Electronic Journal: Southwest Journal of Pure and Applied Mathematics
Internet: http://rattler.cameron.edu/swjpam.html
ISSN 1083-0464
Issue 1, June 2001, pp. 93–101.
Submitted: April 13, 2001. Published: June 30, 2001.

FIXED POINTS FOR W-CONTRACTIVE OR W-EXPANSIVE MAPS IN UNIFORM SPACES: TOWARD A UNIFIED APPROACH

Jaime Rodríguez–Montes And Jairo A. Charris

Abstract A simple condition on the functions $\varphi : \mathbb{R}^+ \to \mathbb{R}^+ = [0, +\infty)$ allows to establish results on fixed and coincidence points for contractive (when $\varphi(t) < t$ for all $t > 0$)or expansive (provided $\varphi(t) > t,\ t > 0$) maps with respect to a w-distance. These generalize recent results of different people, including the authors of the present paper. As in a previous work, an attempt is made toward a unification of the point of view in contractions and expansions.

1991 A.M.S. Subject Classification Codes:. 4H25, 47H10

Key words and phrases:. Metric (distance), w-distance, contractive and expansive maps for a distance or a w-distance, fixed and coincidence points of maps, Cauchy sequence, uniform and complete uniform spaces.

1.Introduction

A fundamental result in fixed point theory is the Banach contraction principle, which has been extended in many different directions ([2], [4], [9]). Of somewhat lesser importance is the expansion principle ([5], [8], [12], [13]). Recently, O. Kada , T. Suzuki and W. Takahashi [7] have introduced the concept of a w-distance on a metric space and have generalized important results in non-convex minimizations and in fixed point theory for both w-contractive or w-expansive maps.

In this paper we follow ideas in [4], [11], [12] to establish some new results on fixed and coincidence points of maps by means of appropriate w-contractive or w-expansive assumptions. These generalize results in [4], [6], [11], [12], [13] and in many other places. As in [12], a unified point of view for contractions and expansions is sought after by means of the same techniques in that paper but now adapted to w-distances.

Universidad Nacional, Bogotá, Colombia
E-mail: jarodrig@matematicas.unal.edu.co

Typeset by $\mathcal{AMS}$-TEX

In this section we review the basic definitions and the known results that are relevant to our purposes. To begin with, we slightly generalize the concept of a w-distance on a metric space in [7] to that of a w-distance on a *uniform space*. For the description of the latter concept and related notions we refer to [1], Chapter II or to [3], Chapter IX. We call *uniform space* (X, ϑ) a nonempty set X endowed of a *uniformity* ϑ, the latter being a special kind of filter on $X \times X$, all whose elements contain the diagonal $\Delta = \{(x,x)|x \in X\}$.

If $V \in \vartheta$ and $(x,y) \in V$, $(y,x) \in V$, x and y are said to be V–close, and a sequence (x_n) in X is a *Cauchy sequence* for ϑ if for any $V \in \vartheta$, $N \geq 1$ exists such that x_m and x_n are V-close for $m,\ n \geq N$. A uniformity ϑ defines a unique topology $\tau(\vartheta)$ on X for which the neighborhoods of $x \in X$ are the sets $V(x) = \{y \in X|(x,y) \in V\}$ when V runs over ϑ, and a subspace Y of X is called *complete* if any Cauchy sequence of Y for ϑ converges to a point of Y for $\tau(\vartheta)$ (as a matter of fact, this is the definition of *sequential completeness*, the more general concept of *completeness* requiring the notion of a *Cauchy filter* ([1] pp. 197, 201), but it is all we need for our purposes). A uniform space (X, ϑ) is said to be *Hausdorff, compact, connected,* etc., if the topological space $(X, \tau(\vartheta))$ has the corresponding property.
We will assume all the relevant uniform spaces (X, ϑ) *to be Hausdorff*, which ocurrs if and only if the intersection of all the $V \in \vartheta$ reduces to the diagonal Δ of X (([1], p. 187); i.e., if $(x,y) \in V$ for all $V \in \vartheta$ implies $x = y$). This guarantees the uniqueness of limits of sequences.

A metric or distance d on X defines a Hausdorff uniformity, called the *uniformity of d*. This is the set ϑ_d of all subsets of $X \times X$ containing a "*band*" $B_\epsilon = \{(x,y)|d(x,y) < \epsilon\}$ for some $\epsilon > 0$. The topology $\tau(\vartheta_d)$ on X is the usual topology τ_d of d. Hence, results on general uniform spaces apply to metric spaces, but many other mathematical objects carry natural uniformities. Such is the case of topological groups, topological vector spaces, general (not necessarily metrizable) compact spaces, and also of subspaces of all these spaces. Thus, results on general uniform spaces apply equally well to these objects, which was the motivation for A. Weil [14] introduction of the concept. This also widens the scope of results on w-contractions or w-expansions.

