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A Modification of the Kantorovich Conditions for the Secant Method

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Abstract

In this work we carry out a modification of the Kantorovich conditions for the semilocal convergence of the Secant method. This modification generalizes the Lipschtz and Hölder continuity conditions. Besides, we apply the result obtained to approximate the solution of a nonlinear boundary value problem.

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1 Introduction

Let X, Y be two Banach spaces and $F : \Omega \subseteq X \to Y$ a nonlinear operator and consider the equation

$$F(x) = 0. \tag{1}$$

Let us denote by $\mathcal{L}(X, Y)$ the space of bounded linear operators from X to Y. An operator $[x, y; F] \in \mathcal{L}(X, Y)$ is called a divided difference of first order for the operator F on the points x and y $(x \neq y)$ if the following equality holds

$$[x, y; F](x - y) = F(x) - F(y).$$

Using this definition, the Secant method is described by the following algorithm:

$$x_{n+1} = x_n - [x_{n-1}, x_n; F]^{-1} F(x_n), \quad x_0,\ x_{-1} \text{ given.} \tag{2}$$

The convergence of (2) to a solution of (1) has been studied by other authors ([1], [2], [3], [4], [5], [6]). The basic assumption made is that the divided difference of F is Lipschitz or Hölder continuous in some ball around the initial iterate. We relax this requierement and we only assume that the divided difference $[x, y; F]$ satisfies

$$\|[x, y; F] - [v, w; F]\| \leq \omega(\|x - v\|, \|y - w\|); \quad x, y, v, w \in \Omega$$

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where $\omega : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous funtion nondecreasing in both components and $\omega(0,0) = 0$. It is evident that this condition generalizes the conditions previously indicaded, by only considering $\omega(u_1, u_2) = k(u_1 + u_2)$ for the Lipschitz continuous case and $\omega(u_1, u_2) = k(u_1^p + u_2^p)$ for the (k,p)-Hölder continuous case.

In this paper, we analyse the semilocal convergence of (2) using a technique that consists of a new system of recurrence relations. Finally, we apply our results to solve a nonlinear boundary value problem of second order.

2 Convergence study

In this paper, we consider Ω as a convex open subset of X and we assume that F is once Fréchet differentiable in Ω. It is known ([3],[4]) that, if for any $x, y \in \Omega$ there is a divided difference operator $[x, y; F]$ which satisfies a Lipschitz or Hölder condition, then the Fréchet derivative of F exists in Ω and verifies

$$[x, x; F] = F'(x), \quad x \in \Omega. \tag{3}$$

Note that, by means of continuity, it is easy to prove that the following result is also verified for our conditions.

Lemma 2.1 *Let Ω be a convex open subset of X and suppose that, for each pair of points $x, y \in \Omega$, there exists a first order divided difference $[x, y; F] \in \mathcal{L}(X, Y)$ such that*

$$\|[x, y; F] - [v, w; F]\| \leq \omega(\|x - v\|, \|y - w\|); \quad x, y, v, w \in \Omega,$$

where $\omega : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a continuos funtion nondecreasing in both components and $\omega(0,0) = 0$. Then, (3) holds.

Now, we are going to study the semilocal convergence of the Secant method.
Let $x_0, x_{-1} \in \Omega$ and assume that

(I) $\|x_0 - x_{-1}\| \leq \alpha$,
(II) there exists $L_0{}^{-1} = [x_{-1}, x_0; F]^{-1}$ such that $\|L_0{}^{-1}\| \leq \beta$,
(III) $\|L_0{}^{-1} F(x_0)\| \leq \eta$,
(IV) $\|[x, y; F] - [v, w; F]\| \leq \omega(\|x - v\|, \|y - w\|); \quad x, y, v, w \in \Omega$, where $\omega : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a continuos funtion nondecreasing in both components and $\omega(0,0) = 0$,
(V) there exists $h : [0,1] \to \mathbb{R}$ such that if $t \in [0,1]$, $\omega(tu_1, tu_2) \leq h(t)\omega(u_1, u_2)$.

