Surveys in Mathematics and its Applications

ISSN 1842-6298 (electronic), 1843-7265 (print)
Volume 8 (2013), 23 – 34

POSITIVE BLOCK MATRICES ON HILBERT AND KREIN C^*-MODULES

M. Dehghani, S. M. S. Modarres and M. S. Moslehian

Abstract. Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be Hilbert C^*-modules. In this paper we give some necessary and sufficient conditions for the positivity of a block matrix on the Hilbert C^*-module $\mathscr{H}_1 \oplus \mathscr{H}_2$. If $(\mathscr{H}_1, J_1)$ and $(\mathscr{H}_2, J_2)$ are two Krein C^*-modules, we study the $\tilde{\mathbf{J}}$-positivity of 2×2 block matrix

$$\begin{pmatrix} A & X \\ X^\sharp & B \end{pmatrix}$$

on the Krein C^*-module $(\mathscr{H}_1 \oplus \mathscr{H}_2, \tilde{\mathbf{J}} = J_1 \oplus J_2)$, where $X^\sharp = J_2 X^* J_1$ is the (J_2, J_1)-adjoint of the operator X. We prove that if A is J_1-selfadjoint and B is J_2-selfadjoint and A is invertible, then the operator $\begin{pmatrix} A & X \\ X^\sharp & B \end{pmatrix}$ is $\tilde{\mathbf{J}}$-positive if and only if $A \geq^{J_1} 0$, $B \geq^{J_2} 0$ and $X^\sharp A^{-1} X \leq^{J_2} B$. We also present more equivalent conditions for the $\tilde{\mathbf{J}}$-positivity of this operator.

1 Introduction and preliminaries

Hilbert C^*-modules are generalizations of Hilbert spaces by allowing the inner product to take values in a C^*-algebra rather than in the field of complex numbers. The theory of Hilbert C^*-modules has applications in the study of locally compact quantum groups, non-commutative geometry and KK-theory. Actually Hilbert C^*-modules can be considered as a 'quantization' of the Hilbert space theory; see e.g. [10].

Let $\mathcal{A}$ be a C^*-algebra. A complex linear space $\mathscr{H}$ is said to be an inner product $\mathcal{A}$-module if $\mathscr{H}$ is a right $\mathcal{A}$-module together with a C^*-valued map $(x, y) \mapsto \langle x, y\rangle : \mathscr{H} \times \mathscr{H} \to \mathcal{A}$ such that

(i) $\langle x, \alpha y + \beta z\rangle = \alpha\langle x, y\rangle + \beta\langle x, z\rangle$ $(x, y, z \in \mathscr{H}, \alpha, \beta \in \mathbb{C})$;
(ii) $\langle x, ya\rangle = \langle x, y\rangle a$ $(x, y \in \mathscr{H}, a \in \mathcal{A})$;
(iii) $\langle y, x\rangle = \langle x, y\rangle^*$ $(x, y \in \mathscr{H})$;
(iv) $\langle x, x\rangle \geq 0$ and if $\langle x, x\rangle = 0$, then $x = 0$ $(x \in \mathscr{H})$.

An inner product $\mathcal{A}$-module $\mathscr{H}$ which is complete with respect to the induced norm $\|x\| = \|\langle x, x\rangle\|^{\frac{1}{2}}$ $(x \in \mathscr{H})$ is called a Hilbert $\mathcal{A}$-module or a Hilbert C^*-module over

2010 Mathematics Subject Classification: Primary 47B50; Secondary 46L08; 46C20; 47B65.
Keywords: Block matrix; Indefinite inner product module; J-positive operator; J-contraction; Krein C^*-module.

$\mathcal{A}$. Every Hilbert space is a Hilbert $\mathbb{C}$-module.
Suppose that $\mathscr{H}_1$ and $\mathscr{H}_2$ are two Hilbert $\mathcal{A}$-modules. We denote by $\mathcal{L}(\mathscr{H}_1, \mathscr{H}_2)$ the set of all bounded linear operators $T : \mathscr{H}_1 \to \mathscr{H}_2$ which are adjointable in the sense that there is a map $T^* : \mathscr{H}_2 \to \mathscr{H}_1$ such that

$$\langle Tx, y\rangle = \langle x, T^*y\rangle \quad (x \in \mathscr{H}_1, y \in \mathscr{H}_2).$$

Let $(\mathscr{H}, \langle\cdot,\cdot\rangle)$ be a Hilbert C^*-module. Then $\mathcal{L}(\mathscr{H}) := \mathcal{L}(\mathscr{H}, \mathscr{H})$ is a C^*-algebra with the identity operator $I_{\mathscr{H}}$. An operator $T \in \mathcal{L}(\mathscr{H})$ is called selfadjoint if $T^* = T$ and is positive if $\langle Tx, x\rangle \geq 0$ for all $x \in \mathscr{H}$. We denote by $T^{\frac{1}{2}}$ the unique positive square root of T. If T is a positive invertible operator we write $T > 0$. For selfadjoint operators T and S on $\mathscr{H}$, we say $T \leq S$ if $S - T \geq 0$. A selfadjoint idempotent operator $T \in \mathcal{L}(\mathscr{H})$ is called a projection.
Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be Hilbert $\mathcal{A}$-modules. The operator $T \in \mathcal{L}(\mathscr{H}_1, \mathscr{H}_2)$ is called a contraction if $T^*T \leq I_{\mathscr{H}_1}$ and is called an isometry if $T^*T = I_{\mathscr{H}_1}$. We write $\mathscr{R}(T)$ and $\mathscr{N}(T)$ for the range and null space of the operator T, respectively.
Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be Hilbert C^*-modules over $\mathcal{A}$. Every operator $\mathbf{A} \in \mathcal{L}(\mathscr{H}_1 \oplus \mathscr{H}_2)$ is uniquely determined by operators $A_{ij} \in \mathcal{L}(\mathscr{H}_j, \mathscr{H}_i)$ $(1 \leq i, j \leq 2)$ defined by $A_{ij} = \pi_i \mathbf{A} \tau_j$, where τ_j is the canonical embedding of $\mathscr{H}_j$ in $\mathscr{H}_1 \oplus \mathscr{H}_2$ and π_i is the natural projection from $\mathscr{H}_1 \oplus \mathscr{H}_2$ onto $\mathscr{H}_i$. Note that $\pi_i^* = \tau_i$. Let us represent $\mathbf{A}$ by the block matrix

