Quasi-Polynomials and the Singular \([Q,R] = 0\) Theorem

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Abstract. In this short note we revisit the ‘shift-desingularization’ version of the \([Q,R] = 0\) theorem for possibly singular symplectic quotients. We take as starting point an elegant proof due to Szenes–Vergne of the quasi-polynomial behavior of the multiplicity as a function of the tensor power of the prequantum line bundle. We use the Berline–Vergne index formula and the stationary phase expansion to compute the quasi-polynomial, adapting an early approach of Meinrenken.

Key words: symplectic geometry; Hamiltonian \(G\)-spaces; symplectic reduction; geometric quantization; quasi-polynomials; stationary phase

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1 Introduction

Let \((M,\omega)\) be a compact connected symplectic manifold equipped with an action of a compact connected Lie group \(G\) by symplectomorphisms. Suppose that the action of \(G\) is Hamiltonian, meaning that there is a \(G\)-equivariant map, the moment map,

\[ \mu_g: M \to \mathfrak{g}^*, \]

where \(\mathfrak{g}^*\) is the dual of the Lie algebra \(\mathfrak{g} = \text{Lie}(G)\), satisfying the moment map condition

\[ \iota(X_M)\omega = -d\langle \mu_g, X \rangle, \quad X \in \mathfrak{g}. \] (1.1)

Let \((L,\nabla^L)\) be a \(G\)-equivariant prequantum line bundle with connection on \(M\), i.e., \(L\) is a \(G\)-equivariant Hermitian line bundle with compatible connection \(\nabla^L\), \((\nabla^L)^2 = -2\pi i\omega\) and the derivative of the \(G\)-action on \(L\) satisfies Kostant’s condition

\[ \mathcal{L}_X - \nabla^L_{X_M} = 2\pi i\langle \mu_g, X \rangle. \]

Choose a compatible almost complex structure \(J\) on \(M\), i.e., \(\omega(Jw,Jv) = \omega(w,v)\) and \(\omega(w,Jv) = g(w,v)\) is a Riemannian metric. Let \(D_L\) denote the Dolbeault–Dirac operator twisted by \((L,\nabla^L)\), an elliptic differential operator acting on sections of the spinor bundle \(\wedge T_{0,1}^* M \otimes L\). The kernel of \(D_L\) carries an action of \(G\), and the \(G\)-equivariant index is defined to be the difference \(\text{index}_G(D_L) := \text{ker}(D_{L}^{\text{even}}) - \text{ker}(D_{L}^{\text{odd}})\) of the kernel of \(D_L\) on even/odd degree forms, regarded as an element of the representation ring \(R(G)\).

The quantization-commutes-with-reduction theorem ([\(Q,R\] = 0 theorem]) describes the multiplicity of the trivial representation in \(\text{index}_G(D_L)\) in terms of the symplectic quotient \(M^\text{red} := \mu_g^{-1}(0)/G\). When 0 is a regular value of \(\mu_g\), \(M^\text{red}\) is an orbifold and the theorem states that \(\text{index}_G(D_L)^G\) equals the index of the twisted Dolbeault–Dirac operator \(D_{L}^\text{red}\) on \(M^\text{red}\). The theorem was first conjectured by Guillemin–Sternberg [3], and the general case \((M, G\) both compact, 0 a regular value) was first proved by Meinrenken [8]. Different proofs of the \([Q,R] = 0\) theorem
were given by Tian–Zhang [15] and Paradan [11]. The theorem has since been extended in various directions.

There are versions of the $[Q,R] = 0$ theorem when 0 is not necessarily a regular value, due to Meinrenken–Sjamaar [10]; below we will give a precise statement of one of these results, involving a partial shift desingularization, i.e., $\text{index}_G(D_L)^G$ is related to the index on the symplectic quotient at a nearby weakly regular value. At the same time, we introduce some shift desingularization involving a partial variation in various directions.

Fix a maximal torus $T$ with Lie algebra $\mathfrak{t}$. Let $\Lambda \subset \mathfrak{t}^*$ be the (real) weight lattice. Given $\lambda \in \Lambda$, the corresponding character $T \to U(1)$ is written $t \mapsto t^\lambda = e^{2\pi i (\lambda, X)}$ where $t = e^X$, $X \in \mathfrak{t}$. Let $R \subset \Lambda$ be the set of roots. We also fix a closed positive Weyl chamber $\mathfrak{t}_+^\ast$. For each relatively open face $\sigma \subset \mathfrak{t}_+^\ast$, the stabilizer $G_\xi$ of points $\xi \in \sigma$ under the coadjoint action, does not depend on $\xi$, and will be denoted $G_\sigma$. If $\sigma_1 \subset \sigma_2$ then $G_{\sigma_1} \supset G_{\sigma_2}$. Note also that $G_\sigma$ is connected and contains the maximal torus $T$. The Lie algebra $\mathfrak{g}_\sigma$ decomposes into its semi-simple and central parts $\mathfrak{g}_\sigma = [\mathfrak{g}_\sigma, \mathfrak{g}_\sigma] \oplus \mathfrak{z}_\sigma$. The subspace $\mathfrak{z}_\sigma^\ast \subset \mathfrak{t}^\ast$ is defined to be the annihilator of $[\mathfrak{g}_\sigma, \mathfrak{g}_\sigma]$, or equivalently the fixed point set of the coadjoint $G_\sigma$ action. The face $\sigma$ is an open subset of $\mathfrak{z}_\sigma^\ast$.

Let $\Delta = \mu_\emptyset(M) \cap \mathfrak{t}_+^\ast$ be the moment polytope. A well-known theorem in symplectic geometry states that there is a unique face $\sigma \subset \mathfrak{t}_+^\ast$ of minimal dimension such that $\Delta \subset \overline{\sigma}$ (briefly, this is a consequence of (1.1), which implies that $d\mu_\emptyset$ has constant rank on the top dimensional $G$-orbit type stratum, and the complement of the latter has codimension at least 2); $\sigma$ is called the principal face or principal wall. The corresponding symplectic cross-section, called the principal cross-section, $Y = \mu_\emptyset^{-1}(\sigma)$ is a Hamiltonian $G_\sigma$-space. Moreover the semi-simple part $[G_\sigma, G_\sigma]$ of $G_\sigma$ acts trivially on $Y$. For further details, see for example [5] and references therein.

Let $I \subset \mathfrak{z}_\sigma^\ast$ be the smallest affine subspace containing $\Delta$. Let $t_I \subset \mathfrak{t}$ be the annihilator of the subspace parallel to $I$, and let $T_I = \exp(t_I) \subset T$ be the corresponding subtorus. By equation (1.1), $t_I$ is the generic infinitesimal stabilizer of $Y$. In particular $T_I$ acts trivially, hence the quotient torus $T/T_I$ acts on $Y$. The moment map $\mu_\emptyset$ may have no non-trivial regular values. But the restriction

$$\mu_\emptyset|_Y : Y \to I$$

viewed as a map with codomain $I$, always has non-trivial regular values, and we will refer to these as weakly-regular values. If $\xi$ is a weakly-regular value, then the reduced space $M_\xi = \mu_\emptyset^{-1}(\xi)/G_\sigma$ is an orbifold. Let $L_\xi = L|_{\mu_\emptyset^{-1}(\xi)/G_\sigma}$ be the corresponding (orbifold) line bundle over $M_\xi$.

Theorem 1.1 ([10], see also [11, 13]). Let $(M, \omega, \mu_\emptyset)$ be a compact connected Hamiltonian $G$-space with moment polytope $\Delta$. If $0 \notin \Delta$ then $\text{index}_G(D_L)^G = 0$. Otherwise for every weakly-regular value $\xi \in \Delta$ sufficiently close to 0, $\text{index}_G(D_L)^G$ equals the index of the Dolbeault–Dirac operator $D_\xi^G$ on the reduced space $M_\xi$.

We will now describe the main result of this article and its relation to Theorem 1.1. Consider tensor powers $L^k$, $k \in \mathbb{Z}_{>0}$ of the prequantum line bundle. For a dominant weight $\lambda$, let $\chi_\lambda \in R(G)$ denote the character of the irreducible representation of $G$ with highest weight $\lambda$. We define the multiplicity function $m_G(k, \lambda)$ by the expression

$$\text{index}_G(D_{L^k}) = \sum_{\lambda \in \Lambda \cap \mathfrak{t}_+^\ast} m_G(k, \lambda) \chi_\lambda. \quad (1.2)$$

An important theme in the work of Szenes–Vergne [14] and also in our approach, is that the function $m_G(k, \lambda)$ has more coherent behavior than its restriction to any fixed value of $k$.

