Symmetry, Integrability and Geometry: Methods and Applications SIGMA **12** (2016), 113, 13 pages

On Free Field Realizations of $W(2,2)$-Modules

Dražen ADAMOVIĆ † and *Gordan RADOBOLJA* ‡

† *Department of Mathematics, University of Zagreb, Bijenička 30, 10 000 Zagreb, Croatia*
E-mail: *adamovic@math.hr*
URL: https://web.math.pmf.unizg.hr/~adamovic

‡ *Faculty of Science, University of Split, Rudera Boškovića 33, 21 000 Split, Croatia*
E-mail: *gordan@pmfst.hr*

Received June 09, 2016, in final form December 03, 2016; Published online December 06, 2016
http://dx.doi.org/10.3842/SIGMA.2016.113

Abstract. The aim of the paper is to study modules for the twisted Heisenberg–Virasoro algebra $\mathcal{H}$ at level zero as modules for the $W(2,2)$-algebra by using construction from [*J. Pure Appl. Algebra* **219** (2015), 4322–4342, arXiv:1405.1707]. We prove that the irreducible highest weight $\mathcal{H}$-module is irreducible as $W(2,2)$-module if and only if it has a typical highest weight. Finally, we construct a screening operator acting on the Heisenberg–Virasoro vertex algebra whose kernel is exactly $W(2,2)$ vertex algebra.

Key words: Heisenberg–Virasoro Lie algebra; vertex algebra; $W(2,2)$ algebra; screening-operators

2010 Mathematics Subject Classification: 17B69; 17B67; 17B68; 81R10

1 Introduction

Lie algebra $W(2,2)$ was first introduced by W. Zhang and C. Dong in [20] as part of a classification of certain simple vertex operator algebras. Its representation theory has been studied in [14, 15, 18, 19] and several other papers. Although $W(2,2)$ is an extension of the Virasoro algebra, its representation theory is very different. This is most notable with highest weight representations. It was shown in [19] that some Verma modules contain a cosingular vector.

Highest weight representation theory of the twisted Heisenberg–Virasoro Lie algebra has also been studied recently. Representations with nontrivial action of C_I have been developed in [6]. Representations at level zero, i.e., with trivial action of C_I were studied in [8] due to their importance in some constructions over the toroidal Lie algebras (see [7, 9]). In this case, a free field realization of highest weight modules along with the fusion rules for a suitable category of modules were obtained in [4].

Irreducible highest weight modules of highest weights $(0,0)$ over these algebras carry the structure of simple vertex operator algebras. Let us denote these vertex operator algebras as $L^{W(2,2)}(c_L,c_W)$ and $L^{\mathcal{H}}(c_L,c_{L,I})$. It was proved in [4] that simple vertex operator algebra $L^{W(2,2)}(c_L,c_W)$ embeds into Heisenberg–Virasoro vertex operator algebra $L^{\mathcal{H}}(c_L,c_{L,I})$ so that $c_W=-24c_{L,I}^2$. As a result each highest weight module over $\mathcal{H}$ is also a $W(2,2)$-module. In this paper we shall completely described the structure of the irreducible highest weight $\mathcal{H}$-modules as $W(2,2)$-modules. We show that in generic case the resulting $W(2,2)$-module is irreducible. However, in case of a module of highest weight such that associated Verma module over $W(2,2)$ contains cosingular vectors (we shall call this kind of weight atypical), irreducible $\mathcal{H}$-module is reducible over $W(2,2)$. We shall denote the irreducible highest weight $\mathcal{H}$-module

$L^{\mathcal{H}}(c_L, 0, c_{L,I}, h, h_I)$ shortly as $L^{\mathcal{H}}(h, h_I)$. We also use the following notation[1]

$$h_{p,r} = \left(1 - p^2\right)\frac{c_L - 2}{24} + p(p-1) + p\frac{1-r}{2}$$

for $p, r \in \mathbb{Z}_{>0}$. Define

$$\mathcal{AT}_{\mathcal{H}}(c_L, c_{L,I}) = \{(h_{p,r}, (1 \pm p)c_{L,I}) \,|\, p, r \in \mathbb{Z}_{>0}\}.$$

We call a weight (h, h_I) *atypical* for $\mathcal{H}$ (resp. *typical*) if $(h, h_I) \in \mathcal{AT}_{\mathcal{H}}(c_L, c_{L,I})$ (resp. $(h, h_I) \notin \mathcal{AT}_{\mathcal{H}}(c_L, c_{L,I})$). We shall refer to a highest weight module over $\mathcal{H}$ as (a)typical if its highest weight is (a)typical for $\mathcal{H}$.

The next theorem gives a main result of the paper.

Theorem 1.1. *Assume that $c_{L,I} \neq 0$.*

(1) *$L^{\mathcal{H}}(h, h_I)$ is irreducible as a $W(2,2)$-module if and only if*

$$(h, h_I) \notin \mathcal{AT}_{\mathcal{H}}(c_L, c_{L,I}).$$

(2) *If $(h, h_I) \in \mathcal{AT}_{\mathcal{H}}(c_L, c_{L,I})$ then $L^{\mathcal{H}}(h, h_I)$ is a non-split extension of two irreducible highest weight $W(2,2)$-modules.*

We recall some aspects of representation theories of infinite-dimensional Lie algebras $\mathcal{H}$ and $W(2,2)$ in Section 2. The main results of the branching rules will be proved in Section 3. From the free field realization in [4] follows that irreducible $\mathcal{H}$-modules are pairwise contragredient. For half of these modules, proofs rely on a $W(2,2)$-homomorphism between Verma modules over $W(2,2)$ and $\mathcal{H}$ which is induced by a homomorphism of vertex operator algebras. The rest is then proved elegantly by passing to contragredients. We also prove a very interesting result that the Verma module for $\mathcal{H}$ with typical highest weight is an infinite direct sum of irreducible $W(2,2)$-modules (cf. Theorem 3.7). This result presents a $W(2,2)$-analogue of certain Feigin–Fuchs modules for the Virasoro algebra (cf. Remark 3.8).

From the results in the paper, we see that the vertex algebra $L^{W(2,2)}(c_L, c_W)$ has many properties similar to the $\mathcal{W}$-algebras appearing in logarithmic conformal field theory (LCFT):

- $L^{W(2,2)}(c_L, c_W)$ admits a free field realization inside of the Heisenberg–Virasoro vertex algebra $L^{\mathcal{H}}(c_L, c_{L,I})$.
- Typical modules are realized as irreducible modules for $L^{\mathcal{H}}(c_L, c_{L,I})$.
- In the atypical case, irreducible $L^{\mathcal{H}}(c_L, c_{L,I})$-modules as $L^{W(2,2)}(c_L, c_W)$-modules have semi–simple rank two.