$$\mathbf{A} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}. \tag{1.1}$$

Clearly the operator $\mathbf{A}$ is selfadjoint if and only if $\mathbf{A}$ is of the form $\begin{pmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{pmatrix}$, where A_{11} and A_{22} are selfadjoint operators on $\mathscr{H}_1$ and $\mathscr{H}_2$, respectively. The diagonal block matrix $\begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix}$ is denoted by $A_{11} \oplus A_{22}$.

Proposition 1. *[7, Lemma 2.1] Suppose that $\mathscr{H}_1$ and $\mathscr{H}_2$ are Hilbert $\mathcal{A}$-modules. Let $A \in \mathcal{L}(\mathscr{H}_1)$, $C \in \mathcal{L}(\mathscr{H}_2, \mathscr{H}_1)$ and $B \in \mathcal{L}(\mathscr{H}_2)$. Then $\begin{pmatrix} A & C \\ C^* & B \end{pmatrix} \geq 0$ if and only if $A \geq 0$, $B \geq 0$ and*

$$|\varphi(\langle Cy, x\rangle)|^2 \leq \varphi(\langle Ax, x\rangle)\varphi(\langle By, y\rangle). \tag{1.2}$$

for all $x \in \mathscr{H}_1$, $y \in \mathscr{H}_2$ and all $\varphi \in \mathcal{S}(\mathcal{A})$, where $\mathcal{S}(\mathcal{A})$ is the state space of $\mathcal{A}$.

Linear spaces with indefinite inner products were used for the first time in the quantum field theory in physics by Dirac [6]. Krein spaces as an indefinite generalization of Hilbert spaces were formally defined by Ginzburg [8]. The notion of a Krein C^*-modules is a natural generalization of a Krein space. In sequel we present the standard terminology and some basic results on Krein spaces and Krein C^*-modules. For a complete exposition on the subject see [1, 2, 9, 11].

Let $(\mathscr{H}, \langle\cdot,\cdot\rangle)$ be a Hilbert C^*-module over a C^*-algebra $\mathcal{A}$. Suppose that a nontrivial selfadjoint involution J on $\mathscr{H}$, i.e. $J = J^* = J^{-1}$, is given to produce an $\mathcal{A}$-valued indefinite inner product

$$[x, y]_J := \langle Jx, y\rangle \qquad (x, y \in \mathscr{H}).$$

Then $(\mathscr{H}, J)$ is called a Krein C^*-module. Trivially a Krein space is a Krein C^*-module over $\mathcal{A} = \mathbb{C}$. The Minkowski space is a well-known Krein space.

Example 2. *Let $M_n(\mathbb{C})$ be the algebra of all complex $n \times n$ matrices and let $\langle\cdot,\cdot\rangle$ be the standard inner product on $\mathbb{C}^n$. For selfadjoint involution*

$$J_0 = \begin{pmatrix} I_{n-1} & 0 \\ 0 & -1 \end{pmatrix},$$

where I_{n-1} denotes the identity of $M_{n-1}(\mathbb{C})$, let us consider the indefinite inner product $[\cdot,\cdot]_{J_0}$ on $\mathbb{C}^n$ given by

$$[x, y]_{J_0} = \langle J_0x, y\rangle = \sum_{k=1}^{n-1} x_k\bar{y}_k - x_n\bar{y}_n$$

for all $x = (x_1, \cdots, x_n)$, $y = (y_1, \cdots, y_n) \in \mathbb{C}^n$. The Krein space $(\mathbb{C}^n, J_0)$ is called a Minkowski space.

Let $(\mathscr{H}_1, J_1)$ and $(\mathscr{H}_2, J_2)$ be Krein C^*-modules. The (J_1, J_2)-adjoint operator of $A \in \mathcal{L}(\mathscr{H}_1, \mathscr{H}_2)$ is defined by

$$[Ax, y]_{J_2} = [x, A^\sharp y]_{J_1} \qquad (x \in \mathscr{H}_1\, ,\, y \in \mathscr{H}_2),$$

which is equivalent to say that $A^\sharp = J_1A^*J_2$. Trivially $(A^\sharp)^\sharp = A$. Let $(\mathscr{H}, J)$ be a Krein C^*-module. An operator $A \in \mathcal{L}(\mathscr{H})$ is said to be J-selfadjoint if $A^\sharp = A$, or equivalently, $A = JA^*J$. For J-selfadjoint operators A and B, the J-order, denoted as $A \leq^J B$, is defined by

$$[Ax, x]_J \leq [Bx, x]_J \qquad (x \in \mathscr{H}).$$

It is easy to see that $A \leq^J B$ if and only if $JA \leq JB$. The J-selfadjoint operator $A \in \mathcal{L}(\mathscr{H})$ is said to be J-positive if $A \geq^J 0$. Note that neither $A \geq 0$ implies $A \geq^J 0$ nor $A \geq^J 0$ implies $A \geq 0$; for instance, let $A = \begin{pmatrix} 1 & -1 \\ 1 & -3 \end{pmatrix}$ in 2-dimensional Minkowski space $(\mathbb{C}^2, J_0)$. Then A is J_0-positive, but A is not positive.

