frac{\sin\eta x_0}{\eta}\partial_{x_1} + \frac{\cos\eta x_0}{\cosh\eta x_1}\partial_{\xi_1}\\
Y^R_{K_2} &= \frac{\cos\eta x_0\tanh\eta x_2}{\eta\cosh\eta x_1}(\partial_{x_0}-\eta\sinh\eta x_1\partial_{\xi_1}) - \frac{\sin\eta x_0\sinh\eta x_1\tanh\eta x_2}{\eta}\partial_{x_1} + \frac{\sin\eta x_0\cosh\eta x_1}{\eta}\partial_{x_2}\\
&\quad + \left(\frac{\sin\eta x_0\sinh\eta x_1\sinh\xi_1+\cos\eta x_0\cosh\xi_1}{\cosh\eta x_2}\right)\partial_{\xi_2}\\
&\quad + \left(\frac{\sin\eta x_0\sinh\eta x_1\cosh\xi_1+\cos\eta x_0\sinh\xi_1}{\cosh\eta x_2\cosh\xi_2}\right)(\partial_\theta-\sinh\xi_2\partial_{\xi_1})\\
Y^R_J &= -\frac{\cosh\eta x_1\tanh\eta x_2}{\eta}\partial_{x_1} + \frac{\sinh\eta x_1}{\eta}\partial_{x_2} - \frac{\cosh\eta x_1}{\cosh\eta x_2}\left(\cosh\xi_1\tanh\xi_2\partial_{\xi_1} - \sinh\xi_1\partial_{\xi_2} - \frac{\cosh\xi_1}{\cosh\xi_2}\partial_\theta\right)
\end{aligned}
$$

where the entries $\Lambda_{\mu\nu}$ parametrise an element of $\mathrm{SO}(2,1)$ and depend only on $(\boldsymbol{\xi},\theta)$. From the group action of $\mathrm{SO}_\omega(2,2)$ on itself via right- and left-multiplication, one obtains expressions for the left- and right-invariant vector fields in terms of the coordinates $(x_0,\mathbf{x},\boldsymbol{\xi},\theta)$, which are displayed in Table 1. We stress that the limit $\eta\to 0$ of these expressions is always well defined and gives the left- and right-invariant vector fields on the Poincaré group $\mathrm{ISO}(2,1)$.

3 Twisted κ-AdS$_\omega$ algebra as a Drinfel'd double

We now consider the Lie bialgebra structures underlying the quantum deformations of the isometry groups $\mathrm{SO}_\omega(2,2)$. It is well-known that the Lie bialgebra structures underlying quantum deformations of semisimple Lie algebras are always coboundary ones [21] and hence characterised by classical r-matrices. This will be the case for all possible quantum deformations of the isometry groups of AdS and dS. Moreover, it is well-known that all quantum deformations of the (2+1)-dimensional Poincaré algebra are also coboundaries [65]. These quantum deformations can therefore be classified by considering the classical r-matrices of AdS$_\omega$ and relating them via the cosmological constant.

The first steps towards such a classification for the Lie algebras AdS$_\omega$ were recently presented in [18], where it became evident that there is a plethora of possible quantum deformations.

Therefore, criteria to select the physically relevant cases are required. In this respect, the Chern–Simons formulation of (2+1)-gravity can be helpful, since Poisson–Lie structures and Lie bialgebra structures arise naturally in the description of its classical phase space. The compatibility of a given classical r-matrix for AdS_ω with the Chern–Simons formulation imposes restrictions on the possible r-matrices [58, 59]. We have recently shown in [16] that these constraints are always fulfilled if the classical r-matrix that defines the deformation is the canonical r-matrix of certain Drinfel'd double structures of the Lie algebra AdS_ω. In this section we show that the *twisted* κ-deformation is one of these compatible structures and thus appears to be the appropriate generalisation of the κ-deformation in the context of (2+1)-gravity.

3.1 Drinfel'd double Lie algebras

A $2d$-dimensional Lie algebra $\mathfrak{a}$ has the structure of a Drinfel'd double (DD) [34] (see also [6, 7, 20]) if there exists a basis $\{Y_1, \ldots, Y_d, y^1, \ldots, y^d\}$ of $\mathfrak{a}$ in which the Lie bracket takes the form

$$[Y_i, Y_j] = c_{ij}^k Y_k, \qquad [y^i, y^j] = f_k^{ij} y^k, \qquad [y^i, Y_j] = c_{jk}^i y^k - f_j^{ik} Y_k. \tag{3.1}$$

Note that this implies that $\{Y_1, \ldots, Y_d\}$ and $\{y^1, \ldots, y^d\}$ span two Lie subalgebras with structure constants c_{ij}^k and f_k^{ij}, respectively. From the form of the "crossed" brackets $[y^i, Y_j]$, it follows that there is an Ad-invariant quadratic form on $\mathfrak{a}$ given by

$$\langle Y_i, Y_j\rangle = 0, \qquad \langle y^i, y^j\rangle = 0, \qquad \langle y^i, Y_j\rangle = \delta_j^i \qquad \forall\, i, j, \tag{3.2}$$

and a quadratic Casimir for $\mathfrak{a}$ is given by

$$C = \frac{1}{2}\sum_{i=1}^{d}\left(y^i Y_i + Y_i y^i\right). \tag{3.3}$$

A Lie algebra $\mathfrak{a}$ with a DD structure can therefore be regarded as a pair of Lie algebras, $\mathfrak{g}$ with basis $\{Y_1, \ldots, Y_d\}$ and $\mathfrak{g}^*$ with basis $\{y^1, \ldots, y^d\}$, together with a specific set of crossed commutation rules (3.1) that ensures the existence of the Ad-invariant symmetric bilinear form (3.2). We shall refer to Lie algebras with a DD structure as *DD Lie algebras* in the following.

A DD Lie algebra $D(\mathfrak{g}) \equiv \mathfrak{a}$ is always endowed with a (quasi-triangular) Lie bialgebra structure $(D(\mathfrak{g}), \delta_D)$ that is generated by the canonical classical r-matrix

$$r = \sum_{i=1}^{d} y^i \otimes Y_i, \tag{3.4}$$

through the coboundary relation

$$\delta_D(X_i) = [X_i \otimes 1 + 1 \otimes X_i, r] \qquad \forall\, X_i \in D(\mathfrak{g}). \tag{3.5}$$

Thus, the cocommutator δ_D takes the form

$$\delta_D\left(y^k\right) = c_{ij}^k y^i \otimes y^j, \qquad \delta_D(Y_n) = -f_n^{lm} Y_l \otimes Y_m.$$

Note that the cocommutator δ_D only depends on the skew-symmetric component of the r-matrix (3.4), namely

$$r' = \frac{1}{2}\sum_{i=1}^{d} y^i \wedge Y_i, \tag{3.6}$$

while the symmetric component of the r-matrix defines a canonical quadratic Casimir element of $D(\mathfrak{g})$ in the form (3.3). This implies that the associated element of $D(\mathfrak{g}) \otimes D(\mathfrak{g})$ given by

$$\Omega = r - r' = \frac{1}{2} \sum_{i=1}^{d} \left(y^i \otimes Y_i + Y_i \otimes y^i \right)$$

is invariant under the action of $D(\mathfrak{g})$

$$[Y_i \otimes 1 + 1 \otimes Y_i, \Omega] = 0 \qquad \forall\, Y_i \in D(\mathfrak{g}).$$

To summarise, if a Lie algebra $\mathfrak{a}$ has a DD structure (3.1), then this implies that $(\mathfrak{a}, \delta_D)$ is a Lie bialgebra with canonical r-matrix (3.4). Therefore, there exists a quantum algebra $(U_z(\mathfrak{a}), \Delta_z)$ whose first-order coproduct is given by δ_D (3.5), and this quantum deformation can be viewed as the quantum symmetry corresponding to the given DD structure for $\mathfrak{a}$.

3.2 The AdS Drinfel'd double

As shown in [63], the Lie algebras $\mathfrak{so}(3,1)$, $\mathfrak{iso}(2,1) = \mathfrak{so}(2,1) \ltimes \mathbb{R}^3$ and $\mathfrak{so}(2,2)$ of the isometry groups of Lorentzian (2+1)-gravity can be described in terms of a common basis in which the cosmological constant Λ plays the role of a structure constant. In terms of the generators T_a $(a = 0, 1, 2)$ of translations and the generators J_a $(a = 0, 1, 2)$ of Lorentz transformations, the Lie bracket then takes the form

$$\begin{array}{lll}
[J_0, J_1] = J_2, & [J_0, J_2] = -J_1, & [J_1, J_2] = -J_0, \\
[J_0, T_0] = 0, & [J_0, T_1] = T_2, & [J_0, T_2] = -T_1, \\
[J_1, T_0] = -T_2, & [J_1, T_1] = 0, & [J_1, T_2] = -T_0, \\
[J_2, T_0] = T_1, & [J_2, T_1] = T_0, & [J_2, T_2] = 0, \\
[T_0, T_1] = -\Lambda J_2, & [T_0, T_2] = \Lambda J_1, & [T_1, T_2] = \Lambda J_0.
\end{array} \tag{3.7}$$

As these are six-dimensional real Lie algebras, they can carry DD structures. In that case, the DD structure provides a canonical r-matrix and, therefore, an associated quantum deformation compatible with the constraints imposed by (2+1)-gravity. The compatible DD structures on these Lie algebras were investigated in [16]. The twisted κ-AdS r-matrix arises as the case F in the classification of admissible r-matrices, which corresponds to the DD $(6_0|5.iii|\lambda)$ in [60] and case (11) in [37]. This DD depends on an essential deformation parameter $\eta \neq 0$ and is explicitly given by

$$\begin{array}{lll}
[Y_0, Y_1] = -Y_2, & [Y_0, Y_2] = -Y_1, & [Y_1, Y_2] = 0, \\
[y^0, y^1] = \eta y^1, & [y^0, y^2] = \eta y^2, & [y^1, y^2] = 0,
\end{array}$$

together with the crossed relations

$$\begin{array}{lll}
[y^0, Y_0] = 0, & [y^0, Y_1] = -\eta Y_1, & [y^0, Y_2] = -\eta Y_2, \\
[y^1, Y_0] = -y^2, & [y^1, Y_1] = \eta Y_0, & [y^1, Y_2] = y^0, \\
[y^2, Y_0] = -y^1, & [y^2, Y_1] = y^0, & [y^2, Y_2] = \eta Y_0.
\end{array}$$

The change of basis that transforms these expressions into (3.7) is

$$\begin{array}{lll}
J_0 = \dfrac{1}{\sqrt{2\eta}} \left(Y_2 - y^1 \right), & J_1 = \dfrac{1}{\sqrt{2\eta}} \left(Y_2 + y^1 \right), & J_2 = -\dfrac{1}{\eta} y^0, \\
T_0 = \sqrt{\dfrac{\eta}{2}} \left(Y_1 - y^2 \right), & T_1 = \sqrt{\dfrac{\eta}{2}} \left(Y_1 + y^2 \right), & T_2 = -\eta Y_0.
\end{array} \tag{3.8}$$

Hence we obtain the AdS Lie algebra $\mathfrak{so}(2,2)$ with negative cosmological constant $\Lambda = -\eta^2$, which is directly related to the deformation parameter η. The canonical pairing (3.2) takes the form

$$\langle J_a, T_b \rangle = g_{ab}, \qquad \langle J_a, J_b \rangle = \langle T_a, T_b \rangle = 0, \tag{3.9}$$

where $g = \mathrm{diag}(-1,1,1)$ denotes the Minkowski metric in three dimensions. We stress that (3.9) was shown in [63] to be the appropriate pairing for the Chern–Simons formulation of (2+1)-gravity, while other choices of pairing lead to a different symplectic structure.