Notice that the last condition is not restrictive, since ω is non-decreciente in its two components, and we obtain that $\omega(tu_1, tu_2) \leq \omega(u_1, tu_2) \leq \omega(u_1, u_2)$, and therefore, in particular, it is always possible to consider $h(t) = 1$. This condition is included in order to can be more precise in certain bounds we shall carry out later on.

In the following result, we use a new technique to analyse the convergence of the Secant method. Taking into account certain recurrence relations and forcing the sequence to be within a particular ball, we obtain a semilocal convergence result under mild conditions.

Before getting the semilocal convergence result for the Secant method under these new conditions, we will introduce some notations that allow us to relax the expressions appearing in its calculation. Thus, we will take into account the following auxiliary functions:

$$a(u) = \frac{\beta\,[\omega(\alpha,0) + M\omega(u,u)]}{1-\beta\omega(\alpha,u)}, \quad b(u) = \frac{\beta\,[\omega(u,0) + M\omega(2u,2u)]}{1-\beta\omega(\alpha+u,u)},$$
$$c(u) = \frac{\beta\,[\omega(2u,0) + M\omega(2u,2u)]}{1-\beta\omega(\alpha+u,u)}, \quad \text{where } M = \int_0^1 h(t)\,dt.$$

Theorem 2.2 *Under conditions (I)-(V), we assume that the equation*

$$u = \left(\frac{b(u)a(u)}{1-c(u)} + a(u) + 1\right)\eta \tag{4}$$

has a minimum positive zero R. If $\beta\omega(\alpha+R,R) < 1$, $c(R) < 1$ and $\overline{B(x_0,R)} \subset \Omega$, then the sequence $\{x_n\}$ given by (2) is well defined, remain in $\overline{B(x_0,R)}$ and converges to a solution x^* of equation (1) in $\overline{B(x_0,R)}$.

Proof. To simplify the notation we denote $a(R) = a$, $b(R) = b$, $c(R) = c$ and $[x_{n-1},x_n;F] = L_n$. Firstly, we prove, by mathematical induction, that the sequence given in (2) is well defined, namely iterative procedure (2) makes sense if, at each step, the operator $[x_{n-1},x_n;F]$ is invertible and the point x_{n+1} lies in Ω.

From the initial hypotheses, it follows that x_1 is well defined and $\|{L_0}^{-1}F(x_0)\| \leq \eta < R$. Therefore, $x_1 \in B(x_0,R) \subseteq \Omega$.

Using (IV) and assuming that ω is nondecreasing, we obtain

$$\|I - {L_0}^{-1}L_1\| \leq \|{L_0}^{-1}\|\;\|L_0 - L_1\| = \|{L_0}^{-1}\|\;\|[x_{-1},x_0;F] - [x_0,x_1;F]\|$$
$$\leq \|{L_0}^{-1}\|\omega(\|x_{-1}-x_0\|,\|x_0-x_1\|) \leq \beta\omega(\alpha,R) \leq \beta\omega(\alpha+R,R) < 1,$$

and, by the Banach lemma, ${L_1}^{-1}$ exists and

$$\|{L_1}^{-1}\| \leq \frac{\beta}{1-\beta\omega(\alpha,R)}.$$

By (2) and Taylor's formula, we have

$$F(x_1) = (F'(x_0) - L_0)(x_1-x_0) + \int_0^1 (F'(x_0+t(x_1-x_0)) - F'(x_0))\;(x_1-x_0)\,dt.$$

As $[x,x;F] = F'(x)$, then, by (IV),

$$\|F(x_1)\| \leq \|[x_0,x_0;F] - [x_{-1},x_0;F]\|\;\|x_1-x_0\|$$
$$+\int_0^1 \|[x_0+t(x_1-x_0),x_0+t(x_1-x_0);F] - [x_0,x_0;F]\|\;\|x_1-x_0\|\,dt$$
$$\leq \left[\omega(\|x_0-x_{-1}\|,0) + \int_0^1 \omega(t\|x_1-x_0\|,t\|x_1-x_0\|)\,dt\right]\|x_1-x_0\|$$
$$\leq \left[\omega(\alpha,0) + \int_0^1 h(t)\omega(\|x_1-x_0\|,\|x_1-x_0\|)\,dt\right]\|x_1-x_0\|$$

$$\leq [\omega(\alpha,0) + M\omega(\|x_1 - x_0\|, \|x_1 - x_0\|)]\,\|x_1 - x_0\| \leq [\omega(\alpha,0) + M\omega(R,R)]\,\|x_1 - x_0\|$$