The singlet vertex algebra $\overline{M(1)}$ has similar properties. $\overline{M(1)}$ is realized as kernel of a screening operator inside the Heisenberg vertex algebra $M(1)$ (cf. [1]). In Section 4 we construct the screening operator

$$S_1\colon\ L^{\mathcal{H}}(c_L, c_{L,I}) \to L^{\mathcal{H}}(1, 0),$$

which commutes with the action of $W(2,2)$-algebra such that

$$\mathrm{Ker}_{L^{\mathcal{H}}(c_L, c_{L,I})}\, S_1 \cong L^{W(2,2)}(c_L, c_W).$$

Our construction uses an extension $\mathcal{V}_{\mathrm{ext}}$ of the vertex algebra $L^{\mathcal{H}}(c_L, c_{L,I})$ by a non-weight module for the Heisenberg–Virasoro vertex algebra. In our forthcoming paper [5], we shall present an explicit realization of $\mathcal{V}_{\mathrm{ext}}$ and apply this construction to the study of intertwining operators and logarithmic modules.

[1] We emphasise a term $\frac{c_L - 2}{24}$ for its importance in a free field realization of $\mathcal{H}$ (see [4] for details).

2 Lie algebra $W(2,2)$ and the twisted Heisenberg–Virasoro Lie algebra at level zero

$W(2,2)$ is a Lie algebra with basis $\{L(n), W(n), C_L, C_W : n \in \mathbb{Z}\}$ over $\mathbb{C}$, and a Lie bracket

$$[L(n), L(m)] = (n-m)L(n+m) + \delta_{n,-m}\frac{n^3-n}{12}C_L,$$
$$[L(n), W(m)] = (n-m)W(n+m) + \delta_{n,-m}\frac{n^3-n}{12}C_W,$$
$$[W(n), W(m)] = [\cdot, C_L] = [\cdot, C_W] = 0.$$

Highest weight representation theory over $W(2,2)$ was studied in [14, 19]. However, representations treated in these papers have equal central charges $C_L = C_W$. These results have recently been generalised to $C_L \neq C_W$ in [15]. Here we state the most important results. Verma module with central charge (c_L, c_W) and highest weight (h, h_W) is denoted by $V^{W(2,2)}(c_L, c_W, h, h_W)$, its highest weight vector by v_{h,h_W} and irreducible quotient module by $L^{W(2,2)}(c_L, c_W, h, h_W)$.

Recall the definition of a cosingular vector. Homogeneous vector $v \in M$ is called cosingular (or subsingular) if it is not singular in M and if there is a proper submodule $N \subset M$ such that $v + N$ is a singular vector in M/N.

Theorem 2.1 ([15, 19])**.** *Let $c_W \neq 0$.*

(i) *Verma module $V^{W(2,2)}(c_L, c_W, h, h_W)$ is reducible if and only if $h_W = \frac{1-p^2}{24}c_W$ for some $p \in \mathbb{Z}_{>0}$. In that case, there exists a singular vector $u'_p \in \mathbb{C}[W(-1), \dots, W(-p)]v_{h,h_W}$ such that $U(W(2,2))u'_p \cong V^{W(2,2)}(c_L, c_W, h+p, h_W)$.*

(ii) *A quotient module*[2]

$$V^{W(2,2)}(c_L, c_W, h, h_W)/U(W(2,2))u'_p =: \widetilde{L}^{W(2,2)}(c_L, c_W, h_{p,r}, h_W)$$

is reducible if and only if $h = h_{p,r}$ for some $r \in \mathbb{Z}_{>0}$. In that case, there is a cosingular vector $u_{rp} \in V^{W(2,2)}(c_L, c_W, h, h_W)_{h+rp}$ such that the short sequence

$$\begin{aligned} 0 \to L^{W(2,2)}(c_L, c_W, h_{p,r}+rp, h_W) &\to \widetilde{L}^{W(2,2)}(c_L, c_W, h_{p,r}, h_W) \\ &\to L^{W(2,2)}(c_L, c_W, h_{p,r}, h_W) \to 0 \end{aligned} \tag{2.1}$$

is exact.

Define

$$\mathcal{AT}_{W(2,2)}(c_L, c_W) = \left\{ \left(h_{p,r}, \frac{1-p^2}{24}c_W \right) \,|\, p, r \in \mathbb{Z}_{>0} \right\}.$$

Remark 2.2. We will refer to the (modules of) highest weights $(h, h_W) \in \mathcal{AT}_{W(2,2)}(c_L, c_W)$ as *atypical* for $W(2,2)$, and otherwise as *typical*. Again, we refer to a highest weight $W(2,2)$-module as (a)typical depending on its highest weight. So a Verma module over $W(2,2)$ contains a nontrivial cosingular vector if and only if it is atypical.

Proposition 2.3. *Let $h_W = \frac{1-p^2}{24}c_W$, $p \in \mathbb{Z}_{>0}$.*

(i) *Let $(h_{p,r}, h_W)$, $r \in \mathbb{Z}_{>0}$ be an atypical weight and $k \in \mathbb{Z}$. Then $(h_{p,r} + kp, h_W)$ is atypical if and only if $k < \frac{r}{2}$.*

[2] This module is denoted by L' in [15, 19]. We change notation to $\widetilde{L}$ due to use of superscript $W(2,2)$.

(*ii*) *Atypical Verma module* $V^{W(2,2)}(h_{p,r}, h_W)$ *contains exactly* $\lfloor \frac{r+1}{2} \rfloor$ *cosingular vectors. The weights of these vectors are* $h_{p,r} + (r-i)p = h_{p,-r+2i}$, $i = 0, \dots, \lfloor \frac{r-1}{2} \rfloor$.

Proof. (i) Directly from Theorem 2.1 since $h_{p,r} + kp = h_{p,r-2k}$.

(ii) Follows from (i) since $V^{W(2,2)}(h_{p,r}, h_W)$ contains a chain of submodules which are isomorphic to $V^{W(2,2)}(h_{p,r} + ip, h_W)$, $i \in \mathbb{Z}_{>0}$. ■

Remark 2.4. Standard PBW basis for $V^{W(2,2)}(c_L, c_W, h, h_W)$ consists of vectors

$$Wt(-m_s) \cdots W(-m_1) L(-n_t) \cdots L(-n_1) v_{h,h_W}$$

such that $m_s \geq \cdots \geq m_1 \geq 1$, $n_t \geq \cdots \geq n_1 \geq 1$. The only nonzero component of u_{rp} belonging to $\mathbb{C}[L(-1), L(-2), \dots]v$ is $L(-p)^r v_{h,h_W}$ [19].

