Issue 1 JULY–DECEMBER 2001 ISSN 1083-0464

SOUTHWEST JOURNAL OF PURE AND APPLIED MATHEMATICS

CAMERON UNIVERSITY
DEPARTMENT OF MATHEMATICAL SCIENCES
LAWTON, OK 73505, U.S.A

Southwest Journal
of
Pure and Applied Mathematics

Editorial Office

SWJPAM
Cameron University
Department of Mathematical Sciences
Lawton, OK 73505, U.S.A.

Publication Program

2 issues per year
July, December

Information for Authors

Scope. SWJPAM is an electronic journal devoted to all aspects of Pure and Applied mathematics, and related topics. Authoritative expository and survey articles on subjects of special interest are also welcomed.

Since SWJPAM aims to serve not only the specialists, but also the user of mathematics (promoting, in such a way, a strong interaction between them), particular emphasis is placed on applications. Thus, SWJPAM publishes papers which, although not directly contributing to mathematics, contain, however, significant results (from any field) obtained by making an essential and well stressed use of the methods and results of mathematics.

Submission of Manuscripts. All manuscripts must be submitted in AMSTEXformat. Papers for publication should be sent as e-mail attachments to swj@rattler.cameron.edu. The articles must be placed in the 'SWJPAM_incoming' directory. The submission should include the title of the paper, the author's name and affiliation, and a short abstract which summarizes the paper; a list of keywords and a classification according to the Mathematical Reviews' Subject Classification (1991). Authors are informed by e-mail of referee communication.

Key Words. Key words should be comprehensive and subject specific. It is not necessary to list the subject area of the journal's coverage as a key word. Four key words should be sufficient to cover the major subjects off a given paper, although more can always be supplied if the author deems it necessary.

(continued on inside back cover)

SOUTHWEST JOURNAL
OF
PURE AND APPLIED MATHEMATICS
ISSUE 1, DECEMBER–JULY 2001

CONTENTS

ISSN 1083-0464

Electronic Journal: Southwest Journal of Pure and Applied Mathematics
Internet: http://rattler.cameron.edu/swjpam.html
ISBN 1083-0464
Issue 1 June 2001, pp. 1 – 12
Submitted: December 22, 2000. Published: June 30, 2001

On the Existence and Stability of Unfoldings of Equivariant Two-Parameter Bifurcation Problems

Bing Li, Jiancheng Zou

Abstract

Under the action of Γ-equivariant **U***-equivalence group, we study the unfoldings of equivariant two-parameter bifurcation problems of finite codimension, where the bifurcating parameters are of vector value. The main results : the existence theorem , stability theorem and the versal unfolding theorem (a lemma is proved and by means of it we give simple proofs of the theorems), the uniqueness theorem.*

A.M.S Classification Codes : 58F14, 58C27.
Keywords and Phrases: Equivariant bifurcation problem, versal unfolding, stability.

Golubitsky and Schaeffer[1][2] introduced the idea of applying the methods and ideas in singularity theory and Lie group theory to study equivariant bifurcation problems. In many bifurcation problems, two or more bifurcation parameters appear. [3] introduced **E**-equivalence and **U**-equivalence for such problems which has one real state variable with two real parameters and considered the classification under **E**-equivalence.

When n bifurcation parameters are considered as a vector, a more general "**E**"-equivalence is employed (cf. [4][5]). But when bifurcation parameters are two vectors and one may be dependent on the other, a more general "**U**"-equivalence (which is still called **U**-equivalence in this paper) should be used. In addition, the **U**-equivalence is more general than **E**-equivalence . In fact, if we let one parameter be zero vector at all time in the **U**-equivalence, we get **E**-equivalence. Furthermore, suppose a compact Lie group Γ acts on the state space. Therefore we will consider Γ-equivariant **U**-equivalence.