The iterate x_2 is well defined since $F(x_1)$ and ${L_1}^{-1}$ are. Moreover,

$$\|x_2 - x_1\| \leq \|{L_1}^{-1}\|\ \|F(x_1)\| \leq \frac{\beta\,[\omega(\alpha,0) + M\omega(R,R)]}{1 - \beta\omega(\alpha,R)}\|x_1 - x_0\| = a\|x_1 - x_0\|. \quad (5)$$

On the other hand, if we take into account that R is a solution of (4),

$$\|x_2 - x_0\| \leq \|x_2 - x_1\| + \|x_1 - x_0\| \leq \left[\frac{\beta\,[\omega(\alpha,0) + M\omega(R,R)]}{1 - \beta\omega(\alpha,R)} + 1\right]\|x_1 - x_0\| \leq (a+1)\eta < R, \quad (6)$$

then, $x_2 \in B(x_0,R) \subseteq \Omega$.

We now assume the linear operators L_j are invertible and $x_{j+1} \in B(x_0,R) \subseteq \Omega$ for all $j = 1,\ldots,n$. Then,

$$\|I - {L_0}^{-1}L_{n+1}\| \leq \|{L_0}^{-1}\|\ \|L_0 - L_{n+1}\| = \|{L_0}^{-1}\|\ \|[x_{-1},x_0;F] - [x_n,x_{n+1};F]\|$$

$$\leq \beta\,\omega(\|x_{-1} - x_n\|, \|x_0 - x_{n+1}\|) \leq \beta\omega(\|x_{-1} - x_0\| + \|x_0 - x_n\|, \|x_0 - x_{n+1}\|)) \leq \beta\omega(\alpha+R,R) < 1,$$

and, according to Banach lemma, ${L_{n+1}}^{-1}$ exists and

$$\|{L_{n+1}}^{-1}\| \leq \frac{\beta}{1 - \beta\omega(\alpha+R,R)}.$$

Taking norms in

$$F(x_{n+1}) = (F'(x_n) - L_n)(x_{n+1} - x_n) + \int_{x_n}^{x_{n+1}} (F'(x) - F'(x_n))\,dx$$

we get

$$\|F(x_{n+1})\| \leq \|[x_n,x_n;F] - [x_{n-1},x_n;F]\|\ \|x_{n+1} - x_n\|$$

$$+\int_0^1 \|[x_n + t(x_{n+1} - x_n), x_n + t(x_{n+1} - x_n); F] - [x_n,x_n;F]\|\ \|x_{n+1} - x_n\|\,dt$$

$$\leq \left[\omega(\|x_n - x_{n-1}\|,0) + \int_0^1 \omega(t\|x_{n+1} - x_n\|, t\|x_{n+1} - x_n\|)\,dt\right]\|x_{n+1} - x_n\|$$

$$\leq \left[\omega(\|x_n - x_{n-1}\|,0) + \int_0^1 h(t)\omega(\|x_{n+1} - x_n\|, \|x_{n+1} - x_n\|)\,dt\right]\|x_{n+1} - x_n\|$$

$$= [\omega(\|x_n - x_{n-1}\|,0) + M\omega(\|x_{n+1} - x_n\|, \|x_{n+1} - x_n\|)]\,\|x_{n+1} - x_n\|$$

$$\leq [\omega(\|x_n - x_{n-1}\|,0) + M\omega(\|x_{n+1} - x_0\| + \|x_0 - x_n\|, \|x_{n+1} - x_0\| + \|x_0 - x_n\|)]$$

$$\|x_{n+1} - x_n\| \leq [\omega(\|x_n - x_{n-1}\|,0) + M\omega(2R,2R)]\,\|x_{n+1} - x_n\|.$$