By inserting the inverse of (3.8) into the canonical classical r-matrix (3.4) and by using the Casimir operator (3.3) in order to get a fully skew-symmetric expression as in (3.6), we find that the AdS deformation induced by this DD structure is generated by

$$r'_{\mathrm{F}} = \tfrac{1}{2}(J_1 \wedge T_0 - J_0 \wedge T_1 + J_2 \wedge T_2). \tag{3.10}$$

Since the Lie algebra elements J_a are the generators of the Lorentz group and the generators T_a generate the (non-commutative) AdS translation sector ($a = 0,1,2$), we conclude that the classical r-matrix (3.10) is just a superposition of the standard deformation of $\mathfrak{so}(2,2)$ [12] generated by $(J_1 \wedge T_0 - J_0 \wedge T_1)$ and a Reshetikhin twist generated by $J_2 \wedge T_2$ (note that J_2 and T_2 commute). Therefore, we have obtained a twisted κ-AdS algebra which is a DD structure.

3.3 The dS Drinfel'd double

The analogous DD deformation of $\mathfrak{so}(3,1)$ is given by case (9) in [37] and case $(7_0|5.ii|\lambda)$ in [60], and corresponds to case C in [16]. It depends again on one essential deformation parameter $\eta \neq 0$, and the Lie bracket is given by

$$\begin{aligned}
&[Y_0, Y_1] = Y_2, &&[Y_0, Y_2] = -Y_1, &&[Y_1, Y_2] = 0,\\
&[y^0, y^1] = -\eta y^1, &&[y^0, y^2] = -\eta y^2, &&[y^1, y^2] = 0,
\end{aligned}$$

with crossed commutators

$$\begin{aligned}
&[y^0, Y_0] = 0, &&[y^0, Y_1] = \eta Y_1, &&[y^0, Y_2] = \eta Y_2,\\
&[y^1, Y_0] = -y^2, &&[y^1, Y_1] = -\eta Y_0, &&[y^1, Y_2] = y^0,\\
&[y^2, Y_0] = y^1, &&[y^2, Y_1] = -y^0, &&[y^2, Y_2] = -\eta Y_0.
\end{aligned}$$

To obtain a Lie algebra isomorphism between this DD Lie algebra and the isometry algebra (3.7) of dS space, we consider the change of basis

$$\begin{aligned}
&J_0 = \frac{1}{\sqrt{2\eta}}\big(Y_1 - y^2\big), &&J_1 = \frac{1}{\sqrt{2\eta}}\big(Y_1 + y^2\big), &&J_2 = \frac{1}{\eta} y^0,\\
&T_0 = \sqrt{\frac{\eta}{2}}\big(Y_2 - y^1\big), &&T_1 = \sqrt{\frac{\eta}{2}}\big(Y_2 + y^1\big), &&T_2 = \eta Y_0.
\end{aligned}$$

This yields the Lie algebra $\mathfrak{so}(3,1)$ with bracket (3.7) and positive cosmological constant $\Lambda = \eta^2$, together with the same canonical pairing (3.9). Using again the Casimir operator (3.3), one obtains the skew-symmetric classical r-matrix from the canonical one (3.4)

$$r'_{\mathrm{C}} = \tfrac{1}{2}(J_1 \wedge T_0 - J_0 \wedge T_1 + J_2 \wedge T_2) \equiv r'_{\mathrm{F}}. \tag{3.11}$$

As the r-matrix $r'_{\mathrm{F}} = r'_{\mathrm{C}}$ does not depend on the deformation parameter η and, consequently, is independent of the cosmological constant, it is a *common* classical r-matrix for the *three* DD structures $\mathfrak{so}(2,2)$, $\mathfrak{so}(3,1)$ and $\mathfrak{iso}(2,1)$ on the (2+1)-dimensional Lorentzian spacetimes $\mathbf{AdS}^{2+1}$, $\mathbf{dS}^{2+1}$ and $\mathbf{M}^{2+1}$, the latter being obtained from the limit $\eta \to 0$.

3.4 The AdS_ω Drinfel'd double in the kinematical basis

We now analyse the DD structures of the Lie algebra AdS_ω in the kinematical basis $\{P_0, \mathbf{P}, J, \mathbf{K}\}$ with commutation relations (2.1). Recall that AdS_ω gives a unified description of the three Lie algebras $\mathfrak{so}(2,2)$, $\mathfrak{so}(3,1)$ and $\mathfrak{iso}(2,1)$, which are, accordingly, parametrised by the constant sectional curvature ω of their corresponding homogeneous spacetimes $\mathbf{AdS}^{2+1}$, $\mathbf{dS}^{2+1}$ and $\mathbf{M}^{2+1}$. In particular, we show that the classical r-matrix (3.11) defines *two* different quantum DD deformations that, according to [11, 12], we shall call *space-like* and *time-like* deformations.

3.4.1 The space-like r-matrix

It is easy to see that the change of basis

$$P_0 = T_0, \qquad P_1 = T_1, \qquad P_2 = T_2, \qquad K_1 = J_2, \qquad K_2 = -J_1, \qquad J = J_0, \tag{3.12}$$

transforms (3.7) into (2.1) provided that $\omega = -\Lambda$. Consequently, the deformation parameter η above coincides with the one introduced in Section 2 in the form $\omega = \eta^2$, assuming that η is a real number for AdS but a purely imaginary one for dS. Using (3.12), one finds that the classical r-matrix (3.11) is given by

$$r'_{\mathrm{F}} = r'_{\mathrm{C}} = \tfrac{1}{2}(P_0 \wedge K_2 + P_1 \wedge J) + \tfrac{1}{2} K_1 \wedge P_2. \tag{3.13}$$

To construct the associated quantum deformation, we scale this r-matrix by a quantum deformation parameter z (different from η) as

$$r_z^{\mathrm{space}} = 2z r'_{\mathrm{F}} = z(P_0 \wedge K_2 + P_1 \wedge J) + z K_1 \wedge P_2, \tag{3.14}$$

This allows us to relate (3.14) to the results in [12]. It becomes apparent that the first term in (3.14) is just a *space-like* r-matrix of type (a) for AdS_ω, while the second one is a twist. From (3.5) it follows that the primitive (non-deformed) generators are P_2 and K_1. Note that for the space-like r-matrix the physical dimension of z is determined by the generator P_2 of spatial translations $[z] = [P_2]^{-1}$, which implies that z has the dimension of a *length*. It is related to the usual deformation parameters κ and q by

$$z = 1/\kappa, \qquad q = \mathrm{e}^z. \tag{3.15}$$

The classical limit of the quantum deformation, therefore, corresponds to $z \to 0$ ($\kappa \to \infty$, $q \to 1$). Accordingly, we shall call (3.14) the *twisted κ-space-like r-matrix*.

Moreover, the Lie algebra isomorphism $\mathfrak{so}(2,2) \simeq \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$ suggests that a (real) change of basis should exist between the AdS_ω basis (2.1) and that of two copies of $\mathfrak{sl}(2,\mathbb{R})$ when $\omega = \eta^2 > 0$ ($\eta \in \mathbb{R}$). A direct computation shows that such an isomorphism is given by

$$\begin{aligned}
&P_0 = \frac{\sqrt{\eta}}{2\sqrt{2}} \left(J_+^1 + J_+^2 - 2\eta\big(J_-^1 + J_-^2\big)\right), \qquad J = \frac{1}{2\sqrt{2\eta}} \left(J_+^1 - J_+^2 + 2\eta\big(-J_-^1 + J_-^2\big)\right), \\
&P_1 = \frac{\sqrt{\eta}}{2\sqrt{2}} \left(J_+^1 + J_+^2 + 2\eta\big(J_-^1 + J_-^2\big)\right), \qquad K_1 = \frac{1}{2}\big(J_3^1 + J_3^2\big), \\
&P_2 = \frac{\eta}{2}\big(J_3^1 - J_3^2\big), \qquad K_2 = \frac{1}{2\sqrt{2\eta}} \left(-J_+^1 + J_+^2 + 2\eta\big(-J_-^1 + J_-^2\big)\right),
\end{aligned}$$

where

$$[J_3^l, J_\pm^l] = \pm 2 J_\pm^l, \qquad [J_+^l, J_-^l] = J_3^l, \qquad l = 1, 2,$$

and all other Lie brackets vanish. In this basis, the classical r-matrix (3.13) reads

$$r'_F = \frac{\eta}{2}\big(-J^1_+ \wedge J^1_- + J^2_+ \wedge J^2_-\big) - \frac{\eta}{4} J^1_3 \wedge J^2_3. \tag{3.16}$$

It becomes apparent that it is the superposition of two standard (Drinfel'd–Jimbo [34, 42]) deformations, $J^l_+ \wedge J^l_-$, on each of the two copies of $\mathfrak{sl}(2,\mathbb{R})$ but with *opposite sign*, and of a twist generated by $J^1_3 \wedge J^2_3$. This identification $U_z(\mathfrak{so}(2,2)) \simeq U_z(\mathfrak{sl}(2,\mathbb{R})) \oplus U_{-z}(\mathfrak{sl}(2,\mathbb{R}))$ was first introduced in [25, 26] and further investigated in [14].

