The Subelliptic Heat Kernel of the Octonionic Anti-De Sitter Fibration

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Received July 31, 2020, in final form January 29, 2021; Published online February 10, 2021
https://doi.org/10.3842/SIGMA.2021.014

Abstract. In this note, we study the sub-Laplacian of the 15-dimensional octonionic anti-de Sitter space which is obtained by lifting with respect to the anti-de Sitter fibration the Laplacian of the octonionic hyperbolic space $\mathbb{O}H^1$. We also obtain two integral representations for the corresponding subelliptic heat kernel.

Key words: sub-Laplacian; 15-dimensional octonionic anti-de Sitter space; the anti-de Sitter fibration

2020 Mathematics Subject Classification: 58J35; 53C17

1 Introduction and results

In this note we study the sub-Laplacian and the corresponding sub-Riemannian heat kernel of the octonionic anti-de Sitter fibration

$$S^7 \hookrightarrow \text{AdS}^{15}(\mathbb{O}) \to \mathbb{O}H^1.$$  

This paper follows the previous works [2, 3, 10] which respectively concerned:

1. The complex anti-de Sitter fibrations:

$$S^1 \hookrightarrow \text{AdS}^{2n+1}(\mathbb{C}) \to \mathbb{C}H^n.$$  

2. The quaternionic anti-de Sitter fibrations:

$$S^3 \hookrightarrow \text{AdS}^{4n+3}(\mathbb{H}) \to \mathbb{H}H^n.$$  

The 15-dimensional anti-de Sitter fibration is the last model space that remained to be studied of a sub-Riemannian manifold arising from a $H$-type semi-Riemannian submersion over a rank-one symmetric space, see the Table 3 in [4].

Similarly to the complex and quaternionic case, the sub-Laplacian is defined as the lift on $\text{AdS}^{15}(\mathbb{O})$ of the Laplace–Beltrami operator of the octonionic hyperbolic space $\mathbb{O}H^1$. However, in the complex and quaternionic case the Lie group structure of the fiber played an important role that we can not use here, since the fiber $S^7$ is not a group. Instead, we make use of some algebraic properties of $S^7$ that were already pointed out and used by the authors in [1] for the study of the octonionic Hopf fibration:

$$S^7 \hookrightarrow S^{15} \to \mathbb{O}P^1.$$  

Let us briefly describe our main results. Due to the cylindrical symmetries of the fibration, the heat kernel of the sub-Laplacian only depends on two variables: the variable $r$ which is the
Riemannian distance on $\mathbb{O}H^1$ (the starting point is specified with inhomogeneous coordinate in Section 3) and the variable $\eta$ which is the Riemannian distance starting at a pole on the fiber $S^7$. We prove in Proposition 3.1 that in these coordinates, the radial part of the sub-Laplacian $\tilde{L}$ writes
\[
\tilde{L} = \frac{\partial^2}{\partial r^2} + (7 \coth r + 7 \tanh r) \frac{\partial}{\partial r} + \tanh^2 r \left( \frac{\partial^2}{\partial \eta^2} + 6 \cot \eta \frac{\partial}{\partial \eta} \right).
\]
As a consequence of this expression for the sub-Laplacian, we are able to derive two equivalent formulas for the heat kernel. The first formula, see Proposition 4.1, reads as follows: for $r \geq 0$, $\eta \in [0, \pi)$, $t > 0$
\[
p_t(r, \eta) = \int_0^\infty s_t(\eta, i u) q_{t,15}(\cosh r \cosh u) \sinh^6 u \, du,
\]
where $s_t$ is the heat kernel of the Jacobi operator
\[
\tilde{\Delta}_{S^7} = \frac{\partial^2}{\partial \eta^2} + 6 \cot \eta \frac{\partial}{\partial \eta}
\]
with respect to the measure $\sin^6 \eta \, d\eta$, and where $q_{t,15}$ is the Riemannian heat kernel on the 15-dimensional real hyperbolic space $\mathbb{H}^{15}$ given in (4.1). The second formula, see Proposition 4.2, writes as follows:
\[
p_t(r, \eta) = \int_0^\pi \int_0^\infty G_t(\eta, \varphi, u) q_{t,9}(\cosh r \cosh u) \sin^5 \varphi \, du \, d\varphi,
\]
where $q_{t,9}$ is Riemannian heat kernel on the 9-dimensional hyperbolic space $\mathbb{H}^9$ and $G_t(\eta, \varphi, u)$ is given in (4.3).