For $n = 1$,

$$\|F(x_2)\| \leq [\omega(R,0) + M\omega(2R,2R)]\,\|x_2 - x_1\|,$$

and

$$\|x_3 - x_2\| \leq \|{L_2}^{-1}\|\ \|F(x_2)\| \leq \frac{\beta\,[\omega(R,0) + M\omega(2R,2R)]}{1 - \beta\omega(\alpha+R,R)}\|x_2 - x_1\| = b\|x_2 - x_1\|. \quad (7)$$

Then, by (4), (5), (6) and (7), we have

$$\|x_3 - x_0\| \leq \|x_3 - x_2\| + \|x_2 - x_0\| \leq b\|x_2 - x_1\| + \|x_2 - x_0\| \leq (ba + a + 1)\|x_1 - x_0\|$$

$$\leq (ba + a + 1)\eta < R.$$

Therefore, $x_3 \in B(x_0, R) \subseteq \Omega$.

For $n > 1$:

$$\|F(x_{n+1})\| \leq [\omega(\|x_n - x_0\| + \|x_0 - x_{n-1}\|, 0) + M\omega(2R, 2R)]\, \|x_{n+1} - x_n\|$$

$$\leq [\omega(2R, 0) + M\omega(2R, 2R)]\, \|x_{n+1} - x_n\|.$$

Thus

$$\|x_{n+2} - x_{n+1}\| \leq \|{L_{n+1}}^{-1}\|\, \|F(x_{n+1})\| \leq \frac{\beta\,[\omega(2R, 0) + M\omega(2R, 2R)]}{1 - \beta\omega(\alpha + R, R)} \|x_{n+1} - x_n\|$$

$$= c\|x_{n+1} - x_n\|. \tag{8}$$

Consequently, from (4), (5), (6), (7) and (8), it follows

$$\|x_{n+2} - x_0\| \leq \|x_{n+2} - x_{n+1}\| + \|x_{n+1} - x_n\| + \cdots + \|x_3 - x_2\| + \|x_2 - x_0\|$$

$$\leq \left[c^{n-1} + c^{n-2} + \cdots + 1\right] \|x_3 - x_2\| + \|x_2 - x_0\| \leq \left[\frac{1 - c^n}{1 - c} ba + a + 1\right] \|x_1 - x_0\|$$

$$< \left[\frac{ba}{1 - c} + a + 1\right] \eta = R.$$

So, $x_{n+2} \in B(x_0, R) \subseteq \Omega$ and the induction is complete.

Secondly, we prove that $\{x_n\}$ is a Cauchy sequence. For $m \geq 1$ and $n \geq 2$ we obtain

$$\|x_{n+m} - x_n\| \leq \|x_{n+m} - x_{n+m-1}\| + \|x_{n+m-1} - x_{n+m-2}\| + \cdots + \|x_{n+1} - x_n\|$$

$$\leq \left[c^{m-1} + c^{m-2} + \cdots + 1\right] \|x_{n+1} - x_n\| \leq \frac{1 - c^m}{1 - c} \|x_{n+1} - x_n\| < \frac{1}{1 - c} c^{n-2} \|x_3 - x_2\|.$$

Therefore, $\{x_n\}$ is a Cauchy sequence and converges to $x^* \in \overline{B(x_0, R)}$.

Finally, we see that x^* is a zero of F. Since

$$\|F(x_n)\| \leq [\omega(2R, 0) + M\omega(2R, 2R)]\, \|x_n - x_{n-1}\|,$$

and $\|x_n - x_{n-1}\| \to 0$ when $n \to \infty$, we obtain $F(x^*) = 0$. ∎

Theorem 2.3 *Under the assumption of theorem 2.2, if* $\beta\,[\omega(\alpha, 0) + \omega(2R, 2R)] < 1$, *the solution* x^* *of equation (1) is unique in* $\overline{B(x_0, R)}$.