3.4.2 The time-like r-matrix

Alternatively, the classical r-matrix (3.11) can be expressed as a superposition of the *time-like* r-matrix for AdS_ω [12] with a twist. This requires the complex change of basis

$$\begin{aligned} &P_0 = \mathrm{i}T_2, \qquad P_1 = -\mathrm{i}T_0, \qquad P_2 = -T_1,\\ &K_1 = J_1, \qquad K_2 = -\mathrm{i}J_0, \qquad J = -\mathrm{i}J_2, \end{aligned} \tag{3.17}$$

which again transforms (3.7) into (2.1) with $\omega = -\Lambda$. The classical r-matrix (3.11) then takes the form

$$r'_\mathrm{F} = r'_\mathrm{C} = \tfrac{\mathrm{i}}{2}(K_1 \wedge P_1 + K_2 \wedge P_2) + \tfrac{1}{2} J \wedge P_0.$$

Introducing again the quantum deformation parameter (3.15) via a rescaling we can write

$$r_z^{\mathrm{time}} = -2\mathrm{i}z r'_\mathrm{F} = z(K_1 \wedge P_1 + K_2 \wedge P_2) - \mathrm{i}zJ \wedge P_0. \tag{3.18}$$

In this case, the first term in (3.18) is the *time-like* r-matrix of type (b) for AdS_ω [12], which coincides with the κ-Poincaré r-matrix [51, 57] when $\omega = \Lambda = \eta = 0$, and the second one is the twist. The primitive generators are now P_0 and J and the dimensions of z are given by those of the generator of time translations P_0 as $[z] = [P_0]^{-1}$. This implies that z has the dimension of a *time*. Accordingly, we shall call (3.18) the *twisted κ-time-like r-matrix*.

Similarly, we can also consider the isomorphism $\mathfrak{so}(2,2) \simeq \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$, and apply the (complex) change of basis

$$\begin{aligned} &P_0 = \mathrm{i}\frac{\eta}{2}\big(J^1_3 - J^2_3\big), \qquad J = -\frac{\mathrm{i}}{2}\big(J^1_3 + J^2_3\big),\\ &P_1 = -\frac{\mathrm{i}}{2}\sqrt{\frac{\eta}{2}}\left(J^1_+ + J^2_+ - 2\eta\big(J^1_- + J^2_-\big)\right), \qquad K_1 = \frac{1}{2\sqrt{2\eta}}\left(J^1_+ - J^2_+ + 2\eta\big(J^1_- - J^2_-\big)\right),\\ &P_2 = -\frac{\mathrm{i}}{2}\sqrt{\frac{\eta}{2}}\left(J^1_+ + J^2_+ + 2\eta\big(J^1_- + J^2_-\big)\right), \qquad K_2 = -\frac{1}{2\sqrt{2\eta}}\left(J^1_+ - J^2_+ - 2\eta\big(J^1_- - J^2_-\big)\right), \end{aligned}$$

which gives rise to the same r-matrix (3.16).

4 Lie bialgebra of the twisted κ-AdS_ω algebra

So far, we have obtained a common classical r-matrix (3.11) from DD structures for the three Lie algebras that form the family AdS_ω. Moreover, we have identified two physically distinct quantum deformations: the twisted κ-space-like deformation and the twisted κ-time-like one. As the latter (without the twist motivated by (2+1)-gravity) has been widely studied in the literature due to the role of the deformation parameter $z = 1/\kappa$ as a fundamental time or energy scale possibly related to the Planck length, we construct its full quantisation in the following. Note, however, that in all the expressions to be presented in the sequel, it will always be possible to obtain the corresponding κ-space-like counterparts by simply applying the map between both bases provided by (3.12) and (3.17), namely

$$\begin{array}{llll}\text{time-like} \to \text{space-like:} & P_0 \to \mathrm{i}P_2, & P_1 \to -\mathrm{i}P_0, & P_2 \to -P_1, \\ & J \to -\mathrm{i}K_1, & K_1 \to -K_2, & K_2 \to -\mathrm{i}J. \end{array}$$

Moreover, in order to highlight the effect of $J \wedge P_0$ in the twisted κ-time-like deformation of AdS_ω with classical r-matrix (3.18), we shall consider the *two-parameter* classical r-matrix given by

$$r = z(K_1 \wedge P_1 + K_2 \wedge P_2) + \vartheta J \wedge P_0, \tag{4.1}$$

where ϑ is a generic deformation parameter associated to the twist, that for $\vartheta = -\mathrm{i}z$ yields the underlying DD structure. We also recall that the r-matrix (4.1) arises as a particular case within the classification of deformations of AdS_ω given in [18] for

$$\begin{array}{lll} a_i = b_i = 0, \quad i = 1, 2, 3, 4, & a_5 = b_5 = -z, & a_6 = b_6 = 0, \\ c_1 = \vartheta, \qquad c_2 = c_3 = 0, & & \end{array}$$

which is a solution of the modified classical Yang–Baxter equation with Schouten bracket

$$[[r, r]] = -z^2(P_0 \wedge P_1 \wedge K_1 + P_0 \wedge P_2 \wedge K_2 + P_1 \wedge P_2 \wedge J) - z^2\omega K_1 \wedge K_2 \wedge J.$$

The first term is just the Schouten bracket for the κ-Poincaré r-matrix, while the second one includes the effect of the curvature or cosmological constant in the AdS and dS cases with $\omega \neq 0$. As expected, the twist does not affect the Schouten bracket.

The Lie bialgebra $(\mathrm{AdS}_\omega, \delta)$ generated by (4.1) can be computed via (3.5), which yields the cocommutator

$$\begin{aligned} &\delta(P_0) = \delta(J) = 0, \\ &\delta(P_1) = z(P_1 \wedge P_0 - \omega K_2 \wedge J) + \vartheta(P_0 \wedge P_2 + \omega K_1 \wedge J), \\ &\delta(P_2) = z(P_2 \wedge P_0 + \omega K_1 \wedge J) - \vartheta(P_0 \wedge P_1 - \omega K_2 \wedge J), \\ &\delta(K_1) = z(K_1 \wedge P_0 + P_2 \wedge J) + \vartheta(P_0 \wedge K_2 - P_1 \wedge J), \\ &\delta(K_2) = z(K_2 \wedge P_0 - P_1 \wedge J) - \vartheta(P_0 \wedge K_1 + P_2 \wedge J). \end{aligned} \tag{4.2}$$

Denoting by $\{x_0, \mathbf{x}, \theta, \boldsymbol{\xi}\}$ the dual non-commutative coordinates of the generators $\{P_0, \mathbf{P}, J, \mathbf{K}\}$, respectively, one obtains from the cocommutators (4.2) the following dual Lie brackets between the non-commutative spacetime coordinates

$$[\hat{x}_0, \hat{x}_1] = -z\hat{x}_1 - \vartheta\hat{x}_2, \qquad [\hat{x}_0, \hat{x}_2] = -z\hat{x}_2 + \vartheta\hat{x}_1, \qquad [\hat{x}_1, \hat{x}_2] = 0, \tag{4.3}$$

as well as

$$\begin{aligned} &[\hat{\theta}, \hat{x}_i] = z\epsilon_{ij}\hat{\xi}_j + \vartheta\hat{\xi}_i \qquad [\hat{\theta}, \hat{\xi}_i] = -\omega\left(z\epsilon_{ij}\hat{x}_j + \vartheta\hat{x}_i\right), \qquad [\hat{\theta}, \hat{x}_0] = 0, \\ &[\hat{x}_0, \hat{\xi}_i] = -z\hat{\xi}_i - \vartheta\epsilon_{ij}\hat{\xi}_j, \qquad [\hat{\xi}_1, \hat{\xi}_2] = 0, \qquad [\hat{x}_i, \hat{\xi}_j] = 0, \qquad i, j = 1, 2. \end{aligned} \tag{4.4}$$

Note that the expressions (4.3) are just the *first-order* of the non-commutative twisted κ-AdS_ω spacetimes, that will be constructed in the following sections. Nevertheless, these expressions are sufficient to analyse the role of both the cosmological constant $\Lambda = -\omega$ and the twist with quantum parameter ϑ and to compare them to the well-known κ-Minkowski spacetime (see [9, 49, 51, 57, 64] and references therein) that is given by

$$[\hat{x}_0, \hat{x}_1] = -z\hat{x}_1, \qquad [\hat{x}_0, \hat{x}_2] = -z\hat{x}_2, \qquad [\hat{x}_1, \hat{x}_2] = 0, \qquad z = 1/\kappa. \tag{4.5}$$

It was already shown in [10, 18] that, at the first-order in the quantum coordinates, both non-commutative AdS and dS spacetimes coincide with the κ-Minkowski one (4.5) and that (see [10]) the parameter $\omega = \eta^2$ only enters the higher-order terms of the corresponding quantum deformation.

5 The twisted κ-AdS$_\omega$ quantum algebra

In this section we construct the twisted κ-AdS$_\omega$ algebra with underlying Lie bialgebra given by (2.1) and (4.2). We first present this algebra in the so-called *symmetrical* kinematical basis [10, 18], in which the corresponding algebra without the ϑ-twist was obtained in [12]. We then present the twisted κ-AdS$_\omega$ quantum deformation in a *bicrossproduct-type* basis, which is characterised by the fact that its limit $\omega \to 0$ gives rise to the κ-Poincaré algebra endowed with a proper bicrossproduct structure [54, 55] (see also [4, 31, 32]). Finally, we compare these results to quantum deformations investigated in the literature [28].

5.1 "Symmetrical" basis

We start by recalling the expressions for the quantum κ-AdS$_\omega$ algebra in terms of the kinematical basis (2.1), as obtained in [12]. The corresponding coproduct and compatible deformed commutation rules for the κ-AdS$_\omega$ algebra read

$$
\begin{aligned}
&\Delta_z(P_0) = 1\otimes P_0 + P_0\otimes 1, \qquad \Delta_z(J) = 1\otimes J + J\otimes 1,\\
&\Delta_z(P_i) = \mathrm{e}^{-\frac{z}{2}P_0}\cosh\left(\frac{z}{2}\eta J\right)\otimes P_i + P_i\otimes \mathrm{e}^{\frac{z}{2}P_0}\cosh\left(\frac{z}{2}\eta J\right)\\
&\qquad\qquad + \eta \mathrm{e}^{-\frac{z}{2}P_0}\sinh\left(\frac{z}{2}\eta J\right)\otimes \epsilon_{ij}K_j - \eta\epsilon_{ij}K_j\otimes \mathrm{e}^{\frac{z}{2}P_0}\sinh\left(\frac{z}{2}\eta J\right),\\
&\Delta_z(K_i) = \mathrm{e}^{-\frac{z}{2}P_0}\cosh\left(\frac{z}{2}\eta J\right)\otimes K_i + K_i\otimes \mathrm{e}^{\frac{z}{2}P_0}\cosh\left(\frac{z}{2}\eta J\right)\\
&\qquad\qquad - \mathrm{e}^{-\frac{z}{2}P_0}\frac{\sinh(\frac{z}{2}\eta J)}{\eta}\otimes \epsilon_{ij}P_j + \epsilon_{ij}P_j\otimes \mathrm{e}^{\frac{z}{2}P_0}\frac{\sinh(\frac{z}{2}\eta J)}{\eta},
\end{aligned}
\tag{5.1}
$$