The statement of the result requires some further background on orbifolds, for which we refer the reader to, for example, [2, Appendix A], [8, Section 2]. A small warning is that we will not
require the action of isotropy groups in orbifold charts to be effective (this is in agreement with the references [2, 8] mentioned above). One advantage of permitting this, is that for a locally free action of a compact Lie group $K$ on a manifold $P$, the corresponding orbifold $P/K$ has orbifold charts given automatically by the slice theorem, with the isotropy groups being simply the isotropy groups for the action of $K$ on $P$.

In fact all the orbifolds that we will encounter arise naturally as such quotients $P/K$, and one could avoid mentioning orbifolds altogether by working instead with suitable $K$-basic structures on $P$. An example is the description of characteristic forms for orbifold vector bundles, which can be defined in terms of orbifold charts for $P/K$, or alternatively in terms of $K$-basic differential forms on $P$. In brief, the latter approach goes as follows. One can take the complex $(\Omega_{bas}(P), d)$ of $K$-basic differential forms on $P$ as a working definition of the de Rham complex of $P/K$ (if $K$ acts freely then $P/K$ is a manifold and pullback of forms from $P/K$ to $P$ is an isomorphism of complexes $(\Omega(P/K), d) \simeq (\Omega_{bas}(P), d)$). A $K$-equivariant vector bundle $E \to P$ determines an orbifold vector bundle $E/K$ over $P/K$. Let $\theta$ be a connection on $P$ with curvature $F_\theta$. The choice of connection determines a Cartan map (cf. [9]) from closed $K$-equivariant forms $\alpha(X)$ on $P$ to closed $K$-basic forms: $\alpha(X) \mapsto \text{Car}_\rho(\alpha) := \Pi_{\text{hor}}\alpha(F_\theta)$, where $\Pi_{\text{hor}}$ is the projection onto the horizontal part relative to the connection. The Cartan map induces an isomorphism from the $K$-equivariant cohomology of $P$ to the cohomology of the complex of basic differential forms on $P$. If $\alpha(X)$ is a $K$-equivariant characteristic form (constructed via the $K$-equivariant analogue of the usual Chern–Weil construction cf. [1, 9]), then one may take $\text{Car}_\rho(\alpha) \in \Omega_{bas}(P)$ as the definition of the corresponding characteristic form for $E/K$.

Let $\xi \in \Delta$ be a weakly-regular value. By the moment map equation (1.1), the action of $K = T/T_I$ on the level set

$$P = \mu^{-1}_g(\xi)$$

is locally free. The set $S_P$ of elements $g \in T/T_I$ such that $P^g \neq \emptyset$ is finite. For each $g \in S_P$, we obtain an orbifold

$$\Sigma_g = P^g/(T/T_I), \quad \Sigma = \bigsqcup_{g \in S_P} \Sigma_g.$$

Note that $\Sigma_1 = P/(T/T_I) = M_\xi$ identifies with the reduced space itself, and more generally $\Sigma_g$ identifies with a symplectic quotient of $Y^g$. For each $g \in S_P$ there is an immersion $\Sigma_g \hookrightarrow \Sigma$ induced by $P^g \hookrightarrow P$. Let $\nu_{\Sigma_g, \Sigma}$ denote the (orbifold) normal bundle (the quotient $\nu_{P^g, P}/(T/T_I)$), which inherits a complex structure from the almost complex structures on $Y$, $Y^g$. Define the characteristic form

$$D^g_C(\nu_{\Sigma_g, \Sigma}) = \det_C(1 - g^{-1}_\nu e^{-\frac{i}{\pi}F_\nu}),$$

where $g_\nu$ denotes the action of $g$ on the normal bundle (defined in terms of an orbifold chart, or in terms of $\nu_{P^g, P}$), and $F_\nu$ denotes the curvature. Taking the quotient of $L_{P^g}$ we obtain (orbifold) line bundles

$$L_{\Sigma_g} = (L_{P^g})/(T/T_I), \quad L_\Sigma = \bigsqcup_{g \in S_P} L_{\Sigma_g}.$$

There is a locally constant function

$$g_L: \Sigma_g \to U(1)$$

giving the phase of the action of $g$ on $L_{\Sigma_g}$ (or equivalently on $L_{P^g}$). Let $d: \Sigma \to \mathbb{Z}$ be the locally constant function giving the size of a generic isotropy group for $\Sigma$ (or equivalently the number of elements in the generic stabilizer for the $T/T_I$ action on $\sqcup P^g$).
Let $\theta$ be a connection for the locally free $K = T/T_t$-action on $\bigcup_{g \in S_p} P^g$. The curvature $F_\theta$ is horizontal and $t/t_t$-valued, hence for any $\lambda \in (t/t_t)^* = I$, the form $\langle \lambda, F_\theta \rangle$ is $K$-basic, hence descends to $\Sigma$. With the preparations above, we can state the main result of this note.

**Theorem 1.2.** If $0 \not\in \Delta$ then $m(k, 0) = 0$ for all $k \geq 1$. If $0 \in \Delta$ then there is a closed polytope $p \subset \Delta$ of the same dimension as $\Delta$ and containing the origin such that the following is true. Let $C_p$ denote the cone

$$C_p = \{(t, t\tau) \mid t \in (0, \infty), \tau \in p\} \subset \mathbb{R} \times t^*.$$  

Fix a weakly regular value $\xi \in \Delta$ sufficiently close to 0 as in Theorem 1.1. Let $P = \mu_q^{-1}(\xi)$ and define $\Sigma$, $L_\Sigma$, etc. as above. Then for all $(k, \lambda) \in (\mathbb{Z}_{>0} \times \Lambda) \cap C_p$,

$$m_G(k, \lambda) = \sum_{g \in S_p} g^{-\lambda} \frac{1}{d} \int_{\Sigma_g} g^k \frac{\text{Ch}(L_\Sigma^k) \text{Td}(\Sigma)}{\text{Ch}_D(\nu_{\Sigma_g, \Sigma})} e^{\langle \lambda, F_\theta \rangle}. \quad (1.3)$$

Of course this result is also originally due to Meinrenken–Sjamaar [10]. Theorem 1.1 follows immediately from Theorem 1.2 by applying Kawasaki’s index theorem for orbifolds to index($D_{L_\xi}^{\text{red}}$) and comparing with the evaluation of (1.3) at $(k, \lambda) = (1, 0)$.

Let us give a brief summary of our approach to deriving Theorem 1.2. Recall that a function $f$ on a lattice Gamma in a real vector space $V$ is said to be quasi-polynomial if there is a sub-lattice $\Gamma'$ with $\Gamma/\Gamma'$ finite and $f$ restricts to a polynomial function on each coset of $\Gamma'$. More generally, one says $f$ is quasi-polynomial on a subset $\Gamma_0 \subset \Gamma$ if $f \mid \Gamma_0 = q \mid \Gamma_0$ for some quasi-polynomial $q$. A fundamental fact, originally derived from Theorem 1.1 by Meinrenken–Sjamaar [10], is that $m_G$ is quasi-polynomial on the subset $C_p \cap (\mathbb{Z}_{>0} \times \Lambda)$. Our first goal, in Section 2, is to give an independent proof of this fact, taking as a starting point a formula for $m_G$ due to Szénes–Vergne [14] (inspired by work of Paradan [11]), which they obtained by a combinatorial rearrangement of the fixed-point formula for the index.

Then in Section 3 we adapt an idea of Meinrenken [7] to compute the quasi-polynomial $m_G \mid C_p$ using the Berline–Vergne index formula and the principle of stationary phase. The output of the stationary phase formula is an asymptotic expansion for $m_G(k, k\xi)$ in powers of $k$ (allowing coefficients that are periodic in $k$). As one knows in advance that $m_G(k, k\xi)$ is quasi-polynomial in $k$, one concludes that the expansion is exact, yielding Theorem 1.2.

The article of Meinrenken–Sjamaar [10] contains, besides Theorem 1.1, a wealth of detailed information about singular reduction and $[Q, R] = 0$. Our goal in this short note is much more modest. We also do not make a great claim of originality, and in particular the debt to [14] and [7] will be apparent. Part of our motivation stems from the hope that the article of Szénes–Vergne [14], in combination with this note, will provide a more elementary treatment of the $[Q, R] = 0$ theorem than was previously available.

