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UPPER AND LOWER BOUNDS OF SOLUTIONS FOR FRACTIONAL INTEGRAL EQUATIONS

Rabha W. Ibrahim and Shaher Momani

Abstract. In this paper we consider the integral equation of fractional order in sense of Riemann-Liouville operator

$$u^m(t) = a(t)I^\alpha[b(t)u(t)] + f(t)$$

with $m \geq 1$, $t \in [0, T]$, $T < \infty$ and $0 < \alpha < 1$. We discuss the existence, uniqueness, maximal, minimal and the upper and lower bounds of the solutions. Also we illustrate our results with examples.

1 Introduction and Preliminaries

Consider the Volterra integral equation of the second kind

$$u(t) - \lambda \int_a^t K(\tau, t)u(\tau)d\tau = f(t)$$

where f, K are given functions, λ is a parameter and u is the solution. This equation arises very often in solving various problems of mathematical physics, especially that describing physical processes after effects [2, 4]. Fractional integral and diffeo-integral equations involving Riemann-Liouville operators of arbitrary order $\alpha > 0$ have been solved by various authors (see [5, 8, 10, 11, 13]), in many techniques, but all of them leading to the solution involving the Mitag-Leffler function [8]. The solution of the first kind Volterra integral equations $I^\alpha u(t) = f(t)$ are well known. When $\alpha = 1/2$, the equation is called Abel integral equation. In this paper, we consider the Volterra fractional integral equations of the form

$$u^m(t) = a(t)I^\alpha[b(t)u(t)] + f(t), \;\; m \geq 1 \tag{1}$$

where $a(t), b(t), f(t)$ are real positive functions in $C[0, T]$, $t \in [0, T]$, and $0 < \alpha < 1$. Equation (1) is solved for $m = 1$ by many authors. Recall the operator A is compact if it is continuous and maps bounded sets into relatively compact ones.

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Definition 1. *The fractional (arbitrary) order integral of the function f of order $\alpha > 1$ is defined by (see[8, 11, 6, 7])*

$$I_a^\alpha f(t) = \int_a^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau) d\tau.$$

When $a = 0$, we write $I_a^\alpha f(t) = f(t) * \phi_\alpha(t)$, where $\phi_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$, $t > 0$ and $\phi_\alpha(t) = 0,\ t \leq 0$ and $\phi_\alpha \to \delta(t)$ as $\alpha \to 0$ where $\delta(t)$ is the delta function.

Definition 2. *The fractional (arbitrary) order derivative of the function f of order $\alpha > 1$ is defined by (see[8, 11, 6, 7])*

$$D_a^\alpha f(t) = \frac{d}{dt} \int_a^t \frac{(t-\tau)^{-\alpha}}{\Gamma(1-\alpha)} f(\tau) d\tau = \frac{d}{dt} I_a^{1-\alpha} f(t). \qquad (2)$$

The proof of the existence solution for the equations (1), depends on Schauder fixed point theorem.

Theorem 3. *(see[1, 3]). Let U be a convex subset of Banach space E and $T : U \to U$ is a compact map. Then T has at least one fixed point in U.*

And the proof of uniqueness theorem, will based on the following Banach theorem.

Theorem 4. *(see[12]) Banach fixed point theorem). If X is a Banach space and $T : X \to X$ is a contraction mapping then T has a unique fixed point.*

2 The Existence and Uniqueness Theorems

In order to discuss the conditions for the existence and uniqueness for the solution of equation (1), let us define $\mathcal{B} := C[0,T]$ to be the Banach space endow with the sup norm, the convex set $U := \{u \in C[0,T] : \|u\|^m \leq l,\ l > 0,\ m \geq 1\}$, and the operator

$$Au^m(t) := \frac{a(t)}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} b(\tau) u(\tau) d\tau + f(t),\ t \in [0,T],\ m \geq 1,\ \alpha > 0, \qquad (3)$$

with $\|a\|\|b\| \leq \frac{\Gamma(\alpha+1)}{2T^\alpha}$, $\|f\| < \frac{l}{2}$. Then the properties of A are in the next lemma.