$$
\begin{aligned}
&[J,P_i] = \epsilon_{ij}P_j, \qquad [J,K_i] = \epsilon_{ij}K_j, \qquad [J,P_0] = 0,\\
&[P_i,K_j] = -\delta_{ij}\frac{\sinh(zP_0)}{z}\cosh(z\eta J), \qquad [P_0,K_i] = -P_i, \qquad [P_0,P_i] = \omega K_i,\\
&[P_1,P_2] = -\omega\cosh(zP_0)\frac{\sinh(z\eta J)}{z\eta}, \qquad [K_1,K_2] = -\cosh(zP_0)\frac{\sinh(z\eta J)}{z\eta},
\end{aligned}
\tag{5.2}
$$

with $\omega = \eta^2 = -\Lambda$. The quantum deformation of the two Casimir invariants (2.2) reads

$$
\begin{aligned}
\mathcal{C} &= 4\cos(z\eta)\left\{\frac{\sinh^2(\frac{z}{2}P_0)}{z^2}\cosh^2\left(\frac{z}{2}\eta J\right) + \frac{\sinh^2(\frac{z}{2}\eta J)}{z^2}\cosh^2\left(\frac{z}{2}P_0\right)\right\}\\
&\quad - \frac{\sin(z\eta)}{z\eta}\left(\mathbf{P}^2 + \omega\mathbf{K}^2\right),\\
\mathcal{W} &= -\cos(z\eta)\frac{\sinh(z\eta J)}{z\eta}\frac{\sinh(zP_0)}{z} + \frac{\sin(z\eta)}{z\eta}(K_1P_2 - K_2P_1).
\end{aligned}
\tag{5.3}
$$

With these results, we construct the twisted (ϑ,κ)-quantum AdS$_\omega$ algebra with general twist parameter ϑ by applying the well-known twisting procedure [33]. This preserves the deformed commutation relations (5.2) and hence the Casimir invariants (5.3). The new coproduct $\Delta_{\vartheta,z}$ is obtained by twisting (5.1) with an element $\mathcal{F}_\vartheta \in \kappa\text{-AdS}_\omega\otimes\kappa\text{-AdS}_\omega$ given by

$$
\Delta_{\vartheta,z}(Y) = \mathcal{F}_\vartheta\Delta_z(Y)\mathcal{F}_\vartheta^{-1} \quad \forall Y\in\kappa\text{-AdS}_\omega, \qquad \text{where} \quad \mathcal{F}_\vartheta = \exp(-\vartheta J\wedge P_0). \tag{5.4}
$$

It satisfies the so-called twisting co-cycle and normalisation conditions

$$
\mathcal{F}_{\vartheta,12}(\Delta_z\otimes\mathrm{id})\mathcal{F}_\vartheta = \mathcal{F}_{\vartheta,23}(\mathrm{id}\otimes\Delta_z)\mathcal{F}_\vartheta, \qquad (\epsilon\otimes\mathrm{id})\mathcal{F}_\vartheta = 1 = (\mathrm{id}\otimes\epsilon)\mathcal{F}_\vartheta,
$$

where $\mathcal{F}_{\vartheta,12} = \mathcal{F}_\vartheta \otimes \mathrm{id}$, $\mathcal{F}_{\vartheta,23} = \mathrm{id} \otimes \mathcal{F}_\vartheta$ and ϵ is the co-unit map, $\epsilon(Y) = 0\ \forall Y \in \mathrm{AdS}_\omega$. The explicit form of the twisted coproduct is obtained through cumbersome computations and reads

$$
\begin{aligned}
\Delta_{\vartheta,z}(P_0) &= 1\otimes P_0 + P_0\otimes 1, \qquad \Delta_{\vartheta,z}(J) = 1\otimes J + J\otimes 1,\\
\Delta_{\vartheta,z}(P_i) &= \Delta_z(P_i) + \mathrm{e}^{-\frac{z}{2}P_0}\cosh(\tfrac{z}{2}\eta J)\,[\cos(\vartheta\eta J)\cos(\vartheta P_0)-1]\otimes P_i\\
&\quad + \mathrm{e}^{-\frac{z}{2}P_0}\cosh(\tfrac{z}{2}\eta J)\sin(\vartheta P_0)\cos(\vartheta\eta J)\otimes\epsilon_{ij}P_j\\
&\quad - \eta\mathrm{e}^{-\frac{z}{2}P_0}\cosh(\tfrac{z}{2}\eta J)\sin(\vartheta\eta J)\cos(\vartheta P_0)\otimes K_i\\
&\quad - \eta\mathrm{e}^{-\frac{z}{2}P_0}\cosh(\tfrac{z}{2}\eta J)\sin(\vartheta\eta J)\sin(\vartheta P_0)\otimes\epsilon_{ij}K_j\\
&\quad + P_i\otimes\mathrm{e}^{\frac{z}{2}P_0}\cosh(\tfrac{z}{2}\eta J)\,[\cos(\vartheta\eta J)\cos(\vartheta P_0)-1]\\
&\quad - \epsilon_{ij}P_j\otimes\mathrm{e}^{\frac{z}{2}P_0}\cosh(\tfrac{z}{2}\eta J)\sin(\vartheta P_0)\cos(\vartheta\eta J)\\
&\quad + \eta K_i\otimes\mathrm{e}^{\frac{z}{2}P_0}\cosh(\tfrac{z}{2}\eta J)\sin(\vartheta\eta J)\cos(\vartheta P_0)\\
&\quad - \eta\epsilon_{ij}K_j\otimes\mathrm{e}^{\frac{z}{2}P_0}\cosh(\tfrac{z}{2}\eta J)\sin(\vartheta\eta J)\sin(\vartheta P_0)\\
&\quad - \mathrm{e}^{-\frac{z}{2}P_0}\sinh(\tfrac{z}{2}\eta J)\sin(\vartheta\eta J)\sin(\vartheta P_0)\otimes P_i\\
&\quad + \mathrm{e}^{-\frac{z}{2}P_0}\sinh(\tfrac{z}{2}\eta J)\sin(\vartheta\eta J)\cos(\vartheta P_0)\otimes\epsilon_{ij}P_j\\
&\quad - \eta\mathrm{e}^{-\frac{z}{2}P_0}\sinh(\tfrac{z}{2}\eta J)\sin(\vartheta P_0)\cos(\vartheta\eta J)\otimes K_i\\
&\quad + \eta\mathrm{e}^{-\frac{z}{2}P_0}\sinh(\tfrac{z}{2}\eta J)[\cos(\vartheta\eta J)\cos(\vartheta P_0)-1]\otimes\epsilon_{ij}K_j\\
&\quad + P_i\otimes\mathrm{e}^{\frac{z}{2}P_0}\sinh(\tfrac{z}{2}\eta J)\sin(\vartheta\eta J)\sin(\vartheta P_0)\\
&\quad + \epsilon_{ij}P_j\otimes\mathrm{e}^{\frac{z}{2}P_0}\sinh(\tfrac{z}{2}\eta J)\sin(\vartheta\eta J)\cos(\vartheta P_0)\\
&\quad - \eta K_i\otimes\mathrm{e}^{\frac{z}{2}P_0}\sinh(\tfrac{z}{2}\eta J)\sin(\vartheta P_0)\cos(\vartheta\eta J)\\
&\quad - \eta\epsilon_{ij}K_j\otimes\mathrm{e}^{\frac{z}{2}P_0}\sinh(\tfrac{z}{2}\eta J)\,[\cos(\vartheta\eta J)\cos(\vartheta P_0)-1]\,,\\
\Delta_{\vartheta,z}(K_i) &= \Delta_z(K_i) + \mathrm{e}^{-\frac{z}{2}P_0}\cosh(\tfrac{z}{2}\eta J)\,[\cos(\vartheta P_0)\cos(\vartheta\eta J)-1]\otimes K_i\\
&\quad + \mathrm{e}^{-\frac{z}{2}P_0}\cosh(\tfrac{z}{2}\eta J)\sin(\vartheta P_0)\cos(\vartheta\eta J)\otimes\epsilon_{ij}K_j\\
&\quad + \mathrm{e}^{-\frac{z}{2}P_0}\cosh(\tfrac{z}{2}\eta J)\frac{\sin(\vartheta\eta J)}{\eta}\cos(\vartheta P_0)\otimes P_i\\
&\quad + \mathrm{e}^{-\frac{z}{2}P_0}\cosh(\tfrac{z}{2}\eta J)\frac{\sin(\vartheta\eta J)}{\eta}\sin(\vartheta P_0)\otimes\epsilon_{ij}P_j\\
&\quad + K_i\otimes\mathrm{e}^{\frac{z}{2}P_0}\cosh(\tfrac{z}{2}\eta J)\,[\cos(\vartheta\eta J)\cos(\vartheta P_0)-1]\\
&\quad - \epsilon_{ij}K_j\otimes\mathrm{e}^{\frac{z}{2}P_0}\cosh(\tfrac{z}{2}\eta J)\sin(\vartheta P_0)\cos(\vartheta\eta J)\\
&\quad - P_i\otimes\mathrm{e}^{\frac{z}{2}P_0}\cosh(\tfrac{z}{2}\eta J)\frac{\sin(\vartheta\eta J)}{\eta}\cos(\vartheta P_0)\\
&\quad + \epsilon_{ij}P_j\otimes\mathrm{e}^{\frac{z}{2}P_0}\cosh(\tfrac{z}{2}\eta J)\frac{\sin(\vartheta\eta J)}{\eta}\sin(\vartheta P_0)\\
&\quad - \mathrm{e}^{-\frac{z}{2}P_0}\sinh(\tfrac{z}{2}\eta J)\sin(\vartheta\eta J)\sin(\vartheta P_0)\otimes K_i\\
&\quad + \mathrm{e}^{-\frac{z}{2}P_0}\sinh(\tfrac{z}{2}\eta J)\sin(\vartheta\eta J)\cos(\vartheta P_0)\otimes\epsilon_{ij}K_j\\
&\quad + \mathrm{e}^{-\frac{z}{2}P_0}\frac{\sinh(\frac{z}{2}\eta J)}{\eta}\sin(\vartheta P_0)\cos(\vartheta\eta J)\otimes P_i\\
&\quad - \mathrm{e}^{-\frac{z}{2}P_0}\frac{\sinh(\frac{z}{2}\eta J)}{\eta}[\cos(\vartheta\eta J)\cos(\vartheta P_0)-1]\otimes\epsilon_{ij}P_j\\
&\quad + K_i\otimes\mathrm{e}^{\frac{z}{2}P_0}\sinh(\tfrac{z}{2}\eta J)\sin(\vartheta\eta J)\sin(\vartheta P_0)\\
&\quad + \epsilon_{ij}K_j\otimes\mathrm{e}^{\frac{z}{2}P_0}\sinh(\tfrac{z}{2}\eta J)\sin(\vartheta\eta J)\cos(\vartheta P_0)
\end{aligned}
$$

$$
\begin{aligned}
&+ P_i \otimes \mathrm{e}^{\frac{z}{2}P_0} \frac{\sinh(\frac{z}{2}\eta J)}{\eta} \sin(\vartheta P_0)\cos(\vartheta\eta J) \\
&+ \epsilon_{ij} P_j \otimes \mathrm{e}^{\frac{z}{2}P_0} \frac{\sinh(\frac{z}{2}\eta J)}{\eta} \left[\cos(\vartheta\eta J)\cos(\vartheta P_0) - 1\right],
\end{aligned}
$$

for $i,j = 1,2$. Note that the limit $\vartheta \to 0$ is always well-defined and gives the "untwisted" coproduct Δ_z in (5.1).

