Surveys in Mathematics and its Applications

ISSN 1842-6298 (electronic), 1843-7265 (print)
Volume 7 (2012), 125 – 136

STUDY ABOUT INCLUSION RELATIONSHIPS AND INTEGRAL PRESERVING PROPERTIES

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Abstract. The object of the present paper is to investigate a family of integral operators defined on the space of meromorphic functions. By making use of these novel integral operators, we introduce and investigate several new subclasses of starlike, convex, close-to-convex, and quasi-convex meromorphic functions. In particular, we establish some inclusion relationships associated with the aforementioned integral operators. Several interesting integral-preserving properties are also considered.

1 Introduction and preliminaries

Let $\sum$ denote the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the punctured unit disk $\mathbb{U}^* = \{z : 0 < |z| < 1\} = \mathbb{U} \setminus \{0\}$ where $\mathbb{U} = \{z : |z| < 1\}$ is the open unit disk.

Next we define some well known subclasses of meromorphic functions.

A function $f \in \sum$ is said to belong to the class of meromorphic starlike functions of order ξ in $\mathbb{U}$, and is denoted by $S^*(\xi)$, if it satisfies

$$\Re\left(-\frac{zf'(z)}{f(z)} \right) > \xi, \quad 0 \le \xi < 1, z \in \mathbb{U}\cdot \tag{1.2}$$

A function $f \in \sum$ is said to belong to the class of meromorphic convex functions of order ξ in $\mathbb{U}$, and is denoted by $C(\xi)$, if it satisfies

$$\Re\left(-1 - \frac{zf''(z)}{f'(z)} \right) > \xi, \quad 0 \le \xi < 1, z \in \mathbb{U}\cdot \tag{1.3}$$

2010 Mathematics Subject Classification: 30C45.
Keywords: Meromorphic functions; Meromorphic starlike functions; Meromorphic convex functions; Integral operators; Integral-preserving properties.

It is easy to observe from (1.2) and (1.3) that

$$f \in C(\xi) \Leftrightarrow -zf'(z) \in S^*(\xi)\cdot$$

Let $f \in \sum$ and $g(z) \in S^*(\xi)$. Then the function $f(z) \in \sum$ is said to belong to the class of close-to-convex of order ρ and type ξ in $\mathbb{U}$, and is denoted by $K(\rho,\xi)$, if and only if

$$\Re\left(\frac{-zf'(z)}{g(z)}\right) > \rho, \quad (0 \le \xi, \rho < 1; z \in \mathbb{U})\cdot \tag{1.4}$$

A function $f \in \sum$ is said to belong to the class of quasi-convex of order ρ and type ξ in $\mathbb{U}$, and is denoted by $K^*(\rho,\xi)$, if there exists a function $g(z) \in C(\xi)$ such that

$$\Re\left(\frac{-(zf'(z))'}{g(z)}\right) > \rho, \quad (0 \le \xi, \rho < 1; z \in \mathbb{U})\cdot \tag{1.5}$$

We have the following result followed by (1.4) and (1.5) such that

$f(z) \in K^*(\rho,\xi)$ if and only if $-zf'(z) \in K(\rho,\xi)\cdot$

For more details about above-defined classes of functions as well as their multivalent generalizations, we refer to study to [5]-[1].

Next we introduce a new differential operator for the meromorphic functions.

For a function $f \in \sum$, we define the following differential operator:

$$\begin{aligned} \Theta^0 f(z) &= f(z), \\ (\alpha+\beta)\Theta^1_\lambda(\alpha,\beta,\mu)f(z) &= (\mu+\alpha+\lambda+\beta)f(z) + (\mu+\lambda)(zf'(z)), \\ \Theta^2_\lambda(\alpha,\beta,\mu)f(z) &= D(\Theta^1_\lambda(\alpha,\beta,\mu)f(z)), \\ &\vdots \\ \Theta^n_\lambda(\alpha,\beta,\mu)f(z) &= D(\Theta^{n-1}_\lambda(\alpha,\beta,\mu)f(z))\cdot \end{aligned} \tag{1.6}$$

If f is given by (1.1) then from (1.6) we have

$$\Theta^n_\lambda(\alpha,\beta,\mu)f(z) = \frac{1}{z} + \sum_{k=0}^{\infty}\left(\frac{\alpha+(\mu+\lambda)(k+1)+\beta}{\alpha+\beta}\right)^n a_k z^k \tag{1.7}$$

$$(f \in \Sigma, -1 \le \alpha \le 1, -1 \le \beta \le 1, -1 \le \mu \le 1, \lambda \ge 0, \alpha+\beta \ne 0, n \in \mathbb{N}_0)$$

Also by specializing the parameters α, β, μ and λ, It generalizes the differential operators of Cho et al. [3, 4], Aouf and Hossen [2], Liu and Owa [9], Liu and

Srivastava [10], Srivastava and Patel [13], Uralegaddi and Somanatha [17] and Ashwah and Aouf [6] respectively.