Define $P_2(n) = \sum\limits_{i=0}^{n} P(n-i)P(i)$ where P is a partition function with $P(0) = 1$. We have the following character formulas [19]

$$\operatorname{char} V^{W(2,2)}(c_L, c_W, h, h_W) = q^h \sum_{n \geq 0} P_2(n) q^n = q^h \prod_{k \geq 1} \left(1 - q^k\right)^{-2},$$

for all $h, h_W \in \mathbb{C}$. If $h_W = \frac{1-p^2}{24} c_W$, then

$$\operatorname{char} \widetilde{L}^{W(2,2)}(c_L, c_W, h, h_W) = q^h \left(1 - q^p\right) \sum_{n \geq 0} P_2(n) q^n = q^h \left(1 - q^p\right) \prod_{k \geq 1} \left(1 - q^k\right)^{-2}.$$

If (h, h_W) is typical for $W(2,2)$, then this is the character of an irreducible highest weight module. Finally, the character of atypical irreducible module is

$$\begin{aligned} \operatorname{char} L^{W(2,2)}(c_L, c_W, h_{p,r}, h_W) &= q^{h_{p,r}} \left(1 - q^p\right) \left(1 - q^{rp}\right) \sum_{n \geq 0} P_2(n) q^n \\ &= q^{h_{p,r}} \left(1 - q^p\right) \left(1 - q^{rp}\right) \prod_{k \geq 1} \left(1 - q^k\right)^{-2}. \end{aligned}$$

The twisted Heisenberg–Virasoro algebra $\mathcal{H}$ is the universal central extension of the Lie algebra of differential operators on a circle of order at most one. It is the infinite-dimensional complex Lie algebra with a basis

$$\{L(n), I(n) \colon n \in \mathbb{Z}\} \cup \{C_L, C_{LI}, C_I\}$$

and commutation relations

$$\begin{aligned} &[L(n), L(m)] = (n-m)L(n+m) + \delta_{n,-m} \frac{n^3 - n}{12} C_L, \\ &[L(n), I(m)] = -mI(n+m) - \delta_{n,-m} \left(n^2 + n\right) C_{LI}, \\ &[I(n), I(m)] = n \delta_{n,-m} C_I, \qquad [\mathcal{H}, C_L] = [\mathcal{H}, C_{LI}] = [\mathcal{H}, C_I] = 0. \end{aligned}$$

The Lie algebra $\mathcal{H}$ admits the following triangular decomposition

$$\begin{aligned} &\mathcal{H} = \mathcal{H}^- \oplus \mathcal{H}^0 \oplus \mathcal{H}^+, \\ &\mathcal{H}^\pm = \operatorname{span}_{\mathbb{C}}\{I(\pm n), L(\pm n) \,|\, n \in \mathbb{Z}_{>0}\}, \qquad \mathcal{H}^0 = \operatorname{span}_{\mathbb{C}}\{I(0), L(0), C_L, C_{L,I}, C_I\}. \end{aligned} \tag{2.2}$$

Although they seem to be two similar extensions of the Virasoro algebra, representation theories of $W(2,2)$ and $\mathcal{H}$ are different. The main reason for that lies in the fact that $I(0)$ is a central element, while $W(0)$ is not. However, applying free field realization, we shall see that highest weight modules over the two algebras are related.

Denote by $V^{\mathcal{H}}(c_L, c_I, c_{L,I}, h, h_I)$ the Verma module and by v_{h,h_I} its highest weight vector. C_L, C_I, $C_{L,I}$, $L(0)$ and $I(0)$ act on v_{h,h_I} by scalars c_L, c_I, $c_{L,I}$, h and h_I, respectively. Then $(c_L, c_I, c_{L,I})$ is called a central charge, and (h, h_I) a highest weight. In this paper we consider central charges $(c_L, 0, c_{L,I})$ such that $c_{L,I} \neq 0$.

Theorem 2.5 ([8]). *Let $c_{L,I} \neq 0$. Verma module $V^{\mathcal{H}}(c_L, 0, c_{L,I}, h, h_I)$ is reducible if and only if $h_I = (1 \pm p)c_{L,I}$ for some $p \in \mathbb{Z}_{>0}$. In that case, there is a singular vector $v_p^{\pm}$ of weight p, which generates a maximal submodule in $V^{\mathcal{H}}(c_L, 0, c_{L,I}, h, h_I)$ isomorphic to $V^{\mathcal{H}}(c_L, 0, c_{L,I}, h+p, h_I)$.*

Remark 2.6. In case $h_I = (1+p)c_{L,I}$ an explicit formula for a singular vector v_p^+ is obtained using Schur polynomials in $I(-1), \dots, I(-p)$. See [4] for details. Suppose that $xv_p^+ \in V^{\mathcal{H}}(c_L, 0, c_{L,I}, h, h_I)$ lies in a maximal submodule. Then x does not have a nontrivial additive component belonging to $\mathbb{C}[L(-1), L(-2), \dots]$ [8].

There is an infinite chain of Verma submodules generated by singular vectors $v_{kp}^{\pm}$, $k \in \mathbb{Z}_{>0}$, with all the subquotients being irreducible. Note that there is no mention of $\widetilde{L}^{\mathcal{H}}$ since there are no cosingular vectors in $V^{\mathcal{H}}$.

The following character formulas were obtained in [8]:

$$\operatorname{char} V^{\mathcal{H}}(c_L, 0, c_L, h, h_I) = q^h \sum_{n \geq 0} P_2(n) q^n = q^h \prod_{k \geq 1} \left(1 - q^k\right)^{-2},$$
$$\operatorname{char} L^{\mathcal{H}}(c_L, 0, c_L, h, h_I) = q^h \left(1 - q^p\right) \sum_{n \geq 0} P_2(n) q^n = q^h \left(1 - q^p\right) \prod_{k \geq 1} \left(1 - q^k\right)^{-2}.$$

Remark 2.7. Throughout the rest of the paper we work with highest weight modules over the Lie algebras $W(2,2)$ and $\mathcal{H}$ so we always denote algebra in superscript. In order to avoid too cumbersome notation, we omit central charges. Therefore, we write $V^{\mathcal{H}}(h, h_I)$ for Verma module over $\mathcal{H}$, $V^{W(2,2)}(h, h_W)$ for Verma module over $W(2,2)$ and so on. We always assume that c_W and $c_{L,I}$ are nonzero. Moreover, if we work with several modules over both algebras, c_L is equal for all modules.

We shall write $\langle x \rangle_{W(2,2)}$ for a cyclic submodule $U(W(2,2))x$ and $\langle x \rangle_{\mathcal{H}}$ for $U(\mathcal{H})x$. Finally, $\cong_{W(2,2)}$ denotes an isomorphism of $W(2,2)$-modules.

3 Irreducible highest weight modules

In this section we present main results of the paper which completely describe the structure of (irreducible) highest weight modules for $\mathcal{H}$ as $W(2,2)$-modules. The main tool is the homomorphism between $W(2,2)$ and the Heisenberg–Virasoro vertex algebras from [4].

$L^{W(2,2)}(c_L, c_W, 0, 0)$ is a simple universal vertex algebra associated to Lie algebra $W(2,2)$ (cf. [19, 20]) which we denote by $L^{W(2,2)}(c_L, c_W)$. It is generated by fields

$$L(z) = Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2}, \qquad W(z) = Y(W, z) = \sum_{n \in \mathbb{Z}} W(n) z^{-n-2}$$

where $\omega = L(-2)\mathbf{1}$ and $W = W(-2)\mathbf{1}$. Each highest weight $W(2,2)$-module is also a module over a vertex operator algebra $L^{W(2,2)}(c_L, c_W)$.