Remark 1.1. If (X, ϑ) is a uniform space and $V \in \vartheta$, also $V^{-1} = \{(y,x)|(x,y) \in V\} \in \vartheta$ ([1], p. 181). If $V \in \vartheta$ and $V^{-1} = V$, V is called *symmetrical.* Observe that if V is symmetrical and $(x,y) \in V$ then $(y,x) \in V$, and x and y are automatically V-close. Since $V \cap V^{-1} \subseteq V$, any $V \in \vartheta$ contains a $W \in \vartheta$, symmetrical, and if $(x,y) \in W$ then x and y are both W and V-close. For many purposes it is then enough to assume $V \in \vartheta$ symmetrical.

Remark 1.2. When topological concepts are mentioned in the context of a uniform space (X, ϑ) (continuity or semi-continuity of maps on X, for example), they always refer to the topological space $(X, \tau(\vartheta))$.

Definition 1.1. *Let* (X, ϑ) *be a uniform space. A function* $\mathfrak{p} : X \times X \longrightarrow \mathbb{R}^+ = [0, \infty)$ *is called a w-distance on* X *(or more precisely, on* (X, ϑ)*), if the following conditions hold for all* x, y, z *in* X*:*

(1) $\mathfrak{p}(x,z) \leq \mathfrak{p}(x,y) + \mathfrak{p}(y,z)$.
(2) $\mathfrak{p}(x, \cdot) : X \longrightarrow \mathbb{R}^+$ *is lower semi-continuous.*

(3) *For any $V \in \vartheta$ there is $\delta > 0$ such that if $\mathfrak{p}(z,x) \leq \delta$ and $\mathfrak{p}(z,y) \leq \delta$ then $(x,y) \in V$.*

For example, if $\mathfrak{p}$ is a distance on X satisfying (2) above and such that $\vartheta \subseteq \vartheta_{\mathfrak{p}}$, then $\mathfrak{p}$ is a w-distance on (X, ϑ). Definition 1.1 is in [7] within the frame of metric spaces.

The following lemma contains some useful properties of w-distances. It is stated in [7] for metric spaces. The proof is straightforward.

Lemma 1.1. *Let (X, ϑ) be a uniform space and $\mathfrak{p}$ be a w-distance on X. Let (x_n), (y_n) be arbitrary sequences in X and (α_n), (β_n) be sequences in $\mathbb{R}^+$ converging to 0. Then, for $x, y, z \in X$, the following holds:*

(a) *If $\mathfrak{p}(x_n, y) \leq \alpha_n$ and $\mathfrak{p}(x_n, z) \leq \beta_n$ for all $n \in \mathbb{N}$, then $y = z$. In particular, if $\mathfrak{p}(x,y) = 0$ and $\mathfrak{p}(x,z) = 0$, then $y = z$.*
(b) *If $\mathfrak{p}(x_n, y_n) \leq \alpha_n$ and $\mathfrak{p}(x_n, z) \leq \beta_n$ for all $n \in \mathbb{N}$, then (y_n) converges to z.*
(c) *If $\mathfrak{p}(x_n, x_m) \leq \alpha_n$ for all $m, n \in \mathbb{N}$ with $m > n$, then (x_n) is a Cauchy sequence in (X, ϑ).*
(d) *If $\mathfrak{p}(y, x_n) \leq \alpha_n$ for all $n \in \mathbb{N}$, then (x_n) is a Cauchy sequence in (X, ϑ).*

Throughout this paper, Φ and Ψ will respectively denote the sets of maps ϕ and ψ of $\mathbb{R}^+$ into itself such that $\phi(t) < t$ and $\psi(t) > t$ for all $t > 0$, and satisfying the following condition:

Condition(A). *For any $\varphi \in \Phi \cup \Psi$ and any decreasing sequence (t_n) in $\mathbb{R}^+$ such that*

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} \varphi(t_n) = t,$$

it follows that $t = 0$.

Remark 1.3. Condition (A) above is the simple condition mentioned in the abstract. In [12], it proved a versatile and valuable instrument in establishing results on contractions and expansions with respect to the metric d in a metric space (X, d). We will see it is also useful in the context of w-distances. Condition (A) is generally satisfied by functions appearing in the framework of contractions or expansions. It is obviously equivalent to the apparently stronger condition below:

Condition (B). *For any $\varphi \in \Phi \cup \Psi$ and any non-increasing sequence (t_n) in $(0, +\infty)$, if*

$$\lim t_n = \lim \varphi(t_n) = t,$$

then $t = 0$.

Remark 1.4. For $\phi \in \Phi$, it is obvious that $\phi(0+) = \lim_{t\to 0+} \phi(t) = 0$. For $\psi \in \Psi$, it may happen that $\psi(0+)$ either does not exist or $\psi(0+) \neq 0$. The assumption $\psi(0+) = 0$ for $\psi \in \Psi$ is, however, frequently needed.