The main results in this paper include: (1) the existence theorem , the stability theorem and the versal unfolding theorem, we prove a lemma by which we can give simple proofs of the theorems. (2) the orbits of two-parameter bifurcation problems are one-to-one corresponding to the equivalent classes of versal k-parameter unfoldings. It is a analogue of Mather's fundamental theorem (cf. [6] or [7]) and is fundamental in the classification.

Changsha University, China
Research partially supported by National NSF of China under grant number 19771035
E-mail: lib@mail.csuep.edu.cn

All mapping germs in this paper are of C^∞.

1 Preliminaries

Let Γ be a compact Lie group, it acts linearly on the state space $\mathbb{R}^n$ and $x = (x_1, \ldots, x_n)$ is the coordinate in it. The bifurcation parameters are $\lambda = (\lambda_1, \ldots, \lambda_s) \in \mathbb{R}^s$ and $\mu = (\mu_1, \ldots, \mu_t) \in \mathbb{R}^t$. Γ also acts on $\mathbb{R}^n \times \mathbb{R}^s \times \mathbb{R}^t$ naturally, the action of it on $\mathbb{R}^s \times \mathbb{R}^t$ is trivial.

Let $\mathbf{E}_{x,\lambda,\mu}(\Gamma) = \{g : (\mathbb{R}^n \times \mathbb{R}^s \times \mathbb{R}^t, 0) \to \mathbb{R}^n |$ germ g satisfies the following condition (1)$\}$, and $\mathbf{E}_{\lambda,\mu}$ is the set of mapping germs $f : (\mathbb{R}^s \times \mathbb{R}^t, 0) \to \mathbb{R}^s$, similarly, $\mathbf{E}_\mu$ is the set of germs $f : (\mathbb{R}^t, 0) \to \mathbb{R}^t$. Let $\vec{\mathbf{E}}_{x,\lambda,\mu}(\Gamma) = \{S : (\mathbb{R}^n \times \mathbb{R}^s \times \mathbb{R}^t, 0) \to gl(n)|$ germ S satisfies the following condition (2)$\}$.

$$g(\gamma x, \lambda, \mu) = \gamma g(x, \lambda, \mu), \forall \gamma \in \Gamma, \tag{1}$$

$$S(\gamma x, \lambda, \mu) = \gamma S(x, \lambda, \mu)\gamma^{-1}, \forall \gamma \in \Gamma. \tag{2}$$

It can be proved that $\mathbf{E}_{x,\lambda,\mu}(\Gamma)$ and $\vec{\mathbf{E}}_{x,\lambda,\mu}(\Gamma)$ are finitely generated modules over the ring $\varepsilon_{x,\lambda,\mu}(\Gamma)$ of Γ-invariant function germs (cf. [8]). If $g \in \mathbf{E}_{x,\lambda,\mu}(\Gamma)$ satisfies $g(0) = 0$ and $(D_x g)_0$=0, we call it a *$\Gamma -$ equivariant bifurcation problem* with two-parameters (or **bifurcation problem** for short), where $(D_x g)_0$ is the derivative $\frac{\partial g}{\partial x}$ at 0. Let $\mathbf{M}^l_{x,\lambda,\mu}(\Gamma)$ be the subspace of $\mathbf{E}_{x,\lambda,\mu}(\Gamma)$, a germ $g \in \mathbf{M}^l_{x,\lambda,\mu}(\Gamma)$ iff all of the derivatives of g at $0 \in \mathbb{R}^n \times \mathbb{R}^s \times \mathbb{R}^t$ with orders less than l are equal to zero. When l=1, $\mathbf{M}^1_{x,\lambda,\mu}(\Gamma)$ is denoted by $\mathbf{M}_{x,\lambda,\mu}(\Gamma)$. We define the subspace $\mathbf{M}^l_{\lambda,\mu}$ of $\mathbf{E}_{\lambda,\mu}$ and $\mathbf{M}^l_\mu$ of $\mathbf{E}_\mu$ similarly. Let