Positivity of 2×2 block matrices of operators on Hilbert spaces have been studied by many authors; see e.g. [4, 5, 12, 13] and references therein. In section 2, we study the positivity of 2×2 block matrices of adjointable operators on Hilbert C^*-modules. We give a necessary and sufficient condition for the contractibility of an adjointable operator on a Hilbert C^*-module via positivity of a certain block matrix. Then we characterize the positive block matrices of adjointable operators on Hilbert C^*-modules (Theorem 6).

In section 3, we assume that $(\mathscr{H}_1, J_1), (\mathscr{H}_2, J_2)$ are Krein C^*-modules and consider the Krein C^*-module $(\tilde{\mathscr{H}} = \mathscr{H}_1 \oplus \mathscr{H}_2, \tilde{\mathbf{J}} = J_1 \oplus J_2)$. We investigate the positivity of 2×2 block matrices on Krein C^*-module $(\tilde{\mathscr{H}}, \tilde{\mathbf{J}})$. We give some necessary and sufficient conditions for the $\tilde{\mathbf{J}}$-positivity of 2×2 block matrix $\begin{pmatrix} A & X \\ X^\sharp & B \end{pmatrix}$ on the Krein C^*-module $(\tilde{\mathscr{H}}, \tilde{\mathbf{J}})$. We also give the relation between contractions and 2×2 block matrices in the setting of Krein C^*-modules.

2 Positivity of block matrices of adjointable operators on Hilbert C^*-modules

The following lemma characterize the relation between contractions and the positivity of a block matrix of operators on Hilbert C^*-modules.

Lemma 3. *Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be Hilbert $\mathcal{A}$-modules. An operator $C \in \mathcal{L}(\mathscr{H}_2, \mathscr{H}_1)$ is a contraction if and only if the block matrix $\begin{pmatrix} I_{\mathscr{H}_1} & C \\ C^* & I_{\mathscr{H}_2} \end{pmatrix} \in \mathcal{L}(\mathscr{H}_1 \oplus \mathscr{H}_2)$ is positive.*

Proof. Suppose that $\begin{pmatrix} I_{\mathscr{H}_1} & C \\ C^* & I_{\mathscr{H}_2} \end{pmatrix} \geq 0$. By the definition, we have

$$\left\langle \begin{pmatrix} I_{\mathscr{H}_1} & C \\ C^* & I_{\mathscr{H}_2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle = \langle x, x\rangle + 2Re\langle Cy, x\rangle + \langle y, y\rangle \geq 0$$

for all $x \in \mathscr{H}_1$, $y \in \mathscr{H}_2$. Now put $x = -Cy$. Then

$$\langle Cy, Cy\rangle - 2Re\langle Cy, Cy\rangle + \langle y, y\rangle \geq 0.$$

It follows that $\langle Cy, Cy\rangle \leq \langle y, y\rangle$ for all $y \in \mathscr{H}_2$. Therefore $C^*C \leq I_{\mathscr{H}_2}$.
Conversely, suppose that $C^*C \leq I_{\mathscr{H}_2}$ and $\varphi \in \mathcal{S}(\mathcal{A})$. The map $(\cdot,\cdot) \mapsto \varphi(\langle\cdot,\cdot\rangle)$ is a positive sesquilinear form. Using the Cauchy–Schwarz inequality we conclude that

$$|\varphi(\langle Cy, x\rangle)|^2 \leq \varphi(\langle Cy, Cy\rangle)\varphi(\langle x, x\rangle) \leq \varphi(\langle y, y\rangle)\varphi(\langle x, x\rangle).$$

Let $A = I_{\mathscr{H}_1}$ and $B = I_{\mathscr{H}_2}$ in (1.2). Then Proposition 1 implies that

$$\begin{pmatrix} I_{\mathscr{H}_1} & C \\ C^* & I_{\mathscr{H}_2} \end{pmatrix} \geq 0.$$

□

A closed submodule $\mathscr{F}$ of a Hilbert C^*-module $\mathscr{H}$ is called orthogonally complemented if $\mathscr{H} = \mathscr{F} \oplus \mathscr{F}^{\perp}$, where $\mathscr{F}^{\perp} = \{x \in \mathscr{H} : \langle x, y\rangle = 0 \text{ for all } y \in \mathscr{F}\}$. It is well-known that the closed submodules of Hilbert C^*-modules are not orthogonally complemented, in general. However, the null space of an element of $\mathcal{L}(\mathscr{H}_1, \mathscr{H}_2)$ with closed range is orthogonally complemented, which can be stated as follows:

Proposition 4. *[10, Theorem 3.2] Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be two Hilbert C^*-modules and let $A \in \mathcal{L}(\mathscr{H}_1, \mathscr{H}_2)$. If $\mathscr{R}(A)$ is closed, then $\mathscr{R}(A^*)$ is closed and the following orthogonal decompositions holds:*

$$\mathscr{H}_1 = \mathscr{N}(A) \oplus \mathscr{R}(A^*), \quad \mathscr{H}_2 = \mathscr{R}(A) \oplus \mathscr{N}(A^*).$$