For $\psi \in \Psi$, we define $\widehat{\psi} : (0, +\infty) \longrightarrow \mathbb{R}^+$ by $\widehat{\psi}(t) = \sup\{x \mid \psi(x) < t\}$ if $\{x \mid \psi(x) < t\} \neq \emptyset$ and $\widehat{\psi}(t) = 0$ otherwise. Clearly $\widehat{\psi}$ is non–decreasing and $0 \leq \widehat{\psi}(t) \leq t$ for any $t > 0$. For $\phi \in \Phi$, we also define $\widetilde{\phi} : (0, +\infty) \to \mathbb{R}^+$ by $\widetilde{\phi}(t) = \sup\{x \mid \phi(x) \leq t\}$. Clearly $\widetilde{\phi}$ is non-decreasing and $\phi(t) \leq \widetilde{\phi}(t) \leq t$ for all $t > 0$.

The next two lemmas appear in [12] and will be repeatedly used in the sequel. For completeness we also include their proofs here.

Lemma 1.2. *If $\psi \in \Psi$ and $\psi(0+) = 0$, then*

(a) $0 < \widehat{\psi}(t) \leq t$, *for all* $t > 0$;
(b) *for any* $\epsilon > 0$ *there is* $0 < t \leq \epsilon$ *such that* $\widehat{\psi}(t) < t$.

Proof. From $\psi(0+) = 0$ it follows that $\widetilde{\psi}(t) > 0$ for $t > 0$. If (b) were not satisfied, $\epsilon > 0$ could be found such that $\widehat{\psi}(t) = t$ for all $0 < t \leq \epsilon$. Let $0 < \alpha < \epsilon$. Since $\widehat{\psi}(\epsilon) = \epsilon$ then, for some t_1, $\alpha < t_1 < \psi(t_1) < \epsilon$, and having selected t_n such that $\alpha < t_n < \psi(t_n)$, we could choose t_{n+1} such that $\alpha < t_{n+1} < \psi(t_{n+1}) < t_n$, from which it follows that $t \geq \alpha > 0$ exists such that $\lim t_n = \lim \psi(t_n) = t > 0$. This is absurd. □

Lemma 1.3. *Let $\phi \in \Phi$. Then, for any $\epsilon > 0$ there is $t \leq \epsilon$ such that $\widetilde{\phi}(t) < t$.*

Proof. Otherwise, for some $\epsilon > 0$, $\widetilde{\phi}(t) = t$ for all $t \leq \epsilon$, and since $\phi(\epsilon) < \epsilon = \widetilde{\phi}(\epsilon)$, for some t_1 in $\mathbb{R}^+$, $\phi(\epsilon) < \phi(t_1) < t_1 < \epsilon$. Thus, having chosen t_n with $\phi(t_{n-1}) < \phi(t_n) < t_n < t_{n-1}$, from $\widetilde{\phi}(t_n) = t_n$ we could also choose t_{n+1} such that $\phi(t_n) < \phi(t_{n+1}) < t_{n+1} < t_n$. Thus, a sequence (t_n) would exist such that $\lim t_n = \lim \phi(t_n) = t > 0$. Since $\phi \in \Phi$, this is contradictory. □

In the next two sections we establish our main results.

2. Existence theorems of the w-contractive type.

Our first lemma adapts Lemma 2.1 in [12] to contractions for a w-distance.

Lemma 2.1. *Let (Y, ϑ) be a uniform space and $X \subseteq Y$ be a complete subspace of Y. Let f be a map of X into Y and $\mathfrak{p}$ be a w-distance on Y. Further assume that:*

i) There is a sequence (x_n) in X such that $x_{n+1} = f(x_n)$ for all $n \geq 1$.
ii) For any $\delta > 0$ there exists $0 < \delta_1 \leq \delta$ such that $\mathfrak{p}(x, f(z)) < \delta$ whenever $\mathfrak{p}(x, f(x)) < \delta_1$ and $\mathfrak{p}(x, z) \leq \delta$.
iii) Given $\delta_1 > 0$, $N \geq 1$ exists such that $\mathfrak{p}(x_N, f(x_N)) < \delta_1$.

Then, there is $x_0 \in X$ such that $f(x_0) = x_0$ and

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} f(x_n) = x_0. \tag{2.1}$$

Proof. Let $V \in \vartheta$, arbitrary, and let $\delta > 0$ small be such that Condition (3) of Definition 1.1 is satisfied. Also let $0 < \delta_1 \leq \delta$ and $N \geq 1$ be such that assertions *(ii)* and *(iii)* above hold. From $\mathfrak{p}(x_N, f(x_N)) = \mathfrak{p}(x_N, x_{N+1}) < \delta_1 \leq \delta$ and *(ii)*, also $\mathfrak{p}(x_N, f(x_{N+1})) < \delta$. Thus, again from $x_{N+2} = f(x_{N+1})$ and *(ii)*, $\mathfrak{p}(x_N, f(x_{N+2})) < \delta$. Continuing this way we conclude that $\mathfrak{p}(x_N, x_{N+p}) < \delta$, $p = 1, 2, 3, ...$, which implies, in view of *(c)* of Lemma 1.1, that (x_n) is a Cauchy sequence. Completeness of X then ensures that (2.1) holds for some $x_0 \in X$.
Now, from $\mathfrak{p}(x_N, x_m) < \delta$ for $m > N$ and the lower semicontinuity of $\mathfrak{p}$ it follows that $\mathfrak{p}(x_N, x_0) \leq \delta$, and hence , by (ii) above, $\mathfrak{p}(x_N, f(x_0)) < \delta$. Condition (3) of Definition 1.1 then guarantees that $(x_0, f(x_0)) \in V$, so that, since V is arbitrary, $f(x_0) = x_0$. Thus, x_0 is a fixed point of f. □