Similarly to [2, 3, 10], it might be expected that explicit integral representations of the heat kernel might be used to study small-time asymptotics, inside and outside of the cut-locus. Integral representations of heat kernels can also be used to obtain sharp heat kernel estimates, see [7]. Those applications of the heat kernel representations we obtain will possibly be addressed in a future research project.

2 The octonionic anti-de Sitter fibration

Let
\[
\mathbb{O} = \left\{ x = \sum_{j=0}^7 x_j e_j, \, x_j \in \mathbb{R} \right\},
\]
be the division algebra of octonions (see [9] for explicit representations of this algebra). We recall that the multiplication rules are given by
\[
\begin{align*}
    e_i e_j &= e_j & \text{if } i = 0, \\
    e_i e_j &= e_i & \text{if } j = 0, \\
    e_i e_j &= -\delta_{ij} e_0 + \epsilon_{ijk} e_k & \text{otherwise},
\end{align*}
\]
where $\delta_{ij}$ is the Kronecker delta and $\epsilon_{ijk}$ is the completely antisymmetric tensor with value 1 when $ijk = 123, 145, 176, 246, 257, 347, 365$ (also see [1]). The octonionic norm is defined for $x \in \mathbb{O}$ by
\[
||x||^2 = \sum_{j=0}^7 x_j^2.
\]
The octonionic anti-de Sitter space $\text{AdS}^{15}(\mathcal{O})$ is the quadric defined as the pseudo-hyperbolic space by:

$$\text{AdS}^{15}(\mathcal{O}) = \{(x, y) \in \mathcal{O}^2, ||(x, y)||_\mathcal{O}^2 = -1\},$$

where

$$||(x, y)||_\mathcal{O}^2 := ||x||^2 - ||y||^2.$$

In real coordinates we have $x = \sum_{j=0}^{7} x_j e_j$, $y = \sum_{j=0}^{7} y_j e_j$, and the pseudo-norm can be written as $x_0^2 + \cdots + x_7^2 - y_0^2 - \cdots - y_7^2$. As such, $\text{AdS}^{15}(\mathcal{O})$ is embedded in the flat 16-dimensional space $\mathbb{R}^{8,8}$ endowed with the Lorentzian real signature $(8, 8)$ metric

$$ds^2 = dx_0^2 + \cdots + dx_7^2 - dy_0^2 - \cdots - dy_7^2.$$

Consequently, $\text{AdS}^{15}(\mathcal{O})$ is naturally endowed with a pseudo-Riemannian structure of signature $(8, 7)$. Let $\mathcal{O}H^1$ denote the octonionic hyperbolic space. The map $\pi: \text{AdS}^{15}(\mathcal{O}) \to \mathcal{O}H^1$, given by $(x, y) \mapsto [x:y] = y^{-1}x$ is a pseudo-Riemannian submersion with totally geodesic fibers isometric to the seven-dimensional sphere $\mathbb{S}^7$. Notice that, as a topological manifold, $\mathcal{O}H^1$ can therefore be identified with the unit open ball in $\mathcal{O}$. The pseudo-Riemannian submersion $\pi$ yields the octonionic anti-de Sitter fibration

$$\mathbb{S}^7 \hookrightarrow \text{AdS}^{15}(\mathcal{O}) \to \mathcal{O}H^1.$$

For further information on semi-Riemannian submersions over rank-one symmetric spaces, we refer to [6].