5.2 "Bicrossproduct-type" basis

Similarly to the previous subsection we now consider the κ-AdS$_\omega$ algebra expressed in the "bicrossproduct-type" basis introduced in [10]. In this basis, the coproduct is given by

$$
\begin{aligned}
&\Delta_z(P_0) = 1\otimes P_0 + P_0 \otimes 1, \qquad \Delta_z(J) = 1\otimes J + J\otimes 1, \\
&\Delta_z(P_i) = \mathrm{e}^{-zP_0}\otimes P_i + P_i \otimes \cosh(z\eta J) - \eta\epsilon_{ij} K_j \otimes \sinh(z\eta J), \\
&\Delta_z(K_i) = \mathrm{e}^{-zP_0}\otimes K_i + K_i \otimes \cosh(z\eta J) + \epsilon_{ij} P_j \otimes \frac{\sinh(z\eta J)}{\eta},
\end{aligned}
\tag{5.5}
$$

and the deformed commutation rules read

$$
\begin{aligned}
&[J,P_i] = \epsilon_{ij}P_j, \qquad [J,K_i] = \epsilon_{ij}K_j, \qquad [J,P_0] = 0, \qquad [P_0,K_i] = -P_i, \\
&[P_0,P_i] = \omega K_i, \qquad [P_1,P_2] = -\omega\frac{\sinh(2z\eta J)}{2z\eta}, \qquad [K_1,K_2] = -\frac{\sinh(2z\eta J)}{2z\eta}, \\
&[P_i,K_j] = \delta_{ij}\left\{ \frac{\mathrm{e}^{-2zP_0} - \cosh(2z\eta J)}{2z} - \frac{\tan(z\eta)}{2\eta}\left(\mathbf{P}^2 + \omega\mathbf{K}^2\right)\right\} + \frac{\tan(z\eta)}{\eta}(P_jP_i + \omega K_iK_j),
\end{aligned}
$$

while the deformed Casimir invariants are given by

$$
\begin{aligned}
\mathcal{C} = {}& 4\cos(z\eta)\left\{ \frac{\sinh^2(\frac{z}{2}P_0)}{z^2}\cosh^2\left(\frac{z}{2}\eta J\right) + \frac{\sinh^2(\frac{z}{2}\eta J)}{z^2}\cosh^2\left(\frac{z}{2}P_0\right)\right\} \\
&- \frac{\sin(z\eta)}{z\eta}\mathrm{e}^{zP_0}\left\{\cosh(z\eta J)\left(\mathbf{P}^2+\omega\mathbf{K}^2\right) - 2\eta\sinh(z\eta J)(K_1P_2 - K_2P_1)\right\}, \\
\mathcal{W} = {}& -\cos(z\eta)\frac{\sinh(z\eta J)}{z\eta}\frac{\sinh(zP_0)}{z} \\
&+ \frac{\sin(z\eta)}{z\eta}\mathrm{e}^{zP_0}\left\{\cosh(z\eta J)(K_1P_2 - K_2P_1) - \frac{\sinh(z\eta J)}{2\eta}\left(\mathbf{P}^2+\omega\mathbf{K}^2\right)\right\}.
\end{aligned}
$$

By applying the twist (5.4) to (5.5) we obtain the two-parameter coproduct

$$
\begin{aligned}
&\Delta_{\vartheta,z}(P_0) = 1\otimes P_0 + P_0\otimes 1, \qquad \Delta_{\vartheta,z}(J) = 1\otimes J + J\otimes 1, \\
&\Delta_{\vartheta,z}(P_i) = \Delta_z(P_i) + \mathrm{e}^{-zP_0}[\cos(\vartheta\eta J)\cos(\vartheta P_0) - 1]\otimes P_i + \mathrm{e}^{-zP_0}\sin(\vartheta P_0)\cos(\vartheta\eta J)\otimes\epsilon_{ij}P_j \\
&\quad - \eta\mathrm{e}^{-zP_0}\sin(\vartheta\eta J)\cos(\vartheta P_0)\otimes K_i - \eta\mathrm{e}^{-zP_0}\sin(\vartheta\eta J)\sin(\vartheta P_0)\otimes\epsilon_{ij}K_j \\
&\quad + P_i\otimes\cosh(z\eta J)\left[\cos(\vartheta\eta J)\cos(\vartheta P_0) - 1\right] - \epsilon_{ij}P_j\otimes\cosh(z\eta J)\sin(\vartheta P_0)\cos(\vartheta\eta J) \\
&\quad + \eta K_i\otimes\cosh(z\eta J)\sin(\vartheta\eta J)\cos(\vartheta P_0) - \eta\epsilon_{ij}K_j\otimes\cosh(z\eta J)\sin(\vartheta\eta J)\sin(\vartheta P_0) \\
&\quad + P_i\otimes\sinh(z\eta J)\sin(\vartheta\eta J)\sin(\vartheta P_0) + \epsilon_{ij}P_j\otimes\sinh(z\eta J)\sin(\vartheta\eta J)\cos(\vartheta P_0) \\
&\quad - \eta K_i\otimes\sinh(z\eta J)\sin(\vartheta P_0)\cos(\vartheta\eta J) - \eta\epsilon_{ij}K_j\otimes\sinh(z\eta J)[\cos(\vartheta\eta J)\cos(\vartheta P_0) - 1], \\
&\Delta_{\vartheta,z}(K_i) = \Delta_z(K_i) + \mathrm{e}^{-zP_0}[\cos(\vartheta P_0)\cos(\vartheta\eta J) - 1]\otimes K_i + \mathrm{e}^{-zP_0}\sin(\vartheta P_0)\cos(\vartheta\eta J)\otimes\epsilon_{ij}K_j \\
&\quad + \mathrm{e}^{-zP_0}\frac{\sin(\vartheta\eta J)}{\eta}\cos(\vartheta P_0)\otimes P_i + \mathrm{e}^{-zP_0}\frac{\sin(\vartheta\eta J)}{\eta}\sin(\vartheta P_0)\otimes\epsilon_{ij}P_j
\end{aligned}
$$

$$
\begin{aligned}
&+ K_i \otimes \cosh(z\eta J)[\cos(\vartheta\eta J)\cos(\vartheta P_0) - 1] - \epsilon_{ij} K_j \otimes \cosh(z\eta J)\sin(\vartheta P_0)\cos(\vartheta\eta J) \\
&- P_i \otimes \cosh(z\eta J)\frac{\sin(\vartheta\eta J)}{\eta}\cos(\vartheta P_0) + \epsilon_{ij} P_j \otimes \cosh(z\eta J)\frac{\sin(\vartheta\eta J)}{\eta}\sin(\vartheta P_0) \\
&+ K_i \otimes \sinh(z\eta J)\sin(\vartheta\eta J)\sin(\vartheta P_0) + \epsilon_{ij} K_j \otimes \sinh(z\eta J)\sin(\vartheta\eta J)\cos(\vartheta P_0) \\
&+ P_i \otimes \frac{\sinh(z\eta J)}{\eta}\sin(\vartheta P_0)\cos(\vartheta\eta J) + \epsilon_{ij} P_j \otimes \frac{\sinh(z\eta J)}{\eta}[\cos(\vartheta\eta J)\cos(\vartheta P_0) - 1].
\end{aligned}
$$

Again, the untwisted coproduct (5.5) is straightforwardly recovered by setting $\vartheta = 0$.

As expected, there exists a nonlinear mapping between the "symmetrical" and the "bicrossproduct" bases. This is just the (z, ϑ)-generalisation of the invertible nonlinear map introduced for κ-Poincaré in [55] and coincides with the one given in [10] for the quantum κ-AdS$_\omega$ algebra without the ϑ-twist. If we denote by $\tilde{Y}_i$ the generators of the quantum twisted κ-AdS$_\omega$ algebra expressed in the above "bicrossproduct-type" basis and by Y_i the corresponding generators in the "symmetrical" basis of Section 5.1, then the nonlinear map between both bases is given by

$$
\begin{aligned}
&\tilde{P}_0 = P_0, \qquad \tilde{J} = J, \\
&\tilde{P}_i = \mathrm{e}^{-\frac{z}{2}P_0}\left(\cosh\left(\frac{z}{2}\eta J\right)P_i - \eta\sinh\left(\frac{z}{2}\eta J\right)\epsilon_{ij}K_j\right), \\
&\tilde{K}_i = \mathrm{e}^{-\frac{z}{2}P_0}\left(\cosh\left(\frac{z}{2}\eta J\right)K_i + \frac{\sinh(\frac{z}{2}\eta J)}{\eta}\epsilon_{ij}P_j\right).
\end{aligned}
\tag{5.6}
$$