The next theorem, of the contractive type, adapts Theorem 2.1 in [12] to w-distances.

Theorem 2.1. *Let f be a selfmap of a uniform space (X, ϑ), and assume that either X or $f(X)$ is complete. Let $\mathfrak{p}$ be a w-distance on X such that for some $\phi \in \Phi$,*

$$\mathfrak{p}(f(x), f(y)) \leq \phi(\mathfrak{p}(x, y)), \quad x, y \in X. \tag{2.2}$$

Then, a unique fixed point x_0 of f exists such that (2.1) holds for any sequence (x_n) in X satisfying $x_{n+1} = f(x_n), \quad n \geq 1$.

Proof. Since $f(X) \subseteq X$, a sequence (x_n) in X such that $x_{n+1} = f(x_n), \quad n \geq 1$, clearly exists. From (2.2) we get, provided $\mathfrak{p}(x_n, x_{n+1}) > 0$ for all $n \geq 1$, *that*

$$\mathfrak{p}(x_{n+1}, x_{n+2}) = \mathfrak{p}(f(x_n), f(x_{n+1})) \leq \phi(\mathfrak{p}(x_n, x_{n+1})) < \mathfrak{p}(x_n, x_{n+1}).$$

Hence, $t_n = \mathfrak{p}(x_n, x_{n+1}), \quad n \geq 1$, is a decreasing sequence such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} \phi(t_n) = t \geq 0$, so that $t = 0$. Let $\delta > 0$ be such that $\widetilde{\phi}(\delta) < \delta$ (Lemma 1.3), and let $\delta_1 = \delta - \widetilde{\phi}(\delta)$. If $\mathfrak{p}(x, f(x)) < \delta_1$ and $\mathfrak{p}(x, z) < \delta$, it follows from $\mathfrak{p}(f(x), f(z)) \leq \widetilde{\phi}(\mathfrak{p}(x, z)) \leq \widetilde{\phi}(\delta)$ that $\mathfrak{p}(x, f(z)) \leq \mathfrak{p}(x, f(x)) + \mathfrak{p}(f(x), f(z)) < \delta_1 + \widetilde{\phi}(\delta) = \delta$. Hence, from Lemma 2.1, there is x_0 in X such that (2.1) holds, and then $f(x_0) = x_0$. Now assume that $\mathfrak{p}(x_N, x_{N+1}) = 0$ for some $N \geq 1$. Then, from (2.2), also $\mathfrak{p}(x_{N+1}, x_{N+2}) = 0$, and since $\mathfrak{p}(x_N, x_{N+2}) \leq \mathfrak{p}(x_N, x_{N+1}) + \mathfrak{p}(x_{N+1}, x_{N+2})$, then $\mathfrak{p}(x_N, x_{N+2}) = 0$. Hence, from Lemma 1.1(a), $x_{N+1} = x_{N+2} = f(x_{N+1})$, and x_{N+1} is a fixed point of f. Finally assume that also $f(\widetilde{x}_0) = \widetilde{x}_0$. From (2.2) we have, provided $\mathfrak{p}(x_0, \widetilde{x}_0) > 0$, that $\mathfrak{p}(x_0, \widetilde{x}_0) = \mathfrak{p}(f(x_0), f(\widetilde{x}_0)) \leq \phi(\mathfrak{p}(x_0, \widetilde{x}_0)) < \mathfrak{p}(x_0, \widetilde{x}_0)$. This is absurd. Thus, $\mathfrak{p}(x_0, \widetilde{x}_0) = 0$. Similarly, $\mathfrak{p}(\widetilde{x}_0, x_0) = 0$, so that $\mathfrak{p}(x_0, x_0) \leq \mathfrak{p}(x_0, \widetilde{x}_0) + \mathfrak{p}(\widetilde{x}_0, x_0) = 0$. Lemma 1.1(a) then guarantees that $\widetilde{x}_0 = x_0$, and thus x_0 is the unique fixed point of f. $\square$

The following result is a generalization of Lemma 2.1 above to a pair of maps.