## 2 Quasi-polynomials and the multiplicity function

The goal of this section is Theorem 2.2 on the quasi-polynomial behavior of the multiplicity function, which we prove using results of Szénes–Vergne [14] reviewed below.

The quotient $g/t$ can be identified with the unique $\text{Ad}(T)$-invariant complement to $t$ in $g$. Let $t \subset h \subset g$ be a $T$-invariant subspace. We may similarly identify $h/t$ and $g/h$ with subspaces of $g$. The choice of positive roots $\mathcal{R}_+$ determines a complex structure on $g/t$, whose $+i$-eigenspace is identified with the direct sum of the positive root spaces:

$$(g/t)^{1,0} \simeq \bigoplus_{\alpha \in \mathcal{R}_+} g_{\alpha}.$$
We obtain similar complex structures on \( g/h, h/t \), whose \( +1 \)-eigenspaces are direct sums of positive roots spaces. We will write \( \det^{g/h}_C(a) \) (resp. \( \det^{h/t}_C(a) \)) for the determinant of a complex linear endomorphism \( a \) of \( g/t \) (resp. \( g/h, h/t \)). An example is the endomorphism \( \text{Ad}_t \) of \( t \in T \) (resp. \( \text{ad}_X, X \in t \)). In this case we will simply write \( \det^{g/t}_C(t) \) (resp. \( \det^{h/t}_C(\text{Ad}_t) \)) (resp. \( \det^{h/t}_C(X) \) instead of \( \det^{g/t}_C(\text{ad}_X) \)), the action of \( T \) (resp. \( t \)) on \( g/t \) being understood. Then for example if \( t = e^X \in T \),

\[
\det^{g/t}_C(1 - t^{-1}) = \prod_{\alpha \in R_+} (1 - t^{-\alpha}) = \prod_{\alpha \in R_+} (1 - e^{-2\pi i \langle \alpha, X \rangle}),
\]

\[
\det^{h/t}_C(-X) = \prod_{\alpha \in R_+} -2\pi i \langle \alpha, X \rangle.
\]

For \( \lambda \in \Lambda \cap t_+^* \), the Weyl character formula says that for \( t \in T \),

\[
\chi_\lambda(t) \cdot \det^{g/t}_C(1 - t^{-1}) = \sum_{w \in W} (-1)^{l(w)} t^{w(\lambda + \rho) - \rho},
\]

where \( W \) is the Weyl group, \( l(w) \) is the length of the element \( w \in W \), and \( \rho \) is the half sum of the positive roots. The right-hand-side is an element of \( R(T) \) with multiplicity function \( m_\lambda \) obtained by Fourier transform. Note that

- \( m_\lambda \) is anti-symmetric under the \( \rho \)-shifted action of the Weyl group:

\[
m_\lambda(w(\mu + \rho) - \rho) = (-1)^{l(w)} m_\lambda(\mu).
\]

- The support of \( m_\lambda|_{\Lambda \cap t_+^*} \) is \( \{ \lambda \} \), where it takes the value 1.

Conversely these two properties determine \( m_\lambda \): it is the unique \( W \)-anti-symmetric function on \( \Lambda \) extending the multiplicity function of \( \chi_\lambda \). Applying these observations to the multiplicity function \( m_G \) defined in (1.2), we make the following definition.

**Definition 2.1.** Let \( m(k, -): \Lambda \rightarrow \mathbb{Z} \) be the unique \( \rho \)-shifted \( W \)-anti-symmetric function such that \( m(k, \lambda) = m_G(k, \lambda) \) for all \( \lambda \in \Lambda \cap t_+^* \). The corresponding character \( Q(k, -): T \rightarrow \mathbb{C} \) is defined as the inverse Fourier transform:

\[
Q(k, t) = \sum_{\lambda \in \Lambda} m(k, \lambda) t^\lambda.
\]

Using the Weyl character formula (2.1) and the definition of \( m_G \), it is easy to verify that

\[
Q(k, t) = \sum_{\lambda \in \Lambda} m(k, \lambda) t^\lambda = \text{index}_T(D_{L,k})(t) \cdot \det^{g/t}_C(1 - t^{-1}).
\]

We define

\[
\mu = \text{pr}_{t^*} \circ \mu_g
\]

to be the composition of the moment map \( \mu_g \) with the projection to \( t^* \). Then \( \mu \) is a moment map for the action of \( T \) on \( M \). Suppose \( t \in T \) is sufficiently generic, so that \( M^t = M^T \). The Atiyah–Bott–Segal formula for the index yields

\[
Q(k, t) = \sum_{F \subseteq M^T} t^{k_F} \int_F e^{k_F Td(F)} \cdot \det^{g/t}_C(1 - t^{-1}),
\]

(2.2)
where the sum is over connected components $F$ of $M^T$, and $\mu_F$ denotes the constant value of the moment map $\mu$ on $F$. The multiplicity $m$ is obtained by Fourier transform of (2.2).

Key to the approach in [14] is a different expression for $m(k,\lambda)$ that we briefly describe here. The formula depends on the choice of an invariant inner product on $\mathfrak{g}$, as well as a generic point $\gamma$ contained in $t^*_\mathfrak{g}$ and sufficiently close to 0 (see [14, Section 4.1] for the meaning of ‘generic’ here).

Using the inner product we identify $t \simeq t^*$.

We need some additional notation:

- Let $\text{Comp}_T(M)$ denote the set of connected components of $M^H$, as $H$ ranges over all (connected) sub-tori of $T$.
- For $C \in \text{Comp}_T(M)$, let $t_C \subset t$ be its generic infinitesimal stabilizer. Let $A_C$ be the smallest affine subspace containing the image $\mu(C)$. In particular $A_M$ is the smallest affine subspace containing $\mu(M)$. Note that $A_C$ is a translate of the annihilator of $t_C$.
- Let $\gamma_C \in A_C$ be the orthogonal projection of $\gamma$ onto $A_C$, and let $\tau_C = \gamma_C - \gamma$.

The Szenes–Vergne–Paradan formula [14, equation (39)] (see also [14, Proposition 41, Theorem 48]) is a sum of contributions:

$$m = \sum_C m_C,$$

where $C$ ranges over components $C \in \text{Comp}_T(M)$ such that $\gamma_C \in \mu_\mathfrak{g}(C)$. Szenes–Vergne derive this formula directly from (2.2) using an interesting combinatorial rearrangement, the main ingredient of which is a decomposition formula for Kostant-type partition functions. The formula is inspired by, and closely related to, the work of Paradan [11]. The fact that only a subset of the components in $\text{Comp}_T(M)$ contribute is non-trivial and quite important for $[Q,R] = 0$. The proof given by Szenes–Vergne involves studying the asymptotic behavior of the $m_C$’s using the Berline–Vergne formula and the principle of stationary phase. It goes back to results of Paradan [11], who proved a closely related result using transversally elliptic symbols and K-theoretic methods. Note that Szenes–Vergne assume for simplicity that $M^T$ consists of isolated fixed points, but it is not difficult to handle the general case with the same methods; see for example [6, Section 7] for some indications of how this can be done.

For the proof of Theorem 2.2 we do not need the precise definition of the terms $m_C$ in (2.3), but we will need the following two crucial properties:

1. The function $m_C$ restricts to a quasi-polynomial on each $\Lambda$-translate of the set $(\mathbb{Z} \times \Lambda) \cap A_C$, where

   $$A_C = \{(t,t\tau) \mid t \in \mathbb{R}_{>0}, \tau \in A_C\} \subset \mathbb{R} \times t^*.$$

2. Let $\text{wt}(\nu_C)$ denote the list of complex weights (for the compatible almost complex structure $J$) for the $t_C$ action on the normal bundle $\nu_C$. If $\lambda \in \Lambda$ is in the support of $m_C(k,-)$ then $\lambda$ satisfies the inequality

   $$\langle \tau_C, \lambda \rangle \geq k \langle \tau_C, \gamma_C \rangle + \langle \tau_C, \sigma_C \rangle,$$

   where

   $$\sigma_C := \sum_{\delta \in \text{wt}(\nu_C), \langle \tau_C, \delta \rangle > 0} \delta - \sum_{\alpha \in \mathbb{R}_{>0}, \langle \tau_C, \alpha \rangle > 0} \alpha.$$

   See the proof of [14, Theorem 49]. Note that, except for the special case $\tau_C = 0$, (2.4) defines a half-space in $t^*$.

