Surveys in Mathematics and its Applications

ISSN 1842-6298 (electronic), 1843-7265 (print)
Volume 13 (2018), 237 – 250

SOME RESULTS OF η-RICCI SOLITONS ON $(LCS)_n$-MANIFOLDS

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Abstract. In this paper, we consider an η-Ricci soliton on the $(LCS)_n$-manifolds (M, ϕ, ξ, η, g) satisfying certain curvature conditions likes: $R(\xi, X) \cdot S = 0$ and $W_2(\xi, X) \cdot S = 0$. We show that on the $(LCS)_n$-manifolds (M, ϕ, ξ, η, g), the existence of η-Ricci soliton implies that (M, g) is a quasi-Einstein. Further, we discuss the existence of Ricci solitons with the potential vector field ξ. In the end, we construct the non-trivial examples of η-Ricci solitons on the $(LCS)_n$-manifolds.

1 Introduction

In 2003, Shaikh [33] introduced the notion of Lorentzian concircular structure manifolds (briefly, $(LCS)_n$-manifold) with an example, which generalize the notion of LP-Sasakian manifolds introduced by Matsumoto [27] and also by Mihai and Rosca [28]. The properties of $(LCS)_n$-manifolds have been studied by many geometer, for instance we refer ([7], [8], [22]-[25], [29], [34], [36], [39]-[42]).

The Ricci solitons are natural generalization of Einstein metrics on a Riemannian manifold, being generalized fixed points of Hamilton's Ricci flow $\frac{\partial}{\partial t} g = -2S$ [20]. The evolution equation defining the Ricci flow is a kind of nonlinear diffusion equation, an analogue of heat equation for metrics. Under Ricci flow, a metric can be improved to evolve into more canonical one by smoothing out its irregularities, depending on the Ricci curvature of the manifold: it will expand in the directions of negative Ricci curvature and shrink in the positive case. The geometrical properties of the Ricci solitons have been studied in ([1]-[5], [7]-[13], [17]-[21], [26], [31], [37], [38], [43]) and by others. In paracontact geometry, the Ricci soliton first appeared in the paper of G. Calvaruso and D. Perrone [6]. C. L. Bejan and M. Crasmareanu studied the properties of Ricci solitons on the 3-dimensional normal paracontact manifolds [3]. A more general notion of a Ricci soliton is that of η-Ricci soliton introduced by J. T. Cho and M. Kimura [18], which was treated by C. Calin and M. Crasmareanu on Hopf hypersurfaces in complex-space-forms [4]. Metrics satisfying

2010 Mathematics Subject Classification: 53C15; 53C21; 53C25.
Keywords: η-Ricci soliton; Quasi-Einstein; $(LCS)_n$-manifold; Ricci tensors; Curvature tensors.

Ricci flow equations are interesting and useful in physics and are often referred as quasi-Einstein ([12]-[16]).

2 $(LCS)_n$-manifolds (M, ϕ, ξ, η, g)

Let M be an n-dimensional smooth connected paracontact Hausdroff manifold equipped with a Lorentzian metric g. Then (M, g) is a Lorentzian manifold, that is, M admits a smooth symmetric tensor field g of type $(0, 2)$ such that for each point $p \in M$, the tensor $g_p : T_pM \times T_pM \to \Re$ is a non degenerate inner product of signature $(-, +, ..., +)$, where T_pM denotes the tangent space of M at p and $\Re$ is the real number. A non-zero vector field $v \in T_pM$ is said to be timelike (resp., non-spacelike, null, and spacelike) if it satisfies $g_p(v, v) < 0$ (resp., $\leq 0, =, > 0$) [30].

Definition 1. *A non-vanishing vector field ρ on a Lorentzian manifold (M, g) defined by $g(X, \rho) = A(X)$, $\forall\ X \in \chi(M)$ is said to be a concircular vector field [41] if*

$$(\nabla_X A)(Y) = \alpha \{g(X, Y) + \omega(X)A(Y)\},$$

where α is a non-zero scalar and ω is a closed 1-form.

If the Lorentzian manifold M admits a unit timelike concircular vector field ξ, called the *generator* of the manifold, then we have

$$g(\xi, \xi) = -1, \ \ g(X, \xi) = \eta(X), \ \ (\nabla_X \eta)(Y) = \alpha \{g(X, Y) + \eta(X)\eta(Y)\}, \qquad (2.1)$$

where $\alpha \neq 0$ and η is a non-zero 1-form. It is obvious from (2.1) that

$$\nabla_X \xi = \alpha\{X + \eta(X)\xi\} \qquad (2.2)$$

for all vector field X on M. Here ∇ denotes the operator of the covariant differentiation with respect to the Lorentzian metric g and α satisfies

$$\nabla_X \alpha = (X\alpha) = d\alpha(X) = \rho\eta(X), \qquad (2.3)$$

ρ being a certain scalar function given by $\rho = -(\xi\alpha)$. If we put

$$\alpha\, \phi X = \nabla_X \xi, \qquad (2.4)$$

then (2.2) and (2.4) give

$$\phi X = X + \eta(X)\xi, \qquad (2.5)$$

where ϕ is a $(1, 1)$-tensor, called the structure tensor of M. Thus the Lorentzian manifold M together with a unit timelike concircular vector field ξ, its associated 1-form η and $(1, 1)$-tensor field ϕ is said to be a Lorentzian concircular structure manifold (briefly $(LCS)_n$-manifold) [33]. Especially, if we take $\alpha = 1$, then we can obtain the LP-Sasakian structure of Matsumoto [27]. For details, we refer [11] and the references therein. In an $(LCS)_n$-manifold, $n > 2$, the following relations