Lemma 2.2. *Let f and g be maps of a set X into a uniform space (Y, ϑ), and assume that $f(X)$ is a complete subspace of Y. Let $\mathfrak{p}$ be a w-distance on Y, and further assume that:*

i) There is a sequence (x_n) in X with $g(x_n) = f(x_{n+1})$, $n \geq 1$.
ii) For any $\delta > 0$ and x, y, z in X, there is $0 < \delta_1 \leq \delta$ such that $\mathfrak{p}(f(x), g(z)) < \delta$ whenever $\mathfrak{p}(f(x), g(x)) < \delta_1$ and $\mathfrak{p}(f(x), f(z)) \leq \delta$.
iii) Given $\delta_1 > 0$, there is $N \geq 1$ such that $\mathfrak{p}(f(x_N), g(x_N)) < \delta_1$.

Then, there is x_0 in X such that

$$\lim f(x_n) = \lim g(x_n) = f(x_0) = g(x_0). \tag{2.3}$$

Proof. Let $V \in \vartheta$ and $\delta > 0$ be as in Definition 1.1(3). Let $0 < \delta_1 \leq \delta$ and $N \geq 1$ be such that assertions *(ii)* and *(iii)* above hold. From $\mathfrak{p}(f(x_N), g(x_N)) = \mathfrak{p}(f(x_N), f(x_{N+1})) < \delta_1 \leq \delta$ and *(ii)*, also $\mathfrak{p}(f(x_N), g(x_{N+1})) = \mathfrak{p}(f(x_N), f(x_{N+2})) < \delta$, and continuing this way, $\mathfrak{p}(f(x_N), f(x_{N+p})) < \delta, \quad p = 1, 2, 3, ...,$ which implies, in view of *(c)* of Lemma 1.1, that $(f(x_n))$ is a Cauchy sequence in $f(X)$, which is complete. Thus, for some $x_0 \in X, \quad \lim f(x_n) = \lim g(x_n) = f(x_0)$, and from Definition 1.1(2), $\mathfrak{p}(f(x_N), f(x_0)) \leq \delta$, and thus, by assumptions *(ii)* and *(iii)* above, $\mathfrak{p}(f(x_N), g(x_0)) \leq \delta$. By Definition 1.1(3), $(f(x_0), g(x_0)) \in V$, so that $f(x_0) = g(x_0)$, and (2.3) holds. $\square$

Theorem 2.2 below extends Corollary 3.3 in [12] to w-distances.

Theorem 2.2. *Let f, g be selfmaps of a uniform space (X, ϑ) such that $g(X) \subseteq f(X)$. Assume either $g(X)$ or $f(X)$ is complete and let $\mathfrak{p}$ be a w-distance on X such that, for some $\phi \in \Phi$,*

$$\mathfrak{p}(g(x), g(y)) \leq \phi(\mathfrak{p}(f(x), f(y))), \quad x, y \in X. \tag{2.4}$$

Then, there is $x_0 \in X$ such that

$$f(x_0) = \lim f(x_n) = \lim g(x_n) = g(x_0) \tag{2.5}$$

for any sequence (x_n) in X such that $g(x_n) = f(x_{n+1})$ for all $n \geq 1$. Furthermore, if f and g commute at their coincidence points, $f(x_0)$ is a unique common fixed point of f and g.

Proof. Since $g(X) \subseteq f(X) \subseteq X$, there is a sequence (x_n) in X such that $g(x_n) = f(x_{n+1})$ for all $n \geq 1$. Then, from (2.4), $\mathfrak{p}(f(x_{n+1}), f(x_{n+2})) = \mathfrak{p}(g(x_n), g(x_{n+1})) \leq \phi(\mathfrak{p}(f(x_n), f(x_{n+1}))) < \mathfrak{p}(f(x_n), f(x_{n+1}))$, provided $t_n = \mathfrak{p}(f(x_n), f(x_{n+1})) > 0$ for all $n \geq 1$. Thus (t_n) is a decreasing sequence in $\mathbb{R}^+$ such that $\lim t_n = \lim \phi(t_n)$, so that $\lim t_n = 0$. Now arguing as in Theorem 2.1, (2.4) ensures that f and g satisfy Lemma 2.2 *(ii)*, which guarantees the existence of $x_0 \in X$ such that

$$f(x_0) = \lim f(x_n) = \lim g(x_n) = g(x_0).$$

The commutativity of f and g at x_0 then ensures that

$$f^2(x_0) = f \circ g \ (x_0) = g \circ f \ (x_0) = g^2(x_0). \tag{2.6}$$

Now, from (2.4) and (2.6),

$$\mathfrak{p}(g(x_0), g(x_0)) \leq \phi(\mathfrak{p}(f(x_0), f(x_0))) < \mathfrak{p}(f(x_0), f(x_0)) = \mathfrak{p}(g(x_0), g(x_0))$$

and

$$\mathfrak{p}(g(x_0), g^2(x_0)) \leq \phi(\mathfrak{p}(f(x_0), f^2(x_0))) < \mathfrak{p}(f(x_0), f^2(x_0)) = \mathfrak{p}(g(x_0), g^2(x_0)),$$