We now define the following subclasses of the meromorphic functions by means of the operator $\Theta_\lambda^n(\alpha,\beta,\mu)f(z)$ given by (1.7).

Definition 1. *Using (1.2) and (1.7)*

$$S_n^*\left(\xi,\lambda,\mu,\alpha,\beta\right)=\left\{f\in\Sigma:\Theta_\lambda^n(\alpha,\beta,\mu)f\in S^*(\xi)\right\}\cdot \tag{1.8}$$

Definition 2. *Using (1.3) and (1.7)*

$$C_n\left(\xi,\lambda,\mu,\alpha,\beta\right)=\left\{f\in\Sigma:\Theta_\lambda^n(\alpha,\beta,\mu)f\in C(\xi)\right\}\cdot \tag{1.9}$$

Definition 3. *Using (1.4) and (1.7)*

$$K_n\left(\rho,\xi,\lambda,\mu,\alpha,\beta\right)=\left\{f\in\Sigma:\Theta_\lambda^n(\alpha,\beta,\mu)f\in K(\rho,\xi)\right\}\cdot \tag{1.10}$$

Definition 4. *Using (1.5) and (1.7)*

$$K_n^*\left(\rho,\xi,\lambda,\mu,\alpha,\beta\right)=\left\{f\in\Sigma:\Theta_\lambda^n(\alpha,\beta,\mu)f\in K^*(\rho,\xi)\right\}\cdot \tag{1.11}$$

Next we investigate various inclusion relationships and integral preserving properties for the meromorphic function classes newly introduced above.

2 The main inclusion relationships

First we state the following lemma which we need for our main results.

Lemma 5. *[11, 12] Let $\varphi(\mu,\nu)$ be a complex function, $\phi:D\to\mathbb{C}$, $D\subset\mathbb{C}\times\mathbb{C}$, and let $\mu=\mu_1+i\mu_1$, $\nu=\nu_1+i\nu_1$. Suppose that $\varphi(\mu,\nu)$ satisfies the following conditions:*

(i) $\varphi(\mu,\nu)$ is continuous in D;

(ii) $(1,0)\in D$ and $\Re\{\varphi(1,0)\}>0$;

(iii) $\Re\{\varphi(i\mu_2,\nu_1)\}\leq 0$ for all $(i\mu_2,\nu_1)\in D$ such that $\nu_1\leq -\frac{1}{2}(1+\mu_2^2)\cdot$

Let $h(z)=1+c_1z+c_2z^2+\cdots$ be analytic in $\mathbb{U}$, such that $(h(z),zh'(z))\in D$ for all $z\in\mathbb{U}$. If $\Re\{\varphi(h(z),zh'(z))\}>0 (z\in\mathbb{U})$, then $\Re\{h(z)\}>0$ for $z\in\mathbb{U}\cdot$

Theorem 6. *If* $f \in \sum$ *and* $-1 \leq \alpha \leq 1, -1 \leq \beta \leq 1, -1 \leq \mu \leq 1, \lambda \geq 0, \alpha+\beta \neq 0, n \in \mathbb{N}_0$. *Then*

$$S^*_{n+1}\Big(\xi,\lambda,\mu,\alpha,\beta\Big) \subseteq S^*_n\Big(\xi,\lambda,\mu,\alpha,\beta\Big) \subseteq S^*_{n-1}\Big(\xi,\lambda,\mu,\alpha,\beta\Big)\cdot \tag{2.1}$$