Lemma 5. *The operator A is completely continuous.*

Proof. In order to show that the equation (1) has a solution we have to show that the operator (3) has a fixed point. $u^m \in U$ because $\|u^m\| \leq \|u\|^m \leq l$. For $u \in U$,

$$\begin{aligned} |Au^m(t)| &\leq \frac{|a(t)|}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} |b(\tau)u(\tau)| d\tau + |f(t)| \\ &\leq \frac{\|a\|\|b\|\|u\|}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} d\tau + \|f\| \\ &\leq \frac{\Gamma(\alpha+1)}{2T^\alpha} . \frac{lT^\alpha}{\Gamma(\alpha+1)} + \frac{l}{2} \\ &= \frac{l}{2} + \frac{l}{2} = l, \end{aligned}$$

proving that A maps U to itself. Moreover, $A(U)$ is bounded operator. To prove that A is continuous. Let $u, v \in U$, then we have

$$\begin{aligned} |Au^m(t) - Av^m(t)| &\leq \frac{|a(t)|}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} |b(\tau)||u(\tau) - v(\tau)| d\tau \\ &\leq \frac{\|a\|\|b\|\|u-v\|}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} d\tau \\ &\leq \frac{\Gamma(\alpha+1)}{2T^\alpha} . \frac{2lT^\alpha}{\Gamma(\alpha+1)} = l, \end{aligned}$$

that is A is continuous. Now, we shall prove that A is equicontinuous. Let $u \in U$ and $t_1, t_2 \in [0, T]$. If we denote $C = \|a\|\|b\|\|u\|$, then

$$\begin{aligned} |Au^m(t_1) - Au^m(t_2)| &\leq \frac{C}{\Gamma(\alpha)} | \int_0^{t_1} (t_1-\tau)^{\alpha-1} d\tau - \int_0^{t_2} (t_2-\tau)^{\alpha-1} d\tau | + |f(t_1) - f(t_2)| \\ &\leq \frac{C}{\Gamma(\alpha+1)} |t_1^\alpha - t_2^\alpha| \leq \frac{\Gamma(\alpha+1)}{2T^\alpha} . \frac{l}{\Gamma(\alpha+1)} |t_1^\alpha + t_2^\alpha| + 2\|f\| \\ &\leq \frac{l}{2T^\alpha} . 2T^\alpha + l = 2l. \end{aligned}$$

which is independent of $u(t)$. Thus A is relatively compact. Arzela-Ascoli Theorem, implies that A is completely continuous. □

Then Schauder fixed point theorem gives that A has a fixed point, which corresponding to the solution of equation (1). Then we have the following theorem.

Theorem 6. *Let $a(t), b(t), f(t)$ are real nonnegative functions in $C[0, T]$ and that $t \in [0, T]$, $0 < \alpha < 1$, with $\|a\|\|b\| \leq \frac{\Gamma(\alpha+1)}{2T^\alpha}$, $\|f\| < \frac{l}{2}$. Then equation (1) has a solution u in a convex set U.*

Theorem 7. *Let the assumptions of Theorem 6 be hold. Then the solution of equation (1) is unique.*

Proof. Since for $u, v \in U$, we have

$$
\begin{aligned}
|Au^m(t) - Av^m(t)| &\le \frac{|a(t)|}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} |b(\tau)||u(\tau) - v(\tau)| d\tau \\
&\le \frac{\|a\|\|b\|\|u-v\|}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} d\tau \\
&\le \frac{T^\alpha \|a\|\|b\|}{\Gamma(\alpha+1)} \|u-v\|.
\end{aligned}
$$

But $\frac{T^\alpha \|a\|\|b\|}{\Gamma(\alpha+1)} < \frac{1}{2}$. Thus A is a contraction mapping, then in view of Theorem 4, A has a unique fixed point corresponds to the unique solution of equation (1). □

As an application of Theorem 6 we have the next result.

Theorem 8. *Let $a(t), f(t)$ and φ_i be positive functions in $C[0,T]$, and $h(t, u(t)) : [0,T] \times C[0,T] \to \mathbb{R}^+$ is a continuous function with $\|h_i(t, u(t))\| \le \varphi_i(t)|u(t)|$. Then equation*

$$
u^m(t) = a(t) I^\alpha [\sum_{i=1}^n h_i(t, u(t))] + f(t), \tag{4}
$$

has a solution in U.