provided $\mathfrak{p}(g(x_0), g(x_0)) \neq 0$ and $\mathfrak{p}(g(x_0), g^2(x_0)) \neq 0$. This is a contradiction. Thus $\mathfrak{p}(g(x_0), g(x_0)) = 0 = \mathfrak{p}(g(x_0), g^2(x_0))$, and from Lemma 1.1(a) we conclude that $g^2(x_0) = g(x_0) = f(x_0) = f^2(x_0)$. Hence, $f(x_0)$ is a common fixed point of f and g. If we now assume that $\mathfrak{p}(f(x_N), f(x_{N+1})) = 0$ for some $N \geq 1$, then $\mathfrak{p}(f(x_{N+1}), f(x_{N+2})) = 0$ and, since $\mathfrak{p}(f(x_N), f(x_{N+2})) \leq \mathfrak{p}(f(x_N), f(x_{N+1})) + \mathfrak{p}(f(x_{N+1}), f(x_{N+2}))$, also $\mathfrak{p}(f(x_N), f(x_{N+2})) = 0$. Thus, as above, $f(x_{N+1}) = f(x_{N+2}) = g(x_{N+1})$ is a fixed point of f and g. Uniqueness follows from an argument entirely similar to that in Theorem 2.1. □

The following example shows that Theorems 2.1 and 2.2 generalize results in [4], [6], [11], [12].

Example 2.1. Let $X = [0, 1]$ and $d(x, y) = |x - y|$. Let $f(x) = x^2$ for $x \in X$, not rational, and $f(x) = 0$ for rational x. It is not difficult to check that no $\phi \in \Phi$ exists such that $d(f(x), f(y)) \leq \phi(d(x, y))$ for all $x, y \in X$. On the other hand, if $\phi(t) = t^2$ for $0 \leq t < 1$ and $\phi(t) = 0$ for $t \geq 1$, then $\phi \in \Phi$. Let $\mathfrak{p}(x, y) = y$ for $x, y \in X$. Then $\mathfrak{p}$ is a w-distance on (X, ϑ_d) and $\mathfrak{p}(f(x), f(y)) \leq \phi(\mathfrak{p}(x, y))$. Theorem 2.1 thus ensures the existence of a unique fixed point of f.

Example 2.2. Let $X, d, \mathfrak{p}$ and ϕ be as in Example 2.1; $f(x) = x^2$, $x \in X$; $g(x) = x^4$, $x \in X - \mathbb{Q}$, and $g(x) = 0$, $x \in X \cap \mathbb{Q}$. Then no $\phi \in \Phi$ exists such $d(g(x), g(y)) \leq \phi(d(f(x), f(y)))$ for all $x, y \in X$, but $\mathfrak{p}(g(x), g(y)) \leq \phi(\mathfrak{p}(f(x), f(y)))$ does hold. Theorem 2.2 thus ensures the existence of a unique common fixed point of f and g. The assertions in this and the above example could not be drawn from results in [12].

3. Fixed point theorems for w-expansive maps

Lemma 3.1. *Let X be a subset of a uniform space (Y, ϑ), f a map of X into Y such that $f(X)$ is complete, $\mathfrak{p}$ a w-distance on Y. Further assume that:*

- *i) There is a sequence (x_n) in X such that $x_n = f(x_{n+1})$, $\quad n \geq 1$*
- *ii) For each $\delta > 0$ there is $0 < \delta_1 \leq \delta$ such that $\mathfrak{p}(f(x), z) < \delta$ whenever $\mathfrak{p}(f(x), x) < \delta_1$ and $\mathfrak{p}(f(x), f(z)) \leq \delta$.*
- *iii) Given $\delta_1 > 0$, there is $N \geq 1$ such that $\mathfrak{p}(f(x_N), x_N) < \delta_1$.*

Then, there is $x_0 \in X$ such that $f(x_0) = x_0$ and

$$\lim x_n = \lim f(x_n) = x_0 = f(x_0). \tag{3.1}$$

Proof. For $V \in \vartheta$, let $\delta > 0$ be as in Definition 1.1(3), and let $0 < \delta_1 \leq \delta$, $N \geq 1$, be such that conditions *(ii)* and *(iii)* above hold. From $\mathfrak{p}(f(x_N), x_N) = \mathfrak{p}(f(x_N), f(x_{N+1})) < \delta_1 \leq \delta$ and *(ii)*, we have that $\mathfrak{p}(f(x_N), x_{N+1}) = \mathfrak{p}(f(x_N), f(x_{N+2})) < \delta$. Also $\mathfrak{p}(f(x_N), x_{N+1}) = \mathfrak{p}(f(x_N), f(x_{N+2})) < \delta$. Continuing this way we conclude that $\mathfrak{p}(x_{N-1}, x_{N+p}) = \mathfrak{p}(f(x_N), x_{N+p}) < \delta$, $p = 0, 1, 2, 3, \ldots$, which implies, by Lemma 1.1 (c), that (x_n) is a Cauchy sequence. Completeness of $f(X)$ then ensures that

$$\lim x_n = \lim f(x_n) = f(x_0)$$

for some $x_0 \in X$. On the other hand, from $\mathfrak{p}(f(x_N), x_m) < \delta$, $m > N$, and the lower semicontinuity of $\mathfrak{p}$, it follows that $\mathfrak{p}(f(x_N), f(x_0)) \leq \delta$, and hence, from assumption *(ii)* above, $\mathfrak{p}(f(x_N), x_0) < \delta$. Definition 1.1(3) thus ensures that $(f(x_0), x_0) \in V$, so that, since V is arbitrary, $f(x_0) = x_0$, and (3.1) holds. $\square$