Likewise (see [7]) $L^{\mathcal{H}}(c_L, 0, c_{L,I}, 0, 0)$ is a simple Heisenberg–Virasoro vertex operator algebra, which we denote by $L^{\mathcal{H}}(c_L, c_{L,I})$. This algebra is generated by the fields

$$L(z) = Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2}, \qquad I(z) = Y(I, z) = \sum_{n \in \mathbb{Z}} I(n) z^{-n-1}$$

where $\omega = L(-2)\mathbf{1}$ and $I = I(-1)\mathbf{1}$. Moreover, highest weight $\mathcal{H}$-modules are modules over a vertex operator algebra $L^{\mathcal{H}}(c_{L,}c_{L,I})$.

It was shown in [4] that there is a monomorphism of vertex operator algebras

$$\begin{aligned} \Psi\colon\ L^{W(2,2)}(c_L, c_W) &\to L^{\mathcal{H}}(c_L, c_{L,I}), \\ \omega &\mapsto L(-2)\mathbf{1}, \\ W &\mapsto (I(-1)^2 + 2c_{L,I}I(-2))\mathbf{1}, \end{aligned} \tag{3.1}$$

where $c_W = -24c_{L,I}^2$. By means of Ψ, each highest weight module over $\mathcal{H}$ becomes an $L^{W(2,2)}(c_L, c_W)$-module and therefore a module over $W(2,2)$. In particular, Ψ induces a nontrivial $W(2,2)$-homomorphism (which we shall denote by the same letter)

$$\Psi\colon\ V^{W(2,2)}(c_L, c_W, h, h_W) \to V^{\mathcal{H}}(c_L, 0, c_{L,I}, h, h_I),$$

where $c_W = -24c_{L,I}^2$ and $h_W = h_I(h_I - 2c_{L,I})$. Ψ maps the highest weight vector v_{h,h_W} to the highest weight vector v_{h,h_I} and the action of $W(-n)$ on $V^{\mathcal{H}}(c_L, 0, c_{L,I}, h, h_I)$ is given by

$$\begin{aligned} W(-n) &\equiv 2c_{L,I}(n-1)I(-n) + \sum_{i\in\mathbb{Z}} I(-i)I(-n+i), \\ W(-n) &\equiv 2c_{L,I}\left(n - 1 + \frac{h_I}{c_{L,I}}\right)I(-n) + \sum_{i\neq 0,n} I(-i)I(-n+i). \end{aligned} \tag{3.2}$$

Note that $h_W = \frac{1-p^2}{24}c_W$ if and only if $h_I = (1 \pm p)c_{L,I}$, so either both of these Verma modules are irreducible, or they are reducible with singular vectors at equal levels. Moreover, $(h, h_W) \in \mathcal{AT}_{W(2,2)}(c_L, c_W)$ if and only if $(h, h_I) \in \mathcal{AT}_{\mathcal{H}}(c_L, c_{L,I})$.

Throughout the rest of this section we assume that $c_W = -24c_{L,I}^2$.

Lemma 3.1 ([4, Lemma 7.2])**.** *Suppose that $h_I \neq (1-p)c_{L,I}$ for all $p \in \mathbb{Z}_{>0}$. Then Ψ is an isomorphism of $W(2,2)$-modules. In particular, if $h_I \neq (1 \pm p)c_{L,I}$ for $p \in \mathbb{Z}_{>0}$, then*

$$L^{\mathcal{H}}(h, h_I) \cong_{W(2,2)} L^{W(2,2)}(h, h_W),$$

where $h_W = h_I(h_I - 2c_{L,I})$.

Lemma 3.2. *Suppose that $x \in V^{\mathcal{H}}(h, h_I)$ is $\mathcal{H}$-singular. Then x is $W(2,2)$-singular as well. In particular, if $x = \Psi(y)$ is an $\mathcal{H}$-singular vector, then y is a (co)singular vector in $V^{W(2,2)}(h, h_W)$.*

Proof. Follows directly from (3.2) since

$$W(n)x = -2c_{L,I}(n+1)I(n)x + \sum_{i\in\mathbb{Z}} I(-i)I(n+i)x.$$

If $I(k)x = 0$ for all $k \in \mathbb{Z}_{>0}$, then $W(n)x = 0$ for all $n \in \mathbb{Z}_{>0}$. If $x = \Psi(y)$, then $W(2,2)_+ y \in \operatorname{Ker}\Psi$ so y is cosingular (or singular if $\operatorname{Ker}\Psi = 0$). ∎

Theorem 3.3. *Let $p \in \mathbb{Z}_{>0}$.*

(*i*) *If $(h, (1+p)c_{L,I})$ is typical for $\mathcal{H}$ (equivalently if $\left(h, \frac{1-p^2}{24}c_W\right)$ is typical for $W(2,2)$) then*

$$L^{\mathcal{H}}(h, (1+p)c_{L,I}) \cong_{W(2,2)} L^{W(2,2)}\left(h, \frac{1-p^2}{24}c_W\right). \tag{3.3}$$

(ii) *If* $(h_{p,r},(1+p)c_{L,I}) \in \mathcal{AT}_{\mathcal{H}}(c_L,c_{L,I})$ *(equivalently if* $(h_{p,r},\frac{1-p^2}{24}c_W) \in \mathcal{AT}_{W(2,2)}(c_L,c_W)$*) then*

$$L^{\mathcal{H}}(h_{p,r},(1+p)c_{L,I}) \cong_{W(2,2)} \widetilde{L}^{W(2,2)}\left(h_{p,r},\frac{1-p^2}{24}c_W\right)$$

and the short sequence of $W(2,2)$*-modules*

$$0 \to L^{W(2,2)}\left(h_{p,r}+rp,\frac{1-p^2}{24}c_W\right) \to L^{\mathcal{H}}(h_{p,r},(1+p)c_{L,I}) \to L^{W(2,2)}\left(h_{p,r},\frac{1-p^2}{24}c_W\right) \to 0 \tag{3.4}$$

is exact.

Proof. By Lemma 3.1, Ψ is an isomorphism of Verma modules and thus by Lemma 3.2 it maps a $W(2,2)$-singular vector u'_p to an $\mathcal{H}$-singular vector v_p^+. If $h \neq h_{p,r}$, both of these vectors generate maximal submodules in respective Verma modules so (3.3) follows.