Proof. Let $f(z) \in S^*_{n+1}\Big(\xi,\lambda,\mu,\alpha,\beta\Big)$, to prove that $f(z) \in S^*_n\Big(\xi,\lambda,\mu,\alpha,\beta\Big)$, it is enough to show that

$$\Re\Big(\frac{z(\Theta^n_\lambda(\alpha,\beta,\mu)f(z))'}{\Theta^n_\lambda(\alpha,\beta,\mu)f(z)}\Big) > -\xi,\ 0 \leq \xi < 1, z \in \mathbb{U}\cdot$$

We assume that

$$\frac{z(D^n_\lambda(\alpha,\beta,\mu)f(z))'}{D^n_\lambda(\alpha,\beta,\mu)f(z)} = -\xi-(1-\xi)h(z),\ 0 \leq \xi < 1, z \in \mathbb{U}\cdot \tag{2.2}$$

Where $h(z) = 1 + c_1 z + c_2 z^2 + \cdots$. Applying simultaneously (1.7) and (2.2) we conclude that

$$\frac{\Theta^{n+1}_\lambda(\alpha,\beta,\mu)f(z)}{\Theta^n_\lambda(\alpha,\beta,\mu)f(z)} = L(-\xi-(1-\xi)h(z)) + L(\frac{1}{L}+1),\ 0 \leq \xi < 1, z \in \mathbb{U}\cdot \tag{2.3}$$

Differentiating (2.3) with respect to z logarithmically, we obtain

$$\frac{z(\Theta^{n+1}_\lambda(\alpha,\beta,\mu)f(z))'}{\Theta^{n+1}_\lambda(\alpha,\beta,\mu)f(z)} + \xi = -(1-\xi)h(z) + \frac{(1-\xi)zh'(z)}{\xi-(\frac{1}{L}+1)+(1-\xi)h(z)}, \tag{2.4}$$

$$L = \frac{\mu+\nu}{\alpha+\beta}, 0 \leq \xi < 1, z \in \mathbb{U}\cdot$$

Taking $h(z) = \mu = \mu_1 + i\mu_2$ and $zh'(z) = \nu = \nu_1 + i\nu_2$, we define the function $\varphi(\mu,\nu)$ by:

$$\varphi(\mu,\nu) = (1-\xi)\mu - \frac{(1-\xi)\nu}{\xi-(\frac{1}{L}+1)+(1-\xi)\mu}\cdot \tag{2.5}$$

Then from (2.5) we have

(i) $\varphi(\mu,\nu)$ is continuous in $D = (\mathbb{C} - \frac{\xi-(\frac{1}{L}+1)}{\xi-1}) \times \mathbb{C}$;

(ii) $(1,0) \in D$ and $\Re\{\varphi(1,0)\} > 1-\xi > 0$;

(iii) For all $(i\mu_2,\nu_1) \in D$ such that $\nu_1 \leq -\frac{1}{2}(1+\mu_2^2)$,

$$\Re\{\varphi(i\mu_2,\nu_1)\} = \Re\{\frac{-(1-\xi)\nu_1}{\xi-(\frac{1}{L}+1)+(1-\xi)i\mu_2}\} = \frac{[(\frac{1}{L}+1-\xi)](1-\xi)\nu_1}{(\xi-(\frac{1}{L}+1))^2+(1-\xi)^2\mu_2^2},$$

implies

$$\Re\{\varphi(i\mu_2,\nu_1)\} = \frac{[(\frac{1}{L}+1-\xi)](1-\xi)\nu_1}{(\xi-(\frac{1}{L}+1))^2+(1-\xi)^2\mu_2^2} \leq -\frac{[(\frac{1}{L}+1-\xi)](1-\xi)(1+\mu_2^2)}{2(\xi-(\frac{1}{L}+1))^2+2(1-\xi)^2\mu_2^2} < 0\cdot$$

Hence, the function $\varphi(\mu,\nu)$ satisfies the conditions of Lemma 5, implies $\Re\{h(z)\} > 0 (z \in \mathbb{U})$, that is, $f(z) \in S_n^*\Big(\xi,\lambda,\mu,\alpha,\beta\Big)\cdot$ This completes the proof of Theorem 6. □

Theorem 7. *If* $f \in \sum$ *and* $-1 \leq \alpha \leq 1, -1 \leq \beta \leq 1, -1 \leq \mu \leq 1, \lambda \geq 0, \alpha+\beta \neq 0, n \in \mathbb{N}_0$. *Then*

$$C_{n+1}\Big(\xi,\lambda,\mu,\alpha,\beta\Big) \subseteq C_n\Big(\xi,\lambda,\mu,\alpha,\beta\Big) \subseteq C_{n-1}\Big(\xi,\lambda,\mu,\alpha,\beta\Big)\cdot \tag{2.6}$$