$$\mathfrak{U} = \{(S, X, \Lambda, M) \in \vec{\mathbf{E}}_{x,\lambda,\mu}(\Gamma) \times \mathbf{M}_{x,\lambda,\mu}(\Gamma) \times \mathbf{M}_{\lambda,\mu} \times \mathbf{M}_\mu;$$

$$S(0) \in \mathfrak{L}_\Gamma(\mathbb{R}^n)^o, (D_x X)_0 \in \mathfrak{L}_\Gamma(\mathbb{R}^n)^o, (D_\lambda \Lambda)_0 \in \mathfrak{L}(\mathbb{R}^s)^o, (D_\mu M)_0 \in \mathfrak{L}(\mathbb{R}^t)^o\}.$$

Where $\mathfrak{L}_\Gamma(\mathbb{R}^n)$ is the linear space of Γ-equivariant linear maps on $\mathbb{R}^n$ and $\mathfrak{L}_\Gamma(\mathbb{R}^n)^o$ is the connected component of $\mathfrak{L}_\Gamma(\mathbb{R}^n) \cap GL(\mathbb{R}^n)$ which contains the unity.

We say bifurcation problems g and h are *$\Gamma -$equivariant* $\mathbf{U}-$*equivalent* (or **equivalent**) if $\exists (S, X, \Lambda, M) \in \mathfrak{U}$ such that

$$h(x, \lambda, \mu) = S(x, \lambda, \mu) g(X(x, \lambda, \mu), \Lambda(\lambda, \mu), M(\mu)).$$

It is clearly a equivalent relation. When g and h are equivalent, the equality above is also denoted by $h = u \cdot g$, $u = (S, X, \Lambda, M) \in \mathfrak{U}$. Let

$$T(g, \mathfrak{U}) = \vec{\mathbf{E}}_{x,\lambda,\mu}(\Gamma) \cdot g + (D_x g)\mathbf{E}_{x,\lambda,\mu}(\Gamma) + (D_\lambda g)\mathbf{E}_{\lambda,\mu} + (D_\mu g)\mathbf{E}_\mu, \tag{3}$$

$$\tilde{T}(g) = \vec{\mathbf{E}}_{x,\lambda,\mu}(\Gamma) \cdot g + (D_x g)\mathbf{E}_{x,\lambda,\mu}(\Gamma), \tag{4}$$

then $\tilde{T}(g)$ is a submodule of $\mathbf{E}_{x,\lambda,\mu}(\Gamma)$ on the ring $\varepsilon_{x,\lambda,\mu}(\Gamma)$. g is of *finite codimension* if $dim_\mathbb{R} \mathbf{E}_{x,\lambda,\mu}(\Gamma)/\tilde{T}(g) < +\infty$. A Γ-equivariant mapping germ $G : (\mathbb{R}^n \times \mathbb{R}^s \times \mathbb{R}^t \times \mathbb{R}^k, 0) \to$

$(\mathbb{R}^n, 0) \in \mathbf{E}_{x,\lambda,\mu\,\alpha}(\Gamma)$, is called a ($k$-parameter) $unfolding$ of g if it satisfies $G(x, \lambda, \mu, 0) = g(x, \lambda, \mu)$. A k-parameter unfolding H of g is called $factors\ through$ another l-parameter unfolding G of g if

$$H(x, \lambda, \mu, \beta) = S(x, \lambda, \mu, \beta)G(X(x, \lambda, \mu, \beta), \Lambda(\lambda, \mu, \beta), M(\mu, \beta), \psi(\beta)), \tag{5}$$

where $S : (\mathbb{R}^n \times \mathbb{R}^s \times \mathbb{R}^t \times \mathbb{R}^k, 0) \to gl(n)$ and $\psi : (\mathbb{R}^k, 0) \to \mathbb{R}^l$ and

$$\psi(0) = 0, M(\mu, 0) = \mu, \Lambda(\lambda, \mu, 0) = \lambda; \tag{6}$$

$$S(x, \lambda, \mu, 0) = I_n, X(x, \lambda, \mu, 0) = x; \tag{7}$$

$$S(\gamma x, \lambda, \mu, \beta) = \gamma S(x, \lambda, \mu, \beta)\gamma^{-1}, X(\gamma x, \lambda, \mu, \beta) = \gamma X(x, \lambda, \mu, \beta), \forall \gamma \in \Gamma. \tag{8}$$