### 3 Cylindrical coordinates and radial part of the sub-Laplacian

The sub-Laplacian $L$ on $\text{AdS}^{15}(\mathcal{O})$ we are interested in is the horizontal Laplacian of the Riemannian submersion $\pi: \text{AdS}^{15}(\mathcal{O}) \to \mathcal{O}H^1$, i.e., the horizontal lift of the Laplace–Beltrami operator of $\mathcal{O}H^1$. It can be written as

$$L = \Box_{\text{AdS}^{15}(\mathcal{O})} + \Delta_V,$$

where $\Box_{\text{AdS}^{15}(\mathcal{O})}$ is the d’Alembertian, i.e., the Laplace–Beltrami operator of the pseudo-Riemannian metric and $\Delta_V$ is the vertical Laplacian. Since the fibers of $\pi$ are totally geodesic and isometric to $\mathbb{S}^7 \subset \text{AdS}^{15}(\mathcal{O})$, we note that $\Box_{\text{AdS}^{15}(\mathcal{O})}$ and $\Delta_V$ are commuting operators, and we can identify

$$\Delta_V = \Delta_{\mathbb{S}^7}.$$

The sub-Laplacian $L$ is associated with a canonical sub-Riemannian structure on $\text{AdS}^{15}(\mathcal{O})$ which is of $H$-type, see [4]. To study $L$, we introduce a set of coordinates that reflect the cylindrical symmetries of the octonionic unit sphere which provides an explicit local trivialization of the octonionic anti-de Sitter fibration. Consider the coordinates $w \in \mathcal{O}H^1$, where $w$ is the inhomogeneous coordinate on $\mathcal{O}H^1$ given by $w = y^{-1}x$, with $x, y \in \text{AdS}^{15}(\mathcal{O})$. Consider the north pole $p \in \mathbb{S}^7$ and take...
$Y_1, \ldots, Y_7$ to be an orthonormal frame of $T_p\mathbb{S}^7$. Let us denote $\exp_p$ the Riemannian exponential map at $p$ on $\mathbb{S}^7$. Then the cylindrical coordinates we work with are given by

$$(w, \theta_1, \ldots, \theta_7) \mapsto \left( \frac{\exp_p \left( \sum_{i=1}^7 \theta_i Y_i \right) w}{\sqrt{1 - \rho^2}}, \frac{\exp_p \left( \sum_{i=1}^7 \theta_i Y_i \right)}{\sqrt{1 - \rho^2}} \right) \in \text{AdS}^{15}(\mathbb{O}),$$

where $\rho = \|w\|$ and $\|\theta\| = \sqrt{\theta_1^2 + \cdots + \theta_7^2} < \pi$.

A function $f$ on $\text{AdS}^{15}(\mathbb{O})$ is called radial cylindrical if it only depends on the two coordinates $(\rho, \eta) \in [0, 1) \times [0, \pi]$ where $\eta = \sqrt{\sum_{i=1}^7 \theta_i^2}$. More precisely $f$ is radial cylindrical if there exists a function $g$ so that

$$f \left( \frac{\exp_p \left( \sum_{i=1}^7 \theta_i Y_i \right) w}{\sqrt{1 - \rho^2}}, \frac{\exp_p \left( \sum_{i=1}^7 \theta_i Y_i \right)}{\sqrt{1 - \rho^2}} \right) = g(\rho, \eta).$$

We denote by $\mathcal{D}$ the space of smooth and compactly supported functions on $[0, 1) \times [0, \pi)$. Then the radial part of $L$ is defined as the operator $\tilde{L}$ such that for any $f \in \mathcal{D}$, we have

$$L(f \circ \psi) = (\tilde{L}f) \circ \psi. \tag{3.3}$$

We now compute $\tilde{L}$ in cylindrical coordinates.

**Proposition 3.1.** The radial part of the sub-Laplacian on $\text{AdS}^{15}(\mathbb{O})$ is given in the coordinates $(r, \eta)$ by the operator

$$\tilde{L} = \frac{\partial^2}{\partial r^2} + (7 \coth r + 7 \tanh r) \frac{\partial}{\partial r} + \tanh^2 r \left( \frac{\partial^2}{\partial \eta^2} + 6 \cot \eta \frac{\partial}{\partial \eta} \right),$$

where $r = \tanh^{-1} \rho$ is the Riemannian distance on $\mathbb{O}H^1$ from the origin.