Proof. Setting $b(t) := \sum_{i=1}^n \varphi_i(t)$. □

Theorem 9. *Let $h_i : [0,T] \times C[0,T] \to \mathbb{R}^+$ be a continuous function and satisfy Lipschits condition in the second argument*

$$
\|h_i(t,u) - h_i(t,v)\| < L_i \|u - v\|,
$$

where L_i is a constant such that $\frac{\|a\| T^\alpha (\sum_{i=1}^n L_i)}{\Gamma(\alpha+1)} < 1$. Then equation (4) has a unique solution.

Proof. For $u \in U$, define an operator B as follows

$$
Bu^m(t) := \frac{a(t)}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} [\sum_{i=1}^n h_i(\tau, u(\tau))] d\tau + f(t), \ t \in [0,T], \ m \ge 1, \ \alpha > 0, \tag{5}
$$

with $\|a\| \sum_{i=1}^{n} \|\varphi_i\| \leq \frac{\Gamma(\alpha+1)}{2T^\alpha}$. First we show that B has a fixed point. For $u \in U$

$$
\begin{aligned}
|Bu^m(t)| &\leq \frac{|a(t)|}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \sum_{i=1}^{n} |h_i(\tau, u(\tau))| d\tau + |f(t)| \\
&\leq \frac{\|a\| \sum_{i=1}^{n} \|\varphi_i\| \|u\|}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} d\tau + \|f\| \\
&\leq \frac{l\Gamma(\alpha+1)}{2T^\alpha} . \frac{T^\alpha}{\Gamma(\alpha+1)} + \frac{l}{2} \\
&= \frac{l}{2} + \frac{l}{2} = l,
\end{aligned}
$$

proving that B maps U to itself. Moreover, $B(U)$ is bounded operator. To prove that B is continuous. Since h and g are continuous functions in a compact set $[0, T] \times [0, l]$, then they are uniformly continuous there. Thus for $u, v \in U$, and given $\epsilon > 0$, we can find $\mu > 0$ such that $\|h_i(t, u) - h_i(t, v)\| < \frac{\Gamma(\alpha+1)\epsilon}{n\|a\|T^\alpha}$ when $\|u - v\| < \mu$. Then

$$
\begin{aligned}
|Bu^m(t) - Bv^m(t)| &\leq \frac{|a(t)|}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} [\sum_{i=1}^{n} |h_i(\tau, u(\tau)) - h_i(\tau, v(\tau))|] d\tau \\
&\leq \frac{\|a\| [\sum_{i=1}^{n} \|h_i(t, u) - h_i(t, v)\|]}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} d\tau \\
&\leq \frac{n\|a\|T^\alpha}{\Gamma(\alpha+1)} \times \frac{\Gamma(\alpha+1)\epsilon}{n\|a\|T^\alpha} = \epsilon,
\end{aligned}
$$

that is B is continuous. Now, we shall prove that B is equicontinuous. Let $u \in U$ and $t_1, t_2 \in [0, T]$. Then

$$
\begin{aligned}
|Bu^m(t_1) - B^m(t_2)| &\leq \frac{\|a\| \sum_{i=1}^{n} \|\varphi_i\| \|u\|}{\Gamma(\alpha+1)} |t_1^\alpha - t_2^\alpha| + 2\|f\| \\
&\leq \frac{l\Gamma(\alpha+1)}{2T^\alpha \Gamma(\alpha+1)} |t_1^\alpha + t_2^\alpha| + 2\|f\| \\
&\leq \frac{2T^\alpha l}{2T^\alpha} + l = 2l,
\end{aligned}
$$

which is independent of $u(t)$, then B is relatively compact. Arzela-Ascoli Theorem, implies that B is completely continuous. Then, Schauder fixed point theorem (Theorem 3 gives that B has a fixed point. Now we show that B is a contraction mapping. Let $u, v \in U$ the we have