Suppose H factors through G and (5)-(8) hold, then we say H and G are $\Gamma - isomorphic$ if ψ is a diffeomorphism. A unfolding G of g is $versal$ if every unfolding H of g factors through G. Suppose G_1, G_2, G_3 are unfoldings of g. If G_1 factors through G_2 and G_2 factors through G_3, then G_1 factors through G_3 .

2 Existence and Uniqueness

Definition 1 *Let f,g be bifurcation problems, F and G be k-parameter unfoldings of f and g respectively. F and G are called* **equivalent** *if F can be written in the form:*

$$F(x, \lambda, \mu, \alpha) = S(x, \lambda, \mu, \alpha)G(X(x, \lambda, \mu, \alpha), \Lambda(\lambda, \mu, \alpha), M(\mu, \alpha), \phi(\alpha)), \tag{9}$$

where $S(x, \lambda, \mu, \alpha), X(x, \lambda, \mu, \alpha)$ satisfy (8) in §1 and

$$(S(\cdot, \cdot, \cdot, 0), X(\cdot, \cdot, \cdot, 0), \Lambda(\cdot, \cdot, 0), M(\cdot, , 0)) \in \mathfrak{U}, \tag{10}$$

$$\phi : (\mathbb{R}^k, 0) \to (\mathbb{R}^k, 0)\ is\ a\ differential\ homeomorphism. \tag{11}$$

Corollary 1 *Let f,g be bifurcation problems, F and G be k-parameter unfoldings of f and g respectively. If F and G are equivalent, then f and g are equivalent.*

Theorem 1 *Let bifurcation problem g be of finite codimension,*

$$l = dim_{\mathbb{R}}\mathbf{E}_{x,\lambda,\mu}(\Gamma)/T(g, \mathfrak{U}).$$

Then g has a versal l-parameter unfolding.

Proof. It follows from Theorem 5 (in §3), we omit the details.

Theorem 2 *Suppose F, G are versal k-parameter unfoldings of $f(x, \lambda, \mu) = F(x, \lambda, \mu, 0)$ and $g(x, \lambda, \mu) = G(x, \lambda, \mu, 0)$ respectively, where f and g are bifurcation problems of finite codimension, $k \in \mathbb{N}$. Then F and G are equivalent if and only if f and g are equivalent.*

Proof. We only need to prove the "if" part. Because f and g are equivalent, there is an $u = (S, X, \Lambda, M) \in \mathfrak{U}$, such that $g = u \cdot f$. Let

$$H(x, \lambda, \mu, \alpha) = S(x, \lambda, \mu)F(X(x, \lambda, \mu), \Lambda(\lambda, \mu), M(\mu), \alpha), \tag{12}$$

where $\alpha \in (\mathbb{R}^k, 0)$. Then H is a k-parameter unfolding of g. We divide the following proof into three steps: 1.H is a versal unfolding of g; 2.H and G are equivalent ; 3.F and G are equivalent.

Fact 1. H is a versal unfolding of g.

By $u = (S, X, \Lambda, M)$, we get a Γ-equivariant diffeomorphism germ $\Phi : (\mathbb{R}^n \times \mathbb{R}^s \times \mathbb{R}^t, 0) \to (\mathbb{R}^n \times \mathbb{R}^s \times \mathbb{R}^t, 0)$, $\Phi(x, \lambda, \mu) = (X(x, \lambda, \mu), \Lambda(\lambda, \mu), M(\mu)) \equiv (x', \lambda', \mu')$. Then the inverse of Φ , $\Phi^{-1} : (\mathbb{R}^n \times \mathbb{R}^s \times \mathbb{R}^t, 0) \to (\mathbb{R}^n \times \mathbb{R}^s \times \mathbb{R}^t, 0)$ is also a Γ-equivariant diffeomorphism germ of the form $(x', \lambda', \mu') \stackrel{\Phi^{-1}}{\longmapsto} (X'(x', \lambda', \mu'), \Lambda'(\lambda', \mu'), M'(\mu'))$. By (12) we have