**Proof.** Note that the radial part of the Laplace–Beltrami operator on the octonionic hyperbolic space $\mathbb{O}H^1$ is

$$\tilde{\Delta}_{\mathbb{O}H^1} = \frac{\partial^2}{\partial r^2} + (7 \coth r + 7 \tanh r) \frac{\partial}{\partial r},$$

and the radial part of the Laplace–Beltrami operator on $\mathbb{S}^7$ is

$$\tilde{\Delta}_{\mathbb{S}^7} = \frac{\partial^2}{\partial \eta^2} + 6 \cot \eta \frac{\partial}{\partial \eta}. \tag{3.4}$$

Since the octonionic anti-de Sitter fibration defines a totally geodesic submersion with base space $\mathbb{O}H^1$ and fiber $\mathbb{S}^7$, the semi-Riemannian metric on $\text{AdS}^{15}(\mathbb{O})$ is locally given by a warped product between the Riemannian metric of $\mathbb{O}H^1$ and the Riemannian metric on $\mathbb{S}^7$. Hence the radial part of the d’Alembertian becomes

$$\Box_{\text{AdS}^{15}(\mathbb{O})} = \frac{\partial^2}{\partial r^2} + (7 \coth r + 7 \tanh r) \frac{\partial}{\partial r} + g(r) \left( \frac{\partial^2}{\partial \eta^2} + 6 \cot \eta \frac{\partial}{\partial \eta} \right), \tag{3.5}$$

for some smooth function $g$ to be computed.

On the other hand, from the isometric embedding $\text{AdS}^{15}(\mathbb{O}) \subset \mathbb{O} \times \mathbb{O}$, the d’Alembertian on $\text{AdS}^{15}(\mathbb{O})$ is a restriction of the d’Alembertian on $\mathbb{O} \times \mathbb{O} \simeq \mathbb{R}^{8,8}$ in the sense that for a smooth $f: \text{AdS}^{15}(\mathbb{O}) \to \mathbb{R}$

$$\Box_{\text{AdS}^{15}(\mathbb{O})} f = \Box_{\mathbb{O} \times \mathbb{O}} f^\prime_{\text{AdS}^{15}(\mathbb{O})}.$$
\[ \Box_{\mathbb{O} \times \mathbb{O}} = \sum_{i=0}^{7} \left( \frac{\partial^2}{\partial x_i^2} - \frac{\partial^2}{\partial y_i^2} \right) \] and for \( x, y \in \mathbb{O} \) such that \( \| y \|^2 - \| x \|^2 > 0 \), \( f^*(x, y) = f \left( \frac{x}{\sqrt{\| y \|^2 - \| x \|^2}}, \frac{y}{\sqrt{\| y \|^2 - \| x \|^2}} \right) \). For the specific choice of the function \( f(x, y) = y_1 \), one easily computes that \( \Box_{\mathbb{O} \times \mathbb{O}} f^*_{/\text{AdS}^{15}(\mathbb{O})}(x, y) = 15 y_1 \), thus
\[ \Box_{\text{AdS}^{15}(\mathbb{O})} f(x, y) = 15 y_1. \]

For the point with coordinates
\[ \left( \exp \left( \frac{\sum_{i=1}^{7} \theta_i Y_i}{\sqrt{1 - \rho^2}} \right), \exp \left( \frac{\sum_{i=1}^{7} \theta_i Y_i}{\sqrt{1 - \rho^2}} \right) \right) \in \text{AdS}^{15}(\mathbb{O}) \]
one has
\[ y_1 = \frac{\cos \eta}{\sqrt{1 - \rho^2}} = \cosh r \cos \eta. \]

We therefore deduce that
\[ \tilde{\Box}_{\text{AdS}^{15}(\mathbb{O})} (\cosh r \cos \eta) = 15 \cosh r \cos \eta. \]

Using the formula (3.5), after a straightforward computation, this yields \( g(r) = -\frac{1}{\cosh^2 r} \) and therefore
\[ \tilde{\Box}_{\text{AdS}^{15}(\mathbb{O})} = \frac{\partial^2}{\partial r^2} + (7 \coth r + 7 \tanh r) \frac{\partial}{\partial r} - \frac{1}{\cosh^2 r} \left( \frac{\partial^2}{\partial \eta^2} + 6 \cot \eta \frac{\partial}{\partial \eta} \right) \]
\[ = \tilde{\Delta}_{\text{O}H^1} - \frac{1}{\cosh^2 r} \tilde{\Delta}_{S^7}. \]