Furthermore, The closeness of any one of the following sets implies the closeness of the remaining three sets:

$$\mathscr{R}(A), \mathscr{R}(A^*), \mathscr{R}(AA^*), \mathscr{R}(A^*A).$$

Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be Hilbert C^*-modules. An operator $U \in \mathcal{L}(\mathscr{H}_2, \mathscr{H}_1)$ is called a partial isometry if $\mathscr{R}(U)$ is orthogonally complemented and there exists an orthogonally complemented submodule $\mathscr{F}$ of $\mathscr{H}_2$ such that U is isometric on $\mathscr{F}$ and $U|_{\mathscr{F}^{\perp}} = 0$. It is well-known that U is a partial isometry if and only if U^*U (or UU^*) is a projection; cf [10, Chapter 3]. To get our next result we need the following lemma, which is a generalization of a known result [3, Lemma 2.4.2].

Lemma 5. *Let $\mathscr{H}$, $\mathscr{H}_1$ and $\mathscr{H}_2$ be Hilbert $\mathcal{A}$-modules. Suppose that $T \in \mathcal{L}(\mathscr{H}, \mathscr{H}_1)$ and $S \in \mathcal{L}(\mathscr{H}, \mathscr{H}_2)$ such that $\overline{\mathscr{R}(T)}$ and $\overline{\mathscr{R}(S)}$ are orthogonally complemented. If $T^*T = S^*S$, then $T = US$ for some partial isometry $U : \mathscr{H}_2 \to \mathscr{H}_1$.*

Proof. Assume that $T^*T = S^*S$. Let $y \in \mathscr{R}(S)$. Then $y = Sx$ for some $x \in \mathscr{H}$. Define $Uy := Tx$. Suppose that $x' \in \mathscr{H}$ and $Sx = Sx'$. Hence $S(x - x') = 0$. We have

$$\begin{aligned}\langle T(x-x'), T(x-x')\rangle &= \langle T^*T(x-x'), x-x'\rangle \\ &= \langle S^*S(x-x'), x-x'\rangle \\ &= \langle S(x-x'), S(x-x')\rangle = 0.\end{aligned}$$

It follows that $Tx = Tx'$. Therefore U is well-defined on $\mathscr{R}(S)$. In addition,

$$\begin{aligned}\|Uy\|^2 &= \|Tx\|^2 \\ &= \|\langle T^*Tx, x\rangle\| \\ &= \|\langle S^*Sx, x\rangle\| \\ &= \|Sx\|^2 = \|y\|^2 \qquad (y \in \mathscr{R}(S)).\end{aligned}$$

So U is an isometry on $\mathscr{R}(S)$.
Next, let $y \in \overline{\mathscr{R}(S)}$ and $x_n \in \mathscr{H}$ such that $y_n = Sx_n$ and $\lim_{n\to\infty} y_n = y$. Then $Uy_n = Tx_n$ for all n and

$$\begin{aligned}\|Uy_n - Uy_m\|^2 &= \|\langle T(x_n - x_m), T(x_n - x_m)\rangle\| \\ &= \|\langle S(x_n - x_m), S(x_n - x_m)\rangle\| \\ &= \|\langle y_n - y_m, y_n - y_m\rangle\| \\ &= \|y_n - y_m\|^2.\end{aligned}$$

As $\{y_n\}$ is a Cauchy sequence, so is $\{Uy_n\}$. Therefore we can define

$$Uy := \lim_{n\to\infty} Uy_n = \lim_{n\to\infty} Tx_n \qquad (y \in \overline{\mathscr{R}(S)}).$$

We define U to be zero from the orthogonal complement of $\overline{\mathscr{R}(S)}$ into the orthogonal complement of $\overline{\mathscr{R}(T)}$. Note that U can be regarded as a diagonal matrix.
Clearly $T = US$. The operator U is a partial isometry, indeed, let $z \in \overline{\mathscr{R}(T)}$. Then $z = \lim_{n\to\infty} Tx'_n$ for some $x'_n \in \mathscr{H}$. Analogue to the above construction, we can define

$$U^*z := \lim_{n\to\infty} U^*z_n := \lim_{n\to\infty} Sx'_n \qquad (z \in \overline{\mathscr{R}(T)})$$

and U^* to be zero from the orthogonal complement of $\overline{\mathscr{R}(T)}$ onto $\overline{\mathscr{R}(S)}$ and conclude that $S = U^*T$. Therefore

$$\begin{aligned}\langle Uy, z\rangle &= \langle \lim_{n\to\infty} Tx_n, z\rangle = \lim_{n\to\infty}\langle x_n, T^*z\rangle = \lim_{n\to\infty}\langle x_n, \lim_{m\to\infty} T^*Tx'_m\rangle \\ &= \lim_{n\to\infty}\langle x_n, \lim_{m\to\infty} S^*Sx'_m\rangle = \lim_{n\to\infty}\langle Sx_n, \lim_{m\to\infty} Sx'_m\rangle = \langle y, U^*z\rangle\end{aligned}$$

for all $y \in \overline{\mathscr{R}(S)}, z \in \overline{\mathscr{R}(T)}$ and so U^* is actually the adjoint of U. Moreover

$$U^*Uy = \lim_{n\to\infty} U^*Tx_n = \lim_{n\to\infty} Sx_n = y \qquad (y \in \overline{\mathscr{R}(S)}).$$

and

$$UU^*z = \lim_{n\to\infty} USx'_n = \lim_{n\to\infty} Tx'_n = z \qquad (z \in \overline{\mathscr{R}(T)}).$$