5.3 Quantum twisted κ-Poincaré algebra

The expressions above possess a well-defined limit $\omega = \eta^2 \to 0$ which yields the corresponding structures for the Poincaré algebra and can be compared with the results obtained in [28]. In particular, the "symmetrical" basis gives rise to the twisted κ-Poincaré algebra with coproduct

$$
\begin{aligned}
\Delta_{\vartheta,z}(P_0) &= 1 \otimes P_0 + P_0 \otimes 1, \qquad \Delta_{\vartheta,z}(J) = 1 \otimes J + J \otimes 1, \\
\Delta_{\vartheta,z}(P_i) &= \Delta_z(P_i) + P_i \otimes \mathrm{e}^{\frac{z}{2}P_0}[\cos(\vartheta P_0) - 1] + \mathrm{e}^{-\frac{z}{2}P_0}[\cos(\vartheta P_0) - 1] \otimes P_i \\
&\quad - \epsilon_{ij}P_j \otimes \mathrm{e}^{\frac{z}{2}P_0}\sin(\vartheta P_0) + \mathrm{e}^{-\frac{z}{2}P_0}\sin(\vartheta P_0) \otimes \epsilon_{ij}P_j, \\
\Delta_{\vartheta,z}(K_i) &= \Delta_z(K_i) + K_i \otimes \mathrm{e}^{\frac{z}{2}P_0}[\cos(\vartheta P_0) - 1] + \mathrm{e}^{-\frac{z}{2}P_0}[\cos(\vartheta P_0) - 1] \otimes K_i \\
&\quad - \epsilon_{ij}K_j \otimes \mathrm{e}^{\frac{z}{2}P_0}\sin(\vartheta P_0) + \mathrm{e}^{-\frac{z}{2}P_0}\sin(\vartheta P_0) \otimes \epsilon_{ij}K_j \\
&\quad - \vartheta P_i \otimes \mathrm{e}^{\frac{z}{2}P_0}J\cos(\vartheta P_0) + \vartheta\mathrm{e}^{-\frac{z}{2}P_0}J\cos(\vartheta P_0) \otimes P_i \\
&\quad + \vartheta\epsilon_{ij}P_j \otimes \mathrm{e}^{\frac{z}{2}P_0}J\sin(\vartheta P_0) + \vartheta\mathrm{e}^{-\frac{z}{2}P_0}J\sin(\vartheta P_0) \otimes \epsilon_{ij}P_j \\
&\quad + \frac{z}{2}P_i \otimes \mathrm{e}^{\frac{z}{2}P_0}J\sin(\vartheta P_0) + \frac{z}{2}\mathrm{e}^{-\frac{z}{2}P_0}J\sin(\vartheta P_0) \otimes P_i \\
&\quad + \frac{z}{2}\epsilon_{ij}P_j \otimes \mathrm{e}^{\frac{z}{2}P_0}J[\cos(\vartheta P_0) - 1] - \frac{z}{2}\mathrm{e}^{-\frac{z}{2}P_0}J[\cos(\vartheta P_0) - 1] \otimes \epsilon_{ij}P_j,
\end{aligned}
$$

deformed commutation rules

$$
\begin{aligned}
&[J, P_i] = \epsilon_{ij}P_j, \qquad [J, K_i] = \epsilon_{ij}K_j, \qquad [J, P_0] = 0, \qquad [P_0, P_i] = 0, \qquad [P_0, K_i] = -P_i, \\
&[P_1, P_2] = 0, \qquad [K_1, K_2] = -J\cosh(zP_0), \qquad [P_i, K_j] = -\delta_{ij}\frac{\sinh(zP_0)}{z},
\end{aligned}
$$

and deformed Casimir operators

$$
\mathcal{C} = 4\frac{\sinh^2(\frac{z}{2}P_0)}{z^2} - \mathbf{P}^2, \qquad \mathcal{W} = -J\frac{\sinh(zP_0)}{z} + K_1P_2 - K_2P_1.
$$

Performing the associated limit for the "bicrossproduct-type" basis, this gives rise to the quantum Poincaré algebra with coproduct

$$
\begin{aligned}
\Delta_{\vartheta,z}(P_0) &= 1\otimes P_0 + P_0\otimes 1, \qquad \Delta_{\vartheta,z}(J) = 1\otimes J + J\otimes 1,\\
\Delta_{\vartheta,z}(P_i) &= \Delta_z(P_i) + P_i\otimes[\cos(\vartheta P_0)-1] + \mathrm{e}^{-zP_0}[\cos(\vartheta P_0)-1]\otimes P_i\\
&\quad - \epsilon_{ij}P_j\otimes\sin(\vartheta P_0) + \mathrm{e}^{-zP_0}\sin(\vartheta P_0)\otimes\epsilon_{ij}P_j,\\
\Delta_{\vartheta,z}(K_i) &= \Delta_z(K_i) + K_i\otimes[\cos(\vartheta P_0)-1] + \mathrm{e}^{-zP_0}[\cos(\vartheta P_0)-1]\otimes K_i\\
&\quad - \epsilon_{ij}K_j\otimes\sin(\vartheta P_0) + \mathrm{e}^{-zP_0}\sin(\vartheta P_0)\otimes\epsilon_{ij}K_j - \vartheta P_i\otimes J\cos(\vartheta P_0)\\
&\quad + \vartheta\mathrm{e}^{-zP_0}J\cos(\vartheta P_0)\otimes P_i + \vartheta\mathrm{e}^{-zP_0}J\sin(\vartheta P_0)\otimes\epsilon_{ij}P_j + \vartheta\epsilon_{ij}P_j\otimes J\sin(\vartheta P_0)\\
&\quad + zP_i\otimes J\sin(\vartheta P_0) + z\epsilon_{ij}P_j\otimes J[\cos(\vartheta P_0)-1],
\end{aligned}
$$

commutation relations

$$
\begin{aligned}
&[J,P_i] = \epsilon_{ij}P_j, \qquad [J,K_i] = \epsilon_{ij}K_j, \qquad [J,P_0] = 0, \qquad [P_0,P_i] = 0, \qquad [P_0,K_i] = -P_i,\\
&[P_1,P_2] = 0, \qquad [K_1,K_2] = -J, \qquad [P_i,K_j] = \delta_{ij}\left(\frac{\mathrm{e}^{-2zP_0}-1}{2z} - \frac{z}{2}\mathbf{P}^2\right) + zP_jP_i,
\end{aligned}
$$

and deformed Casimir operators

$$
\mathcal{C} = 4\frac{\sinh^2(\frac{z}{2}P_0)}{z^2} - \mathrm{e}^{zP_0}\mathbf{P}^2, \qquad \mathcal{W} = -J\frac{\sinh(zP_0)}{z} + \mathrm{e}^{zP_0}\left(K_1P_2 - K_2P_1 - \frac{z}{2}J\mathbf{P}^2\right).
$$

Note that the two bases are related via the Poincaré contraction $\eta\to 0$ of the map (5.6), namely

$$
\tilde{P}_0 = P_0, \qquad \tilde{J} = J, \qquad \tilde{P}_i = \mathrm{e}^{-\frac{z}{2}P_0}P_i, \qquad \tilde{K}_i = \mathrm{e}^{-\frac{z}{2}P_0}\left(K_i + \frac{z}{2}\epsilon_{ij}JP_j\right),
$$

and the correspondence with the results in [28] is obtained by setting $1/\kappa = 2\mathrm{i}\vartheta$.

6 Twisted κ-AdS$_\omega$ Poisson–Lie group

It is well-known that, given a classical r-matrix $r = r^{ij}Y_i\otimes Y_j$ for a Lie algebra $\mathfrak{a}$, the Poisson–Lie (PL) brackets on the algebra of smooth functions on the associated PL group A are given by the Sklyanin bracket [33]

$$
\{f,g\} = r^{ij}\big(Y_i^L f Y_j^L g - Y_i^R f Y_j^R g\big), \qquad f,g\in\mathrm{Fun}(A), \tag{6.1}
$$

where Y_i^L and Y_i^R are the left- and right-invariant vector fields on A. In the case at hand, we have $\mathfrak{a} = \mathrm{AdS}_\omega$, and $A = \mathrm{SO}_\omega(2,2)$. By inserting the vector fields from Table 1 and the classical r-matrix (4.1) into (6.1), we obtain the PL brackets between the six *commutative* group coordinates $\{x_0,\mathbf{x},\theta,\boldsymbol{\xi}\}$ that are dual to the basis elements $\{P_0,\mathbf{P},J,\mathbf{K}\}$. The PL subalgebra for the space-time local coordinates is given by

$$
\begin{aligned}
&\{x_0,x_1\} = -z\frac{\tanh\eta x_1}{\eta\cosh^2\eta x_2} - \vartheta\cosh\eta x_1\frac{\tanh\eta x_2}{\eta},\\
&\{x_0,x_2\} = -z\frac{\tanh\eta x_2}{\eta} + \vartheta\frac{\sinh\eta x_1}{\eta}, \qquad \{x_1,x_2\} = 0,
\end{aligned} \tag{6.2}
$$

and the "crossed" PL brackets read

$$
\{x_1,\xi_1\} = \frac{z}{\cosh\eta x_2}\left(\frac{\cosh\eta x_2}{\cosh\eta x_1} - \frac{\cosh\xi_1}{\cosh\xi_2} + \tanh\eta x_1\sinh\eta x_2 A\right),
$$

$$
\begin{aligned}
&\{x_1,\xi_2\} = -z\cosh\xi_2 B, \qquad \{x_2,\xi_2\} = z\left(\frac{\cosh\eta x_1}{\cosh\eta x_2}\cosh\xi_1 - \cosh\xi_2\right),\\
&\{x_2,\xi_1\} = -zA, \qquad \{\xi_1,\xi_2\} = z\eta\sinh\eta x_1\left(C - \frac{\tanh\xi_2}{\cosh^2\eta x_2}\right),\\
&\{x_0,\theta\} = -\frac{zB}{\cosh\eta x_1} + \frac{\vartheta}{2}\frac{\cosh\xi_1(\cosh 2\eta x_1 - \cosh 2\xi_2)}{\cosh\eta x_1\cosh\eta x_2\cosh\xi_2},\\
&\{x_0,\xi_1\} = z\left(\frac{\sinh\xi_2}{\cosh\eta x_1}B - \frac{\sinh\xi_1\cosh\xi_2}{\cosh\eta x_1\cosh\eta x_2}\right) - \vartheta\frac{\cosh\eta x_1\cosh\xi_1\tanh\xi_2}{\cosh\eta x_2},\\
&\{x_0,\xi_2\} = -zC + \vartheta\frac{\cosh\eta x_1\sinh\xi_1}{\cosh\eta x_2}, \qquad \{\theta,x_1\} = z\frac{\cosh\eta x_1}{\cosh\xi_2}C + \vartheta\frac{\sinh\xi_1\cosh\xi_2}{\cosh\eta x_2},\\
&\{\theta,x_2\} = -z\frac{\cosh\eta x_1\sinh\xi_1}{\cosh\eta x_2\cosh\xi_2} + \vartheta\sinh\xi_2,\\
&\{\theta,\xi_1\} = -z\eta(\tanh\eta x_2 + \tanh\eta x_1 B) - \vartheta\frac{\eta\tanh\eta x_1\cosh\xi_1\cosh\xi_2}{\cosh\eta x_2},\\
&\{\theta,\xi_2\} = \frac{z\eta\sinh\eta x_1}{\cosh^2\eta x_2\cosh\xi_2} - \vartheta\eta\tanh\eta x_2\cosh\xi_2,
\end{aligned}
\tag{6.3}
$$

where the functions A, B and C are given by

$$
\begin{aligned}
A &= \frac{\sinh\eta x_1\sinh\eta x_2 + \cosh\eta x_1\sinh\xi_1\tanh\xi_2}{\cosh\eta x_2},\\
B &= \frac{\sinh\eta x_1\tanh\eta x_2\cosh\xi_1 + \sinh\xi_1\sinh\xi_2}{\cosh\eta x_2\cosh\xi_2},\\
C &= \frac{\sinh\eta x_1\tanh\eta x_2\sinh\xi_1 + \cosh\xi_1\sinh\xi_2}{\cosh\eta x_1\cosh\eta x_2}.
\end{aligned}
$$

Note that in lowest order, these PL brackets reduce to the first-order commutators (4.3) and (4.4). On the other hand, if the ϑ-twist vanishes, we recover the PL brackets on Fun(AdS$_\omega$) presented in [10].