We will refer to these two properties as ‘property (a)’, ‘property (b)’ in the proof of the next result. Theorem 2.2 is a strengthening of [14, Theorem 49] (which says that the function $k \mapsto m(k,0)$ is quasi-polynomial), and our arguments are based on their elegant approach.
We conclude that if $0 \notin \Delta$ then $m(k, 0) = 0$ for all $k \geq 1$. If $0 \in \Delta$ then there is a closed polytope $p \subset \Delta$ of the same dimension as $\Delta$ and containing the origin such that $m(k, \lambda)$ is quasi-polynomial on the set of integral points $(\mathbb{Z} \times \Lambda) \cap C_p$ contained in the cone

$$C_p = \{(t, t\tau) \mid t \in (0, \infty), \tau \in p\} \subset \mathbb{R} \times t^*.$$ 

**Proof.** The strategy is based on choosing a suitable $\gamma \in t^*_+$ and then analyzing the supports of the contributions $mc$ to $m$ in the corresponding Szenes–Vergne–Paradan formula (2.3) using property (b). The contribution $m_C$ appears in (2.3) only if $\gamma_C \in \mu_g(M) \cap t^* = W - \Delta \subset W - I$ (recall by definition $I$ is the smallest affine subspace containing $\Delta$). Because $\gamma$ is chosen generically, the only $C \in \text{Comp}_T(M)$ which may contribute to (2.3) are those such that the affine subspace $A_C$ is entirely contained in $I$ or one of its Weyl reflections, and throughout the proof we assume this is the case.

Suppose $0 \in \Delta$. We argue that by a suitable choice of $\gamma$, one can arrange that for all but one

the contributions, (i) $\langle \tau_C, \gamma_C \rangle \geq 0$ with equality if and only if $0 \in A_C$, (ii) $\langle \tau_C, \gamma_C \rangle > \langle \tau_C, \gamma_I \rangle$, where $\gamma_I$ is the orthogonal projection of $\gamma$ onto $I$, and (iii) $\langle \tau_C, \sigma_C \rangle > 0$. The one special contribution is denoted $mc_I$ below and corresponds to the subspace $A_{C_I} = I$. By property (b), (i) and (ii) imply that for $C \neq C_I$, the support of $m_C(k, -)$ lies outside $kH_C$ where $H_C$ is the half-space

$$H_C = \{\xi \mid \langle \tau_C, \xi \rangle \leq \langle \tau_C, \gamma_C \rangle\}.$$ 

Let $p$ be the intersection of $I$ with all of the half-spaces $H_C$ for $C \neq C_I$. By (ii), the relative interior of $p$, viewed as a polytope in $I$, contains the point $\gamma_I$, hence in particular is non-empty. By construction $m \upharpoonright C_p = m_{C_I} \upharpoonright C_p$. Then property (a) implies that $m_{C_I}$ is quasi-polynomial on $C_p$, hence the result.

We claim that one can ensure (i) holds for all $C$ by choosing $\gamma \in t^*_+$ sufficiently close to 0. Indeed let $A^0_C$ be the subspace parallel to $A_C$, and let $a_C \in A_C$ be the nearest point in $A_C$ to 0. Then $\gamma_C - a_C \in A^0_C$, while $\tau_C$, $a_C$ are both orthogonal to $A^0_C$, hence $\langle \tau_C, \gamma_C - a_C \rangle = 0 = \langle a_C, \gamma_C - a_C \rangle$. These imply $\langle \tau_C, \gamma_C \rangle = \|a_C\|^2 - \langle a_C, \gamma \rangle$. If $0 \in A_C$ then $a_C = 0$ and this vanishes. Otherwise we can ensure $\langle \tau_C, \gamma_C \rangle > 0$ by choosing $\|\gamma\| < \|a_C\|$. Since only finitely many $C$ occur, we can choose $\gamma$ such that this holds for all $C$ with $0 \notin A_C$. We now turn to verifying (ii), (iii), and also handle the case $0 \notin \Delta$ along the way.

Suppose $\gamma_C \in \mu_g(C)$, so that $mc$ indeed appears in (2.3). If $\alpha \in \mathcal{R}_+$ and $\langle \tau_C, \alpha \rangle > 0$, then since $\gamma \in t^*_+$ it follows that $\langle \gamma_C, \alpha \rangle > 0$. It is a consequence of the cross-section theorem (cf. [5]) that $\alpha|_{t_C}$ appears in the list of weights $\text{wt}(\nu_C)$. Hence

$$\sigma_C = \sum_{\delta \in \text{wt}(\nu_C) - \mathcal{R}_+^C} \delta, \quad \langle \tau_C, \delta \rangle > 0$$

where $\mathcal{R}_+^C$ denotes the set of positive roots $\alpha$ such that $\langle \tau_C, \alpha \rangle > 0$, and $\text{wt}(\nu_C) - \mathcal{R}_+^C$ denotes the list of weights on $\nu_C$ with one copy of $\alpha|_{t_C}$ removed for each $\alpha \in \mathcal{R}_+$ satisfying $\langle \tau_C, \alpha \rangle > 0$. Hence

$$\langle \tau_C, \sigma_C \rangle \geq 0$$

and the inequality is strict if at least one weight $\delta$ contributes in (2.5).

If $0 \notin \Delta$ then, choosing $\gamma$ sufficiently close to 0, we can ensure that for each $C$ such that $0 \in A_C$ we have $\gamma_C \notin \mu_g(M)$ (a fortiori $\gamma_C \notin \mu_g(C)$), hence $mc$ does not appear in (2.3) at all. On the other hand, by (i), (2.6) and property (b), if $0 \notin A_C$ then $m_C(k, 0) = 0$ for all $k \geq 1$. We conclude that if $0 \notin \Delta$ then $m(k, 0) = 0$ for all $k \geq 1.$
We turn to the case $0 \in \Delta \subset I$. In this case we may choose $\gamma$ such that it is simultaneously close to 0 and arbitrarily close to $\gamma_I$, the orthogonal projection of $\gamma$ onto $I$. Since $\tau_C = \gamma_C - \gamma$, $\langle \tau, \gamma \rangle \leq \langle \tau_C, \gamma_C \rangle$ with equality if and only if $\gamma_C = \gamma$. By taking $\gamma$ sufficiently close to $I$, one can ensure that $\langle \tau_C, \gamma_I \rangle \leq \langle \tau_C, \gamma_C \rangle$ with equality if and only if $\gamma_C = \gamma_I$.

We first consider contributions from components $C \in \text{Comp}_T(M)$ such that $\gamma_C \notin t^*_+$. In this case there exists a negative root $\alpha \in \mathcal{R}_+$ such that $\langle \gamma_C, \alpha \rangle > 0$. It follows from the cross-section theorem that $\alpha|_{\nu} \in \text{wt}(\nu_C)$. Since $\gamma \in t^*_+$, $\langle \gamma, \alpha \rangle \leq 0$ and so

$$\langle \tau_C, \gamma_I \rangle = \langle \gamma_C, \alpha \rangle - \langle \gamma, \alpha \rangle > 0.$$ 

As $\alpha \notin \mathcal{R}_+$, we see that $\delta = \alpha$ indeed contributes in (2.5), hence $\langle \tau_C, \sigma_C \rangle > 0$. Moreover since $\gamma_C \notin t^*_+$, $\gamma_C \neq \gamma_I$, hence $\langle \tau_C, \gamma_I \rangle < \langle \tau_C, \gamma_C \rangle$. This establishes (ii), (iii) for this case.

We are left to consider contributions from $C \in \text{Comp}_T(M)$ such that $\gamma_C \in \Delta = \mu_0(M) \cap t^*_+$. Let $\Delta_{\text{reg}} \subset \Delta$ be the relatively open dense subset of weakly regular values. The connected components of $\Delta_{\text{reg}}$ are relatively open polytopes inside the subspace $I$. Choose a connected component $\sigma \subset \Delta_{\text{reg}}$ containing 0 in its closure. We may choose $\gamma \in t^*_+$ such that the orthogonal projection $\gamma_I$ onto $I$ lies in $\sigma$. The fibre $\mu_0^{-1}(\gamma_I)$ is connected and contained in $M^T$, hence there is a unique connected component $C_I \subset M^T$ containing $\mu_0^{-1}(\gamma_I)$. Then $A_{C_I} = I$ and by property (a), $m_{C_I}$ is quasi-polynomial on the set of integral points in $A_C = \{(t, t\tau) \mid t > 0, \tau \in \mathbb{R} \} \supset C_p$.