Proof. Let us assume that there exists $y^* \in \overline{B(x_0, R)}$ such that $F(y^*) = 0$. Moreover,

$$L_0^{-1}[F(y^*) - F(x^*)] = \int_{x^*}^{y^*} L_0^{-1} F'(x)\, dx = \int_0^1 L_0^{-1} F'(x^* + t\ (y^* - x^*))\ (y^* - x^*)\, dt.$$

Denoting $A = \int_0^1 {L_0}^{-1} F'(x^* + t\ (z^* - x^*))\, dt$, we have

$$\|A - I\| \leq \|{L_0}^{-1}\| \int_0^1 \|F'(x^* + t\ (y^* - x^*)) - L_0\|\, dt$$

$$\leq \|L_0{}^{-1}\| \int_0^1 (\|F'(x^* + t\ (y^* - x^*)) - F'(x_0)\| + \|F'(x_0) - L_0\|)\ dt$$

$$\leq \beta\ \left(\int_0^1 \omega\,(\|x^* + t\ (y^* - x^*) - x_0\|, \|x^* + t\ (y^* - x^*) - x_0\|)\ dt + \|F'(x_0) - L_0\| \right)$$

$$\leq \beta \left(\int_0^1 \omega\,(\ (1-t)\ \|x^* - x_0\| + t\ \|y^* - x_0\|, (1-t)\ \|x^* - x_0\| + t\ \|y^* - x_0\|\ )\ dt + \omega(\alpha, 0) \right)$$

$$\leq \beta \left(\int_0^1 \omega\,(\ \|x^* - x_0\| + \|y^* - x_0\|, \|x^* - x_0\| + \|y^* - x_0\|)\ dt + \omega(\alpha, 0) \right)$$

$$\leq \beta\,(\omega(\alpha, 0) + \omega(2R, 2R)) < 1.$$

In consequence, A is invertible and $y^* = x^*$. ■

3 Numerical Example

Now we consider a situation where if divided differences are Lipschitz or (k,p)-Hölder continuous, then we cannot work. Now we apply the semilocal convergence result given above to the following boundary value problem:

$$\begin{cases} x'' + x^{1+p} + x^2 = 0, & p \in [0,1] \\ x(0) = x(1) = 0. \end{cases} \tag{9}$$

To solve this problem by finite differences, we start drawing the usual grid line with the grid points $t_i = ih$, where $h = 1/n$ and n is an appropriate integer. Denote $x(t_i) = x_i$, notice that x_0 and x_n are given by the boundary conditions, then $x_0 = 0 = x_n$. We first approximate the second derivative $x''(t)$ by

$$\begin{aligned} x''(t) &\approx [x(t+h) - 2x(t) + x(t-h)]/h^2, \\ x''(t_i) &= (x_{i+1} - 2x_i + x_{i-1})/h^2, \quad i = 1, 2, \ldots, n-1. \end{aligned}$$

Substituting this expression into the differential equation, we have the following system of nonlinear equations

$$\begin{cases} 2x_1 - h^2 x_1^{1+p} - h^2 x_1^2 - x_2 = 0, \\ -x_{i-1} + 2x_i - h^2 x_i^{1+p} - h^2 x_i^2 - x_{i+1} = 0, \quad i = 2, 3, \ldots, n-2, \\ -x_{n-2} + 2x_{n-1} - h^2 x_{n-1}^{1+p} - h^2 x_{n-1}^2 = 0. \end{cases} \tag{10}$$

We can then consider an operator $F : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ such that $F(x) = H(x) - h^2 g(x) - h^2 f(x)$, where

$$g(x) = \left(x_1^{1+p}, x_2^{1+p}, \ldots, x_{n-1}^{1+p} \right)^t, \quad f(x) = \left(x_1^2, x_2^2, \ldots, x_{n-1}^2 \right)^t,$$

and

$$H = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 2 \end{pmatrix}.$$

Thus

$$F'(x) = H - h^2(1+p)\begin{pmatrix} x_1^p & 0 & \cdots & 0 \\ 0 & x_2^p & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_{n-1}^p \end{pmatrix} - 2h^2\begin{pmatrix} x_1 & 0 & \cdots & 0 \\ 0 & x_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_{n-1} \end{pmatrix}$$