Finally, to conclude, one notes that the sub-Laplacian \( L \) is given by the difference between the Laplace–Beltrami operator of \( \text{AdS}^{15}(\mathbb{O}) \) and the vertical Laplacian. Therefore by (3.1) and (3.2),
\[ \tilde{L} = \tilde{\Box}_{\text{AdS}^{15}(\mathbb{O})} + \tilde{\Delta}_{S^7} = \frac{\partial^2}{\partial r^2} + (7 \coth r + 7 \tanh r) \frac{\partial}{\partial r} + \tanh^2 r \left( \frac{\partial^2}{\partial \eta^2} + 6 \cot \eta \frac{\partial}{\partial \eta} \right). \]

**Remark 3.2.** As a consequence of the previous result, we can check that the Riemannian measure of \( \text{AdS}^{15}(\mathbb{O}) \) in the coordinates \((r, \eta)\), which is the symmetric and invariant measure for \( \tilde{L} \) is given by
\[ d\mu = \frac{\pi^7}{90} \sinh^7 r \cosh^7 r \sin^6 \eta \, dr \, d\eta. \]

(See also Remark 2 in [1], which corresponds to the case of the octonionic Hopf fibration.)

### 4 Integral representations of the subelliptic heat kernel

In this section, we give two integral representations of the subelliptic heat kernel associated with \( \tilde{L} \). We denote by \( p_t(r, \eta) \) the heat kernel of \( \tilde{L} \) issued from the point \( r = \eta = 0 \) with respect to the measure (3.6). We remark that studying the subelliptic heat kernel associated with \( \tilde{L} \) is enough to study the heat kernel of \( L \), because due to (3.3) the heat kernel \( h_t(w, \theta) \) of \( L \) issued from the point with cylindric coordinates \( w = 0, \theta = 0 \) is then given by
\[ h_t(w, \theta) = p_t \left( \tanh^{-1} \|w\|, \|\theta\| \right). \]
4.1 First integral representation

We denote by \( s_t \) the heat kernel of the operator

\[
\tilde{\Delta}_\mathbb{S}^7 = \frac{\partial^2}{\partial \eta^2} + 6 \cot \eta \frac{\partial}{\partial \eta}
\]

with respect to the reference measure \( \sin^6 \eta \, d\eta \). The operator \( \tilde{\Delta}_\mathbb{S}^7 \) belongs to the family of Jacobi diffusion operators which have been extensively studied in the literature, see for instance the appendix in [5] and the references therein. In particular, the spectrum of \( \tilde{\Delta}_\mathbb{S}^7 \) is given by

\[
\text{Sp}(\tilde{\Delta}_\mathbb{S}^7) = \{ m(m + 6), \ m \in \mathbb{N} \},
\]

and the eigenfunction corresponding to the eigenvalue \( m(m + 6) \) is \( P_m^{5/2,5/2}(\cos \eta) \) where \( P_m^{5/2,5/2} \) is the Jacobi polynomial

\[
P_m^{5/2,5/2}(x) = \frac{(-1)^m}{2^m m!} \frac{d^m}{dx^m} (1 - x^2)^{5/2-m}.
\]

As a consequence, one has the following spectral decomposition for the heat kernel:

\[
s_t(\eta, u) = \frac{1}{\pi} \sum_{m=0}^{\infty} 2^{4m+7} m!(m+5)![(m+3)!]^2 e^{-m(m+6)t} P_m^{5/2,5/2}(\cos \eta) P_m^{5/2,5/2}(\cos u).
\]

**Proposition 4.1.** For \( r \geq 0, \ \eta \in [0, \pi], \) and \( t > 0 \) we have

\[
p_t(r, \eta) = \int_0^\infty s_t(\eta, u) q_{t,15}(\cosh r \cosh u) \sinh^6 u \, du,
\]

where

\[
q_{t,15}(\cosh s) := \frac{e^{-49t}}{(2\pi)^7} \sqrt{4\pi t} \left( -\frac{1}{\sinh \sqrt{s}} \frac{d}{ds} \right)^7 e^{-s^2/4t}
\]

is the Riemannian heat kernel on the 15-dimensional real hyperbolic space \( \mathbb{H}^{15} \).