The final situation to consider consists of the contributions from $C \in \text{Comp}_T(M)$ such that $\gamma_C \in \Delta \setminus \Delta_{\text{reg}}$. In particular $\gamma_C \neq \gamma_I$ hence

$$\langle \tau_C, \gamma_I \rangle < \langle \tau_C, \gamma_C \rangle$$

establishing (ii) for this case. Let $\sigma$ be the face of $t^*_+$ containing $\gamma_C$. The subset

$$U = G_{\sigma} \cdot \bigcup_{\tau \supset \sigma} \tau,$$

where the union is taken over relatively open faces of $t^*_+$ whose closure contains $\sigma$, is a slice for the coadjoint $G_{\sigma}$-action. Let $Y = \mu_0^{-1}(U)$ be the corresponding symplectic cross-section, cf. [5, Remark 3.7, Theorem 3.8]. Consider the function $f = \langle \tau_C, \mu \rangle|_Y : Y \to \mathbb{R}$, for which $C \cap Y \subset Y^{\tau_C} = \text{Crit}(f)$ is a critical submanifold. Note that $f|_{C \cap Y} = \langle \tau_C, \gamma_C \rangle$. A result from symplectic geometry says that in a suitable tubular neighborhood of $C \cap Y$, the function $f$ takes the form

$$f(z_1, \ldots, z_n) = \langle \tau_C, \gamma_C \rangle - \pi \sum_j |z_j|^2 \langle \tau_C, \delta_j \rangle,$$

where $\delta_j \in \text{wt}(\nu_{C \cap Y})$, $\langle \tau_C, \delta_j \rangle \neq 0$, $z_j$ is a vector in the subbundle of $\nu_{C \cap Y}$, $t_C$ acts with weight $\delta_j$, and $|z_j|$ denotes its norm with respect to a suitable Hermitian structure.

Let $S$ be the line segment with endpoints $\gamma_I$ and $\gamma_C$. By convexity $S \subset \Delta$. The inverse image $\mu_0^{-1}(S) \subset Y$ is connected since $\mu_0$ has connected fibres. By (2.7), along the line segment $S$, $f$ varies between its absolute minimum $\langle \tau_C, \gamma_I \rangle$ on the fibre $\mu_0^{-1}(\gamma_I)$ and its absolute maximum $\langle \tau_C, \gamma_C \rangle$ on the fibre $\mu_0^{-1}(\gamma_C)$. By connectedness of $\mu_0^{-1}(S)$ and equation (2.8), there must exist a $\delta_j$ such that $\langle \tau_C, \delta_j \rangle > 0$.

By the cross-section theorem $\nu_{C,T} = (C \cap Y) \times \mathfrak{g}_{\gamma_C}^\perp$, where the orthogonal complement $\mathfrak{g}_{\gamma_C}^\perp$ is embedded in $TM_{\gamma_C}$ as the orbit directions. The weights $\mathcal{R}_+^g$ which are removed in (2.5) can be identified with the weights of the $t_C$-action on $\nu_{C,T} = (C \cap Y) \times \mathfrak{g}_{\gamma_C}^\perp$. With this understanding we have $\text{wt}(\nu_{C \cap Y}) \subset \text{wt}(\nu_C) - \mathcal{R}_+^g$. Thus $\delta_j$ indeed contributes to (2.5), establishing (iii) for this case. This completes the proof.
Corollary 2.3. Suppose $0 \in \Delta$ and let $\mathfrak{p} \subset \Delta$ be as in Theorem 2.2. If $\xi \in \mathfrak{p}$ is rational and $n_\xi \in \mathbb{Z}_{>0}$ is the least positive integer such that $n_\xi \xi \in \Lambda$, then the function

$$f_\xi: n_\xi \cdot \mathbb{Z}_{>0} \to \mathbb{Z}, \quad f_\xi(k) = m(k, k\xi)$$

is quasi-polynomial. Moreover $m \mid C_\mathfrak{p}$ is the unique quasi-polynomial function such that $m(k, k\xi) = f_\xi(k)$ for all rational, weakly regular values $\xi$ in the relative interior of $\mathfrak{p}$.

Remark 2.4. A suitable finite collection of the functions $f_\xi$ already fully determines $m \mid C_\mathfrak{p}$.

3 Stationary phase calculation

Assume $0 \in \Delta$ and let $\mathfrak{p} \subset \Delta$ be as in Theorem 2.2, so that $m \mid C_\mathfrak{p}$ is quasi-polynomial. By Corollary 2.3, $m \mid C_\mathfrak{p}$ is completely determined by the collection of quasi-polynomial functions $f_\xi(k) = m(k, k\xi)$, for $\xi$ ranging over rational, weakly regular values of $\mu$ lying in the relative interior of $\mathfrak{p}$. In this section we use the Berline–Vergne index formula and the stationary phase expansion to compute the functions $f_\xi$, and hence also $m \mid C_\mathfrak{p}$. The end result will be the formula (1.3) in Theorem 1.2.

Let $t \in T$. By the Berline–Vergne formula, for $X \in t$ sufficiently small one has $Q(k, te^X) = Q_t(k, X)$ where

$$Q_t(k, X) := \int_{M^t} \frac{\tau^{k, e^{t(\omega + 2\pi i(\mu, X))}} \text{Td}(M^t, \frac{2\pi}{t} X)}{\mu_{M^t, M, \frac{2\pi}{t} X}} \det_{\mathcal{C}}^t(1 - t^{-1}e^{-X}),$$

and $\text{Td}(M^t, \frac{2\pi}{t} X)$, $D^t_{\mathcal{C}}(\nu_{M^t, M, \frac{2\pi}{t} X})$ denote equivariant extensions of the usual Chern–Weil forms, closed with respect to the differential $d + 2\pi i(X_M)$, obtained by replacing curvatures with equivariant curvatures (evaluated at $\frac{2\pi}{t} X$) in the usual formulas (cf. [1] for details, although note that we are using the topologist’s convention for characteristic classes).

Let $B_r$ denote the ball of radius $r > 0$ around the origin in $\mathfrak{g}/t$. Let $\mu_{\mathfrak{g}/t}$ denote the composition of $\mu$ with the quotient map $\mathfrak{g} \to \mathfrak{g}/t$. Let $\mathfrak{g}/t \subset \mathfrak{g}$ denote the fixed-point set of $Ad_t$. Then $B_r^t$ is a neighborhood of 0 in $\mathfrak{g}/t$. Recall $\mathfrak{g}/t$, $\mathfrak{g}/t'$ are equipped with complex structures such that their $+i$-eigenspaces are identified with sums of positive root spaces. Equip $\mathfrak{g}/t$ with the orientation induced by the complex structure, and let $\tau^{\mathfrak{g}/t}(X)$ be a $T$-equivariant Thom form with support contained in $B_r^t$, closed for the differential $d - i(X_M)$. Consider the $T$-equivariant differential form on $\mathfrak{g}/t$ (closed for the differential $d + 2\pi i(X_{\mathfrak{g}/t})$) given by

$$\text{Ch}^t(b, \frac{2\pi}{t} X) = \det_{\mathcal{C}}^{\mathfrak{g}/t}(1 - t^{-1}e^{-X}) \det_{\mathcal{C}}^{\mathfrak{g}/t}(\frac{1 - e^{-X}}{X}) \tau^{\mathfrak{g}/t}(\frac{2\pi}{t} X),$$

The map $\mu_{\mathfrak{g}/t}$ restricts to a map $M^t \to \mathfrak{g}/t$, which we use to pull back the form $\text{Ch}^t(b, \frac{2\pi}{t} X)$.

Lemma 3.1.

$$Q_t(k, X) = \int_{\mu_{\mathfrak{g}/t}^{-1}(B_r^t)} \frac{\mu_{\mathfrak{g}/t}^k e^{k(\omega + 2\pi i(\mu, X))}}{D^t_{\mathcal{C}}(\nu_{M^t, M, \frac{2\pi}{t} X})} \text{Ch}^t(b, \frac{2\pi}{t} X).$$

Proof. The pullback of $\tau^{\mathfrak{g}/t}(X)$ to $0 \in \mathfrak{g}/t$ is the equivariant Euler class, which (since 0 is just a point) is the function

$$\prod_{\alpha \in R^\vee_{\mathfrak{g}/t}} - (\alpha, X) = \det_{\mathcal{C}}^{\mathfrak{g}/t}(\frac{i}{2\pi} X),$$
where $\mathcal{R}^g \subset \mathcal{R}_+$ is a set of positive roots for $\mathfrak{g}$. Note also that $t$ acts trivially on $\mathfrak{g}/t$, since $\mathfrak{g}$ is the fixed point subspace under the adjoint action. It follows that the pullback of $\text{Ch}^f(b, \frac{2\pi}{t}X)$ to $0 \in \mathfrak{g}/t$ is the function $\det(1 - t^{-1}e^{-X})$. Since pullback to $\{0\} = (\mathfrak{g}/t)^T$ is injective on equivariant cohomology classes, $\text{Ch}^f(b, \frac{2\pi}{t}X)$, $\det(1 - t^{-1}e^{-X})$ determine the same class in $T$-equivariant cohomology of $\mathfrak{g}/t$. As $M$ is compact, we may make this replacement in (3.1) without changing the value of the integral. 