Now suppose that $h = h_{p,r}$. We need to show that a cosingular vector u_{rp} is not mapped to a maximal submodule of $V^{\mathcal{H}}(h_{p,r},h_I)$. But u_{rp} has $L(-p)^r v$ as an additive component (see Remark 2.4), and by construction (3.1), $\Psi(u_{rp})$ also must have this additive component. However, $\Psi(u_{rp})$ can not lie in a maximal $\mathcal{H}$-submodule of $V^{\mathcal{H}}(h,h_I)$ (see Remark 2.6). This means that isomorphism Ψ of Verma modules induces a $W(2,2)$-isomorphism of $\widetilde{L}^{W(2,2)}(h,h_W)$ and $L^{\mathcal{H}}(h,h_I)$ for all $h \in \mathbb{C}$. Exactness of (3.4) is just an application of (2.1). ■

Remark 3.4. Note that the image $\Psi(u_{rp})$ of a $W(2,2)$-cosingular vector is neither $\mathcal{H}$-singular, nor $\mathcal{H}$-cosingular in $V^{\mathcal{H}}(h_{p,r},(1+p)c_{L,I})$. For example, $L(-1)v_{0,0}$ in $V^{\mathcal{H}}(0,2c_{L,I})$ is $W(2,2)$-cosingular, but not $\mathcal{H}$-singular since $I(1)L(-1)v_{0,0} = 2c_{L,I}v_{0,0}$.

If $h_I = (1-p)c_{L,I}$, then Ψ is not an isomorphism. We shall present a $W(2,2)$-structure of Verma module later. In order to examine irreducible $W(2,2)$-modules we apply the properties of contragredient modules.

Let us recall the definition of contragredient module (see [12]). Assume that (M,Y_M) is a graded module over a vertex operator algebra V such that $M = \oplus_{n=0}^{\infty} M(n)$, $\dim M(n) < \infty$ and suppose that there is $\gamma \in \mathbb{C}$ such that $L(0)|M(n) \equiv (\gamma+n)\,\mathrm{Id}$. The contragredient module (M^*,Y_{M^*}) is defined as follows. For every $n \in \mathbb{Z}_{>0}$ let $M(n)^*$ be the dual vector space and let $M^* = \oplus_{n=0}^{\infty} M(n)^*$ be a restricted dual of M. Consider the natural pairing $\langle\cdot,\cdot\rangle : M^* \otimes M \to \mathbb{C}$. Define the linear map $Y_{M^*} \colon V \to \operatorname{End} M^*[[z,z^{-1}]]$ such that

$$\langle Y_{M^*}(v,z)m',m\rangle = \left\langle m', Y_M\left(e^{zL(1)}\left(-z^{-2}\right)^{L(0)}v,z^{-1}\right)m\right\rangle \tag{3.5}$$

for each $v \in V$, $m \in M$, $m' \in M^*$. Then (M^*,Y_{M^*}) is a V-module.

In particular, choosing $v = \omega = L_{-2}\mathbf{1}$ in (3.5) one gets

$$\langle L(n)m',m\rangle = \langle m',L(-n)m\rangle.$$

Simple calculation with $I \in L^{\mathcal{H}}(c_L,c_{L,I})$ and $W \in L^{W(2,2)}(c_L,c_W)$ shows that

$$\langle I(n)m',m\rangle = \langle m',-I(-n)m+\delta_{n,0}2c_{L,I}\rangle, \qquad \langle W(n)m',m\rangle = \langle m',W(-n)m\rangle.$$

Therefore we get the following result (the first and third relations were given in [4]):

Lemma 3.5.

$$L^{\mathcal{H}}(h,h_I)^* \cong L^{\mathcal{H}}(h,-h_I+2c_{L,I}), \qquad L^{W(2,2)}(h,h_W)^* \cong L^{W(2,2)}(h,h_W).$$

In particular,

$$L^{\mathcal{H}}(h,(1\pm p)c_{L,I})^* \cong L^{\mathcal{H}}(h,(1\mp p)c_{L,I}).$$

Directly from Theorem 3.3 and Lemma 3.5 follows

Corollary 3.6. *Let* $p \in \mathbb{Z}_{>0}$.

(i) *If* $(h,(1-p)c_{L,I})$ *is typical for* $\mathcal{H}$ *(equivalently if* $\left(h, \frac{1-p^2}{24}c_W\right)$ *is typical for* $W(2,2)$*) then*

$$L^{\mathcal{H}}(h,(1-p)c_{L,I}) \cong_{W(2,2)} L^{W(2,2)}\left(h, \frac{1-p^2}{24}c_W\right).$$

(ii) *If* $(h_{p,r},(1-p)c_{L,I}) \in \mathcal{AT}_{\mathcal{H}}(c_L,c_{L,I})$ *(equivalently if* $\left(h_{p,r}, \frac{1-p^2}{24}c_W\right) \in \mathcal{AT}_{W(2,2)}(c_L,c_W)$*) then*

$$L^{\mathcal{H}}(h_{p,r},(1-p)c_{L,I}) \cong_{W(2,2)} \widetilde{L}^{W(2,2)}\left(h_{p,r}, \frac{1-p^2}{24}c_W\right)^*$$

and the short sequence of $W(2,2)$*-modules*

$$0 \to L^{W(2,2)}\left(h_{p,r}, \frac{1-p^2}{24}c_W\right) \to L^{\mathcal{H}}\left(h_{p,r},(1-p)c_{L,I}\right) \to L^{W(2,2)}\left(h_{p,r}+rp, \frac{1-p^2}{24}c_W\right) \to 0$$

is exact.

From Lemma 3.1, Theorem 3.3 and Corollary 3.6 follow assertions of Theorem 1.1.

Finally, we show that Verma module over $\mathcal{H}$ is an infinite direct sum of irreducible $W(2,2)$-modules. Recall that $V^{\mathcal{H}}(h,(1-p)c_{L,I})$ has a series of singular vectors v^-_{ip}, $i \in \mathbb{Z}_{\geq 0}$ (for $i=0$, we set $v^-_0 = v_{h,h_I}$) which generate a descending chain of Verma submodules over $\mathcal{H}$:

$$\begin{aligned}
\langle v_{h,h_I}\rangle_{\mathcal{H}} &= V^{\mathcal{H}}(h,h_I)\\
&\cup\!\!|\\
\langle v^-_{p}\rangle_{\mathcal{H}} &\cong V^{\mathcal{H}}(h+p,h_I)\\
&\cup\!\!|\\
&\vdots\\
&\cup\!\!|\\
\langle v^-_{ip}\rangle_{\mathcal{H}} &\cong V^{\mathcal{H}}(h+ip,h_I)\\
&\cup\!\!|\\
\langle v^-_{(i+1)p}\rangle_{\mathcal{H}} &\cong V^{\mathcal{H}}(h+(i+1)p,h_I)\\
&\cup\!\!|\\
&\vdots
\end{aligned}$$

Therefore one may identify $V^{\mathcal{H}}(h+ip,h_I)$ with a submodule of $V^{\mathcal{H}}(h,h_I)$ and a singular vector $v^-_{ip} \in V^{\mathcal{H}}(h,h_I)$ with the highest weight vector $v_{h+ip,h_I} \in V^{\mathcal{H}}(h+ip,h_I)$. We will prove that in a typical case each of those vectors generates an irreducible $W(2,2)$-submodule.