Proof. Let

$$f \in C_{n+1}\Big(\xi,\lambda,\mu,\alpha,\beta\Big) \Rightarrow \Theta_\lambda^{n+1}(\alpha,\beta,\mu)f \in C(\xi),$$

$$\Leftrightarrow -z(\Theta_\lambda^{n+1}(\alpha,\beta,\mu)f)' \in S^*(\xi) \Leftrightarrow \Theta_\lambda^{n+1}(\alpha,\beta,\mu)(-zf') \in S^*(\xi),$$

$$\Leftrightarrow -zf' \in S_{n+1}^*\Big(\xi,\lambda,\mu,\alpha,\beta\Big) \subset S_n^*\Big(\xi,\lambda,\mu,\alpha,\beta\Big),$$

$$\Rightarrow -zf' \in S_n^*\Big(\xi,\lambda,\mu,\alpha,\beta\Big) \Leftrightarrow \Theta_\lambda^n(\alpha,\beta,\mu)(-zf') \in S^*(\xi),$$

$$\Leftrightarrow -z(\Theta_\lambda^n(\alpha,\beta,\mu)f))' \in S^*(\xi) \Rightarrow \Theta_\lambda^n(\alpha,\beta,\mu)f \in C(\xi),$$

$$\Rightarrow f \in C_n(\xi,\lambda,\alpha,\beta,\mu) \Rightarrow C_{n+1}(\xi,\lambda,\alpha,\beta,\mu) \subset C_n(\xi,\lambda,\alpha,\beta,\mu)\cdot$$

. □

Theorem 8. *If* $f \in \sum$ *and* $-1 \leq \alpha \leq 1, -1 \leq \beta \leq 1, -1 \leq \mu \leq 1, \lambda \geq 0, \alpha+\beta \neq 0, n \in \mathbb{N}_0$. *Then*

$$K_{n+1}\Big(\rho,\xi,\lambda,\mu,\alpha,\beta\Big) \subseteq K_n\Big(\rho,\xi,\lambda,\mu,\alpha,\beta\Big) \subseteq K_{n-1}\Big(\rho,\xi,\lambda,\mu,\alpha,\beta\Big)\cdot \tag{2.7}$$

Proof. Let $f(z) \in K_{n+1}(\rho, \xi, \lambda, \alpha, \beta, \mu)$, so by definition there exist $k(z) \in S^*(\xi)$ such that

$$\Re(\frac{z(\Theta_\lambda^{n+1}(\alpha,\beta,\mu)f(z))'}{k(z)}) > -\rho,\ 0 \le \rho < 1, z \in \mathbb{U}\cdot$$

Taking the function $g(z)$ which satisfies $\Theta_\lambda^{n+1}(\alpha,\beta,\mu)g(z) = k(z)$, implies $g(z) \in S^*_{n+1}(\xi)$ and

$$\Re\left(\frac{z(\Theta_\lambda^{n+1}(\alpha,\beta,\mu)f(z))'}{\Theta_\lambda^{n+1}(\alpha,\beta,\mu)g(z)}\right) > -\rho,\ 0 \le \rho < 1, z \in \mathbb{U}\cdot$$

We assume that

$$\left(\frac{z(\Theta_\lambda^{n}(\alpha,\beta,\mu)f(z))'}{\Theta_\lambda^{n}(\alpha,\beta,\mu)g(z)}\right) = -\rho - (1-\rho)h(z),\ 0 \le \rho < 1, z \in \mathbb{U}\cdot$$

Where $h(z) = 1 + c_1 z + c_2 z^2 + \cdots$. Using (1.7), we have

$$L\Big(\Theta_\lambda^{n+1}(\alpha,\beta,\mu)zf'(z)\Big) = z\Big(\Theta_\lambda^{n}(\alpha,\beta,\mu)zf'(z)\Big)' + (1+L)\Big(\Theta_\lambda^{n}(\alpha,\beta,\mu)zf'(z)\Big) \quad (2.8)$$