Lemma 3.1 above is Lemma 2.1 in [12] for the case of w-distances. Theorem 3.1, next, also adapts Theorem 2.1 in the same paper to this more general situation.

Theorem 3.1. *Let X be a subset of a uniform space (Y, ϑ) and let f be a map of X into Y such that $X \subseteq f(X)$. Also assume that either X or $f(X)$ is complete. Let $\mathfrak{p}$ be a w-distance on Y such that, for some $\psi \in \Psi$ with $\psi(0+) = 0$,*

$$\mathfrak{p}(f(x), f(y)) \geq \psi(\mathfrak{p}(x, y)) \quad x, y \in X. \tag{3.2}$$

Then, f has a unique fixed point x_0 satisfying (3.1) for any sequence (x_n) in X such that $x_n = f(x_{n+1})$ for all $n \geq 1$.

Proof. Since $X \subseteq f(X)$, there is a sequence (x_n) in X such that $x_n = f(x_{n+1})$ for all $n \geq 1$, and by (3.2),if we assume $\mathfrak{p}(x_n, x_{n+1}) > 0$ for all $n \geq 1$ then

$$\mathfrak{p}(x_{n+1}, x_{n+2}) < \psi(\mathfrak{p}(x_{n+1}, x_{n+2})) \leq \mathfrak{p}(f(x_{n+1}), f(x_{n+2})) = \mathfrak{p}(x_n, x_{n+1}),$$

so that $t_n = \mathfrak{p}(x_n, x_{n+1})$, $n \geq 1$, is a decreasing sequence such that $\lim t_n = \lim \psi(t_n) = t$ for some $t \geq 0$. Since $\psi \in \Psi$, then $t = 0$.
Let $\delta > 0$ be such that $0 < \widehat{\psi}(\delta) < \delta$ (Lemma 1.2) and $\delta_1 = \delta - \widehat{\psi}(\delta)$. If $\mathfrak{p}(f(x), x) < \delta_1$ and $\mathfrak{p}(f(x), f(z)) < \delta$, it follows from $\widehat{\psi}(\mathfrak{p}(x,z)) \leq \mathfrak{p}(f(x), f(z)) < \delta$ that $\mathfrak{p}(x,z) \leq \widehat{\psi}(\delta)$, so that $\mathfrak{p}(f(x), z) \leq \mathfrak{p}(f(x), x) + \mathfrak{p}(x,z) < \delta_1 + \widehat{\psi}(\delta) = \delta$. Lemma 3.1 then ensures the existence of x_0 in X such that (3.1) holds. If $\mathfrak{p}(x_N, x_{N+1}) = 0$ for some $N \geq 1$ then, from (3.2), $\mathfrak{p}(x_{N+1}, x_{N+2}) = 0$, and, as in Theorem 2.1, $x_{N+1} = x_{N+2} = f(x_{N+2})$, so that x_{N+2} is a fixed point of f. Uniqueness follows easily from (3.2) and the argument in Theorem 2.1. □

Lemma 3.2. *Let f and g be maps of a set X into a uniform space (Y, ϑ). Assume that $g(X)$ is complete and let $\mathfrak{p}$ be a w-distance on Y. Also assume that:*

i) There is a sequence (x_n) in X with $f(x_n) = g(x_{n+1})$, $n \geq 1$.
ii) Given $\delta > 0$, there is $0 < \delta_1 \leq \delta$ such that $\mathfrak{p}(g(x), f(z)) < \delta$, provided $\mathfrak{p}(g(x), f(x)) < \delta_1$ and $\mathfrak{p}(g(x), g(z)) < \delta$.
iii) For any $\delta_1 > 0$ there is $N \geq 1$ such that $\mathfrak{p}(g(x_N), f(x_N)) < \delta_1$.

