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On Self-Similar Centralizers of Circle Maps

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Abstract

In this paper we introduce the notion of self-similar centralizers and give sufficient conditions for certain circle maps not to have a self-similar centralizer (*we say f has a self-similar centralizer if $g \in Z(f) \Rightarrow Z(g) \cong Z(f)$*).

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Let $Imm^n(S^1)$ be the space of C^n immersions of the circle (i.e.maps of S^1 onto itself for which $f'(z) \neq 0$ for for all $z \in S^1$) endowed with the C^n topology where $1 \leq n \leq \infty$. For $f \in Imm^n(S^1)$ let $Z(f)$ denote the set of elements of $Imm^n(S^1)$ that commute with f. This semi-group is called the centralizer of f. We say that $Z(f)$ is trivial if it is just the itterates of f (i.e $Z(f) = \{f^m : m \in \mathbf{N}\}$).Wesaythat f $\in Imm^n(S^1)$ is expanding if $|f'(z)| > 1$ for all $z \in S^1$ and these maps are denoted by $Exp^n(S^1)$. The expanding map $z \rightarrow z^m$ will be denoted by $T_m(z)$. In this paper sufficient conditions for an element of $Exp^n(S^1)$ not to have a self-similar centralizer are given. We shall need the following theorem:

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THEOREM 1 [4]

Let f_1, f_2 be expanding orientation-preserving maps of S^1 that commute, have degrees p and q respectively and are of class $C^{n-}(n \geq 2)$. Then if p and q are coprime, there is a diffeomorphism h of class C^{n-} satisfying $h(f_1(z)) = T_p(h(z))$ and $h(f_2(z)) = T_q(h(z))$ for all $z \in S^1$.

We now state a sufficient condition for a centralizer semi-group not to be self-similar:

THEOREM 2

Let $f \in Exp^n(S^1)$ be orientation preserving with degree p. If these exists $g \in Z(f)$ with degree q where p and q are coprime then $Z(f)$ is not self-similar.

Proof: From Theorem 1 there exists a diffeomorphism h of S^1 such that $h(f(z)) = T_p(h(z))$ and $h(g(z)) = T_q(h(z))$ for all $z \in S^1$. In [1] it was shown that if h is the homeomorphism conjugating f and T_p (whose existence is guarenteed by a Theorem of Shub in [5]) then $Z(f)$ is a subset of $\{h^{-1}\ o\ \theta T_m\ o\ h : m \in \mathbf{N}, \theta^{\mathrm{p}-1} = 1\}$. From above, we have h is a diffeomorphism of the same class as f and so:

$$Z(f) = \{h^{-1}\ o\ \theta T_m\ o\ h : m \in \mathbf{N} \text{ and } \theta^{p-1} = 1\}.$$

Similarly we have that:

$$Z(g) = \{h^{-1}\ o\ \omega T_m\ o\ h : m \in \mathbf{N} \text{ and } \omega^{q-1} = 1\}.$$

We now show (by contradiction) that there is no bijective semi-group homomorhism between $Z(f)$ and $Z(g)$:

Suppose $Z(f) \cong Z(g)$ and $\Phi : Z(f) \to Z(g)$ is the isomorphism. Assume $p > q$ (otherwise consider a homomorhism from $Z(g)$ to $Z(f)$). Let $\{\theta_i : 1 \leq i \leq p-1\}$ be the $p-1$ roots of unity and let $\{\omega_i : 1 \leq i \leq q-1\}$ be the $q-1$ roots of unity. The identity element of both the semi-groups $Z(f), Z(g)$ is $h^{-1}\ o\ T_1\ o\ h$ and so $\Phi(h^{-1}\ o\ T_1\ o\ h) = h^{-1}\ o\ T_1\ o\ h$. Let $i \in \{1, ..., p-1\}$ be arbitrary and denote $\Phi(h^{-1}\ o\ \theta_i T_1\ o\ h)$ by $h^{-1}\ o\ \omega_{j(i)} T_{k(i)}\ o\ h$. Now:
$h^{-1}\ o\ T_1\ o\ h$
$= \Phi(h^{-1}\ o\ T_1\ o\ h)$

$= \Phi((h^{-1}\ o\ \theta_i T_1\ o\ h)\ o\ ...\ o\ (h^{-1}\ o\ \theta_i T_1\ o\ h))$
(where we take $p-1$ compositions of $h^{-1}\ o\ \theta_i T_1\ o\ h$)
$= \Phi(h^{-1}\ o\ \theta_i T_1\ o\ h)\ o\ ...\ o\ \Phi(h^{-1}\ o\ \theta_i T_1\ o\ h)$
$= h^{-1}\ o\ \omega_{j(i)} T_{k(i)}\ o\ h\ o\ ...\ o\ h^{-1}\ o\ \omega_{j(i)} T_{k(i)}\ o\ h$
$= h^{-1}\ o\ \omega_{j(i)}^{1+k(i)+k(i)^2+...+k(i)^{p-2}} T_{k(i)^{p-1}}\ o\ h.$
Thus $(k(i))^{p-1} = 1 \Rightarrow k(i) = 1$, and so $\Phi(h^{-1}\ o\ \theta_i T_1\ o\ h) = h^{-1}\ o\ \omega_{j(i)} T_1\ o\ h$. By injectivity of Φ we must have that $\omega_{j(i)} \neq \omega_{j(s)}$ if $i \neq s$ for all $i, s \in \{1, ..., p-1\}$. But this is impossible since $p > q$ and so $Z(f)$ is not isomorphic to $Z(g)$ and the Theorem is proved.

A semi-group of $\mathbf{N}$ is called nonlacunary if it is not contained in a single generated semi-group. For example 2 and 6 form a nonlacunary semi-group but 4 and 8 do not. It was shown in [2] that if f and g have a common fixed point then the condition that p and q be coprime in Theorem 1 can be replaced by the weaker condition that p and q form a nonlacunary semi-group of $\mathbf{N}$. Thus we have:

COROLLARY 1

Let $f \in Exp^n(S^1)$ be orientation preserving with degree p. If there exists $g \in Z(f)$ with degree q and:
(i) p and q form a nonlacunary semi-group of $\mathbf{N}$
(ii) f and g have a common fixed point.
Then $Z(f)$ is not self-similar.

We now give an example of a map in $Exp^n(S^1)$ whose centralizer is not self-similar:

EXAMPLE

Let $f : S^1 \to S^1$ be given by $f(z) = \dfrac{(z - \frac{4}{21})(z + \frac{4}{13})}{(1 - \frac{4}{21}z)(1 + \frac{4}{13}z)}$. Then $f(z)$ is the restriction to S^1 of a Blaschke product defined on the closed unit disk so $f(z)$ is anlytic on S^1. The map $f(z)$ has degree 2 and as

$$\frac{1 - \frac{4}{21}}{1 + \frac{4}{21}} + \frac{1 - \frac{4}{13}}{1 + \frac{4}{13}} = \frac{17}{25} + \frac{11}{17} = \frac{564}{425} > 1$$

we have that $f(z)$ is expanding (c.f [3]). Let $h(z)$ be the Möbius map on S^1 defined by $h(z) = \dfrac{z + \frac{1}{16}}{1 + \frac{1}{16}z}$. As the measure theoretic entopy of $f(z)$ is equal to the topological entopy of f (where the measure is derived from using the Poisson kernal of the unique fixed point of $f(z)$ in the open unit disk) we have that $h \ o \ f \ o \ h^{-1} = cT_2$ for some $c \in S^1$ (c.f [3]). Evaluating this equation

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