And

$$L\Big(\Theta_\lambda^{n+1}(\alpha,\beta,\mu)g(z)\Big) = z\Big(\Theta_\lambda^{n}(\alpha,\beta,\mu)g(z)\Big)' + (1+L)\Big(\Theta_\lambda^{n}(\alpha,\beta,\mu)g(z)\Big)\cdot \quad (2.9)$$

Using (2.8) and (2.9) we conclude that

$$\frac{z(\Theta_\lambda^{n+1}(\alpha,\beta,\mu)f(z))'}{\Theta_\lambda^{n+1}(\alpha,\beta,\mu)g(z)} = \frac{z\Big(\Theta_\lambda^{n}(\alpha,\beta,\mu)zf'(z)\Big)' + (1+L)\Big(\Theta_\lambda^{n}(\alpha,\beta,\mu)zf'(z)\Big)}{z\Big(\Theta_\lambda^{n}(\alpha,\beta,\mu)g(z)\Big)' + (1+L)\Big(\Theta_\lambda^{n}(\alpha,\beta,\mu)g(z)\Big)} \quad (2.10)$$

Since $g(z) \in S^*_{n+1}(\xi) \subset S^*_n(\xi)$, so let

$$\frac{z(\Theta_\lambda^{n}(\alpha,\beta,\mu)g(z))'}{\Theta_\lambda^{n}(\alpha,\beta,\mu)g(z)} = -\xi - (1-\xi)H(z)\cdot$$

So from (2.10) we have

$$\frac{z(\Theta_\lambda^{n+1}(\alpha,\beta,\mu)f(z))'}{\Theta_\lambda^{n+1}(\alpha,\beta,\mu)g(z)} = \frac{\frac{z(\Theta_\lambda^{n}(\alpha,\beta,\mu)zf'(z))'}{\Theta_\lambda^{n}(\alpha,\beta,\mu)g(z)} + (1+L)(-\rho-(1-\rho)h(z))}{-\xi-(1-\xi)H(z)+(1+L)}\cdot \quad (2.11)$$

As

$$(\Theta_{\lambda}^{n}(\alpha,\beta,\mu)zf'(z)) = (\Theta_{\lambda}^{n}(\alpha,\beta,\mu)g(z))[-\rho-(1-\rho)h(z)],\ 0\le\rho<1, z\in\mathbb{U}\cdot$$

After differentiating with respect to z and some calculation we get

$$\frac{z((\Theta_{\lambda}^{n}(\alpha,\beta,\mu)zf'(z)))}{(\Theta_{\lambda}^{n}(\alpha,\beta,\mu)g(z))} = [-\rho-(1-\rho)h(z)][-\xi-(1-\xi)H(z)]-(1-\rho)zh'(z) \tag{2.12}$$

After substituting the result of (2.12) in (2.11) we have

$$\frac{z(\Theta_{\lambda}^{n+1}(\alpha,\beta,\mu)f(z))'}{\Theta_{\lambda}^{n+1}(\alpha,\beta,\mu)g(z)}+\rho = -(1-\rho)h(z)+\frac{(1-\rho)zh'(z)}{\xi-1-L+(1-\xi)H(z)}\cdot \tag{2.13}$$

Taking $h(z)=\mu=\mu_1+i\mu_2$ and $zh'(z)=\nu=\nu_1+i\nu_2$, we define the function $\varphi(\mu,\nu)$ by:

$$\varphi(\mu,\nu) = (1-\rho)\mu-\frac{(1-\rho)\nu}{\xi-1-L+(1-\xi)H(z)}\cdot \tag{2.14}$$

It is easy to see that the function $\varphi(\mu,\nu)$ satisfies the conditions (i) and (ii) of Lemma 2.1 in $D=\mathbb{C}\times\mathbb{C}$. To verify the condition (iii), we proceed as follows;

$$\Re\{\varphi(i\mu_2,\nu_1)\} = \frac{\nu_1(1-\rho)[(L+1-\xi)-(1-\xi)h_1(x_1,y_1)]}{[(\xi+(1-\xi)h_1(x_1,y_1)-L-1]^2+[(1-\xi)h_2(x_2,y_2)]^2}\cdot$$