Hence $U^*U = \mathcal{P}_{\overline{\mathscr{R}(S)}}$ and $UU^* = \mathcal{P}_{\overline{\mathscr{R}(T)}}$ are projections onto $\overline{\mathscr{R}(S)}$ and $\overline{\mathscr{R}(T)}$, respectively. It follows that U is a partial isometry. □

A characterization of positive 2×2 block matrices can be obtained by using Lemma 3 as follows:

Theorem 6. *Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be Hilbert C^*-modules and let $A \in \mathcal{L}(\mathscr{H}_1)$ and $B \in \mathcal{L}(\mathscr{H}_2)$ such that $\mathscr{R}(A)$ and $\mathscr{R}(B)$ be closed submodules of $\mathscr{H}_1$ and $\mathscr{H}_2$, respectively. Then the block matrix* $\begin{pmatrix} A & C \\ C^* & B \end{pmatrix} \in \mathcal{L}(\mathscr{H}_1 \oplus \mathscr{H}_2)$ *is positive if and only if $A \geq 0$, $B \geq 0$ and there exists a contraction G such that $C = A^{\frac{1}{2}} G B^{\frac{1}{2}}$.*

Proof. Let $A \geq 0$ and $B \geq 0$ and let $C = A^{\frac{1}{2}} G B^{\frac{1}{2}}$ for some contraction G. Then Lemma 3 forces that $\begin{pmatrix} I_{\mathscr{H}_1} & G \\ G^* & I_{\mathscr{H}_2} \end{pmatrix} \geq 0$. It follows from

$$\begin{pmatrix} A & C \\ C^* & B \end{pmatrix} = \begin{pmatrix} A^{\frac{1}{2}} & 0 \\ 0 & B^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} I_{\mathscr{H}_1} & G \\ G^* & I_{\mathscr{H}_2} \end{pmatrix} \begin{pmatrix} A^{\frac{1}{2}} & 0 \\ 0 & B^{\frac{1}{2}} \end{pmatrix}.$$

that $\begin{pmatrix} A & C \\ C^* & B \end{pmatrix}$ is positive.

Conversely, suppose that $\mathbf{M} = \begin{pmatrix} A & C \\ C^* & B \end{pmatrix} \geq 0$. Then $\mathbf{M} = \mathbf{N}^*\mathbf{N}$ for some $\mathbf{N} \in \mathcal{L}(\mathscr{H}_1 \oplus \mathscr{H}_2)$. We can write $\mathbf{N} = \begin{pmatrix} P & Q \end{pmatrix}$, where $P \in \mathcal{L}(\mathscr{H}_1, \mathscr{H}_1 \oplus \mathscr{H}_2)$ and $Q \in \mathcal{L}(\mathscr{H}_2, \mathscr{H}_1 \oplus \mathscr{H}_2)$ are defined by $P = \mathbf{N}\tau_1$ and $Q = \mathbf{N}\tau_2$, respectively. To see this, let $x_1 \in \mathscr{H}_1$ and $x_2 \in \mathscr{H}_2$. Then

$$\begin{pmatrix} P & Q \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = Px_1 + Qx_2 = \mathbf{N}(x_1, 0) + \mathbf{N}(0, x_2) = \mathbf{N}(x_1, x_2).$$

Therefore

$$\mathbf{M} = \begin{pmatrix} A & C \\ C^* & B \end{pmatrix} = \mathbf{N}^*\mathbf{N} = \begin{pmatrix} P^* \\ Q^* \end{pmatrix} \begin{pmatrix} P & Q \end{pmatrix} = \begin{pmatrix} P^*P & P^*Q \\ Q^*P & Q^*Q \end{pmatrix}.$$

It follows that $A = P^*P \geq 0$, $B = Q^*Q \geq 0$ and $C = P^*Q$.
Due to $\mathscr{R}(A)$ is closed, $\mathscr{R}(A^{\frac{1}{2}}) = \mathscr{R}(A)$ is closed. Moreover, it follows from $\mathscr{R}(A) = \mathscr{R}(P^*P)$ and Proposition 4 that $\mathscr{R}(P)$ is closed. Similarly, $\mathscr{R}(B^{\frac{1}{2}})$ and $\mathscr{R}(Q)$ are closed. Since $A^{\frac{1}{2}}A^{\frac{1}{2}} = A = P^*P$ and $B^{\frac{1}{2}}B^{\frac{1}{2}} = B = Q^*Q$ Lemma 5 implies that there exist partial isometries U_1 and U_2 such that $P = U_1 A^{\frac{1}{2}}$, $Q = U_2 B^{\frac{1}{2}}$ and $U_1 U_1^* = \mathcal{P}_{\mathscr{R}(P)}$, $U_2^* U_2 = \mathcal{P}_{\mathscr{R}(B)}$ are projections onto $\mathscr{R}(P)$ and $\mathscr{R}(B)$, respectively. Therefore $C = P^*Q = A^{\frac{1}{2}} U_1^* U_2 B^{\frac{1}{2}}$. Set $G := U_1^* U_2$. Then

$$G^*G = U_2^* U_1 U_1^* U_2 = U_2^* \mathcal{P}_{\mathscr{R}(P)} U_2 \leq U_2^* I_{\mathscr{H}_1 \oplus \mathscr{H}_2} U_2 = U_2^* U_2 = \mathcal{P}_{\mathscr{R}(B)} \leq I_{\mathscr{H}_2}$$