$$\begin{aligned} |Bu^m(t) - Bv^m(t)| &\leq \frac{|a(t)|}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} [\sum_{i=1}^n |h_i(\tau, u(\tau)) - h_i(\tau, v(\tau))|] d\tau \\ &\leq \frac{\|a\| [\sum_{i=1}^n \|h_i(t,u) - h_i(t,v)\|]}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} d\tau \\ &\leq \frac{\|a\| T^\alpha}{\Gamma(\alpha+1)} (\sum_{i=1}^n L_i) \|u - v\|, \end{aligned}$$

then by Theorem 4 we obtain the result. □

3 The Upper and Lower Estimates for Solutions

In this section we discuss the upper and the lower bounds of solutions for equations (1) and (4). Moreover we use the results again to determined the conditions for the uniqueness. Let us illustrate the following assumption:

$$min_{t\in[0,T]} a(t) := a, \;\; min_{t\in[0,T]} b(t) := b, \;\; min_{t\in[0,T]} f(t) := k. \tag{\textbf{H1}}$$

Theorem 10. *Let the assumption (**H1**) be hold. If equation (1) is solvable in $C[0,T]$, then its solution satisfies the inequality*

$$u(t) \geq (\frac{ab}{\Gamma(\alpha)} k^{1/m})^{1/m} t^{(\alpha-1)/m}. \tag{6}$$

Proof. Consequently to the fact that $u^m > f \;\; \Rightarrow f^{1/m} < u$ then $f^{1/m} \in C[0,T]$. According to Definition 1 and assumption (**H1**) we have

$$u^m(t) = \frac{a(t)}{\Gamma(\alpha)} t^{\alpha-1} u(t) b(t) + f(t) \geq \frac{ab}{\Gamma(\alpha)} t^{\alpha-1} u(t) \geq \frac{ab}{\Gamma(\alpha)} t^{\alpha-1} f^{1/m} \geq \frac{ab}{\Gamma(\alpha)} t^{\alpha-1} k^{1/m} \tag{7}$$

then we have the result. Substituting again inequality (6) in (7), we obtain

$$u(t) \geq \{k^{1/m}\}^{1/m} \{(\frac{ab}{\Gamma(\alpha)})^{1/m}\}^{1/m+1} \{t^{(\alpha-1)/m}\}^{1/m+1}.$$

Repeating this operator n-times, we find

$$u(t) \geq \{k^{1/m}\}^{1/m} \{(\frac{ab}{\Gamma(\alpha)})^{1/m}\}^{1/m^n+1/m^{n-1}+\ldots+1/m+1} \{t^{(\alpha-1)/m}\}^{1/m^n+1/m^{n-1}+\ldots+1/m+1}.$$

Taking the limit as $n \to \infty$, we arrive at the inequality (6) which complete the proof. □

Corollary 11. *Under the assumption of Theorem 10, if equation (1) has a solution, then asymptotic behavior of this solution is of the form*

$$u(t) = ct^{\gamma} + O(t^{\alpha}),\ c > 0,\ \gamma \leq \frac{\alpha - 1}{m}.$$

Corollary 12. *Under the assumption of Theorem 10, and that $t, \alpha \to 0$ then $u(t) \geq f^{1/m}$.*
Similarly for equation (4).

Theorem 13. *Denotes by*

$$min_{(t,u)\in[0,T]\times C[0,T]} h_i(t, u(t)) := \overline{h}_i.$$

If equation (4) is solvable then its solution satisfies

$$u(t) \geq \{\frac{a}{\Gamma(\alpha)}(\sum_{i=1}^{n} \overline{h}_i)\}^{1/m} t^{(\alpha-1)/m}.$$

Now we discuss the upper bounds for solution of equations (1) and (4).