Where $H(z)=h_1(x_1,y_1)+ih_2(x_2,y_2)$, $h_1(x_1,y_1)$ and $h_2(x_2,y_2)$ being functions of x and y and $\Re(h_1(x_1,y_1))>0$. By putting $\nu_1\le-\frac{1}{2}(1+\mu_2^2)$, we obtain

$$\Re\{\varphi(i\mu_2,\nu_1)\} = -\frac{1}{2}\frac{(1+\mu_2^2)(1-\rho)[(L+1-\xi)-(1-\xi)h_1(x_1,y_1)]}{[(\xi+(1-\xi)h_1(x_1,y_1)-L-1]^2+[(1-\xi)h_2(x_2,y_2)]^2}<0\cdot$$

Hence, the function $\varphi(\mu,\nu)$ satisfies the conditions of Lemma 5, Implies $\Re\{h(z)\}>0(z\in\mathbb{U})$, that is, $f(z)\in K_n(\rho,\xi,\lambda,\alpha,\beta,\mu)\cdot$ This completes the proof of Theorem 8. Similarly we can prove the following theorem.. □

Theorem 9. *Let* $f\in\Sigma$, $0\le\xi<1$, $\alpha,\beta,\mu,\lambda\ge0,\mu+\lambda\ne0,\alpha+\beta\ne0,n\in N_o$, $L=(\alpha+\beta)/(\mu+\lambda)$ *then*

$$K_{n+1}^{*}(\rho,\xi,\lambda,\mu,\alpha,\beta)\subset K_{n}^{*}(\rho,\xi,\lambda,\mu,\alpha,\beta)\subset K_{n-1}^{*}(\rho,\xi,\lambda,\mu,\alpha,\beta).$$

3 A set of integral-preserving properties

In this section, we present several integral-preserving properties of the meromorphic function classes introduced here. We first recall a familiar integral operator $L_c(f)$ defined by

$$L_c(f) = \frac{c}{z^{c+1}} \int_0^z t^c f(t)dt, \;\; (c > 0; f \in \Sigma), \tag{3.1}$$

which satisfies the following relationship:

$$z(\Theta_\lambda^n(\alpha,\beta,\mu)L_cf(z))' = (c)(\Theta_\lambda^n(\alpha,\beta,\mu)f(z)) - (c+1)(\Theta_\lambda^n(\alpha,\beta,\mu)L_cf(z))\cdot \tag{3.2}$$

Theorem 10. *Let $c > 0$, $f \in \Sigma$, $0 \leq \xi < 1$, $\alpha,\beta,\mu,\lambda \geq 0, \mu+\lambda \neq 0, \alpha+\beta \neq 0$, and $n \in N_o$. If $f(z) \in S_n^*(\xi,\lambda,\mu,\alpha,\beta)$, then $L_c(f) \in S_n^*(\xi,\lambda,\mu,\alpha,\beta)\cdot$*

Proof. Let $f(z) \in S_n^*(\xi,\lambda,\mu,\alpha,\beta)$. To prove that $L_c(f) \in S_n^*(\xi,\lambda,\alpha,\beta,\mu)\cdot$ It is enough to show

$$\frac{z(\Theta_\lambda^n(\alpha,\beta,\mu)L_cf(z))'}{\Theta_\lambda^n(\alpha,\beta,\mu)L_cf(z)} > -\xi\cdot$$

We suppose that

$$\frac{z(\Theta_\lambda^n(\alpha,\beta,\mu)L_cf(z))'}{\Theta_\lambda^n(\alpha,\beta,\mu)L_cf(z)} = -\xi - (1-\xi)h(z)\cdot \tag{3.3}$$

Where $h(z) = 1 + c_1z + c_2z^2 + \cdots$.

Using (3.2) and (3.3) we conclude that

$$\frac{z(\Theta_\lambda^n(\alpha,\beta,\mu)f(z))'}{\Theta_\lambda^n(\alpha,\beta,\mu)f(z)} + \xi = -(1-\xi)h(z) + \frac{(1-\xi)zh'(z)}{\xi - c - 1 + (1-\xi)h(z)}\cdot$$

Taking $h(z) = \mu = \mu_1 + i\mu_2$ and $zh'(z) = \nu = \nu_1 + i\nu_2$, we define the function $\varphi(\mu,\nu)$ by:

$$\varphi(\mu,\nu) = (1-\xi)\mu - \frac{(1-\xi)\nu}{\xi - c - 1 + (1-\xi)\mu}\cdot \tag{3.4}$$