and $C = A^{\frac{1}{2}} G B^{\frac{1}{2}}$. □

3 Positivity of block matrices of operators on Krein C^*-modules

In this section we study the positivity of a block matrix of operators acting on Krein C^*-modules.
Let $(\mathscr{H}_1, J_1)$ and $(\mathscr{H}_2, J_2)$ be Krein C^*-modules. Note that a selfadjoint involution on $\mathscr{H}_1 \oplus \mathscr{H}_2$ may be defined in some different ways. It is easy to see that $\tilde{\mathbf{J}} = J_1 \oplus J_2$ is a selfadjoint involution on $\mathscr{H}_1 \oplus \mathscr{H}_2$. Let $\mathbf{A}$ be the block matrix introduced in (1.1). Then we have

$$\mathbf{A}^\sharp = \tilde{\mathbf{J}}\mathbf{A}^*\tilde{\mathbf{J}} = \begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix} \begin{pmatrix} A_{11}^* & A_{21}^* \\ A_{12}^* & A_{22}^* \end{pmatrix} \begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix} = \begin{pmatrix} J_1 A_{11}^* J_1 & J_1 A_{21}^* J_2 \\ J_2 A_{12}^* J_1 & J_2 A_{22}^* J_2 \end{pmatrix}.$$

Therefore $\mathbf{A}$ is $\tilde{\mathbf{J}}$-selfadjoint if and only if $\mathbf{A} = \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^\sharp & A_{22} \end{pmatrix}$ in which A_{11} is J_1-selfadjoint and A_{22} is J_2-selfadjoint.
To get our next result we need the following lemma.

Lemma 7. *Suppose that R and S are J-selfadjoint operators on a Krein C^*-module $(\mathscr{H}, J)$. Then $R \geq^J S$ if and only if $W^\sharp R W \geq^J W^\sharp S W$ for all $W \in \mathcal{L}(\mathscr{H})$. Specially $R \geq^J 0$ if and only if $W^\sharp R W \geq^J 0$ for all $W \in \mathcal{L}(\mathscr{H})$.*

Proof. Clear. □

Theorem 8. *Let $(\mathscr{H}_1, J_1)$ and $(\mathscr{H}_2, J_2)$ be Krein C^*modules. Suppose that A is J_1-selfadjoint and B is J_2-selfadjoint. If A is invertible, then the operator $\begin{pmatrix} A & X \\ X^\sharp & B \end{pmatrix}$ is $\tilde{\mathbf{J}}$-positive if and only if $A \geq^{J_1} 0$, $B \geq^{J_2} 0$ and $X^\sharp A^{-1} X \leq^{J_2} B$.*

Proof. By the assumptions, A is an invertible J_1-selfadjoint operator. It follows that AJ_1 is invertible and selfadjoint. Then $AJ_1 = (AJ_1)^* = J_1 A^*$. It follows that $J_1 A^{-1} = (A^{-1})^* J_1$. Therefore A^{-1} is J_1-selfadjoint. Hence $X^\sharp A^{-1} X$ is J_2-selfadjoint. By the definition, we have

$$\begin{aligned} \begin{pmatrix} I_{\mathscr{H}_1} & -A^{-1}X \\ 0 & I_{\mathscr{H}_2} \end{pmatrix}^\sharp &= \begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix} \begin{pmatrix} I_{\mathscr{H}_1} & 0 \\ -(A^{-1}X)^* & I_{\mathscr{H}_2} \end{pmatrix} \begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix} \\ &= \begin{pmatrix} I_{\mathscr{H}_1} & 0 \\ -X^\sharp A^{-1} & I_{\mathscr{H}_2} \end{pmatrix}. \end{aligned}$$

Therefore

$$\begin{pmatrix} I_{\mathscr{H}_1} & -A^{-1}X \\ 0 & I_{\mathscr{H}_2} \end{pmatrix}^\sharp \begin{pmatrix} A & X \\ X^\sharp & B \end{pmatrix} \begin{pmatrix} I_{\mathscr{H}_1} & -A^{-1}X \\ 0 & I_{\mathscr{H}_2} \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & B - X^\sharp A^{-1} X \end{pmatrix}.$$

From this relation, Lemma 7 and taking into account that the operator

$$\begin{pmatrix} I_{\mathscr{H}_1} & -A^{-1}X \\ 0 & I_{\mathscr{H}_2} \end{pmatrix}$$

is invertible, we deduce that the operator $\begin{pmatrix} A & X \\ X^\sharp & B \end{pmatrix}$ is $\tilde{\mathbf{J}}$-positive if and only if the operator $\begin{pmatrix} A & 0 \\ 0 & B - X^\sharp A^{-1}X \end{pmatrix}$ is $\tilde{\mathbf{J}}$-positive. Therefore $\begin{pmatrix} A & X \\ X^\sharp & B \end{pmatrix}$ is $\tilde{\mathbf{J}}$-positive if and only if $A \geq^{J_1} 0$, $B \geq^{J_2} 0$ and $X^\sharp A^{-1}X \leq^{J_2} B$. □

The following corollary is well-known for operators on Hilbert C^*-modules which we present it as a result of Theorem 8.