Theorem 3.7. *Let $p \in \mathbb{Z}_{>0}$. Suppose that $(h,(1-p)c_{L,I}) \notin \mathcal{AT}_{\mathcal{H}}(c_L, c_{L,I})$. Then we have the following isomorphism of $W(2,2)$-modules*

$$V^{\mathcal{H}}(h,(1-p)c_{L,I}) \cong_{W(2,2)} \bigoplus_{i\geq 0} L^{W(2,2)}\left(h+ip, \frac{1-p^2}{24}c_W\right).$$

Proof. First we notice that the vertex algebra homorphism $\Psi\colon L^{W(2,2)}(c_W, c_L) \to L^{\mathcal{H}}(c_W, c_L)$, for every $i \in \mathbb{Z}_{\geq 0}$ induces the following non-trivial homomorphism of $W(2,2)$-modules:

$$\Psi^{(i)}\colon\ V^{W(2,2)}\left(h+ip, \frac{1-p^2}{24}c_W\right) \to \langle v_{ip}^{-}\rangle_{W(2,2)} \subset V^{\mathcal{H}}(h+ip, h_I),$$

which maps the highest weight vector of $V^{W(2,2)}\big(h+ip, \frac{1-p^2}{24}c_W\big)$ to v_{ip}^{-}. Since $(h, \frac{1-p^2}{24}c_W)$ is typical it follows from Proposition 2.3(i) that $(h+ip, \frac{1-p^2}{24}c_W)$ are typical for all $i \in \mathbb{Z}_{>0}$ as well.

Let $h_W = \frac{1-p^2}{24}c_W$. Consider the homomorphism $\Psi^{(i)}\colon V^{W(2,2)}(h+ip, h_W) \to V^{\mathcal{H}}(h+ip, h_I)$ above. Applying (3.2), we get

$$\Psi^{(i)}(W(-p)v_{h+ip,h_W}) = \sum_{i=1}^{p-1} I(-i)I(i-p)v_{h+ip,h_I},$$

so $I(-p)v_{h+ip,h_I} \notin \operatorname{Im}\Psi^{(i)}$. Since the Verma modules $V^{W(2,2)}(h+ip, h_W)$ and $V^{\mathcal{H}}(h+ip, h_I)$ have equal characters, it follows that $\operatorname{Ker}\Psi^{(i)}$ contains a singular vector in $V^{W(2,2)}(h+ip, h_W)$ of conformal weight $h+(i+1)p$. Since the weight $(h+ip, h_W)$ is typical, the maximal submodule in $V^{W(2,2)}(h+ip, h_W)$ is generated by this singular vector so we conclude that $\operatorname{Ker}\Psi^{(i)}$ is the maximal submodule in $V^{W(2,2)}(h+ip, h_W)$. Therefore

$$\operatorname{Im}\Psi^{(i)} = \langle v_{h+ip,h_I}\rangle_{W(2,2)} \cong L^{W(2,2)}(h+ip, h_W).$$

In this way we get a series of $W(2,2)$-monomorphisms

$$L^{W(2,2)}(h+ip, h_W) \hookrightarrow V^{\mathcal{H}}(h,(1-p)c_{L,I}), \qquad i \in \mathbb{Z}_{\geq 0} \tag{3.6}$$

mapping v_{h+ip,h_W} to a singular vector v_{ip}^{-}. Let v_{jp}^{-} be an $\mathcal{H}$-singular vector in $V^{\mathcal{H}}(h+ip,(1-p)c_{L,I})$ of weight $h+jp$, for $j>i$. By Lemma 3.2, v_{jp}^{-} is singular for $W(2,2)$ and therefore $v_{jp}^{-} \notin \langle v_{h+ip,h_I}\rangle_{W(2,2)}$ for $j>i$. We conclude that the images of morphisms (3.6) have trivial pairwise intersections (since these images are non-isomorphic irreducible modules), so their sum is direct. The assertion follows from the observation that the character of this sum is

$$\sum_{i=0}^{\infty} q^{h+ip}(1-q^p)\prod_{k\geq 1}\left(1-q^k\right)^{-2} = q^h \prod_{k\geq 1}\left(1-q^k\right)^{-2} = \operatorname{char} V^{\mathcal{H}}(h,(1-p)c_{L.I}). \qquad \blacksquare$$

From the previous theorem follows

$$\begin{aligned}
&V^{\mathcal{H}}(h, h_I)\\
&\quad \parallel\\
\langle v_{h,h_I}\rangle_{W(2,2)} &= L^{W(2,2)}(h, h_I)\\
&\quad \oplus\\
\langle v_p^{-}\rangle_{W(2,2)} &\cong L^{W(2,2)}(h+p, h_I)\\
&\quad \oplus
\end{aligned}$$

$$\begin{aligned} &\vdots \\ &\oplus \\ \langle v^-_{ip}\rangle_{W(2,2)} \cong &L^{W(2,2)}(h+ip,h_I) \\ &\oplus \\ &\vdots \end{aligned}$$

Remark 3.8. It is interesting to notice that our Theorem 3.7 shows that $V^{\mathcal{H}}(h,h_I)$ can be considered as a $W(2,2)$-analogue of certain Feigin-Fuchs modules for the Virasoro algebra which are also direct sums of infinitely many irreducible modules (cf. [11], [2, Theorem 5.1]).

In atypical case however, these irreducible $W(2,2)$-submodules intertwine as follows. Consider $V^{\mathcal{H}}(h_{p,r},h_I)$ where $(h_{p,r},h_I)\in\mathcal{AT}_{\mathcal{H}}(c_L,c_{L,I})$. Then Ψ maps a cosingular vector $u_{rp}\in V^{W(2,2)}(h_{p,r},h_W)$ to a singular vector v^-_{rp}. In other words we have

$$\langle v^-_{rp}\rangle_{W(2,2)} \subseteq \langle v_{h_{p,r},h_I}\rangle_{W(2,2)}.$$

Using the same argument in view of Proposition 2.3 we see that

$$\langle v^-_{(r-i)p}\rangle_{W(2,2)} \subseteq \langle v^-_{ip}\rangle_{W(2,2)}, \qquad i=1,\ldots,\lfloor\frac{r-1}{2}\rfloor.$$

In this case, $I(-p)^{r-i}v_{h_{p,r},h_I}$ are $W(2,2)$-subsingular vectors in $V^{\mathcal{H}}(h_{p,r},h_I)$.

Example 3.9. Consider $p=1$ case. Singular vector in $V^{\mathcal{H}}(h,0)$ is $u'_1=\big(L(-1)+\frac{h}{c_{L,I}}I(-1)\big)v_{h,0}$, and u'_1 generates a copy of $V^{\mathcal{H}}(h+1,0)$.

$r=1$: $\Psi\colon V^{W(2,2)}(0,0)\to V^{\mathcal{H}}(0,0)$ maps a singular vector $u'_1=W(-1)v_{0,0}$ to 0 and a cosingular vector $u_1=L(-1)v_{0,0}$ to $\mathcal{H}$-singular vector $v^-_1=L(-1)v_{0,0}$. We get the short exact sequence of $W(2,2)$-modules

$$0\to L^{W(2,2)}(0,0)\to L^{\mathcal{H}}(0,0)\to L^{W(2,2)}(1,0)\to 0,$$

which is an expansion of (3.1) considered as a homomorphism of $W(2,2)$-modules. The rightmost module is generated by a projective image of $I(-1)v_{0,0}$. Therefore, $L^{\mathcal{H}}(c_L,c_{L,I})$ is generated over $W(2,2)$ by $v_{0,0}$ and $I(-1)v_{0,0}$.