Let $x \in \mathbb{R}^{n-1}$, and choose the norm $\|x\| = \max_{1\le i\le n-1} |x_i|$, the corresponding norm on $A \in \mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$ is

$$\|A\| = \max_{1\le i\le n-1} \sum_{j=1}^{n-1} |a_{ij}|.$$

It is known (see [5]) that F has a divided difference at the points $x, y \in \mathbb{R}^{n-1}$, which is defined by the matrix whose entries are

$$[x,y;F]_{ij} = \frac{1}{x_j - y_j}\,\left(F_i(x_1,\dots,x_j,y_{j+1},\dots,y_{n-1}) - F_i(x_1,\dots,x_{j-1},y_j,\dots,y_{n-1})\right).$$

Consequently,

$$[x,y;F] = H - h^2\begin{pmatrix} \frac{x_1^{1+p}-y_1^{1+p}+x_1^2-w_1^2}{x_1-y_1} & 0 & \cdots & 0 \\ 0 & \frac{x_2^{1+p}-y_2^{1+p}+x_2^2-y_2^2}{x_2-y_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{x_{n-1}^{1+p}-y_{n-1}^{1+p}+x_{n-1}^2-y_{n-1}^2}{x_{n-1}-y_{n-1}} \end{pmatrix}.$$

In this case, we have that $[x,y;F] = \int_0^1 F'(x+t(y-x))\,dt$. So we study the value $\|F'(x) - F'(v)\|$ to obtain a bound for $\|[x,y;F] - [v,w;F]\|$.

For all $x, v \in \mathbb{R}^{n-1}$ with $|x_i| > 0, |v_i| > 0,\ (i = 1,2,\dots,n-1)$, and taking into account the max-norm it follows

$$\|F'(x) - F'(v)\| = \|\mathrm{diag}\{h^2(1+p)\ (v_i^p - x_i^p) + 2h^2(v_i - x_i)\}\|$$

$$= \max_{1\le i\le n-1} |h^2(1+p)\ (v_i^p - x_i^p) + 2h^2(v_i - x_i)| \le (1+p)h^2 \max_{1\le i\le n-1} |v_i^p - x_i^p| + 2h^2 \max_{1\le i\le n-1} |v_i - x_i|$$

$$\le (1+p)h^2\ [\ \max_{1\le i\le n-1} |v_i - x_i|\ ]^p + 2h^2\|v-x\| = (1+p)h^2\|v-x\|^p + 2h^2\|v-x\|.$$

Therefore

$$\|[x,y;F] - [v,w;F]\| \le \int_0^1 \|F'\,(x+t(y-x)) - F'\,(u+t(w-v))\,\|\,dt$$

$$\le h^2\int_0^1 ((1+p)\|(1-t)(x-v)+t(y-w)\|^p + 2\|(1-t)(x-v)+t(y-w)\|)\ dt$$

$$\le h^2(1+p)\int_0^1 ((1-t)^p\|x-v\|^p + t^p\|y-w\|^p)\ dt + 2h^2\int_0^1 ((1-t)\|x-v\| + t\|y-w\|)\ dt$$

$$= h^2\left(\|x-v\|^p + \|y-w\|^p + \|x-v\| + \|y-w\|\right)$$

From (IV), we consider the function $\omega(u_1,u_2) = h^2(u_1^p + u_2^p + u_1 + u_2)$ and $h(t) = t^p$. So, $M = \frac{1}{1+p}$.