Corollary 9. *Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be Hilbert C^*-modules and let $A \in \mathcal{L}(\mathscr{H}_1)$, $B \in \mathcal{L}(\mathscr{H}_2)$ such that $A > 0$ and $B \geq 0$. The block matrix $\begin{pmatrix} A & C \\ C^* & B \end{pmatrix} \in \mathcal{L}(\mathscr{H}_1 \oplus \mathscr{H}_2)$ is positive if and only if $C^*A^{-1}C \leq B$.*

Proof. Let $\tilde{\mathbf{J}} = \begin{pmatrix} I_{\mathscr{H}_1} & 0 \\ 0 & I_{\mathscr{H}_2} \end{pmatrix}$ in Theorem 8. □

Theorem 10. *Let $(\mathscr{H}_1, J_1)$ and $(\mathscr{H}_2, J_2)$ be Krein C^*-modules. Suppose that $A \in \mathcal{L}(\mathscr{H}_1)$ is J_1-positive and $B \in \mathcal{L}(\mathscr{H}_2)$ is J_2-positive. If A and B are invertible and $\mathscr{R}(J_1A)$ and $\mathscr{R}(J_2B)$ are closed submodules, then the following statements are equivalent.*

(i) $\begin{pmatrix} A & X \\ X^\sharp & B \end{pmatrix}$ *is $\tilde{\mathbf{J}}$-positive.*

(ii) $(J_1A)^{-\frac{1}{2}}J_1X(J_2B)^{-\frac{1}{2}}$ *is a contraction.*

(iii) $X^\sharp A^{-1}X \leq^{J_2} B$.

Proof. $(i) \Rightarrow (ii)$.

By the definition, $\begin{pmatrix} J_1A & J_1X \\ (J_1X)^* & J_2B \end{pmatrix} \geq 0$. Then Theorem 6 implies that $J_1X = (J_1A)^{\frac{1}{2}}G(J_2B)^{\frac{1}{2}}$ for some contraction G. Since A and B are invertible we conclude that $G = (J_1A)^{-\frac{1}{2}}J_1X(J_2B)^{-\frac{1}{2}}$ is a contraction.

$(ii) \Rightarrow (iii)$.

The condition (ii) is equivalent to

$$(J_2B)^{-\frac{1}{2}}X^*J_1A^{-1}X(J_2B)^{-\frac{1}{2}} = (J_2B)^{-\frac{1}{2}}(J_1X)^*(J_1A)^{-\frac{1}{2}}(J_1A)^{-\frac{1}{2}}(J_1X)(J_2B)^{-\frac{1}{2}} \leq I_{\mathscr{H}_2}.$$

It follows that $X^*J_1A^{-1}X \leq J_2B$. Therefore $X^\sharp A^{-1}X \leq^{J_2} B$.

$(iii) \Rightarrow (i)$.

It follows from Theorem 8. □

An operator $X \in \mathcal{L}(\mathscr{H}_2, \mathscr{H}_1)$ is called a (J_2, J_1)-contraction if $X^\sharp X \leq^{J_2} I_{\mathscr{H}_2}$, or equivalently, $X^* J_1 X \leq J_2$.

Remark 11. *The $\tilde{\mathbf{J}}$-positivity of block matrix $\begin{pmatrix} I_{\mathscr{H}_1} & X \\ X^\sharp & I_{\mathscr{H}_2} \end{pmatrix} \in \mathcal{L}(\mathscr{H}_1 \oplus \mathscr{H}_2)$ implies that $J_1 \geq 0$ and $J_2 \geq 0$ which is impossible. Therefore in contrast to operators on Hilbert C^*-modules Lemma 3 is not valid in the setting of Krein C^*-modules. Moreover the following example show that the (J_2, J_1)-contractibility of X, i.e. $X^\sharp X \leq^{J_2} I_{\mathscr{H}_2}$ does not imply the $\tilde{\mathbf{J}}$-positivity of block matrix $\begin{pmatrix} I_{\mathscr{H}_1} & X \\ X^\sharp & I_{\mathscr{H}_2} \end{pmatrix}$.*

Example 12. *Consider the Minkowski space $(\mathbb{C}^2, J_0)$ with $J_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Let $X = \begin{pmatrix} i & i \\ i & 2i \end{pmatrix}$. Then*

$$X^\sharp = J_0 X^* J_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -i & -i \\ -i & -2i \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -i & i \\ i & -2i \end{pmatrix}$$

and

$$J_0 - X^* J_0 X = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \geq 0.$$

Therefore $X^\sharp X \leq^{J_0} I$, where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. It means that X is a J_0-contraction. Now let $\tilde{\mathbf{J_0}} = J_0 \oplus J_0$ and $\mathbf{T} = \begin{pmatrix} I & X \\ X^\sharp & I \end{pmatrix}$. Then

$$\tilde{\mathbf{J_0}}\mathbf{T} = \begin{pmatrix} J_0 & 0 \\ 0 & J_0 \end{pmatrix} \begin{pmatrix} I & X \\ X^\sharp & I \end{pmatrix} = \begin{pmatrix} J_0 & J_0 X \\ J_0 X^\sharp & J_0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & i & i \\ 0 & -1 & -i & -2i \\ -i & i & 1 & 0 \\ -i & 2i & 0 & -1 \end{pmatrix}.$$

The matrix $\tilde{\mathbf{J_0}}\mathbf{T}$ is not positive, because it has negative eigenvalues. It follows that $\mathbf{T}$ is not $\tilde{\mathbf{J_0}}$-positive, while X is a J_0-contraction.

In the following theorem we introduce a good candidate for description of contractions by means of $\tilde{\mathbf{J}}$-positive 2×2 block matrices.