$$\eta(\xi) = -1, \quad \phi\xi = 0, \quad \phi^2 X = X + \eta(X)\xi,$$
$$\eta(\phi X) = 0, \quad g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{2.6}$$
$$\eta(R(X,Y)Z) = (\alpha^2 - \rho)\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}, \tag{2.7}$$
$$R(X,Y)\xi = (\alpha^2 - \rho)\{\eta(Y)X - \eta(X)Y\}, \tag{2.8}$$
$$R(\xi,X)Y = (\alpha^2 - \rho)\{g(X,Y)\xi - \eta(Y)X\}, \tag{2.9}$$
$$(\nabla_X \phi)(Y) = \alpha\{g(X,Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X\}, \tag{2.10}$$
$$S(X,\xi) = (n-1)(\alpha^2 - \rho)\eta(X), \tag{2.11}$$
$$S(\phi X, \phi Y) = S(X,Y) + (n-1)(\alpha^2 - \rho)\eta(X)\eta(Y), \tag{2.12}$$
$$(X\rho) = d\rho(X) = \beta\eta(X), \tag{2.13}$$

hold for any vector fields X, Y, Z on M, $\beta = -(\xi\rho)$ is a scalar function [34]. Here R is the curvature tensor corresponding to the Lorentzian metric g and S is the Ricci tensor corresponding to the Ricci operator Q, that is, $S(X,Y) = g(QX,Y)$.

3 η-Ricci solitons on $(LCS)_n$-manifolds (M, ϕ, ξ, η, g)

Let (M, ϕ, ξ, η, g) be an $(LCS)_n$-manifold, then the quartet (g, ξ, λ, μ) on M is said to be an η-Ricci soliton [18] if it satisfies

$$L_\xi g + 2S + 2\lambda g + 2\mu\, \eta \otimes \eta = 0, \tag{3.1}$$

where L_ξ is the Lie-derivative operator along the vector field ξ, λ and μ are real constants. We write $L_\xi g$ in term of the Levi-Civita connection ∇ as:

$$(L_\xi g)(X,Y) = g(\nabla_Y \xi, X) + g(Y, \nabla_X \xi) = 2\alpha[g(X,Y) + \eta(X)\eta(Y)], \tag{3.2}$$

where equations (2.1) and (2.2) are used. In view of (3.1) and (3.2), we get

$$QX = -(\alpha + \lambda)X - (\alpha + \mu)\eta(X)\xi, \tag{3.3}$$
$$r = -n\lambda - (n-1)\alpha + \mu, \tag{3.4}$$
$$S(X,Y) = -(\alpha + \lambda)g(X,Y) - (\alpha + \mu)\eta(X)\eta(Y), \tag{3.5}$$
$$S(X,\xi) = S(\xi,X) = (\mu - \lambda)\eta(X), \tag{3.6}$$
$$\mu - \lambda = (n-1)(\alpha^2 - \rho) \tag{3.7}$$

for any $X, Y \in \chi(M)$. Here r is the scalar curvature of (M,g) and is defined by $r = S(e_i, e_i)_{i=1}^n$, where $\{e_1, e_2, ..., e_n\}$ is a set of linearly independent vector fields on M. In particular, if $\mu = 0$ then the triplet (g, ξ, λ) is a Ricci soliton [20] and it is called shrinking, steady or expanding according as λ is negative, zero or positive, respectively [19].

Proposition 2. *The following relations hold on an $(LCS)_n$-manifold (M, ϕ, ξ, η, g)*

$$(i)\ \ \eta(\nabla_X \xi) = 0, \ \ (ii)\ \ \nabla_\xi \xi = 0, \ \ (iii)\ \ \nabla_\xi \eta = 0, \ (iv)\ \ L_\xi \phi = 0,$$

$$(v)\ \ L_\xi \eta = 0, \ \ (vi)\ \ L_\xi(\eta \otimes \eta) = 0, \ \ (vii)\ \ L_\xi g = 2\alpha(g + \eta \otimes \eta).$$

Also, if η is closed the distribution is involuntary and the Nijenhuis tensor of ϕ vanishes identically, i.e., the structure is normal.

Proof. Since $(\nabla_X \phi)(Y) = \alpha\{g(X, Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X\}$ and therefore

$$\nabla_X \phi Y - \phi(\nabla_X Y) = \alpha\{g(X, Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X\}.$$

Taking $Y = \xi$ in the above equation, we have $\phi(\nabla_X \xi) = \alpha\phi X$. Applying ϕ on either sides, we get

$$\nabla_X \xi + \eta(\nabla_X \xi)\xi = \alpha\{X + \eta(X)\xi\}.$$

Since $X(g(\xi, \xi)) = 2g(\nabla_X \xi, \xi)$ and $\nabla_X \xi = \alpha\phi X$, therefore $\eta(\nabla_X \xi) = 0$, and hence $\nabla_\xi \xi = 0$. As we know that $\eta(X) = g(X, \xi)$ and ∇ is metric, then we have $\nabla_\xi \eta = 0$. The Lie-derivative of ϕ along ξ gives

$$(L_\xi \phi)(X) = [\xi, \phi X] - \phi([\xi, X]) = \nabla_\xi \phi X - \phi(\nabla_\xi X) = (\nabla_\xi \phi)(X) = 0, i.e., L_\xi \phi = 0.$$