Now we apply the Secant method to approximate the solution of $F(x) = 0$. We choose $p = 1/2$ and if $n = 10$, then (10) gives 9 equations. Since a solution of (9) would vanish at the end points and be positive in the interior, a reasonable choice of initial approximation seems to be 10sinπt. This approximate gives us the following vector:

$$y_{-1} = \begin{pmatrix} 3.090169943749474 \\ 5.877852522924731 \\ 8.090169943749475 \\ 9.51056516295136 \\ 10.0000000000000 \\ 9.51056516295136 \\ 8.090169943749475 \\ 5.877852522924731 \\ 3.090169943749474 \end{pmatrix}.$$

Choose y_0 by setting $y_0(t_i) = y_{-1}(t_i) - 10^{-5}, i = 1, 2, \ldots, 9$ and using iteration (2), after two iterations, we obtain:

$$y_1 = \begin{pmatrix} 2.453176290658909 \\ 4.812704101582601 \\ 6.8481873135861 \\ 8.252997367741953 \\ 8.75737771678512 \\ 8.252997367741953 \\ 6.8481873135861 \\ 4.812704101582601 \\ 2.453176290658909 \end{pmatrix} \quad \text{and} \quad y_2 = \begin{pmatrix} 2.404324055268407 \\ 4.713971539035271 \\ 6.7003394962933925 \\ 8.066765882171131 \\ 8.556329565792526 \\ 8.066765882171131 \\ 6.7003394962933924 \\ 4.713971539035271 \\ 2.404324055268407 \end{pmatrix}.$$

Now we apply (2), then we take $x_{-1} = y_1$ and $x_0 = y_2$. With the notation of teorem 2.2 we can easily obtain the following results: $\alpha = 0.201048$, $\beta = 15.319$, $\eta = 0.0346555$. Since $h^2 = 0.01$, in this particular case, the solution of equation (4) is $R = 0.0416182$. Besides, $\beta\omega(\alpha + R, R) = 0.150264 < 1$ and $c = 0.156374 < 1$. Therefore, the hypotheses of theorem 2.2 are fulfilled, what ensures that a solution of the equation (1) exists in $\overline{B(x_0, R)}$. This solution is also unique in $\overline{B(x_0, R)}$, since $\beta[\omega(\alpha, 0) + \omega(2R, 2R)] = 0.213381 < 1$.

We obtain the vector x^* as the solution of system (10), after eleven iterations:

$$x^* = \begin{pmatrix} 2.394640794786742 \\ 4.694882371216001 \\ 6.672977546934751 \\ 8.033409358893319 \\ 8.520791423704788 \\ 8.033409358893319 \\ 6.67297754693475 \\ 4.694882371216 \\ 2.394640794786742 \end{pmatrix}.$$

If x^* is now interpolated, its approximation $\bar{x}^*$ to the solution of (9) with $p = 1/2$ is that appearing in figure 1.

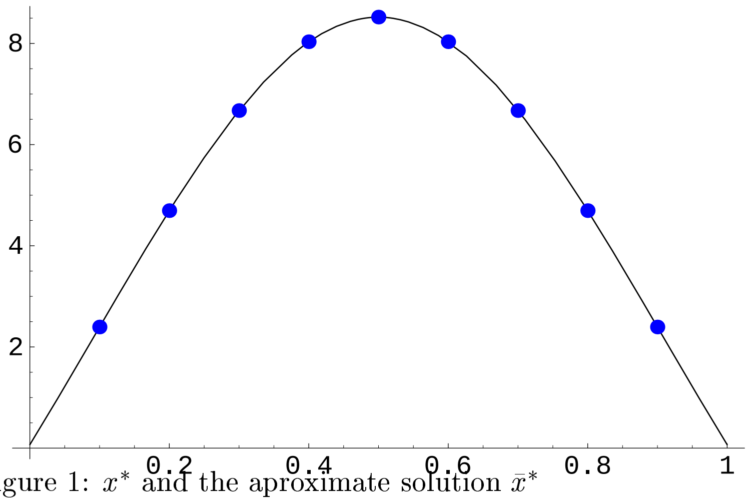

Figure 1: x^* and the aproximate solution $\bar{x}^*$

Note that in this example, we cannot guarantee the convergence to the solution through the classical Lipschitz or Hölder conditions for the divided differences.

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