Theorem 14. *If equation (1) is solvable in U, then its solution satisfies*

$$u(t) \leq (\frac{\|a\|\|b\|l}{\Gamma(\alpha+1)} T^{\alpha} + \|f\|)^{1/m}.$$

Proof.

$$\begin{aligned}
|u^m(t)| &\leq \frac{|a(t)|}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} |b(\tau)u(\tau)| d\tau + |f(t)| \\
&\leq \frac{\|a\|\|b\|\|u\|}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} d\tau + \|f\| \\
&\leq \frac{\|a\|\|b\|l}{\Gamma(\alpha+1)} t^{\alpha} + \|f\| \\
&\leq \frac{\|a\|\|b\|l}{\Gamma(\alpha+1)} T^{\alpha} + \|f\|,
\end{aligned}$$

then we obtain the result. □

Theorem 15. *If equation (4) is solvable in $C[0,T]$ then its solution satisfies*

$$u(t) \leq (\frac{\|a\| l \sum_{i=1}^{n} \|\varphi_i\|}{\Gamma(\alpha+1)} T^{\alpha} + \|f\|)^{1/m}.$$

Now, we discuss the uniqueness for solution of equations (1) and (4) using Theorem 10. For this purpose, we illustrate the following assumption:

$$\text{Assume } m>1. \text{ Denote } N:=\frac{ab}{\Gamma(\alpha)}k^{\frac{1}{m-1}} \ \textit{and} \ M:=\|a\|\|b\|t<mN\Gamma(\alpha+1). \quad (\mathbf{H2})$$

Theorem 16. *Let assumption* (**H2**) *be hold with* $a(t), b(t) \in C[0,T]$. *If equation* (1) *is solvable then its solution is unique in* $C[0,T]$.

Proof. Let u, v be two solutions for equation (1) in $C[0,T]$. Since $m>1$ then by mean value Theorem

$$|u^m(t)-v^m(t)| \ge m|u(t)-v(t)|(min(u,v))^{m-1}.$$

On the other hand by equation (1) we have

$$|u^m(t)-v^m(t)| \le \frac{a(t)}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}b(\tau)|u(\tau)-v(\tau)|d\tau.$$

According to Theorem 10, and assumption (**H2**), we have

$$|u^m(t)-v^m(t)| \ge mNt^{\alpha-1}|u(t)-v(t)|, \ \ \textit{then}$$

$$mNt^{\alpha-1}|u(t)-v(t)| \le \frac{a(t)}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}b(\tau)|u(\tau)-v(\tau)|d\tau. \tag{8}$$

Let us denote $x(t)$ by

$$x(t) := b(t)|u(t)-v(t)| \tag{9}$$

then the inequality (8) can be written as

$$x(t) \le \frac{a(t)b(t)}{mN\Gamma(\alpha)}t^{1-\alpha}\int_0^t (t-\tau)^{\alpha-1}x(\tau)d\tau. \tag{10}$$

Let $t_0 \in [0,T]$ and x_0 be the max. point of $x(t)$ in $[0,T] : x(t_0) = max_{0\le t\le t_0}x(t)$. Then

$$\int_0^t (t-\tau)^{\alpha-1}x(\tau)d\tau \le \int_0^t (t-\tau)^{\alpha-1}d\tau.x(t_0) = \frac{t^\alpha}{\alpha}x_0. \tag{11}$$

Substituting into (10) we obtain

$$x(t) \le \frac{a(t)b(t)x_0}{m\Gamma(\alpha+1)}t.$$

Again by assumption (**H2**) we have

$$x_0 \le \frac{Mx_0}{mN\Gamma(\alpha+1)}$$

since $M < mN\Gamma(\alpha+1)$ then $x_0 = 0$ at the max point on arbitrary interval $[0,t_0]$, then $x(t) \equiv 0, \ \forall t \in [0,T]$ which leads to $u(t) = v(t)$, which complete the proof. □

Similarly for equation (4), with the assumption

$$\text{Assume } m > 1. \text{ Denote } \overline{N} := \frac{a}{\Gamma(\alpha)}(\sum_{i=1}^{n} \overline{h}_i) \text{ and } \overline{M} := \|a\| \sum_{i=1}^{n} \|h_i\| t < m\overline{N}\Gamma(\alpha+1), \tag{H3}$$

we can prove the following theorem.

Theorem 17. *Let assumption* (**H3**) *be hold. If equation* (4) *is solvable then its solution is unique in* $C[0, T]$.