**Remark 3.2.** The reason for the notation is that $\text{Ch}^f(b, \frac{2\pi}{t}X)$ is a representative for the $t$-twisted Chern character of a Bott element $b \in K^0(\mathfrak{g}/t)$, which generates the latter as an $R(T) = K^0_R(pt)$-module. To be more precise, $b$ is the generator whose pullback to $0 \in \mathfrak{g}/t$ is $[\wedge^{ev} n_-] - [\wedge^{odd} n_-] \in K^0_R(pt)$, $n_-$ being the direct sum of the negative root spaces.

Since $T$ is compact, there exists a finite set $S \subset T$ and an open cover $\{U_t \mid t \in S\}$ of $T$ where $U_t$ is a small open ball around $t$ in $T$ such that $Q(k, te^X) = Q_t(k, X)$ for $te^X \in U_t$. Let $\sigma_t, t \in S$ be bump functions on $t$ such that $\hat{\sigma}_t \sigma_t \mid t \in S$ is a partition of unity subordinate to the cover, where $\hat{\sigma}$ is the map

$$\hat{\sigma} : t \rightarrow T, \quad X \mapsto te^X,$$

which we may assume restricts to a diffeomorphism of a small ball around $0 \in t$ onto $U_t$. By equations (3.1) and (3.2)

$$Q = \sum_{t \in S} \hat{\sigma}_t(\sigma_t Q_t).$$

The multiplicity function $m$ is the Fourier transform of $Q$:

$$m(k, \lambda) = \sum_{t \in S} \int_{t} \sigma_t(X)(te^X)^{-\lambda} Q_t(k, X).$$

To do the stationary phase calculation (for $k \rightarrow \infty$) following the approach outlined at the beginning of this section, we now set $\lambda = k\xi$ where $\xi \in (\Lambda \otimes \mathbb{Q}) \cap \mathfrak{p}$ is a rational, weakly regular value of $\mu_0$ contained in the relative interior of $\mathfrak{p} \subset \Delta$ as in Corollary 2.3, $k \in n_0\mathbb{Z}_{>0}$ and $n_0$ is the least positive integer such that $n_0\xi \in \Lambda$. Thus

$$m(k, k\xi) = \sum_{t} t^{-k\xi} \int_{t} \text{d}X \sigma_t(X) \int_{\mu^{-1}_0(B_t)} \frac{1}{\nu_{M_t^t}^{1/2} X} t^L Td(M^t, \frac{2\pi}{t} X) \text{Ch}^f(b, \frac{2\pi}{t} X) e^{k(\omega + 2\pi i(-\xi, X)), (3.3)}$$

Let $f(m, X) = \langle \mu(m) - \xi, X \rangle$ viewed as a real-valued function on $\mu_0^{-1}(B_t) \times t$. According to the principle of stationary phase, we can include a bump function supported in a small neighborhood of the critical set of $f$ in the integrand of (3.3), and the error will be $o(k^{-\infty})$. The derivative

$$d_{(m, X_0)} f = \langle d_m \mu, X_0 \rangle + \langle \mu(m) - \xi, d_X X \rangle$$

and in particular $\text{Crit}(f) \subset \mu^{-1}(\xi) \times t$. Let $\chi$ be the pullback by $\mu$ of a bump function in $t^*$ supported in a small neighborhood of $\xi$. Thus

$$m(k, k\xi) \sim \sum_{t} t^{-k\xi} \int_{t} \text{d}X \sigma_t(X) \times \int_{\mu^{-1}_0(B_t)} \chi \frac{1}{\nu_{M_t^t}^{1/2} X} \text{Ch}^f(b, \frac{2\pi}{t} X) e^{k(\omega + 2\pi i(-\xi, X)), (3.4)}$$

where $\sim$ denotes equality modulo an $o(k^{-\infty})$ error.
Let $Y = \mu^{-1}_\sigma(\sigma)$ be the cross-section for the principal face. By the cross-section theorem, a neighborhood $N$ of $Y$ in $M$ is $G_\sigma$-equivariantly diffeomorphic to

$$Y \times g/g_\sigma,$$

where $g/g_\sigma \simeq g^+_{\sigma^t}$ is embedded in the orbit directions. Since $\mu^{-1}(\xi) \cap \mu^{-1}_\sigma(t^2) = \mu^{-1}_\sigma(\xi) \subset Y$, by taking $r$ and $\text{supp}(\chi)$ sufficiently small, we can assume that $\text{supp}(\chi) \cap \mu^{-1}_\sigma(B_r)$ is contained in a small neighborhood of $\mu^{-1}_\sigma(\xi)$ where the local model $Y \times g/g_\sigma$ is valid, and so we may replace $\mu^{-1}_\sigma(B_r)$ with $N^t$ in equation (3.4). In the next lemma we use the Thom form to integrate over the $(g/g_\sigma)^t$ directions.

**Lemma 3.3.**

$$m(k, k\xi) \sim \sum_{t \in S} t^{-k\xi} \int dX \sigma_t(X)$$

$$\times \int_{Y_{\nu}} (Y^t, \frac{2\pi}{T} X) \chi(X) \bar{D}_{\nu} (\nu_{Y_{\nu}, Y_{\nu}}, \frac{2\pi}{T} X) e^{k(\omega+2\pi i (\mu-\xi, X))} \det_{g_{\sigma}}^{g_{\sigma}^t/(1-t^{-1} e^{-X})}. \quad (3.5)$$

**Proof.** The neighborhood $N^t$ of $Y^t$ in $M^t$ is $T$-equivariantly diffeomorphic to

$$Y^t \times (g/g_\sigma)^t = Y^t \times g^t/g^t_\sigma,$$

where $g^t_\sigma = (g_\sigma)^t$ is the subspace of $g_\sigma$ fixed by $t$. Moreover the almost complex structure on $N^t$ is homotopic to a product almost complex structure, where $Y^t$ is equipped with an almost complex structure compatible with the symplectic form in the cross-section, and $g^t/g^t_\sigma$ is equipped with the almost complex structure whose $+i$-eigenspace is identified with a sum of positive root spaces. Let

$$p: N^t \rightarrow Y^t$$

denote the projection. For the normal bundle

$$\nu_{M^t, M^t} | N^t \simeq p^* \nu_{Y^t, Y} \oplus (g/g_\sigma)/(g/g_\sigma)^t = p^* \nu_{Y^t, Y} \oplus g/(g^t + g_\sigma),$$

and again the almost complex structure is homotopic to a product one, using a compatible almost complex structure on the symplectic vector bundle $\nu_{Y^t, Y}$, and an almost complex structure on $g/(g^t + g_\sigma)$ whose $+i$-eigenspace is identified with a sum of positive root spaces. Using the identifications above we obtain, up to equivariantly exact forms:

$$\text{Td}(M^t, \frac{2\pi}{T} X) | N^t = \text{Td}(Y^t, \frac{2\pi}{T} X) \det_{g_\sigma}^{g^t_\sigma} \left( \frac{X}{1-e^{-X}} \right), \quad (3.6)$$

$$\bar{D}_{\nu} (\nu_{M^t, M^t}, \frac{2\pi}{T} X) | N^t = \bar{D}_{\nu} (\nu_{Y^t, Y^t}, \frac{2\pi}{T} X) \det_{g_\sigma}^{g^t_\sigma/(g^t_\sigma + g_\sigma)}(1-t^{-1} e^{-X}).$$

Since $Y^t \subset \mu^{-1}_\sigma(t^2)$, the pullback of the equivariant Thom form $\tau_{g^t_\sigma/(t^2)}(X)$ to $Y^t$ is just the function

$$\det_{g^t_\sigma/(t^2)}^{g_\sigma}(\frac{i}{2\pi} X) = \det_{g^t_\sigma}^{g_\sigma}(\frac{i}{2\pi} X) \det_{C}^{g^t_\sigma/(t^2)}(\frac{i}{2\pi} X).$$

We recognize $\det_{C}^{g^t_\sigma/(t^2)}(\frac{i}{2\pi} X)$ as the equivariant Euler class of the trivial bundle $Y^t \times g^t/g^t_\sigma$. Thus up to an equivariantly exact form, we have

$$\tau_{g^t_\sigma/(t^2)}(X) = \tau_{p}(X) \det_{g^t_\sigma/(t^2)}^{g^t_\sigma/(t^2)}(\frac{i}{2\pi} X), \quad (3.7)$$

where $\tau_{p}(X)$ is an equivariant Thom form for the vector bundle $p: N^t = Y^t \times g^t/g^t_\sigma \rightarrow Y^t$. 