**Proof.** Since \( \pi: \text{AdS}^{15}(\mathbb{O}) \to \mathbb{O}H^1 \) is a (semi-Riemannian) totally geodesic submersion, the operators \( \tilde{\Delta}_{\text{AdS}^{15}(\mathbb{O})} \) and \( \tilde{\Delta}_\mathbb{S}^7 \) commute. Thus

\[
e^{t\tilde{L}} = e^{t(\tilde{\Delta}_{\text{AdS}^{15}(\mathbb{O})} + \tilde{\Delta}_\mathbb{S}^7)} = e^{t\tilde{\Delta}_\mathbb{S}^7} e^{t\tilde{\Delta}_{\text{AdS}^{15}(\mathbb{O})}}.
\]

We deduce that the heat kernel of \( \tilde{L} \) can be written as

\[
p_t(r, \eta) = \int_0^\pi s_t(\eta, u) p_t^{\text{AdS}^{15}(\mathbb{O})}(r, u) \sinh^6 u \, du,
\]

where \( s_t \) is the heat kernel of (3.4) with respect to the measure \( \sin^6 \eta \, d\eta, \ \eta \in [0, \pi] \), and

\[
p_t^{\text{AdS}^{15}(\mathbb{O})}(r, u) \text{ the heat kernel at } (0, 0) \text{ of } \tilde{\Delta}_{\text{AdS}^{15}(\mathbb{O})} \text{ with respect to the measure in (3.6), i.e.,}
\]

\[
d\mu(r, u) = \frac{\pi^7}{90} \sinh^7 r \cosh^7 r \sinh^6 u \, dr \, du, \quad r \in [0, \infty), \quad u \in [0, \pi].
\]

In order to write (4.2) more precisely, let us consider the analytic change of variables \( \tau: (r, \eta) \to (r, i\eta) \) that will be applied on functions of the type \( f(r, \eta) = h(r)e^{-i\lambda \eta} \), with \( h \) smooth and
compactely supported on $[0, \infty)$ and $\lambda > 0$. Then as we saw in the proof of Proposition 3.1 one can see that

$$\Box_{\text{AdS}^{15}(\mathcal{O})}(f \circ \tau) = (\tilde{\Delta}_{\mathbb{H}^{15}} f) \circ \tau,$$

where

$$\tilde{\Delta}_{\mathbb{H}^{15}} = \tilde{\Delta}_{\text{H}^1} + \frac{1}{\cosh^2 r} \Delta_p, \quad \Delta_p = \frac{\partial^2}{\partial \eta^2} + 6 \coth \eta \frac{\partial}{\partial \eta}.$$

Then, one deduces

$$e^{t\tilde{L}}(f \circ \tau) = e^{t\tilde{\Delta}_{\mathbb{H}^{15}}(0)}(f \circ \tau) = e^{t\tilde{\Delta}_{\mathbb{H}^{15}}((e^{t\tilde{\Delta}_{\mathbb{H}^{15}} f) \circ \tau) = (e^{-t\Delta_p} e^{t\tilde{\Delta}_{\mathbb{H}^{15}} f) \circ \tau}.$$

Now, since for every $f(r, \eta) = h(r) e^{-i\lambda \eta}$,

$$(e^{t\tilde{\Delta}_{\text{AdS}^{15}}(0)} f)(0, 0) = (e^{t\tilde{\Delta}_{\mathbb{H}^{15}}})((f \circ \tau^{-1})(0, 0),$$

one deduces that for a function $h$ depending only on $u$,

$$\int_0^\pi h(u)p_t(\tilde{\Delta}_{\text{AdS}^{15}}(0)) (r, u) \sin^6 u \, du = \int_0^\infty h(-iu)q_{t,15}(\cosh r \cosh u) \sinh^6 u \, du.$$

Therefore, coming back to (4.2), one infers that using the analytic extension of $s_t$ one must have

$$\int_0^\pi s_t(\eta, u)p_t(\tilde{\Delta}_{\text{AdS}^{15}}(0)) (r, u) \sin^6 u \, du = \int_0^\infty s_t(\eta, -iu)q_{t,15}(\cosh r \cosh u) \sinh^6 u \, du,$$

where $q_{t,15}$ is the Riemannian heat kernel on the real hyperbolic space $\mathbb{H}^{15}$ given in (4.1).