Definition 18. *Let* M *be a solution of the equation* (1) *then* M *is said to be a maximal solution of* (1), *if for every solution* u *of* (1) *existing on* $[0, T]$, *the inequality* $u(t) \leq M(t), t \in [0, T]$ *holds. A minimal solution may be define similarly by reversing the last inequality.*

In order to discuss the maximal and the minimal solution of equation (1) and (4), we study the maximal and the minimal solution of equation

$$u^m(t) = a(t)I^{\alpha}[h(t, u(t))] + f(t). \tag{12}$$

We need to the following assumption:

(**H4**)

1. $f(t) \geq 0, \forall t \in [0, T]$.
2. h is continuous nondecreasing function in the first argument $t \in [0, T]$.
3. There exist two positive constants μ, γ with $\mu < \gamma$ such that

$$\frac{\mu}{min_{t\in[o,T]} f(t) + \frac{||a||T^{\alpha}}{\Gamma(\alpha+1)} h(t, \mu)} < \frac{\gamma}{||f|| + \frac{||a||T^{\alpha}}{\Gamma(\alpha+1)} h(t, \gamma)},$$

Theorem 19. *Let assumption* (**H4**) *be hold. Then there exists a maximal and minimal solution of the integral equation* (12) *on* $[0, T]$.

Proof. Consider the fractional order integral equation

$$u^m(t) = \epsilon + a(t)I^{\alpha}[h(t, u(t))] + f(t). \tag{13}$$

Then for some positive constants μ, ν

$$\frac{\mu}{\epsilon + min_{t\in[o,T]} f(t) + \frac{||a||T^{\alpha}}{\Gamma(\alpha+1)} h(t, \mu)} < \frac{\gamma}{\epsilon + ||f|| + \frac{||a||T^{\alpha}}{\Gamma(\alpha+1)} h(t, \gamma)}.$$

Now, let $0 < \epsilon_2 < \epsilon_1 \leq \epsilon$. Then we have $u_{\epsilon_2}(0) < u_{\epsilon_1}(0)$. Thus we can prove that

$$u^m_{\epsilon_2}(t) < u^m_{\epsilon_1}(t), \;\; or \;\; u_{\epsilon_2}(t) < u_{\epsilon_1}(t), \;\; \forall t \in [0, T]. \tag{14}$$

Assume that it is false. Then there exist a t_1 such that

$$u^m_{\epsilon_2}(t_1) = u^m_{\epsilon_1}(t_1) \Rightarrow u_{\epsilon_2}(t_1) = u_{\epsilon_1}(t_1) \ \ and \ \ u_{\epsilon_2}(t) < u_{\epsilon_1}(t) \ \ \forall\, t \in [0, t_1).$$

Since f is monotonic non-decreasing in u, it follows that $h(t, u_{\epsilon_2}(t)) \le h(t, u_{\epsilon_1}(t))$. Consequently, using equation (13), we get

$$\begin{aligned} u^m_{\epsilon_2}(t_1) &= \epsilon_2 + a(t_1)I^\alpha[h(t_1, u_{\epsilon_2}(t_1))] + f(t_1) \\ &< \epsilon_1 + a(t_1)I^\alpha[h(t_1, u_{\epsilon_1}(t_1))] + f(t_1) \\ &= u^m_{\epsilon_1}(t_1). \end{aligned}$$

Which contradict the fact that $u_{\epsilon_2}(t_1) = u_{\epsilon_1}(t_1)$. Hence the inequality (14) is true. That is, there exist a decreasing sequence ϵ_n such that $\epsilon_n \to 0$ as $n \to \infty$ and $lim_{n\to\infty} u_{\epsilon_n}(t)$ exist uniformly in $[0, T]$. We denote this limiting value by $M(t)$. Obviously, the uniform continuity of h then the equation

$$u^m_{\epsilon_n}(t) = \epsilon_n + a(t_1)I^\alpha[h(t, u_{\epsilon_n}(t))] + f(t),$$

yield that M is a solution of equation (12). To show that M is a maximal solution of equation (12), let u be any solution of equation (12). Then

$$u^m(t) < \epsilon + a(t)I^\alpha[h(t, u(t))] + f(t) = u^m_\epsilon(t).$$

Since the maximal solution is unique (see [9]), it is clear that $u_\epsilon(t)$ tend to $M(t)$ uniformly in [0,T] as $\epsilon \to 0$. Which proves the existence of maximal solution to the equation (12). A similar argument holds for the minimal solution. □