6.1 Non-commutative (anti-)de Sitter and Minkowski spacetimes

The first naive possibility for the quantisation of a PL structure is given by the Weyl prescription and consists of replacing the initial PL brackets between commutative group coordinates y^i by Lie brackets between the corresponding non-commutative coordinates $\hat{y}^i$. This, indeed, works for linear PL structures, such as the κ-Poincaré group [49, 55, 57, 64], in which the full set of PL brackets for the local spacetime coordinates is linear in the deformation parameter $z = 1/\kappa$. However, for a generic nonlinear PL bracket, this strategy will not work in general due to ordering ambiguities.

In the case of AdS$_\omega$, the PL brackets (6.2) and (6.3) should be quantised to give rise to the non-commutative quantum group Fun$_{z,\vartheta}$(AdS$_\omega$). In particular, the non-commutative spacetime will arise as the quantisation of the Poisson algebra (6.2). In this case, although the brackets are nonlinear, we have $\{x_1,x_2\} = 0$, and if we assume that the two corresponding quantum coordinates commute $[\hat{x}_1,\hat{x}_2] = 0$, the quantisation of the full PL bracket can be achieved via the Weyl prescription. As the only potential ordering ambiguities involve the coordinates $\hat{x}_1$ and $\hat{x}_2$, it follows that the quantum twisted AdS$_\omega$ spacetime is given by

$$
\begin{aligned}
[\hat{x}_0,\hat{x}_1] &= -z\frac{\tanh\eta\hat{x}_1}{\eta\cosh^2\eta\hat{x}_2} - \vartheta\cosh\eta\hat{x}_1\frac{\tanh\eta\hat{x}_2}{\eta}\\
&= -z\left(\hat{x}_1 - \frac{1}{3}\omega\hat{x}_1^3 - \omega\hat{x}_1\hat{x}_2^2\right) - \vartheta\left(\hat{x}_2 + \frac{1}{2}\omega\hat{x}_1^2\hat{x}_2 - \frac{1}{3}\omega\hat{x}_2^3\right) + \mathcal{O}(\omega^2),
\end{aligned}
$$

$$\begin{aligned}
[\hat{x}_0, \hat{x}_2] &= -z\frac{\tanh \eta\hat{x}_2}{\eta} + \vartheta\frac{\sinh \eta\hat{x}_1}{\eta} \\
&= -z\left(\hat{x}_2 - \frac{1}{3}\omega\hat{x}_2^3\right) + \vartheta\left(\hat{x}_1 + \frac{1}{6}\omega\hat{x}_1^3\right) + \mathcal{O}(\omega^2), \\
[\hat{x}_1, \hat{x}_2] &= 0.
\end{aligned} \tag{6.4}$$

Thus, we have shown how the first-order non-commutative space-time (4.3), which is common to the *three* quantum twisted κ-AdS$_\omega$ algebras, is generalised with an explicit dependence on the curvature or cosmological constant $\omega = -\Lambda$.

The asymmetric form of (6.4) with respect to the $\hat{x}_1$ and $\hat{x}_2$ quantum coordinates could be expected from the beginning, as we are dealing with *local* coordinates. However, if we consider non-commutative ambient (Weierstrass) coordinates $(s_3, s_0, \mathbf{s})$, defined in terms of $(x_0, \mathbf{x})$ through (2.7), we obtain that the PL twisted κ-AdS$_\omega$ spacetime can be written as a quadratic (and much more symmetric) Poisson algebra:

$$\begin{aligned}
&\{s_0, s_i\} = -s_3\left(zs_i + \vartheta\epsilon_{ij}s_j\right), \qquad \{s_1, s_2\} = 0, \\
&\{s_3, s_0\} = zw\mathbf{s}^2, \qquad \{s_3, s_i\} = ws_0\left(zs_i + \vartheta\epsilon_{ij}s_j\right),
\end{aligned}$$

whose limit $\omega \to 0$ is given by $(s_3, s_0, \mathbf{s}) \to (1, x_0, \mathbf{x})$. In fact, the quantisation of this Poisson algebra in a way consistent with the relations (2.7) will provide an alternative description for the non-commutative spacetime (6.4).

7 Twisted κ-AdS quantum group

In this section, we relate the quantum deformation of AdS$_\omega$ to the standard quantum deformations of $\mathfrak{sl}(2,\mathbb{R})$ via the Lie algebra isomorphism $\mathfrak{so}(2,2) \simeq \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$. As in Section 3.4.1, we therefore consider two copies of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ with bases $\{J_+^l, J_-^l, J_3^l\}$, group coordinates $(a_{+,l}, a_{-,l}, \chi_l)$ and (a_l, b_l, c_l, d_l) $(l = 1, 2)$. To exhibit the relevant structures more clearly, we consider the *three-parametric* r-matrix

$$r_{\alpha,\beta,\delta} = \alpha J_+^1 \wedge J_-^1 + \beta J_+^2 \wedge J_-^2 + \tfrac{\delta}{2} J_3^1 \wedge J_3^2, \tag{7.1}$$

which coincides with the classical r-matrix r'_F from (3.16) for $\alpha = \delta = -\frac{\eta}{2}$ and $\beta = \frac{\eta}{2}$. This allows one to determine the role of each term in the construction of the corresponding quantum AdS group.

7.1 Quantum standard $\mathrm{SL}(2,\mathbb{R})$ group

To quantise the PL bracket defined by the r-matrix (7.1), it is worth recalling the well-known construction of the standard quantum $\mathrm{SL}(2,\mathbb{R})$ group (see, for instance, [61]). For this, consider the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ with Lie bracket

$$[J_3, J_\pm] = \pm 2J_\pm, \qquad [J_+, J_-] = J_3,$$

and its fundamental representation given by

$$J_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad J_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad J_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

This enables one to parametrise elements of the group $\mathrm{SL}(2,\mathbb{R})$ near the unit element according to

$$T = \mathrm{e}^{a_- J_-}\mathrm{e}^{a_+ J_+}\mathrm{e}^{\chi J_3} = \begin{pmatrix} \mathrm{e}^{\chi} & a_+\mathrm{e}^{-\chi} \\ a_-\mathrm{e}^{\chi} & (1 + a_- a_+)\mathrm{e}^{-\chi} \end{pmatrix} \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad a\,d - bc = 1, \tag{7.2}$$

where (a_+, a_-, χ) are local coordinates. The corresponding left- and right-invariant vector fields of $\mathrm{SL}(2,\mathbb{R})$ are given by

$$
\begin{aligned}
&Y^L_{J_+} = \mathrm{e}^{2\chi}\partial_{a_+}, && Y^L_{J_-} = a_+^2\mathrm{e}^{-2\chi}\partial_{a_+} + \mathrm{e}^{-2\chi}\partial_{a_-} + a_+\mathrm{e}^{-2\chi}\partial_\chi,\\
&Y^L_{J_3} = \partial_\chi, && Y^R_{J_+} = (1+2a_-a_+)\partial_{a_+} - a_-^2\partial_{a_-} + a_-\partial_\chi,\\
&Y^R_{J_-} = \partial_{a_-}, && Y^R_{J_3} = -2a_-\partial_{a_-} + 2a_+\partial_{a_+} + \partial_\chi.
\end{aligned}
\tag{7.3}
$$

The Sklyanin bracket (6.1) is induced by the standard (Drinfel'd–Jimbo) classical r-matrix [34, 42] given by

$$
r^{\mathrm{DJ}} = zJ_+ \wedge J_-, \qquad z = \mathrm{e}^q,
$$

which yields the following Sklyanin brackets in terms of the local coordinates (a_+, a_-, χ):

$$
\{\chi, a_+\} = -za_+, \qquad \{\chi, a_-\} = -za_-, \qquad \{a_+, a_-\} = -2za_-a_+.
$$

Passing to the coordinates given by the matrix entries (a,b,c,d) from (7.2), one finds that this PL structure is homogeneous quadratic

$$
\begin{aligned}
&\{b,a\} = zab, && \{c,a\} = zac, && \{c,b\} = 0,\\
&\{d,b\} = zbd, && \{d,c\} = zcd, && \{d,a\} = 2zbc,
\end{aligned}
\tag{7.4}
$$

and $C = ad - bc$ is a Casimir function.

The quantisation of the PL algebra (7.4) in terms of the non-commutative coordinates $(\hat a, \hat b, \hat c, \hat d)$ takes the form

$$
\begin{aligned}
&\hat b\hat a - q\hat a\hat b = 0, && \hat c\hat a - q\hat a\hat c = 0, && [\hat c, \hat b] = 0,\\
&\hat d\hat b - q\hat b\hat d = 0, && \hat d\hat c - q\hat c\hat d = 0, && [\hat d, \hat a] = \left(q - q^{-1}\right)\hat b\hat c,
\end{aligned}
\tag{7.5}
$$

and is compatible with the coproduct for the quantum $\mathrm{SL}(2,\mathbb{R})$ group that is induced by the group multiplication $\Delta(\hat T) = \hat T\dot\otimes\hat T$, namely

$$
\begin{aligned}
&\Delta(\hat a) = \hat a\otimes\hat a + \hat b\otimes\hat c, && \Delta(\hat b) = \hat a\otimes\hat b + \hat b\otimes\hat d,\\
&\Delta(\hat c) = \hat c\otimes\hat a + \hat d\otimes\hat c, && \Delta(\hat d) = \hat c\otimes\hat b + \hat d\otimes\hat d.
\end{aligned}
\tag{7.6}
$$

Moreover, the deformed Casimir operator is just the "quantum determinant" of $\hat T$

$$
C_q = \det_q(T) = \hat a\hat d - q^{-1}\hat b\hat c.
$$

Note that the commutation relations (7.5) are consistent with the Poisson brackets (7.4), as we have $[\cdot,\cdot] = z\{\cdot,\cdot\} + \mathcal{O}[z^2]$.