Again, $(L_\xi \eta)(X) = \xi(\eta(X) - \eta([\xi, X])) = g(X, \nabla_\xi \xi) + g(\nabla_X \xi, \xi) = 0$, *i.e.*, $L_\xi \eta = 0$. Also, if $L_\xi \eta = 0$, then $L_\xi \eta \otimes \eta = 0$, as $L_\xi \eta \otimes \eta = (L_\xi \eta) \otimes \eta + \eta \otimes (L_\xi \eta)$. Again $(L_\xi g)(X, Y) = \xi\, g(X, Y) - g([\xi, X], Y) - g(X, [\xi, Y])$, implies that

$$(L_\xi g)(X, Y) = \alpha[g(\phi X, Y) + g(X, \phi Y)].$$

Using (2.5), we get

$$L_\xi g = 2\alpha(g + \eta \otimes \eta).$$

It is well known that

$$(d\eta)(X, Y) = X(\eta(Y)) - Y(\eta(X)) - \eta([X, Y])$$

implies that

$$(d\eta)(X, Y) = g(Y, \nabla_X \xi) - g(X, \nabla_Y \xi)$$
$$= \alpha\{g(Y, X) + \eta(X)\eta(Y)\} - \alpha\{g(X, Y) + \eta(X)\eta(Y)\} = 0, i.e., d\eta = 0.$$

Finally,

$$N_\phi(X, Y) = \phi^2[X, Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y]$$

yields that

$$N_\phi(X, Y) = \phi^2(\nabla_X Y) - \phi^2(\nabla_Y X) - \phi(\nabla_X \phi Y) + \phi(\nabla_Y \phi X)$$
$$+\nabla_{\phi X} \phi Y - \phi(\nabla_{\phi X} Y) - \nabla_{\phi Y} \phi X + \phi(\nabla_{\phi Y} X) = 0,$$

i.e., the structure is normal. □

In [7] and [8], Shaikh et al. proved that a second order parallel symmetric tensor on a Lorentzian concircular structure manifold with $\alpha^2-\rho \neq 0$ is a constant multiple of the Ricci tensor. Thus we apply this concept for η-Ricci soliton and prove the following results.

Theorem 3. *Let (M, ϕ, ξ, η, g) is an $(LCS)_n$-manifold. If the symmetric tensor field $h = L_\xi g + 2S + 2\mu\, \eta \otimes \eta$ of type $(0, 2)$ is parallel with respect to the Levi-Civita connection ∇, then (g, ξ, λ) on M yields an η-Ricci soliton.*

Proof. In consequence of (3.2), we have

$$h(X, Y) = 2\alpha\, g(X, Y) + 2S(X, Y) + 2(\alpha + \mu)\eta(X)\eta(Y).$$

Replacing X and Y with ξ in the above equation, we get

$$h(\xi, \xi) = (L_\xi g)(\xi, \xi) + 2S(\xi, \xi) + 2\mu\eta(\xi)\eta(\xi) = 2\lambda,$$

and therefore

$$\lambda = \frac{1}{2}h(\xi, \xi).$$

From [7] and [8], we have

$$h(X, Y) = -h(\xi, \xi)g(X, Y),\ \forall\, X,\, Y \in \chi(M).$$

Thus, $L_\xi g + 2S + 2\mu\eta \otimes \eta = -2\lambda g$. Hence the statement of the theorem. □

If $\mu = 0$, it follows that $L_\xi g + 2S + 2(n-1)(\alpha^2 - \rho)g = 0$. Thus we conclude the following corollary:

Corollary 4. *On an $(LCS)_n$-manifold (M, ϕ, ξ, η, g) with the property that a symmetric tensor field $h = L_\xi g + 2S$ of type $(0, 2)$ is parallel with respect to the Levi-Civita connection associated to g, then the equation (3.1), for $\mu = 0$ and $\lambda = (n-1)(\alpha^2-\rho)$, define a Ricci soliton.*

An $(LCS)_n$ manifold (M, ϕ, ξ, η, g) is said to be quasi-Einstein if its Ricci tensor S is a linear combination (with real scalars λ and $\mu(\neq 0)$) of g and the tensor product of a non-zero 1-form η satisfying (2.1) and for an Einstein if S is collinear with g [6]. From (3.5), we state the results in the form of corollary as:

Corollary 5. *If the equation (3.5) define an η-Ricci soliton on an $(LCS)_n$-manifold, then (M, g) is quasi-Einstein.*

Next, we prove the following theorem as:

Theorem 6. *Let (g, ξ, λ, μ) is an η-Ricci soliton on an $(LCS)_n$-manifold (M, ϕ, ξ, η, g). If the Ricci tensor S of M is*
(i) cyclic parallel, then $\mu = -\alpha - \frac{\rho}{2\alpha}$, and $\lambda = -\frac{\rho}{2\alpha}(1-2\alpha(n-1)) - \alpha(1+(n-1)\alpha)$.
(ii) cyclic parallel η-recurrent, then there does not exist an η-Ricci soliton or a Ricci soliton with the potential vector field ξ on M.