$r\in\mathbb{Z}_{>0}$: In general, a cosingular vector $u_{rp}\in V^{W(2,2)}\left(\frac{1-r}{2},0\right)$ maps to a singular vector $v^-_r\in V^{\mathcal{H}}\left(\frac{1-r}{2},0\right)$ of weight $\frac{1+r}{2}$.

$$v^-_r=\prod_{i=0}^{r-1}\left(L(-1)+\frac{1-r+2i}{2c_{L,I}}I(-1)\right)v_{\frac{1-r}{2},0}.$$

4 Screening operators and $W(2,2)$-algebra

We think that the vertex algebra $L^{W(2,2)}(c_L,c_W)$ is a very interesting example of non-rational vertex algebra, which admits similar fusion ring of representations as some $\mathcal{W}$-algebras appearing in LCFT (cf. [1, 2, 10, 13]). Since $\mathcal{W}$-algebras appearing in LCFT are realized as kernels of screening operators acting on certain modules for Heisenberg vertex algebras, it is natural to ask if $L^{W(2,2)}(c_L,c_W)$ admits similar realization. In [4] we embedded the $W(2,2)$-algebra as a subalgebra of the Heisenberg–Virasoro vertex algebra. In this section we shall construct a screening operator S_1 such that the kernel of this operator is exactly $L^{W(2,2)}(c_L,c_W)$.

Let us first construct a non-semisimple extension of the vertex algebra $L^{\mathcal{H}}(c_L, c_{L,I})$. Recall that the Lie algebra $\mathcal{H}$ admits the triangular decomposition (2.2). Let $E = \mathrm{span}_{\mathbb{C}}\{v^0, v^1\}$ be 2-dimensional $\mathcal{H}^{\geq 0} = \mathcal{H}^0 \oplus \mathcal{H}^+$-module such that $\mathcal{H}^+$ acts trivially on E and

$$L(0)v^i = v^i, \qquad i = 0,1, \qquad I(0)v^1 = v^0, \qquad I(0)v^0 = 0,$$
$$C_L v^i = c_L v^i, \qquad C_{L,I} v^i = c_{L,I} v^i, \qquad C_I v^i = 0, \qquad i = 1,2.$$

Consider now induced $\mathcal{H}$-module

$$\widetilde{E} = U(\mathcal{H}) \otimes_{U(\mathcal{H}^{\geq 0})} E.$$

By construction, $\widetilde{E}$ is a non-split self-extension of the Verma module $V^{\mathcal{H}}(1,0)$:

$$0 \to V^{\mathcal{H}}(1,0) \to \widetilde{E} \to V^{\mathcal{H}}(1,0) \to 0.$$

Moreover, $\widetilde{E}$ is a restricted module for $\mathcal{H}$ and therefore it is a module over vertex operator algebra $L^{\mathcal{H}}(c_L, c_{L,I})$. Since

$$\widetilde{E} \cong E \otimes U(\mathcal{H}^-)$$

as a vector space, the operator $L(0)$ defines a $\mathbb{Z}_{\geq 0}$-gradation on $\widetilde{E}$.

Note that $(L(-1) + I(-1)/c_{L,I})v_0$ is a singular vector in $\widetilde{E}$ and it generates the proper submodule. Finally we define the quotient module

$$\mathcal{U} = \frac{\widetilde{E}}{U(\mathcal{H}).(L(-1) + I(-1)/c_{L,I})v_0}.$$

Proposition 4.1. *$\mathcal{U}$ is a $\mathbb{Z}_{\geq 0}$-graded module for the vertex operator algebra $L^{\mathcal{H}}(c_L, c_{L,I})$:*

$$\mathcal{U} = \bigoplus_{m \in \mathbb{Z}_{\geq 0}} \mathcal{U}(m), \qquad L(0)|\mathcal{U}(m) \equiv (m+1)\,\mathrm{Id}\,.$$

The lowest component $\mathcal{U}(0) \cong E$. Moreover, $\mathcal{U}$ is a non-split extension of the Verma module $V^{\mathcal{H}}(1,0)$ by the simple highest weight module $L^{\mathcal{H}}(1,0)$:

$$0 \to L^{\mathcal{H}}(1,0) \to \mathcal{U} \to V^{\mathcal{H}}(1,0) \to 0.$$

Proof. By construction $\mathcal{U}$ is a graded quotient of a $\mathbb{Z}_{\geq 0}$-graded $L^{\mathcal{H}}(c_L, c_{L,I})$-module $\widetilde{E}$. The lowest component is $\mathcal{U}(0) \cong E$. Submodule $U(\mathcal{H}).v^0$ is isomorphic to $L^{\mathcal{H}}(1,0)$, and the projective image of v^1 generates the Verma module $V^{\mathcal{H}}(1,0)$ since $I(0)v^1 = v^0$. For the same reason, this exact sequence does not split. ∎

Now we consider $L^{\mathcal{H}}(c_L, c_{L,I})$-module

$$\mathcal{V}^{\mathrm{ext}} := L^{\mathcal{H}}(c_L, c_{L,I}) \oplus \mathcal{U}.$$

By using [16, Theorem 4.8.1] (see also [3, 17]) we have that $\mathcal{V}^{\mathrm{ext}}$ has the structure of a vertex operator algebra with vertex operator map Y_{ext} defined as follows:

$$Y_{\mathrm{ext}}(a_1 + w_1, z)(a_2 + w_2) = Y(a_1, z)(a_2 + w_2) + e^{zL(-1)}Y(a_2, -z)w_1,$$

where $a_1, a_2 \in L^{\mathcal{H}}(c_L, c_{L,I})$, $w_1, w_2 \in \mathcal{U}$.

Take now $v^i \in E \subset \mathcal{U}$, $i = 0,1$ as above and define

$$S_i(z) = Y_{\mathrm{ext}}\big(v^i, z\big) = \sum_{n \in \mathbb{Z}} S_i(n) z^{-n-1}.$$

By construction

$$S_1(z) \in \mathrm{End}\big(L^{\mathcal{H}}(c_L, c_{L,I}), L^{\mathcal{H}}(1,0)\big)((z)).$$

Proposition 4.2. *For all $n, m \in \mathbb{Z}$ we have:*

$$[L(n), S_i(m)] = -mS_i(n+m), \qquad i = 0, 1,$$
$$[W(n), S_0(m)] = 0, \qquad [W(n), S_1(m)] = 2mc_{L,I}S_0(n+m).$$

In particular, $S_0(0)$ and $S_1(0)$ are screening operators. Moreover,

$$S_1 = S_1(0)\colon\ L^{\mathcal{H}}(c_L, c_{L,I}) \to L^{\mathcal{H}}(1, 0)$$

is nontrivial and $S_1(0)I(-1)\mathbf{1} = -v_0$.