7.2 Quantisation of the AdS group in the $\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{sl}(2,\mathbb{R})$ basis

The PL brackets for the r-matrix (7.1) are readily obtained by inserting the vector fields (7.3) into the Sklyanin bracket (6.1). A straightforward computation shows that, in terms of the $\mathrm{SO}(2,2)$ coordinates (a_l, b_l, c_l, d_l) $(l = 1,2)$, this yields

$$
\begin{aligned}
&\{a_1, b_1\} = -\alpha a_1 b_1, && \{a_2, b_2\} = -\beta a_2 b_2,\\
&\{a_1, c_1\} = -\alpha a_1 c_1, && \{a_2, c_2\} = -\beta a_2 c_2,\\
&\{a_1, d_1\} = -2\alpha b_1 c_1, && \{a_2, d_2\} = -2\beta b_2 c_2,\\
&\{b_1, c_1\} = 0, && \{b_2, c_2\} = 0,
\end{aligned}
$$

$$
\{b_1, d_1\} = -\alpha b_1 d_1, \qquad \{b_2, d_2\} = -\beta b_2 d_2,
$$
$$
\{c_1, d_1\} = -\alpha c_1 d_1, \qquad \{c_2, d_2\} = -\beta c_2 d_2,
$$

which correspond to the terms with parameters α and β in the r-matrix (7.1). The "crossed terms" are induced by the twist, which is labelled by the parameter δ

$$
\begin{aligned}
&\{a_1, a_2\} = 0, && \{c_1, a_2\} = \delta c_1 a_2,\\
&\{a_1, b_2\} = -\delta a_1 b_2, && \{c_1, b_2\} = 0,\\
&\{a_1, c_2\} = \delta a_1 c_2, && \{c_1, c_2\} = 0,\\
&\{a_1, d_2\} = 0, && \{c_1, d_2\} = -\delta d_2 c_1,\\
&\{b_1, a_2\} = -\delta b_1 a_2, && \{d_1, a_2\} = 0,\\
&\{b_1, b_2\} = 0, && \{d_1, b_2\} = \delta d_1 b_2,\\
&\{b_1, c_2\} = 0, && \{d_1, c_2\} = -\delta c_2 d_1,\\
&\{b_1, d_2\} = \delta d_2 b_1, && \{d_1, d_2\} = 0.
\end{aligned}
$$

It is immediate to check that $C_1 = a_1 d_1 - b_1 c_1$ and $C_2 = a_2 d_2 - b_2 c_2$ are Casimir functions for this multi-parametric Sklyanin bracket.

The quantisation of this PL algebra is the quantum group $\mathrm{SO}_{q_\alpha, q_\beta, q_\delta}(2,2)$, where the (non-intertwined) deformation parameters are $q_\alpha = \mathrm{e}^\alpha$, $q_\beta = \mathrm{e}^\beta$ and $q_\delta = \mathrm{e}^\delta$. Evidently, the quantum group coproduct is given by a copy of (7.6) for each of the two non-commutative sets of coordinates $(\hat{a}_l, \hat{b}_l, \hat{c}_l, \hat{d}_l)$ $(l = 1, 2)$. The associated q-commutation rules are the ones for two copies of the quantum $\mathrm{SL}(2, \mathbb{R})$ group with independent parameters

$$
\begin{aligned}
&\hat{b}_1 \hat{a}_1 - q_\alpha \hat{a}_1 \hat{b}_1 = 0, && \hat{c}_1 \hat{a}_1 - q_\alpha \hat{a}_1 \hat{c}_1 = 0, && [\hat{c}_1, \hat{b}_1] = 0,\\
&\hat{d}_1 \hat{b}_1 - q_\alpha \hat{b}_1 \hat{d}_1 = 0, && \hat{d}_1 \hat{c}_1 - q_\alpha \hat{c}_1 \hat{d}_1 = 0, && [\hat{d}_1, \hat{a}_1] = \big(q_\alpha - q_\alpha^{-1}\big) \hat{b}_1 \hat{c}_1,\\
&\hat{b}_2 \hat{a}_2 - q_\beta \hat{a}_2 \hat{b}_2 = 0, && \hat{c}_2 \hat{a}_2 - q_\beta \hat{a}_2 \hat{c}_2 = 0, && [\hat{c}_2, \hat{b}_2] = 0,\\
&\hat{d}_2 \hat{b}_2 - q_\beta \hat{b}_2 \hat{d}_2 = 0, && \hat{d}_2 \hat{c}_2 - q_\beta \hat{c}_2 \hat{d}_2 = 0, && [\hat{d}_2, \hat{a}_2] = \big(q_\beta - q_\beta^{-1}\big) \hat{b}_2 \hat{c}_2.
\end{aligned}
$$

Additionally, the quantum algebra exhibits "crossed relations" that are governed by the twist parameter:

$$
\begin{aligned}
&[\hat{a}_1, \hat{a}_2] = 0, \qquad [\hat{b}_1, \hat{b}_2] = 0, \qquad [\hat{c}_1, \hat{c}_2] = 0, \qquad [\hat{d}_1, \hat{d}_2] = 0,\\
&[\hat{a}_1, \hat{d}_2] = 0, \qquad [\hat{a}_2, \hat{d}_1] = 0, \qquad [\hat{b}_1, \hat{c}_2] = 0, \qquad [\hat{b}_2, \hat{c}_1] = 0,\\
&\hat{a}_2 \hat{b}_1 - q_\delta \hat{b}_1 \hat{a}_2 = 0, \qquad \hat{b}_2 \hat{a}_1 - q_\delta \hat{a}_1 \hat{b}_2 = 0,\\
&\hat{a}_1 \hat{c}_2 - q_\delta \hat{c}_2 \hat{a}_1 = 0, \qquad \hat{c}_1 \hat{a}_2 - q_\delta \hat{a}_2 \hat{c}_1 = 0,\\
&\hat{b}_1 \hat{d}_2 - q_\delta \hat{d}_2 \hat{b}_1 = 0, \qquad \hat{d}_1 \hat{b}_2 - q_\delta \hat{b}_2 \hat{d}_1 = 0,\\
&\hat{c}_2 \hat{d}_1 - q_\delta \hat{d}_1 \hat{c}_2 = 0, \qquad \hat{d}_2 \hat{c}_1 - q_\delta \hat{c}_1 \hat{d}_2 = 0.
\end{aligned}
$$

To conclude the discussion, we consider the structure dual to the quantum group $\mathrm{SO}_{q_\alpha, q_\beta, q_\delta}(2,2)$ above, namely the Hopf algebra structure of the associated quantum algebra. Its coproduct is given by a formal series in the deformation parameters, and its first-order is the cocommutators obtained via (3.5) for the classical r-matrix (7.1):

$$
\begin{aligned}
&\delta\big(J_3^1\big) = 0, \qquad \delta\big(J_3^2\big) = 0,\\
&\delta\big(J_+^1\big) = J_+^1 \wedge \big(\alpha J_3^1 - \delta J_3^2\big), \qquad \delta\big(J_+^2\big) = J_+^2 \wedge \big(\beta J_3^2 + \delta J_3^1\big),\\
&\delta\big(J_-^1\big) = J_-^1 \wedge \big(\alpha J_3^1 + \delta J_3^2\big), \qquad \delta\big(J_-^2\big) = J_-^2 \wedge \big(\beta J_3^2 - \delta J_3^1\big).
\end{aligned}
$$

Thus, we are effectively dealing with two almost-disjoint copies of $\mathfrak{sl}(2,\mathbb{R})$ (not truly independent due to the δ-mixed terms in the cocommutators), and one readily obtains the full coproduct:

$$
\begin{aligned}
&\Delta\big(J_3^1\big) = J_3^1\otimes 1 + 1\otimes J_3^1, \qquad \Delta\big(J_3^2\big) = J_3^2\otimes 1 + 1\otimes J_3^2,\\
&\Delta\big(J_+^1\big) = J_+^1\otimes \mathrm{e}^{\frac12(\alpha J_3^1-\delta J_3^2)} + \mathrm{e}^{-\frac12(\alpha J_3^1-\delta J_3^2)}\otimes J_+^1,\\
&\Delta\big(J_-^1\big) = J_-^1\otimes \mathrm{e}^{\frac12(\alpha J_3^1+\delta J_3^2)} + \mathrm{e}^{-\frac12(\alpha J_3^1+\delta J_3^2)}\otimes J_-^1,\\
&\Delta\big(J_+^2\big) = J_+^2\otimes \mathrm{e}^{\frac12(\beta J_3^2+\delta J_3^1)} + \mathrm{e}^{-\frac12(\beta J_3^2+\delta J_3^1)}\otimes J_+^2,\\
&\Delta\big(J_-^2\big) = J_-^2\otimes \mathrm{e}^{\frac12(\beta J_3^2-\delta J_3^1)} + \mathrm{e}^{-\frac12(\beta J_3^2-\delta J_3^1)}\otimes J_-^2.
\end{aligned}
$$

The deformed commutation relations for this quantum algebra can be calculated straightforwardly since the presence of the twist $\frac{\delta}{2}J_3^1\wedge J_3^2$ in the classical r-matrix (7.1) does not affect them. Explicitly, they are given by

$$
\big[J_+^l, J_-^l\big] = \frac{\sinh z_l J_3^l}{z_l}, \qquad \big[J_3^l, J_\pm^l\big] = \pm 2J_\pm^l, \qquad l=1,2,
$$

with $z_1=\alpha$ and $z_2=\beta$, while the basis elements of the different copies commute.

8 Concluding remarks

In this article, we have constructed the full quantum algebra as well as the associated non-commutative spacetimes for the family of Lie algebras AdS_ω with associated DD structures. In these quantum algebras, the cosmological constant Λ plays the role of a deformation parameter in addition to the energy scale given by κ, and a further deformation parameter ϑ parametrises a twist that is motivated by the compatibility of this quantum deformation with (2+1)-gravity.

It would be interesting to investigate the impact of this twist in more detail by considering multi-particle models in which the momenta are added via the coproduct of the quantum algebra. While a twist does not affect the commutation relations of the quantum algebra, it manifests itself in the coproduct, and, consequently, different values of the twist parameter ϑ lead to different momentum addition laws for point particles. It would be interesting to see if the precise value of this parameter that ensures the compatibility with (2+1)-gravity is also motivated by physical considerations in the context of multi-particle models and whether it has a geometrical interpretation.

It would also be desirable to investigate in more depth the role of the cosmological constant or curvature in these models and the spectra of the associated quantum operators. At least in the AdS case, where the quantum algebra is obtained via a twist from two commuting copies of the standard quantum deformation of $\mathrm{SL}(2,\mathbb{R})$, known results about the representation theory of this standard deformation [27, 34, 42] should permit one to work out in detail the spectrum of the associated quantum operators.

Finally, the introduction of another graded contraction parameter associated to the involution Π in (2.3) would allow one to compute the non-relativistic limits of all the quantum twisted κ-AdS_ω algebras presented in the paper. This would lead to twisted κ-deformations of both the Newton–Hooke algebra and the Galilean one [28, 29, 30].

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