Proof. It is well known that

$$(\nabla_X S)(Y,Z) = X(S(Y,Z)) - S(\nabla_X Y, Z) - S(Y, \nabla_X Z). \tag{3.8}$$

In view of (2.2), (2.3) and (3.5), the equation (3.8) reduces to

$$(\nabla_X S)(Y,Z) = -\rho g(\phi Y, \phi Z)\eta(X) - \alpha(\alpha+\mu)\{g(\phi X, \phi Z)\eta(Y) + g(\phi X, \phi Y)\eta(Z)\}. \tag{3.9}$$

If possible, we suppose that the Ricci tensor S of M is cyclic parallel, that is, $(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(Z,Y) = 0 \ \forall \ X,\ Y,\ Z \in \chi(M)$. The cyclic sum of (3.9) together with the last argument give

$$\begin{aligned} &-\rho\{g(\phi Y,\phi Z)\eta(X) + g(\phi X,\phi Z)\eta(Y) + g(\phi Y,\phi X)\eta(Z)\} \\ &\quad -2\alpha(\alpha+\mu)\{g(\phi X,\phi Z)\eta(Y) + g(\phi X,\phi Y)\eta(Z) + g(\phi Y,\phi Z)\eta(X)\} = 0. \end{aligned} \tag{3.10}$$

Replacing $Z = \xi$ in (3.10), we have

$$(\rho + 2\alpha(\alpha+\mu))g(\phi X, \phi Y) = 0$$

for any $X, Y \in \chi(M)$. It follows that $\rho + 2\alpha(\alpha+\mu) = 0$ and thus (3.7) gives $\mu = -\alpha - \frac{\rho}{2\alpha}$, and $\lambda = -\frac{\rho}{2\alpha}(1-2\alpha(n-1)) - \alpha(1+(n-1)\alpha)$. To prove the result (ii), we suppose that M is η-recurrent, that is, $(\nabla_X S)(Y,Z) = \eta(X)S(Y,Z) \ \forall \ X,\ Y,\ Z \in \chi(M)$. If the Ricci tensor S of the η-recurrent $(LCS)_n$-manifold is cyclic parallel, then

$$\begin{aligned} &\eta(X)S(Y,Z) + \eta(Y)S(Z,X) + \eta(Z)S(X,Y) \\ &\quad = -\rho\{g(\phi Y,\phi Z)\eta(X) + g(\phi X,\phi Z)\eta(Y) + g(\phi Y,\phi X)\eta(Z)\} \\ &\quad -2\alpha(\alpha+\mu)\{g(\phi X,\phi Z)\eta(Y) + g(\phi X,\phi Y)\eta(Z) + g(\phi Y,\phi Z)\eta(X)\} = 0 \end{aligned} \tag{3.11}$$

for any $X, Y, Z \in \chi(M)$. Taking $Y = Z = \xi$ in (3.11) and then using (3.5) and (3.6), we get $3(\mu - \lambda)\eta(X) = 0$ for any $X \in \chi(M)$. It follows that $\lambda = \mu$, which is a contradiction. Thus the statements of the theorem are proved. □

In view of the Theorem 6, we can state the following corollaries.

Corollary 7. *In an $(LCS)_n$-manifold (M, ϕ, ξ, η, g) equipped with a cyclic parallel Ricci tensor, there is no Ricci soliton with the potential vector field ξ.*

Corollary 8. *If an $(LCS)_n$-manifold (M, ϕ, ξ, η, g) possesses a cyclic parallel η-recurrent Ricci tensor, then M does not admit η-Ricci soliton or Ricci soliton with the potential vector field ξ.*

Theorem 9. *Let (g, ξ, λ, μ) be an η-Ricci soliton on an $(LCS)_n$-manifold (M, ϕ, ξ, η, g). If the Ricci tensor S of M satisfies*
(i) $\nabla S = 0$, then $\mu = -\alpha + \frac{\xi\alpha}{\alpha}$, and $\lambda = \frac{\xi\alpha}{\alpha} - \alpha - (n-1)(\alpha^2 - \rho)$.
(ii) $\nabla S = \eta \otimes S$, then there does not exist η-Ricci soliton or Ricci soliton with the potential vector field ξ on M.

Proof. Let us suppose that the Ricci tensor S of M satisfies $\nabla S = 0$, that is, M is Ricci symmetric $(LCS)_n$-manifold. Replacing Z by ξ in (3.10), we obtain

$$\{\alpha(\alpha+\mu)+\rho\}g(\phi X, \phi Y) = 0, \quad \forall \ X, Y \in \chi(M).$$

It follows that $\mu = -\alpha + \frac{\xi\alpha}{\alpha}$, and $\lambda = \frac{\xi\alpha}{\alpha} - \alpha - (n-1)(\alpha^2 - \rho)$, the statement (i). Let M is η-recurrent $(LCS)_n$-manifold, that is, $\nabla S = \eta \otimes S$. From (3.5) we obtain $\lambda = \mu$, which is not possible. Thus our theorem is proved. □

In consequence of the Theorem 9, we state the following corollaries.

Corollary 10. *If an $(LCS)_n$-manifold (M, ϕ, ξ, η, g) is Ricci symmetric, then there is no Ricci soliton with the potential vector field ξ on M.*

Corollary 11. *If an $(LCS)_n$-manifold (M, ϕ, ξ, η, g) is admitting an η-recurrent Ricci tensor, then there does not exist η-Ricci soliton or Ricci soliton with the potential vector field ξ on M.*

4 η-Ricci solitons satisfying certain curvature conditions on the $(LCS)_n$-manifolds (M, ϕ, ξ, η, g)

In 1970, Pokhariyal et al. [32], defined and studied the properties of W_2-curvature tenor, and is given by

$$W_2(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}\{g(X,Z)QY - g(Y,Z)QX\} \tag{4.1}$$

for X, Y, $Z \in \chi(M)$.