We next want to make the replacements (3.6), (3.7) in equation (3.4), and then use the Thom form \( \tau_p(X) \) to integrate over the fibres of \( p: N^t \to Y^t \). In the integral over \( N^t \) in (3.4), the integrand has compact support and all terms in the integrand are equivariantly closed except for the bump function \( \chi \). By Stokes’ theorem, replacing a form by a cohomologous form in the integrand leads to an error term containing \( d\chi \); but \( d\chi \) vanishes near \( \mu^{-1}(\xi) \), so the principle of stationary phase implies the error will be \( o(k^{-\infty}) \). Let \( \iota_{Y^t}: Y^t \hookrightarrow N^t \) denote the inclusion. Similarly the formula \( p_*(\tau_p(X)\alpha(X)) = \iota_{Y^t}^*\alpha(X) \) applies when \( \alpha(X) \) is equivariantly closed. But writing \( \chi = 1 - (1 - \chi) \), the principle of stationary phase again shows that we can make this replacement up to an \( o(k^{-\infty}) \) error term.

After making these replacements and integrating over the fibre, the form \( \tau_p(\frac{2\pi i}{t}X) \) disappears:

\[
\frac{\det_{\mathbb{C}}^{g^t/\ell}(1 - t^{-1}e^{-X})}{\det_{\mathbb{C}}^{g^t/(g^t + g_\sigma)}(1 - t^{-1}e^{-X})} \det_{\mathbb{C}}^{g^t/\ell}(1 - e^{-X}/X) \det_{\mathbb{C}}^{\chi/\ell}(X) \det_{\mathbb{C}}^{\chi/\ell} \left( \frac{X}{1 - e^{-X}} \right),
\]

which simplify to \( \det_{\mathbb{C}}^{g^t/\ell}(1 - t^{-1}e^{-X}) \) (one uses that \( t \) acts trivially on \( g^t/\ell \) and that \( (g^t + g_\sigma)/\ell \simeq g_\sigma/g^t_\sigma \)).

Choose a complementary subtorus \( T'_I \) so that \( T \simeq T_I \times T'_I \). The quotient map \( T \to T/T_I \) induces an isomorphism of groups \( T'_I \to T/T_I \). By adding additional points if necessary, we may assume the finite subset \( S \subset T \) is a product \( S_I \times S'_I \), where \( S_I \subset T_I, S'_I \subset T'_I \) and that the image of \( S'_I \) in \( T'/T_I \) contains the set \( S_T \) from the introduction. Thus we will write elements of \( S \) as products \( h g \) with \( h \in S_I \subset T_I \) and \( g \in S'_I \subset T'_I \). We may assume the bump function \( \sigma_t \) is a product \( \sigma_h \cdot \sigma_g \), where \( \sigma_h \) (resp. \( \sigma_g \)) is a bump function on \( t_I \) (resp. \( t'_I \)), satisfying

\[
\sum_{h \in S_I} \hat{h}_* \sigma_h = 1, \quad \sum_{g \in S'_I} \hat{g}_* \sigma_g = 1.
\] (3.8)

The next lemma gives a further simplification of (3.5).

Lemma 3.4.

\[
m(k, k\xi) \sim \sum_{g \in S'_I} g^{-k\xi} \int_{t'_I} dX \sigma_g(X) \int_{Y^g} \frac{g^k Td(Y^g, \frac{2\pi i}{t}X)}{\mathcal{D}_C^{\theta}(Y^g, \frac{2\pi i}{t}X)} \zeta^{k(\omega + 2\pi i(\mu - \xi, X))}. \]

(3.9)

Proof. As \( T_I \) acts trivially on \( Y \) and \( \mu(Y) \subset I \), the characteristic forms in (3.5) only depend on the component of \( X \) (resp. \( t \)) in \( t'_I \) (resp. \( T'_I \)). Likewise as \( \xi \in (\Lambda \otimes \mathbb{Q}) \cap I \), \( t^{-k\xi} \) only depends on the component \( g \) of \( t \) in \( T'_I \). This means the following expression can be split off from (3.5) and evaluated separately:

\[
\sum_{h \in S_I} \int_{t_I} dX \sigma_h(X) \det_{\mathbb{C}}^{\chi/\ell}(1 - h^{-1}g^{-1}e^{-X}). \]

(3.10)

The determinant is given by a product:

\[
\prod_{\alpha \in \mathcal{R}_+^{\theta}} (1 - h^{-\alpha}g^{-\alpha}e^{-2\pi i(\alpha, X)}).
\]

When the product over \( \mathcal{R}_+^{\theta} \) is expanded, we obtain an alternating sum of terms of the form \( h^{-\zeta}g^{-\zeta}e^{-2\pi i(\zeta, X)} \), where \( \zeta \) is a sum of a subset of \( \mathcal{R}_+^{\theta} \). The elements of \( \mathcal{R}_+^{\theta} \) lie in \( \text{ann}(3g) \), the annihilator of \( 3g \) in \( t^* \). Since \( t^* = 3g^* \oplus \text{ann}(3g) \) and \( I \subset 3g^* \), it follows that either \( \zeta = 0 \) or else \( \zeta \notin I \).
We claim that if $\zeta \neq 0$, then the corresponding contribution to (3.10) is 0. Indeed taking the Fourier transform of the first equation in (3.8), we find that for any $[\zeta] \in \Lambda/(\Lambda \cap I)$, the weight lattice of $T_I$, we have

$$
\sum_{h \in S_I} h^{-|\zeta|} \int_{I_i} \sigma_h(X)e^{-2\pi i(|\zeta|,X)}\,dX = \delta_0(|\zeta|),
$$

where $\delta_0$ is the function on $\Lambda/(\Lambda \cap I)$ equal to 1 at 0 and 0 otherwise, obtained by Fourier transform of the constant function 1 on $T_I$. Thus for $\zeta \in \Lambda$,

$$
\sum_{h \in S_I} h^{-\zeta} \int_{I_i} \sigma_h(X)e^{-2\pi i(\zeta,X)}\,dX = \delta_{\Lambda \cap I}(\zeta),
$$

where $\delta_{\Lambda \cap I}$ is the function on $\Lambda$ equal to 1 on $\Lambda \cap I$ and 0 otherwise. In particular if $\zeta \notin I$ we see that the corresponding contribution in (3.10) vanishes.

On the other hand, using equation (3.8), the contribution from $\zeta = 0$ to (3.10) is

$$
\sum_{h \in S_I} \int_{I_i} dX \sigma_h(X) = 1.
$$

This yields the expression on the right-hand-side of (3.9). 