4.2 Second integral representation

**Proposition 4.2.** For $r \geq 0$, $\eta \in [0, \pi]$, and $t > 0$ we have

$$p_t(r, \eta) = \int_0^\pi \int_0^\infty G_t(\eta, \varphi, u) q_{t,9}(\cosh r \cosh u) \sin^5 \varphi \, du \, d\varphi,$$

where $q_{t,9}$ is the 9-dimensional Riemannian heat kernel on the hyperbolic space $\mathbb{H}^9$:

$$q_{t,9}(\cosh s) := e^{-16t} \frac{1}{(2\pi)^4 \sqrt{4\pi t}} \left( \frac{1}{\sinh s} \frac{d}{ds} \right)^4 e^{-s^2/4t},$$

and

$$G_t(\eta, \varphi, u) = \frac{15}{8} \sum_{m \geq 0} e^{-(m(m+6)+33)t}(\cos \eta + i \sin \eta \cos \varphi)^m \cosh((m + 3)u).$$

**Proof.** The strategy of the following method appeals to some results proved in [8]. Firstly, we decompose the subelliptic heat kernel in the $\eta$ variable with respect to the basis of normalized eigenfunctions of $\tilde{\Delta}_\text{S}^7 = \frac{\partial^2}{\partial \eta^2} + 6 \cot \eta \frac{\partial}{\partial \eta}$. Accordingly,

$$p_t(r, \eta) = \sum_{m \geq 0} f_m(t, r) h_m(\eta),$$

where for each $m$, $h_m$ is given by

$$h_m(\eta) = \frac{15}{16} \int_0^\pi (\cos \eta + i \sin \eta \cos \varphi)^m \sin^5 \varphi \, d\varphi.$$
and $f_m(t, \cdot)$ solves the following heat equation
\[
\frac{\partial}{\partial t} f_m(t, r) = \left( \frac{\partial^2}{\partial r^2} + (7 \coth r + 7 \tanh r) \frac{\partial}{\partial r} - m(m + 6) \tanh^2 r \right) f_m(t, r)
\]
\[
= \left( \frac{\partial^2}{\partial r^2} + (7 \coth r + 7 \tanh r) \frac{\partial}{\partial r} + \frac{m(m + 6)}{\cosh^2 r} - m(m + 6) \right) f_m(t, r).
\]