Theorem 12. *If an $(LCS)_n$-manifold (M, ϕ, ξ, η, g) equipped with an η-Ricci soliton (g, ξ, λ, μ) satisfies $R(\xi, X) \cdot S = 0$, then $\mu = -\alpha$ and $\lambda = -\alpha - (n-1)(\alpha^2 - \rho)$.*

Proof. Suppose M satisfies $R(\xi, X) \cdot S = 0$. Then we have

$$S(R(\xi, X)Y, Z) + S(Y, R(\xi, X)Z) = 0$$

for any X, Y, $Z \in \chi(M)$. Using (2.9) and (3.5) in the above equation, we yield

$$(\alpha^2 - \rho)(\mu + \alpha)\{g(X,Y)\eta(Z) + g(X,Z)\eta(Y) + 2\eta(X)\eta(Y)\eta(Z)\} = 0.$$

For $Z = \xi$, we have

$$(\alpha^2 - \rho)(\mu + \alpha)\{g(X,Y) + \eta(X)\eta(Y)\} = 0.$$

It is obvious from the above equation that $\mu = -\alpha$, provided $\alpha^2 - \rho \neq 0$. Equation (3.7) together with the last result give $\lambda = -\alpha - (n-1)(\alpha^2 - \rho)$. Hence the statement of the theorem is proved. □

With the help of the Theorem 12, we state the following corollaries.

Corollary 13. *Let an $(LCS)_n$-manifold (M, ϕ, ξ, η, g) equipped with the η-Ricci soliton satisfies $R(\xi, X) \cdot S = 0$. Then there is no Ricci soliton on M with the potential vector field ξ.*

Corollary 14. *An $(LCS)_n$-manifold (M, ϕ, ξ, η, g) together with the η-Ricci soliton (g, ξ, λ, μ) and $R(\xi, X) \cdot S = 0$ is Einstein.*

Theorem 15. *If an $(LCS)_n$-manifold (M, ϕ, ξ, η, g) with an η-Ricci soliton satisfies $W_2(\xi, X) \cdot S = 0$, then either $\mu = -\alpha$, $\lambda = \alpha - (n-1)(\alpha^2 - \rho)$ or $\lambda = -\alpha$, $\mu = -\alpha + (n-1)(\alpha^2 - \rho)$.*

Proof. If possible, we assume that the $(LCS)_n$-manifolds endowed with the η-Ricci solitons are W_2-Ricci symmetric, that is, $W_2(\xi, X) \cdot S = 0$. Thus we have

$$S(W_2(\xi, X)Y, Z) + S(Y, W_2(\xi, X)Z) = 0 \tag{4.2}$$

for any X, Y, $Z \in \chi(M)$. Using (3.5) and (4.1) in (4.2), we get

$$\begin{aligned} &(\alpha^2 - \rho)\left[g(X,Y)S(\xi,Z) + g(X,Z)S(Y,\xi) - S(X,Z)\eta(Y) - S(X,Y)\eta(Z)\right] \\ &-\tfrac{1}{n-1}[(\alpha+\lambda)\{S(X,Z)\eta(Y) + \eta(Z)S(Y,X)\} + (\alpha+\mu)\{\eta(X)\eta(Y)S(\xi,Z) \\ &+\eta(X)\eta(Z)S(Y,\xi)\} + (\mu-\lambda)\{g(X,Y)S(\xi,Z) + g(X,Z)S(Y,\xi)\}] = 0. \end{aligned} \tag{4.3}$$

In consequence of (3.5)-(3.7), equation (4.3) consider the form

$$\frac{(\alpha+\mu)(\alpha+\lambda)}{n-1}\{\eta(Y)g(X,Y) + \eta(Z)g(X,Y) + 2\eta(X)\eta(Y)\eta(Z)\} = 0. \tag{4.4}$$

Taking Z $= \xi$ in (4.4), we yield

$$\frac{(\alpha+\mu)(\alpha+\lambda)}{n-1} g(\phi X, \phi Y) = 0 \tag{4.5}$$

for any X, $Y \in \chi(M)$. In general, $g \neq 0$ on M, therefore (4.5) shows that either $\mu = -\alpha$ or $\lambda = -\alpha$, for $n > 1$. These results together with (3.7) reflect that either $\mu = -\alpha$, $\lambda = \alpha - (n-1)(\alpha^2 - \rho)$ or $\lambda = -\alpha$, $\mu = -\alpha + (n-1)(\alpha^2 - \rho)$ on M. □

Corollary 16. *If an $(LCS)_n$-manifold (M, ϕ, ξ, η, g) satisfies $W_2(\xi, X) \cdot S = 0$, then there is no Ricci soliton with the potential vector field ξ on M.*

5 Examples of η-Ricci soliton on $(LCS)_n$-manifolds

Example 17. *Let a 3-dimensional manifold $M = \left\{(x, y, z) \in \Re^3 : z \neq 0\right\}$, where (x, y, z) are the standard coordinates in $\Re^3$. Let $\{E_1, E_2, E_3\}$ be a linearly independent global frame on M given by*

$$E_1 = e^z\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right), \qquad E_2 = e^z\frac{\partial}{\partial y}, \qquad E_3 = e^{2z}\frac{\partial}{\partial z}.$$

Assume that g be the Lorentzian metric on M, and is defined by

$$g(E_1, E_3) = g(E_2, E_3) = g(E_1, E_2) = 0, \; g(E_1, E_1) = g(E_2, E_2) = 1, \; g(E_3, E_3) = -1.$$

Let η be the 1-form defined by $\eta(V) = g(V, E_3)$ for any $V \in \chi(M)$ and ϕ is a $(1,1)$-tensor field defined by $\phi E_1 = E_1$, $\phi E_2 = E_2$, $\phi E_3 = 0$. Then using the linearity of ϕ and g we have

$$\eta(E_3) = -1, \quad \phi^2 V = V + \eta(V) E_3, \; g(\phi V, \phi W) = g(V, W) + \eta(V)\eta(W)$$

for any $V, W \in \chi(M)$. Let ∇ be the Levi-Civita connection with respect to the Lorentzian metric g and R be the curvature tensor of g. Then we obtain

$$[E_1, E_2] = -e^z E_2, \qquad [E_1, E_3] = -e^{2z} E_1, \qquad [E_2, E_3] = -e^{2z} E_2.$$