Theorem 13. *Let $(\mathscr{H}_1, J_1)$ and $(\mathscr{H}_2, J_2)$ be Krein C^*-modules.Then $\begin{pmatrix} J_1 & X \\ X^\sharp & J_2 \end{pmatrix}$ is $\tilde{\mathbf{J}}$-positive if and only if X is a contraction.*

Proof. Let $\mathbf{T} = \begin{pmatrix} J_1 & X \\ X^\sharp & J_2 \end{pmatrix}$. By the definition, $\mathbf{T} \geq^{\tilde{\mathbf{J}}} 0$ if and only if $\tilde{\mathbf{J}}\mathbf{T} \geq 0$. It means that

$$\begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix} \begin{pmatrix} J_1 & X \\ X^\sharp & J_2 \end{pmatrix} = \begin{pmatrix} I_{\mathscr{H}_1} & J_1 X \\ (J_1X)^* & I_{\mathscr{H}_2} \end{pmatrix} \geq 0. \tag{3.1}$$

Lemma 3 forces that (3.1) is equivalent to $(J_1X)^*(J_1X) \leq I_{\mathscr{H}_2}$. Also we have $X^*X = X^*J_1^2X = (J_1X)^*(J_1X) \leq I_{\mathscr{H}_2}$. □

Acknowledgement. This work was written whilst the first author was visiting Ferdowsi University of Mashhad during his short sabbatical leave provided by the Ministry of Science, Research and Technology. The authors would like to sincerely thank the referee for several valuable comments improving the manuscript.

References

[1] T. Ando, *Linear operators on Krein spaces,* Hokkaido University, Sapporo, Japan, 1979. MR560903(81e:47032). Zbl 0429.47016.

[2] T. Ya. Azizov, I. S. Iokhvidov, *Linear Operators in Spaces with an Indefinite Metric,* Nauka, Moscow, 1986 English translation: Wiley, New York, 1989. MR1033489(90j:47042). Zbl 0714.47028.

[3] M. Bakonyi, H. J. Woerdeman, *Matrix completions, moments, and sums of Hermitian squares,* Princeton University Press, Princeton, NJ, 2011. MR2807419(2012d:47003). Zbl 1236.15056.

[4] R. Bhatia, *Positive Definite Matrices,* Princeton University Press, Princeton, 2007. MR2284176(2007k:15005). Zbl 1125.15300.

[5] M. D. Choi, *Some assorted inequalities for positive linear maps on C^*-algebras,* J. Operator Theory **4**, 271–285 1980:224. MR595415(82c:46073). Zbl 0511.46051.

[6] P. Dirac, *The Physical Interpretation of Quantum Mechanics,* Pro. Roy. Soc. London, Ser. A **180**, 1-40 1942. MR0010295(5,277c). Zbl 0060.45501.

[7] X. Fang, J. Yu, H. Yao, *Solutions to operator equations on Hilbert C^*-modules,* Linear Algebra Appl. **431**, 2142-2153 (2009). MR2567821(2010j:46114). Zbl 1175.47014.

[8] Yu. P. Ginzburg, *On J-contractive Operator Functions,* Dokl. Akad. Nauk SSSR, **117**, 171-173 (1957). MR0094691(20#1203). Zbl 0084.10802.

[9] I. Gohberg, P. Lancaster, L. Rodman, *Indefinite linear algebra and applications,* Birkhäuser Verlag, Basel, 2005. MR2186302(2006j:15001). Zbl 1084.15005.

[10] E. C. Lance, *Hilbert C^*-Modules. A toolkit for operator algebraists,* London Math. Soc. Lecture Note Series, vol. 210, Cambridge Univ. Press, 1995. MR1325694(96k:46100). Zbl 0822.46080.

[11] M. Langer, A. Luger, H. Woracek, *Operator theory and indefinite inner product spaces* Lectures presented on the occasion of the retirement of Heinz Langer in the Colloquium on Operator Theory held at the Vienna University of Technology, Vienna, March 2004. Edited by Operator Theory: Advances and Applications, 163. Birkhäuser Verlag, Basel, 2006. MR2216680(2006k:47001). Zbl 1086.47007.

[12] M. S. Moslehian, J.I. Fujii, *Operator inequalities related to weak 2-positivity,* J. Math. Inequal. 7, No. 2, 175-182 (2013). Zbl 06182918.

[13] M. S. Moslehian, *Trick with 2×2 matrices over C^*-algebras,* Austral. Math. Soc. Gaz. **30**, no. 3, 150-157 2003. MR1988521(2004c:46108). Zbl 1058.46036.

Mohammad Sal Moslehian
Department of Pure Mathematics,
Center of Excellence in Analysis on
Algebraic Structures (CEAAS),
Ferdowsi University of Mashhad,
P. O. Box 1159, Mashhad 91775, Iran.
e-mail: moslehian@um.ac.ir,
moslehian@member.ams.org

Mahdi Dehghani
Department of Pure and Applied Mathematics,
Yazd University,
Yazd P.O. Box 89195-741, Iran.
Department of Pure Mathematics,
Center of Excellence in Analysis on
Algebraic Structures (CEAAS),
Ferdowsi University of Mashhad,
P. O. Box 1159, Mashhad 91775, Iran.
e-mail: e.g.mahdi@gmail.com

Seyed Mohammad Sadegh Modarres
Department of Pure and Applied Mathematics,
Yazd University,
Yazd P. O. Box 89195-741, Iran.
e-mail: smodares@yazduni.ac.ir