It is easy to see that the function $\varphi(\mu,\nu)$ satisfies the conditions (i) and (ii) of Lemma 5 in $D = \left(\mathbb{C} - \{\frac{\xi-c-1}{\xi-1}\}\right) \times \mathbb{C}$. To verify the condition (iii), we proceed as follows;

$$\Re\{\varphi(i\mu_2,\nu_1)\} = \frac{(c+1-\xi)(1-\xi)\nu_1}{[\xi-c-1]^2 + [(1-\xi)\mu_2]^2} \leq \frac{-(c+1-\xi)(1-\xi)(1+\mu_2^2)}{2[\xi-c-1]^2 + 2[(1-\xi)\mu_2]^2} < 0\cdot \tag{3.5}$$

Hence, the function $\varphi(\mu,\nu)$ satisfies the conditions of Lemma 5, implies $\Re\{h(z)\} > 0 (z \in \mathbb{U})$, that is, $L_c(f) \in S_n^*(\xi,\lambda,\alpha,\beta,\mu)\cdot$ This completes the proof of Theorem 10. □

Theorem 11. *Let* $c > 0$*,* $f \in \Sigma$*,* $0 \le \xi < 1$*,* $\alpha, \beta, \mu, \lambda \ge 0, \mu + \lambda \ne 0, \alpha + \beta \ne 0$*, and* $n \in N_o$*. If* $f(z) \in C_n\big(\xi, \lambda, \mu, \alpha, \beta\big)$*, then* $L_c(f) \in C_n\big(\xi, \lambda, \mu, \alpha, \beta\big)\cdot$

Proof. Since $f(z) \in C_n(\xi, \lambda, \alpha, \beta, \mu) \Leftrightarrow -zf'(z) \in S_n^*(\xi, \lambda, \alpha, \beta, \mu) \Rightarrow L_c(-zf') \in S_n^*(\xi, \lambda, \alpha, \beta, \mu) \Leftrightarrow -z(L_c(f))' \in S_n^*(\xi, \lambda, \alpha, \beta, \mu) \Leftrightarrow L_c(f) \in C_n(\xi, \lambda, \alpha, \beta, \mu)\cdot$ □

Theorem 12. *Let* $c > 0$*,* $f \in \Sigma$*,* $0 \le \xi < 1$*,* $\alpha, \beta, \mu, \lambda \ge 0, \mu + \lambda \ne 0, \alpha + \beta \ne 0$*, and* $n \in N_o$*. If* $f(z) \in K_n\big(\rho, \xi, \lambda, \mu, \alpha, \beta\big)$*, then* $L_c(f) \in K_n\big(\rho, \xi, \lambda, \mu, \alpha, \beta\big)\cdot$

Proof. Since $f(z) \in K_n(\rho, \xi, \lambda, \alpha, \beta, \mu)$ implies $\Theta_\lambda^n(\alpha, \beta, \mu)f \in K(\rho, \xi)$ or

$$\Re\left(\frac{z(\Theta_\lambda^n(\alpha, \beta, \mu)f(z))'}{\Theta_\lambda^n(\alpha, \beta, \mu)g(z)}\right) > -\rho\cdot$$

We suppose that

$$\left(\frac{z(\Theta_\lambda^n(\alpha, \beta, \mu)L_cf(z))'}{\Theta_\lambda^n(\alpha, \beta, \mu)L_cg(z)}\right) = -\rho - (1-\rho)h(z), \;\; z \in \mathbb{U}\cdot \tag{3.6}$$

Where $h(z) = 1 + c_1z + c_2z^2 + \cdots$.
Since we have

$$z(\Theta_\lambda^n(\alpha, \beta, \mu)L_cf(z))' = (c)(\Theta_\lambda^n(\alpha, \beta, \mu)f(z)) - (c+1)(\Theta_\lambda^n(\alpha, \beta, \mu)L_cf(z))\cdot$$

Therefore

$$\left(\frac{z(\Theta_\lambda^n(\alpha, \beta, \mu)f(z))'}{\Theta_\lambda^n(\alpha, \beta, \mu)g(z)}\right) = \frac{\frac{z(\Theta_\lambda^n(\alpha,\beta,\mu)L_c(zf'(z))'}{\Theta_\lambda^n(\alpha,\beta,\mu)L_cg(z)} + (c+1)\frac{(\Theta_\lambda^n(\alpha,\beta,\mu)L_c(zf'(z))}{\Theta_\lambda^n(\alpha,\beta,\mu)L_cg(z)}}{\frac{z(\Theta_\lambda^n(\alpha,\beta,\mu)L_c(g(z))'}{\Theta_\lambda^n(\alpha,\beta,\mu)L_cg(z)} + c + 1}\cdot \tag{3.7}$$