Taking $E_3 = \xi$ and using the Koszul's formula for the Lorentzian metric g, we have

$$\nabla_{E_1} E_3 = -e^{2z} E_1, \qquad \nabla_{E_1} E_1 = -e^{2z} E_3, \qquad \nabla_{E_1} E_2 = 0,$$
$$\nabla_{E_2} E_3 = -e^{2z} E_2, \qquad \nabla_{E_3} E_2 = 0, \qquad \nabla_{E_2} E_1 = e^z E_2,$$
$$\nabla_{E_3} E_3 = 0, \qquad \nabla_{E_2} E_2 = e^{2z} E_3 - e^z E_1, \qquad \nabla_{E_3} E_1 = 0.$$

From the above equations, it can be easily seen that $E_3 = \xi$ is a unit timelike concircular vector field and hence the structure (ϕ, ξ, η, g) is an $(LCS)_3$-structure on M. Consequently, $M^3(\phi, \xi, \eta, g)$ is an $(LCS)_3$-manifold with $\alpha = -e^{2z} \neq 0$ such that $(X\alpha) = \rho\eta(X)$, where $\rho = 2e^{4z}$. Using the above relations, we can easily calculate the non-vanishing components of the curvature tensor R and the Ricci tensor S as follows:

$$R(E_2, E_3)E_3 = -e^{4z} E_2, \quad R(E_1, E_3)E_3 = -e^{4z} E_1, \quad R(E_1, E_2)E_2 = \{e^{4z} - e^{2z}\} E_1,$$

$$R(E_2, E_3)E_2 = e^{4z} E_3 - e^{3z} E_1, \; R(E_1, E_3)E_1 = -e^{4z} E_3, \; R(E_2, E_1)E_1 = \{e^{4z} - e^{2z}\} E_2,$$
$$S(E_1, E_1) = -e^{2z}, \qquad S(E_2, E_2) = -e^{2z}, \qquad S(E_3, E_3) = -2e^{4Z}.$$

Also from the equation (3.5), we can see that

$$S(E_1, E_1) = -(\alpha + \lambda), \qquad S(E_2, E_2) = -(\alpha + \lambda), \qquad S(E_3, E_3) = (\lambda - \mu).$$

Thus we conclude from the last two expressions that for $\alpha = -e^{2z}$, $\lambda = 2e^{2z}$ and $\mu = 2\{e^{2z} + e^{4z}\}$, the structure (g, ξ, λ, μ) is an η-Ricci soliton on $M^3(\phi, \xi, \eta, g)$.

Example 18. *Let a 3-dimensional manifold $M = \left\{(x, y, z) \in \Re^3 : z \neq 0\right\}$, where (x, y, z) are the standard coordinates in $\Re^3$. In [35], Shaikh defined the linearly independent vector fields $\{E_1, E_2, E_3\}$ on M as:*

$$E_1 = e^{-z}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right), \qquad E_2 = e^{-z}\frac{\partial}{\partial y}, \qquad E_3 = e^{-2z}\frac{\partial}{\partial z}.$$

Let g be the Lorentzian metric defined by

$$g(E_1, E_3) = g(E_2, E_3) = g(E_1, E_2) = 0,\ g(E_1, E_1) = g(E_2, E_2) = 1,\ g(E_3, E_3) = -1.$$

Let η be the 1-form defined by $\eta(V) = g(V, E_3)$ for any $V \in \chi(M)$. Let ϕ be the $(1,1)$-tensor field defined by $\phi E_1 = E_1$, $\phi E_2 = E_2$, $\phi E_3 = 0$. Then using the linearity of ϕ and g we have

$$\eta(E_3) = -1, \quad \phi^2 V = V + \eta(V)E_3,\ g(\phi V, \phi W) = g(V, W) + \eta(V)\eta(W),$$

for any $V, W \in \chi(M)$. Let ∇ be the Levi-Civita connection with respect to the Lorentzian metric g and R be the curvature tensor of g. Then we obtain

$$[E_1, E_2] = -e^{-z}E_2, \qquad [E_1, E_3] = -e^{-2z}E_1, \qquad [E_2, E_3] = -e^{-2z}E_2.$$

Taking $E_3 = \xi$ and using the Koszul's formula for the Lorentzian metric g, we have

$$\nabla_{E_1}E_3 = e^{-2z}E_1, \qquad \nabla_{E_1}E_1 = e^{-2z}E_3, \qquad \nabla_{E_1}E_2 = 0,$$

$$\nabla_{E_2}E_3 = e^{-2z}E_2, \qquad \nabla_{E_3}E_2 = 0, \qquad \nabla_{E_2}E_1 = e^{-2z}E_2,$$

$$\nabla_{E_3}E_3 = 0, \qquad \nabla_{E_2}E_2 = e^{-2z}E_3 - e^{-z}E_1, \qquad \nabla_{E_3}E_1 = 0.$$