We can now complete the proof of Theorem 1.2. The fibre $P = \mu^{-1}(\xi) \subset Y$ is smooth, and the quotient $\Sigma_\xi := M_\xi = P/G_\sigma = P/T_I'$ is an orbifold ($T_I'$ acts locally freely on $P$). By the coisotropic embedding theorem, a neighborhood of $P$ in $Y$ is $T$-equivariantly symplectomorphic to

$$P \times B_I \subset P \times I,$$

where $B_I$ is a small ball around $\xi$ in the subspace $I \subset t^*$, the moment map $\mu$ is projection to the second factor, and the symplectic form

$$\omega|_{P \times B_I} = \omega_\xi + d(\eta - \xi, \theta) = \omega_\xi + (d\eta, \theta) + (\eta - \xi, F_\theta),$$

where $\omega_\xi$ is the pullback of the symplectic form on the reduced space $M_\xi$, $\eta$ is the variable in $B_I$, $\theta \in \Omega^1(P, t^*)^T$ is a connection on $P$ with curvature $F_\theta = d\theta$, and here as well as below we have omitted pullbacks from the notation. A neighborhood of $P^g$ in $Y^g$ is $T$-equivariantly symplectomorphic to

$$P^g \times B_I^g = P^g \times B_I,$$

and $T_I'$ acts locally freely on $P^g$, with the quotient $\Sigma_\xi = P^g/T_I'$ being an orbifold. On the same neighborhood we have

$$\text{Td}(Y^g, \frac{2\pi}{I} X) = \text{pr}_1^* \text{Td}(P^g, \frac{2\pi}{I} X),$$

$$\nu_{Y^g, Y} = \text{pr}_1^* \nu_{P^g, P} \quad \Rightarrow \quad D^g_C(\nu_{Y^g, Y}, \frac{2\pi}{I} X) = \text{pr}_1^* D^g_C(\nu_{P^g, P}, \frac{2\pi}{I} X).$$

Below we will omit $\text{pr}_1^*$ from the notation.

Take the bump function $\chi$ to have its support contained in the neighborhood of $P$ where the above local normal forms are valid. We may then integrate over $I$ instead of $B_I$, since $\chi$ vanishes outside of $P \times B_I$ by assumption. On $\text{supp}(\chi)$,

$$e^{k(\omega + 2\pi i(\mu - \xi, X))} = e^{k(\omega_\xi + (d\eta, \theta) + (\eta - \xi, F_\theta) + 2\pi i(\eta - \xi, X))}.$$
Only the top degree part of \( e^{k(d\eta,\theta)} \) contributes to the integral over \( I \); this top degree part is 
\((-1)^{(n-1)/2}k^n d\eta \cdot \Theta \), where \( n = \dim(I) \), \( d\eta = \Pi d\eta^a \), \( \Theta = \Pi \theta_a \) in terms of coordinates on \( I \). The sign \((-1)^{(n-1)/2}\) relates the symplectic and product orientations for \( P^g \times I \), so will be absorbed when we use Fubini’s theorem to write the integral over \( P^g \times I \) as an iterated integral. Let \( \chi(\eta) = \chi(\eta + \xi) \), a bump function on \( I \) supported near 0. Making these substitutions, as well as a change of variables \( \eta \sim \eta + \xi \) in the integral over \( I \), the asymptotic expression (3.9) for \( m(k,k\xi) \) simplifies to

\[
k^n \sum_g g^{-k\xi} \int_{I'_I \times I} dX d\eta e^{2\pi i k(\eta,X)} \sigma_g(X) \chi(\eta) \int_{P^g} \Theta \frac{g^k_{L} \text{Td}(P^g, 2\pi X)}{\mathcal{D}^g(\nu_{P^g}, 2\pi X)} e^{k(\omega_{\xi} + (\eta,F_{\theta})�)}.
\]

We need the following special case of the stationary phase expansion.

**Proposition 3.5** (stationary phase expansion, cf. [4, Lemma 7.7.3]). Let \( u(X,\eta) \) be a Schwartz function. We have the following asymptotic expansion in \( k \):

\[
\int_{I'_I \times (I'_I)^*} dX d\eta e^{2\pi i k(\eta,X)} u(X,\eta) \sim \frac{1}{k^n} \sum_{j=0}^{\infty} \frac{1}{j!} \left( \sum_{a} \frac{i}{2\pi k} \frac{\partial}{\partial \eta_a} \frac{\partial}{\partial X^a} \right)^j u(0,0).
\]

**Remark 3.6.** To obtain the expression here from the expression appearing in *loc. cit.*, one sets \( x = (X,\eta) \in \mathbb{R}^{2n} \) and \( A(X,\eta) = (\eta,X) \). Note also that in Hörmander’s notation \( D = -i(d/dx) \).

We apply this to the smooth compactly supported function

\[
u(X,\eta) = \sigma_g(X) \chi(\eta) \int_{P^g} \Theta \frac{g^k_{L} \text{Td}(P^g, 2\pi X)}{\mathcal{D}^g(\nu_{P^g}, 2\pi X)} e^{k(\omega_{\xi} + (\eta,F_{\theta})�)}.
\]

Although this function depends on \( k \), the dependence is quasi-polynomial, and so the expansion still applies. Since \( \sigma_g(X) \), \( \chi(\eta) \) equal 1 in a neighborhood of 0, they have no effect on the expansion. The \( \eta \) derivatives \( k^{-1} \partial_{\eta_a} \) operate only on the factor \( e^{k(\eta,F_{\theta})�} \). The combined effect of the operator \( \Sigma_a (i/2\pi k) \partial_{\eta_a} \partial_{X^a} \) is to replace \( X \) with \((i/2\pi)F_{\theta} \), yielding the asymptotic expansion

\[
m(k,k\xi) \sim \sum_g g^{-k\xi} \int_{P^g} \Theta \frac{g^k_{L} \text{Td}(P^g, F_{\theta})}{\mathcal{D}^g(\nu_{P^g}, F_{\theta})} e^{k(\omega_{\xi} + (\eta,F_{\theta})�)}.
\]

(By substituting \( F_{\theta} \) for \( X \) in \( \text{Td}(P^g, X), \mathcal{D}^g(\nu_{P^g}, P, X)^{-1} \), we mean to take the Taylor expansion around \( X = 0 \) and substitute the differential form \( F_{\theta} \).) At this stage we see that the contribution of \( g \in S'_I \) vanishes unless \( P^g \neq \emptyset \), so that \( S'_I = S_P \) (\( S_P \) is as in Theorem 1.2). As the characteristic forms \( \text{Td}(P^g, F_{\theta}), \mathcal{D}^g(\nu_{P^g}, P, F_{\theta}) \) appear multiplied by the form \( \Theta \), which has top degree in the \( T'_I \) orbit directions, we can replace these characteristic forms with their horizontal parts. Substituting \( F_{\theta} \) for \( X \) and taking the horizontal part is the definition of the Cartan map \( \text{Car}_g \) for the locally free action of \( T'_I \) on the space \( P^g \), hence the result is the pullback along the map \( P^g \to \Sigma_g = P^g/T'_I \) of the form

\[
\frac{\text{Td}(\Sigma_g)}{\mathcal{D}^g(\nu_{\Sigma_g}, \Sigma_g)}.
\]

(See our remarks in the introduction regarding characteristic forms for orbifolds.) Similarly the \( 1 \)st Chern form \( c_1(L_{\Sigma}) \) is obtained by applying the Cartan map to the equivariant symplectic form \( \omega_1(X) = \omega - (\mu, X) \), and results in \( c_1(L_{\Sigma}) = \omega_{\xi} - (\xi, F_{\theta}) \). Hence \( \text{Ch}(L_{\Sigma}) = e^{c_1(L_{\Sigma})} = e^{\omega_{\xi} - (\xi, F_{\theta})} \). The integral over the fibres of \( P^g \to \Sigma_g \) then gives \( 1/d \), where \( d : \Sigma = \sqcup \Sigma_g \to \mathbb{Z} \) is the locally
constant function giving the size of the generic stabilizer for the $T'_I \simeq T/T_I$ action on $\sqcup P^g \to \Sigma$. Equation (3.11) becomes

$$m(k, k\xi) \sim \sum_{g \in S_P} g^{-k\xi} \int_{\Sigma_g} \frac{1}{D^g_\nu(\nu_{g, \Sigma})} \text{Ch}(L_{\Sigma})^k \text{Td}(\Sigma) e^{k(\xi, F_0)}.$$  

By Corollary 2.3, $m(k, k\xi)$ is a quasi-polynomial function of $k$, hence the asymptotic expansion must be exact, or in other words, ‘$\sim$’ in equation (3.12) can be replaced with ‘$=$’. Thus setting $\lambda = k\xi$ we have

$$m(k, \lambda) = \sum_{g \in S_P} g^{-\lambda} \int_{\Sigma_g} \frac{1}{D^g_\nu(\nu_{g, \Sigma})} \text{Ch}(L_{\Sigma})^k \text{Td}(\Sigma) e^{(\lambda, F_0)}.$$  

The right-hand-side of equation (3.13) is quasi-polynomial in $(k, \lambda)$. Hence by Corollary 2.3, equation (3.13) holds on all of $C_p$ (and not only at points $(k, \lambda)$ with $\lambda = k\xi$, $\xi$ a rational, weakly regular value in the relative interior of $p$). This completes the proof of Theorem 1.2.

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References