Example 20. *For the integral equation*

$$u^2(t) = tI^{1/2}\frac{t^2}{8}u(t) + \frac{1}{16}t, \ \ J := [0, 1], l = \frac{1}{2} \tag{15}$$

with $\|a\|\|b\| \le \frac{1}{8} < \frac{\Gamma(3/2)}{2} = \frac{\sqrt{\pi}}{4} = 0.443$ *and* $\|f\| < \frac{1}{16}$. *Then in view of Theorem 6, equation (15) has a solution which is unique in* $U := \{u \in C[0, 1] : \|u\|^2 \le \frac{1}{2}\}$.

Example 21. *For the integral equation*

$$u^3(t) = tI^{1/2}\frac{\sqrt{t}}{10}u(t) + \frac{\cos t}{4}, \ \ J := [0, 1], l = 1 \tag{16}$$

with $\|a\|\|b\| \le \frac{1}{10} < \frac{\Gamma(3/2)}{2} = \frac{\sqrt{\pi}}{4} = 0.443$ *and* $\|f\| < \frac{1}{4} < \frac{1}{2}$. *Then in view of Theorem 6, equation (16) has a solution which is unique in* $U := \{u \in C[0, 1] : \|u\|^3 \le 1\}$.

References

[1] K. Balachandran and J. P. Dauer, *Elements of Control Theory*, New Delhi, Narosa Publishing House, 1999. Zbl 0965.93002

[2] P. Butzer and L. Westphal, *An introduction to fractional calculus*. Hilfer, R. (ed.), Applications of fractional calculus in physics. Singapore: World Scientific. (2000), 1-85. MR1890105(2003g:26007). Zbl 0987.26005

[3] K. Deimling, *Nonlinear Functional Analysis*, Berlin, Springer-Verlag, 1985. MR0787404(86j:47001). Zbl 0559.47040.

[4] R. Gorenflo and S. Vessella, *Abel integral equations*. Analysis and applications. Lecture Notes in Mathematics, **1461** Springer-Verlag, Berlin, 1991. MR1095269(92e:45003). Zbl 0717.45002

[5] E. Hille and J. Tamarkin, *On the theory of linear integral equations*, Ann. of Math. (2) **31** (1930), 479-528. MR1502959. JFM 56.0337.01.

[6] V. Kiryakova, *Generalized Fractional Calculus and Applications*, Pitman Research Notes in Mathematics Series, **301**. Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, Inc., New York, 1994. MR1265940(95d:26010). Zbl 0882.26003.

[7] K. S. Miller and B. Ross, *An introduction to the fractional calculus and fractional differential equations*, John Wiley & Sons, Inc., 1993. MR1219954(94e:26013). Zbl 0789.26002.

[8] I. Podlubny, *Fractional differential equations. An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*. Mathematics in Science and Engineering, **198**. Academic Press, 1999. MR1658022(99m:26009). Zbl 0924.34008.

[9] M. R. Rao, *Ordinary Differential Equations. Theory and applications*, New Delhi-Madras: Affiliated East-West Press, 1980. Zbl 0482.34001.

[10] B. Ross and B. K. Sachdeva, *The solution of certain integral equations by means of operators of arbitrary order*, Amer. Math. Monthly **97** (1990), 498-502. MR1055906(91e:45005). Zbl 0723.45002.

[11] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional Integrals and Derivatives. Theory and Applications*. Gorden and Breach, New York, 1993. MR1347689(96d:26012).

[12] D. R. Smart, *Fixed Point Theorems*, Cambridge University Press, 1980. Zbl 0427.47036.

[13] H. M. Srivastava and R. G. Buschman, *Theory and Applications of Convolutions Integral Equations*, Kluwer Acad., Dordrecht,1992. MR1205580(94a:45002). Zbl 0755.45002.

Rabha W. Ibrahim
P.O. Box 14526, Sana'a,
Yemen.
e-mail: rabhaibrahim@yahoo.com

Shaher Momani
Department of Mathematics, Mutah University,
P.O. Box 7, Al-Karak, Jordan.
e-mail: shahermm@yahoo.com