From the above equations, it can be easily seen that $E_3 = \xi$ is a unit timelike concircular vector field and hence (ϕ, ξ, η, g) is an $(LCS)_3$-structure on M. Thus $M^3(\phi, \xi, \eta, g)$ is an $(LCS)_3$-manifold with $\alpha = e^{-2z} \neq 0$ such that $(X\alpha) = \rho\eta(X)$, where $\rho = 2e^{-4z}$. Using the above relations, we can easily calculate the non-vanishing components of the curvature tensor R and the Ricci tensor S as follows:

$$R(E_2, E_3)E_3 = e^{-4z}E_2,\ \ R(E_1, E_3)E_3 = e^{-4z}E_1,\ \ R(E_1, E_2)E_2 = \{e^{-4z} - e^{-2z}\}E_1,$$

$$R(E_2, E_3)E_2 = e^{-4z}E_3,\ \ R(E_1, E_3)E_1 = e^{-4z}E_3,\ \ R(E_1, E_2)E_1 = \{-e^{-4z} - e^{-2z}\}E_2,$$

$$S(E_1, E_1) = 2e^{-4z} - e^{-2z}, \qquad S(E_2, E_2) = 2e^{-4z} - e^{-2z}, \qquad S(E_3, E_3) = 2e^{-4z}.$$

Also from (3.5), we calculate that

$$S(E_1, E_1) = -(\alpha + \lambda), \qquad S(E_2, E_2) = -(\alpha + \lambda), \qquad S(E_3, E_3) = (\lambda - \mu).$$

We conclude from (3.5) that for $\alpha = e^{2z}, \lambda = -2e^{-4z}$ and $\mu = -4e^{-4z}$, the data (g, ξ, λ, μ) admits an η-Ricci soliton on $M^3(\phi, \xi, \eta, g)$.

Acknowledgement. The authors are thankful to the Editor and anonymous referees for their valuable comments.

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References

[1] C. S. Bagewadi and G. Ingalahalli, *Ricci soliton in Lorentzian α-Sasakian manifolds*, Acta Math. Acad. Paedagog. Nyházi. (N.S.) **28**(2012), no. 1, 59-68. MR2942704. Zbl 1265.53036.

[2] C. S. Bagewadi, G. Ingalahalli and S. R. Ashoka, *A Study of Ricci soliton in Kenmotsu manifolds,* ISRN Geometry, (2013). Article ID 412593, 6 pages. MR3092012. Zbl 1266.53032.

[3] C. L. Bejan and M. Crasmareanu, *Second order parallel tensor and Ricci solitons in 3-dimensional normal paracontact geometry*, Anal. Global Anal. Geom. **46** (2014), 117–127. DOI:10.1007/s10455-014-9414-4. MR3239277. Zbl 1312.53049.

[4] C. Calin and M. Crasmareanu, *Eta-Ricci soliton on Hopf hypersurfaces in complex space forms*, Rev. Roumaine Math. Pures Appl. **57**(2012), no. 1, 55-63. MR3051979. Zbl 1389.53084.

[5] C. Calin, M. Crasmareanu, *From the Eisenhart problem to Ricci soliton in f-Kenmotsu manifolds*, Bull. Malaysian Math. Sci. Soc. **33**(2010), no. 3, 361-368. MR2732157. Zbl 1204.53024.

[6] G. Calvaruso and D. Perrone, *Geometry of H-paracontact metric manifolds.* Publ. Math. Debrecen **86** (2015), no. 3-4, 325–346. MR3346090. Zbl 1374.53111.

[7] S. Chandra, S. K. Hui and A. A. Shaikh, *Second order parallel tensors and Ricci solitons on $(LCS)_n$-manifolds,* Commun. Korean Math. Soc. **30**(2015), no. 2, 123-130. MR3346486. Zbl 1338.53055.

[8] S. K. Chaubey and A. A. Shaikh, *On 3-dimensional lorentzian concircular structure manifolds*, Commun. Korean Math. Soc., (2018) https://doi.org/10.4134/CKMS.c180044.

[9] S. K. Chaubey and S. K. Yadav, *W-semisymmetric generalized Sasakian-space-forms*, Adv. Pure Appl. Math. (2018) https://doi.org/10.1515/apam-2018-0032.

[10] S. K. Chaubey, *On special weakly Ricci-symmetric and generalized Ricci-recurrent trans-Sasakian manifolds*, Thai Journal of Mathematics, **18** (2018), no. 3 (in press).

[11] S. K. Chaubey, *Some properties of LP-Sasakian manifolds equipped with m-projective curvature tensor*, Bull. Math. Anal. Appl. **3** (2011), no. 4, 50–58. MR 2955373. Zbl 1314.53127.

[12] S. K. Chaubey, *Certain results on N(k)-quasi Einstein manifolds*, Afr. Mat. (2018). https://doi.org/10.1007/s13370-018-0631-z.

[13] S. K. Chaubey, K. K. Baishya and M. Danish Siddiqi, *Existence of some classes of $N(k)$-quasi Einstein manifolds*, Bol. Soc. Paran. Mat., doi:10.5269/bspm.41450.

[14] S. K. Chaubey, *On weakly m-projectively symmetric manifolds*, Novi Sad J. Math. **42** (2012), no. 1, 67-79. MR2986147. Zbl 1313.53060.

[15] S. K. Chaubey, *Existence of $N(k)$-quasi Einstein manifolds*, Facta Universitatis (NIŠ) Ser. Math. Inform. **32** (2017), no. 3 , 369–385. MR3719806.

[16] T. Chave and G. Valent, *Quasi-Einstein metrics and their renormalizability properties*, Helv. Phys. Acta. **69** (1996), 344-347. MR1440692. Zbl 0865.53039.

[17] B. Y. Chen and S. Deshmukh, *Geometry of compact shrinking Ricci solitons*, Balkan J. Geom. Appl. **19**(2014), no. 1, 13-21. MR3223305.