Then, there is x_0 in X such that

$$f(x_0) = \lim f(x_n) = \lim g(x_n) = g(x_0). \tag{3.3}$$

Proof. Now based on Lemma 3.1, the proof is entirely similar to that of Lemma 2.2. □

Lemma 3.2 extends Lemma 3.1 to a pair of maps. We finally have:

Theorem 3.2. *Let f and g be maps of a set X into a uniform space (Y, ϑ) such that $f(X) \subseteq g(X)$. Assume either $f(X)$ or $g(X)$ is complete and let $\mathfrak{p}$ be a w-distance on Y such that*

$$\mathfrak{p}(g(x), g(y)) \geq \psi(\mathfrak{p}(f(x), f(y))), \quad x, y \in X \tag{3.4}$$

for some $\psi \in \Psi$ with $\psi(0+) = 0$. Then, there is x_0 in X such that (3.3) holds for any sequence (x_n) in X such that $f(x_n) = g(x_{n+1})$ for all $n \geq 1$. Furthermore, if $g(X) \subseteq X$ and f and g commute at any of their coincidence points, then $f(x_0)$ is the unique common fixed point of f and g.

Proof. As in Theorem 2.2, from $f(X) \subseteq g(X)$ it follows that there is a sequence (x_n) in X such that $f(x_n) = g(x_{n+1})$, $n \geq 1$, and (3.4) then ensures that $(f(x_n))$ is a Cauchy sequence for ϑ. Therefore $x_0 \in X$ exists such that (3.3) holds. Commutativity of f and g at their coincidence points and (3.4) then guarantee that $f(x_0)$ is the unique common fixed point of f and g. □

The following examples show that Theorems 3.1 and 3.2 generalize results in [5], [12] and [13].

Example 3.1. Let $X = [0,1)$ and $d(x,y) = |x - y|$. Also let $f(x) = 2x$ for $0 \leq x \leq \frac{1}{2}$ and $f(x) = 1$ if $\frac{1}{2} < x < 1$. Then no $\psi \in \Psi$ exists such that $d(f(x), f(y)) \geq \psi(d(x,y))$ for all x, y in X. Let $\psi(t) = f(t)$ if $0 \leq t < 1$ and $\psi(t) = 2t$ for $t \geq 1$. If $\mathfrak{p}(x,y) = y$, then $\mathfrak{p}(f(x), f(y)) \geq \psi(\mathfrak{p}(x,y))$ for x, y in X. Theorem 3.1 thus ensures the existence of a unique fixed point for f.

Example 3.2. Let X, d and $\mathfrak{p}$ be as in Example 3.1. Also let $g(x) = 2x$ for $0 \leq x \leq \frac{1}{2}$ and $g(x) = 1$ for $\frac{1}{2} < x < 1$, and $f(x) = \frac{x}{2}$ for $x \in X$. Then, no $\psi \in \Psi$ exists such that $d(g(x), g(y)) \geq \psi(d(f(x), f(y)))$ for all x, y in X, but $\mathfrak{p}(g(x), g(y)) \geq \psi(\mathfrak{p}(f(x), f(y)))$ holds for ψ as in Example 3.1. Therefore, Theorem 3.2 ensures the existence of a unique fixed point for f and g.

Acknowledgments

The first author acknowledges partial support from the Facultad de Ciencias to pursue graduate (Ph.D) studies.

References

1. N. Bourbaki, *Topologie Générale*, Chaps. I, II, 4 ed., Hermann, Paris, 1965.
2. K.M. Das and V. Naik, *Common fixed points theorems for commuting maps on metric space*, Proc. Amer. Math Soc., **77** (1979), 269–373.
3. J. Dugundji, *Topology*, Allyn and Bacon, Boston, 1966.
4. J. Dugundji and A. Granas, *Fixed Point Theory*, vol. I, Polish. Sci. Pub., Warszawa, 1982.
5. J. Jachymski, *Fixed points theorems for expansive mappings*, Math Japon. **42** (1995), 131–136.
6. G. Jungck, *Commuting mappings and fixed points*, Amer. Math. Monthly. **83** (1976), 261–263.
7. O. Kada, T. Suzuki and W. Takahashi, *Non convex minimization theorems and fixed point theorems in complete metric spaces*, Math. Japon. **44** (1996), 381–391.
8. S. M. Kang, *Fixed points for expansions mappings*, Math. Japon. **38** (1993), 713–717.
9. B. E. Rhoades, *A comparison of various definitions of contractive mapping*, Trans. Amer. Math. Soc. **226** (1977), 257–290.
10. J. Rodríguez-Montes, *Aplicaciones no conmutativas y puntos fijos*, Rev. Colombiana. Mat. **21** (1987), 257–262.
11. J. Rodríguez-Montes, *Some results on fixed and coincidence points for pairs of maps in metric spaces*, Rev. Colombiana. Mat. **27** (1993), 249–252.
12. J. Rodríguez-Montes and J. A. Charris, *Fixed points for contractive and expansive maps in metric space: Toward a unified approach*, submitted, Reporte interno No.73, Departamento de Matemáticas y Estadística, Universidad Nacional, Bogotá, Colombia, 2000.
13. S. Z. Wang, B. Y. Li, Z. M. Gao and K. Iseki, *Some fixed point theorems on expansion mappings*, Math. Japon. **29** (1984), 631–636.
14. A. Weil, *Sur les espaces á structure uniforme et sur la topologie générale*, Hermann, Paris, 1937.