Proof. Since $L(k)v^i = \delta_{k,0}v^i$ for $k \geq 0$, commutator formula gives that

$$[L(n), S_i(m)] = -mS_i(n+m).$$

Next we calculate $[W(n), S_1(m)]$. We have

$$W(-1)v^1 = 2I(-1)v^0 = -2c_{L,I}L(-1)v^0,$$
$$W(0)v^1 = -2c_{L,I}v^0, \qquad W(n)v^1 = 0, \qquad n \geq 0.$$

This implies that

$$[W(n), S_1(m)] = 2c_{L,I}mS_0(n+m).$$

Since $W(n)v^0 = 0$ for $n \geq -1$ we get

$$[W(n), S_0(m)] = 0.$$

Therefore we have proved that $S_i(0)$, $i = 0, 1$ are screening operators. Next we have

$$S_1(0)I(-1)\mathbf{1} = \operatorname{Res}_z Y_{\text{ext}}\big(v^1, z\big)I(-1) = \operatorname{Res}_z e^{zL(-1)}Y(I(-1)\mathbf{1}, -z)v^1 = -v_0.$$

The proof follows. ∎

Theorem 4.3. *S_1 is a derivation of the vertex algebra $\mathcal{V}^{\text{ext}}$ and we have*

$$\operatorname{Ker}_{L^{\mathcal{H}}(c_L, c_{L,I})} S_1 \cong L^{W(2,2)}(c_L, c_W).$$

Proof. By construction $S_1 = \operatorname{Res}_z Y_{\text{ext}}(v^1, z)$, so S_1 is a derivation so $\overline{W} = \operatorname{Ker}_{L^{\mathcal{H}}(c_L, c_{L,I})} S_1$ is a vertex subalgebra of $L^{\mathcal{H}}(c_L, c_{L,I})$. Since

$$S_1L(-2)\mathbf{1} = S_1W(-2)\mathbf{1} = 0$$

we have that $L^{W(2,2)}(c_L, c_W) \subset \overline{W}$. Since $S_1I(-1)\mathbf{1} \neq 0$, we have that $I(-1)\mathbf{1}$ does not belong to $\overline{W}$. By using the fact that $L^{\mathcal{H}}(c_L, c_{L,I})$ is as $W(2,2)$-module generated by singular vector $\mathbf{1}$ and cosingular vector $I(-1)\mathbf{1}$ (see Example 3.9) we get that $\overline{W} = L^{W(2,2)}(c_L, c_W)$. The proof follows. ∎

Remark 4.4. Of course, every $\mathcal{V}_{\text{ext}}$-module becomes a $W(2,2)$-module with screening operator S_1. Similar statement holds for intertwining operators. Constructions of such modules and intertwining operators require different techniques which we will present in our forthcoming paper [5].

Acknowledgements

The authors are partially supported by the Croatian Science Foundation under the project 2634 and by the Croatian Scientific Centre of Excellence QuantixLie.

References

[1] Adamović D., Classification of irreducible modules of certain subalgebras of free boson vertex algebra, *J. Algebra* **270** (2003), 115–132, math.QA/0207155.

[2] Adamović D., Milas A., Logarithmic intertwining operators and $W(2,2p-1)$ algebras, *J. Math. Phys.* **48** (2007), 073503, 20 pages, math.QA/0702081.

[3] Adamović D., Milas A., Lattice construction of logarithmic modules for certain vertex algebras, *Selecta Math. (N.S.)* **15** (2009), 535–561, arXiv:0902.3417.

[4] Adamović D., Radobolja G., Free field realization of the twisted Heisenberg–Virasoro algebra at level zero and its applications, *J. Pure Appl. Algebra* **219** (2015), 4322–4342, arXiv:1405.1707.

[5] Adamović D., Radobolja G., Self-dual and logarithmic representations of the twisted Heisenberg–Virasoro algebra at level zero, in preparation.

[6] Arbarello E., De Concini C., Kac V.G., Procesi C., Moduli spaces of curves and representation theory, *Comm. Math. Phys.* **117** (1988), 1–36.

[7] Billig Y., Energy-momentum tensor for the toroidal Lie algebra, math.RT/0201313.

[8] Billig Y., Representations of the twisted Heisenberg–Virasoro algebra at level zero, *Canad. Math. Bull.* **46** (2003), 529–537, math.RT/0201314.

[9] Billig Y., A category of modules for the full toroidal Lie algebra, *Int. Math. Res. Not.* **2006** (2006), 68395, 46 pages, math.RT/0509368.

[10] Creutzig T., Milas A., False theta functions and the Verlinde formula, *Adv. Math.* **262** (2014), 520–545, arXiv:1309.6037.

[11] Feigin B.L., Fuchs D.B., Representations of the Virasoro algebra, in Representation of Lie Groups and Related Topics, *Adv. Stud. Contemp. Math.*, Vol. 7, Gordon and Breach, New York, 1990, 465–554.

[12] Frenkel I.B., Huang Y.-Z., Lepowsky J., On axiomatic approaches to vertex operator algebras and modules, *Mem. Amer. Math. Soc.* **104** (1993), viii+64 pages.

[13] Gainutdinov A.M., Semikhatov A.M., Tipunin I.Yu., Feigin B.L., The Kazhdan–Lusztig correspondence for the representation category of the triplet W-algebra in logorithmic conformal field theories, *Theoret. Math. Phys.* **148** (2006), 1210–1235, math.QA/0512621.

[14] Jiang W., Pei Y., On the structure of Verma modules over the W-algebra $W(2,2)$, *J. Math. Phys.* **51** (2010), 022303, 8 pages.

[15] Jiang W., Zhang W., Verma modules over the $W(2,2)$ algebras, *J. Geom. Phys.* **98** (2015), 118–127.

[16] Lepowsky J., Li H., Introduction to vertex operator algebras and their representations, *Progress in Mathematics*, Vol. 227, Birkhäuser Boston, Inc., Boston, MA, 2004.

[17] Li H.S., Symmetric invariant bilinear forms on vertex operator algebras, *J. Pure Appl. Algebra* **96** (1994), 279–297.

[18] Liu D., Zhu L., Classification of Harish-Chandra modules over the W-algebra $W(2,2)$, arXiv:0801.2601.

[19] Radobolja G., Subsingular vectors in Verma modules, and tensor product of weight modules over the twisted Heisenberg–Virasoro algebra and $W(2,2)$ algebra, *J. Math. Phys.* **54** (2013), 071701, 24 pages, arXiv:1302.0801.

[20] Zhang W., Dong C., W-algebra $W(2,2)$ and the vertex operator algebra $L(\frac{1}{2},0)\otimes L(\frac{1}{2},0)$, *Comm. Math. Phys.* **285** (2009), 991–1004, arXiv:0711.4624.