[18] J. T. Cho and M. Kimura, *Ricci soliton and real hypersurfaes in a complex space form,* Tohoku Math. J. **61**(2009), no. 2, 205-212. MR2541405. Zbl 1172.53021.

[19] B. Chow, P. Lu and L. Ni, *Hamilton's Ricci flow*, Graduate Studies in Mathematics, 77, AMS, Providence, RI, USA (2006). MR2274812.

[20] R. S. Hamilton, *The Ricci flow on surfaces*, Math. and general relativity (Santa Cruz, CA, 1986), 237-262, Contemp. Math. 71, AMS (1988). MR954419. Zbl 0663.53031.

[21] C. He and M. Zhu, *The Ricci soliton on Sasakian manifold*, arXiv: 1109.4404v2. (2011). MR3719806. Zbl 1265.53036.

[22] S. K. Hui, R. S. Lemence and D. Chakraborty, *Ricci solitons on Ricci pseudosymmetric $(LCS)_n$-manifolds*, Honam Mathematical journal, **40** (2018), 325-346. MR3821461.

[23] S. K. Hui, R. Prasad and T. Pal, *Ricci solitons on submanifolds of $(LCS)_n$-manifolds*, Ganita, **68** (2018), no. 1, 53-63. MR3814887.

[24] S. K. Hui and M. Atceken, *Contact warped product semi-slant submanifolds of $(LCS)_n$-manifolds*, Acta Univ. Sapientiae Math. **3**(2011), no. 2, 212-224. MR2915836. Zbl 1260.53081.

[25] S. K. Hui, *On ϕ-pseudo symmetries of $(LCS)_n$-manifolds,* Kyungpook Math. J. **53**(2013), no. 2, 285-294. MR3078089. Zbl 1286.53038.

[26] G. Ingalahalli and C. S. Bagewadi, *Ricci soliton on α-Sasakian manifolds,* ISRN Geometry, Article ID 421384, (2012), 13 pages. http://dx.doi.org/10.5402/2012/421384.

[27] K. Matsumoto, *On Lorentzian almost paracontact manifolds*, Bull. Yamagata Univ. Nature. Sci. **12** (1989), 151-156.

[28] I. Mihai and R. Rosca, *On Lorentzian para-Sasakian manifolds,* Classical Anal. (1992), 155-169, W.S. Publ., Singapore.

[29] D. Narain and S. Yadav, *On weak Symmetric of Lorentzian concircular structure manifolds*, CUBO A Mathematical Journal **15** (2013), no. 2, 33-42. MR3114862. Zbl 1279.53017.

[30] B. O'Neill, *Semi Riemannian geometry with applications to relativity,* Academic Press, New York, (1983).

[31] G. P. Pokhariyal, S. Yadav and S. K. Chaubey, *Ricci solitons on trans-Sasakian manifolds, Differential Geometry-Dynamical Systems* **20** (2018), pp. 138-158. MR3847743. Zbl 1395.53089.

[32] G. P. Pokhariyal and R. S. Mishra, *The curvature tensor and their relativistic significance*, Yokohama Math. J. **18** (1970), 105-108.

[33] A. A. Shaikh, *On Lorentzian almost paracontact manifolds with a structure of the concircular type,* Kyungpook Math. J. **43** (2003), 305-314. MR1983436. Zbl 1054.53056.

[34] A. A. Shaikh, T. Basu and S. Eyasmin, *On locally ϕ-symmetric$(LCS)_n$-manifolds*, Int. J. Pure Appl. Math. **41** (2007), no. 8, 1161-1170. MR2384602. Zbl 1136.53030.

[35] A. A. Shaikh, *Some results on $(LCS)_n$-manifolds*, J. Korean Math. Soc. **46** (2009), no. 3, 449-461. MR2515129. Zbl 1193.53094.

[36] A. A. Shaikh, T. Basu and S. Eyasmin, *On the existence of φ-recurrent $(LCS)_n$-manifolds*, Extr. Math. **23** (2008), no. 1, 71-83. Zbl 1165.53013.

[37] R. Sharma, *Certain results on K-contact and (κ, μ)-contact manifolds,* J. of Geometry, **89** (2008), 138-147. MR2457028. Zbl 1175.53060.

[38] M. M. Tripathi, *Ricci solitons in contact metric manifolds*, (2008) http//arxiv.org/abs/0801.4222.

[39] S. Yadav, D. L. Suthar and P. K. Dwivedi, *Some results on $(LCS)_{2n+1}$-manifolds* , International journal of Mathematics, Engineering & Technology **6** (2013), 73-84.

[40] S. Yadav, P. K. Dwivedi and D. L. Suthar, *On $(LCS)_{2n+1}$- manifolds satisfying certain conditions on the concircular curvature tensor*, Thai Journal of Mathematics **9** (2011), 597-603. MR2862053. Zbl 1261.53023.

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[41] S. K. Yadav, D. L. Suthar and M. Hailu, *On extended generalized φ-recurrent $(LCS)_{2n+1}$-manifolds,* Bol. Soc. Paran. Mat. (3s.) v. **37** (2019), no. 2, 9-21. MR3845155.

[42] S. K. Yadav, S. K. Chaubey and D. L. Suthar, *Some geometric properties of η-Ricci solitons and gradient Ricci solitons on $(lcs)_n$-manifolds*, CUBO A Mathematical Journal, **19** (2017), no. 2, 33-48. MR3800489.

[43] S. K. Yadav, S. K. Chaubey and D. L. Suthar, *Certain results on almost Kenmotsu (κ, μ, ν)-spaces*, Konuralp Journal of Mathematics **6** (2018), no. 1, 128-133. MR3800439.

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