Since $g(z) \in S_n^*(\xi, \lambda, \alpha, \beta, \mu)$ implies $L_c(g(z)) \in S_n^*(\xi, \lambda, \alpha, \beta, \mu)\cdot$ Let

$$\frac{z(\Theta_\lambda^n(\alpha, \beta, \mu)L_c(g(z))'}{\Theta_\lambda^n(\alpha, \beta, \mu)L_cg(z)} = -\xi - (1-\xi)H(z), \;\; \Re(H(z)) > 0, \;\; z \in \mathbb{U}\cdot$$

Using (3.2) and (3.6) we conclude that

$$\frac{z(\Theta_\lambda^n(\alpha, \beta, \mu)L_c(zf'(z))'}{\Theta_\lambda^n(\alpha, \beta, \mu)L_cg(z)} = [-\rho - (1-\rho)h(z)][-\xi - (1-\xi)H(z)] + [-(1-\rho)zh'(z)] \tag{3.8}$$

Simultaneously solving (3.7) and (3.8) we get

$$\left(\frac{z(\Theta_\lambda^n(\alpha, \beta, \mu)f(z))'}{\Theta_\lambda^n(\alpha, \beta, \mu)g(z)}\right) - \rho = -(1-\rho)h(z) + \frac{(1-\rho)zh'(z)}{\xi - c - 1 + (1-\xi)H(z)} \tag{3.9}$$

Taking $h(z) = \mu = \mu_1 + i\mu_2$ and $zh'(z) = \nu = \nu_1 + i\nu_2$, we define the function $\varphi(\mu,\nu)$ by:

$$\varphi(\mu,\nu) = (1-\rho)\mu - \frac{(1-\rho)\nu}{\xi - c - 1 + (1-\xi)H(z)}\cdot \tag{3.10}$$

It is easy to see that the function $\varphi(\mu,\nu)$ satisfies the conditions (i) and (ii) of Lemma 5 in $D = \mathbb{C}\times\mathbb{C}$. To verify the condition (iii), we proceed as follows;

$$\Re\{\varphi(i\mu_2,\nu_1)\} = \frac{\nu_1(1-\rho)[(c+1-\xi-(1-\xi)h_1(x_1,y_1)]}{[(\xi-c-1)+(1-\xi)h_1(x_1,y_1)]^2 + [(1-\xi)h_2(x_2,y_2)]^2}\cdot$$

Where $H(z) = h_1(x_1,y_1) + ih_2(x_2,y_2)$, $h_1(x_1,y_1)$ and $h_2(x_2,y_2)$ being functions of x and y and $\Re(h_1(x_1,y_1)) > 0$. By putting $\nu_1 \le -\frac{1}{2}(1+\mu_2^2)$, we obtain

$$\Re\{\varphi(i\mu_2,\nu_1)\} = -\frac{1}{2}\frac{(1+\mu_2^2)(1-\rho)[(c+1-\xi-(1-\xi)h_1(x_1,y_1)]}{[(\xi-c-1)+(1-\xi)h_1(x_1,y_1)]^2 + [(1-\xi)h_2(x_2,y_2)]^2} < 0\cdot$$

Hence, the function $\varphi(\mu,\nu)$ satisfies the conditions of Lemma 5, implies $\Re\{h(z)\} > 0 (z \in \mathbb{U})$, that is, $L_c(f) \in K_n(\rho,\xi,\lambda,\alpha,\beta,\mu)\cdot$ This completes the proof of Theorem 12. Similarly we can prove the following theorem.. □

Theorem 13. *Let $c > 0$, $f \in \Sigma$, $0 \le \xi < 1$, $\alpha,\beta,\mu,\lambda \ge 0, \mu+\lambda \ne 0, \alpha+\beta \ne 0$, and $n \in N_o$. If $f(z) \in K_n^*(\rho,\xi,\lambda,\mu,\alpha,\beta)$, then $L_c(f) \in K_n^*(\rho,\xi,\lambda,\mu,\alpha,\beta)\cdot$*

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