We consider then the operator
\[
L_m := \frac{\partial^2}{\partial r^2} + (7 \coth r + 7 \tanh r) \frac{\partial}{\partial r} + \frac{m(m + 6)}{\cosh^2 r} + 49,
\]
which was studied in [8, p. 229]. From [8, Theorem 2], with $\alpha = 3 + \frac{m}{2}$, $\beta = -\frac{m}{2}$, we deduce that the solution to the wave Cauchy problem associated with the subelliptic Laplacian is given $f \in C_0^\infty (\mathbb{O}H^1)$ by
\[
\cos \left( s \sqrt{-L_m} \right)(f)(w) = -\sinh s \left( \frac{1}{2\pi} \right)^4 \int_{\mathbb{O}H^1} K_m(s, w, y)f(y) \frac{dy}{(1 - ||y||^2)^{8/4}},
\]
where
\[
K_m(s, w, y) = \frac{(1 - (w, y)^3 + m/2)}{(1 - (w, y)^m/2)} \frac{1}{\cosh^3(d(w, y)) \sqrt{\cosh^2(s) - \cosh^2(d(w, y))}} \times _2F_1 \left( m + 3, -m - 3, \frac{1}{2}; \frac{\cosh(d(w, y)) - \cosh(s)}{2 \cosh(d(w, y))} \right),
\]
where $_2F_1$ is the Gauss hypergeometric function and $dy$ stands for the Lebesgue measure in $\mathbb{R}^8$. Using the spectral formula
\[
e^{tL} = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-s^2/(4t)} \cos \left( s \sqrt{-L} \right) ds,
\]
which holds for any non positive self-adjoint operator, we deduce that the solution to the heat Cauchy problem associated with $L_m$:
\[
e^{tL_m}(f)(w) = \frac{e^{-m(m+6)t-7t^2}}{\sqrt{4\pi t}(2\pi)^4} \int_{\mathbb{R}} ds (-\sinh s)e^{-s^2/(4t)}
\]
\[
\times \left( \frac{1}{\sinh s ds} \right)^4 \int_{\mathbb{O}H^1} K_m(s, w, y)f(y) \frac{dy}{(1 - ||y||^2)^{8/4}}.
\]
Performing integration by parts 4-times,
\[
\int_{\mathbb{R}} ds (-\sinh s) \left( \frac{1}{\sinh s ds} \right)^4 e^{-s^2/(4t)} \int_{\mathbb{O}H^1} K_m(s, w, y)f(y) \frac{dy}{(1 - ||y||^2)^{8/4}}
\]
\[
= \int_{\mathbb{O}H^1} f(y) \frac{dy}{(1 - ||y||^2)^{8/4}} \int_{\mathbb{R}} ds (-\sinh s)K_m(s, w, y) \left( \frac{1}{\sinh s ds} \right)^4 e^{-s^2/4t}
\]
\[
= 2 \int_{\mathbb{O}H^1} f(y) \frac{dy}{(1 - ||y||^2)^{8/4}} \int_{d(w,y)}^\infty d(cosh(s))K_m(s, w, y) \left( \frac{1}{\sinh s ds} \right)^4 e^{-s^2/4t}.
\]
Thus we get
\[
e^{tL_m}(f)(0) = 2e^{-(m(m+6)+33)t} \int_{\mathbb{O}H^1} f(y) \frac{dy}{(1 - ||y||^2)^{8/4}} \int_{d(0,y)}^\infty d(cosh(s))K_m(s, 0, y)q_{t,9}(cosh s).
As a result, the subelliptic heat kernel of $L_m$ reads
\[
\frac{dy}{(1-||y||^2)^8} \int_0^\infty d(cosh s)K_m(s,0,y)q_{t,9}(cosh s)
\]
\[
= dr \sinh^7 r \cosh^7 r \int_r^\infty d(cosh s)K_m(s,0,y)q_{t,9}(cosh s).
\]

By changing the variable $cosh s = cosh r \cosh u$ for $u \geq 0$, the last expression becomes
\[
dr \sinh^7 r \cosh^7 r \int_0^\infty 2F_1\left(m+3,-m-3,\frac{1}{2};\frac{1-cosh u}{2}\right)q_{t,9}(cosh r \cosh u) du.
\]

Therefore $p_t(r,\eta)$ has the integral representation
\[
2 \sum_{m \geq 0} e^{-(m(m+6)+33)t}h_m(\eta) \int_0^\infty 2F_1\left(m+3,-m-3,\frac{1}{2};\frac{1-cosh u}{2}\right)q_{t,9}(cosh r \cosh u) du.
\]

Now, notice that $2F_1\left(m+3,-m-3,\frac{1}{2};\frac{1-cosh u}{2}\right)$ is simply the Chebyshev polynomial of the first kind
\[
T_{m+3}(x) = 2F_1\left(m+3,-m-3,\frac{1}{2};\frac{1-x}{2}\right),
\]
for all $x \in \mathbb{C}$. Therefore, one has
\[
2F_1\left(m+3,-m-3,\frac{1}{2};\frac{1-cosh u}{2}\right) = T_{m+3}(cosh u) = cosh((m+3)u),
\]
and the proof is over.

\section*{Acknowledgements}

F.B. is partially funded by the NSF grant DMS-1901315.

\section*{References}