$$F(x', \lambda', \mu', \alpha) = S^{-1}(\Phi^{-1}(x', \lambda', \mu'))H(X'(x', \lambda', \mu'), \Lambda'(\lambda', \mu'), M'(\mu'), \alpha), \tag{13}$$

where S^{-1} is the inverse matrix of S.

Let $K(x, \lambda, \mu, \beta)$ be a l-parameter unfolding of g. We prove that K factors through H. Because $u \cdot f = g = K(x, \lambda, \mu, 0)$, we have

$$S^{-1}(\Phi^{-1}(x', \lambda', \mu'))K(\Phi^{-1}(x', \lambda', \mu'), 0) = f(x', \lambda', \mu').$$

Then $S^{-1}(\Phi^{-1}(x', \lambda', \mu'))K(\Phi^{-1}(x', \lambda', \mu'), \beta)$ is an unfolding of f. Since $F(x', \lambda', \mu', \alpha)$ is a versal unfolding of f, so $S^{-1}(\Phi^{-1}(x', \lambda', \mu'))K(\Phi^{-1}(x', \lambda', \mu'), \beta)$ factors through F:

$$\begin{aligned} &S^{-1}(\Phi^{-1}(x', \lambda', \mu'))K(\Phi^{-1}(x', \lambda', \mu'), \beta) = \\ &\widetilde{S}(x', \lambda', \mu', \beta)F(\widetilde{X}(x', \lambda', \mu', \beta), \widetilde{\Lambda}(\lambda', \mu', \beta), \widetilde{M}(\mu', \beta), \widetilde{\phi}(\beta)), \end{aligned} \tag{14}$$

where $\widetilde{S}, \widetilde{X}, \widetilde{\Lambda}, \widetilde{M}, \widetilde{\phi}$ satisfy the conditions corresponding to (6),(7),(8). By (13) and (14) we have

$$\begin{aligned} &S^{-1}(\Phi^{-1}(x', \lambda', \mu'))K(\Phi^{-1}(x', \lambda', \mu'), \beta) = \\ &\widetilde{S}(x', \lambda', \mu', \beta)S^{-1}(\Phi^{-1}(\widetilde{X}, \widetilde{\Lambda}, \widetilde{M}))H(X'(\widetilde{X}, \widetilde{\Lambda}, \widetilde{M}), \Lambda'(\widetilde{\Lambda}, \widetilde{M}), M'(\widetilde{M}), \widetilde{\phi}(\beta)), \end{aligned} \tag{15}$$

where $\widetilde{X} = \widetilde{X}(x', \lambda', \mu', \beta), \widetilde{\Lambda} = \widetilde{\Lambda}(\lambda', \mu', \beta), \widetilde{M} = \widetilde{M}(\mu', \beta)$. Let $S_1(x, \lambda, \mu, \beta) = S(x, \lambda, \mu)\widetilde{S}(\Phi(x, \lambda, \mu), \beta)S^{-1}(\Phi^{-1}(\widetilde{X}(\Phi(x, \lambda, \mu), \beta), \widetilde{\Lambda}(\Lambda(\lambda, \mu), M(\mu), \beta), \widetilde{M}(M(\mu), \beta)))$, therefore we have

$$K(x, \lambda, \mu, \beta) = S_1(x, \lambda, \mu, \beta)H(\Psi(x, \lambda, \mu, \beta)), \tag{16}$$

where Ψ is determined by the following commutative diagram:

$$
\begin{tikzcd}
\mathbb{R}^n \times \mathbb{R}^s \times \mathbb{R}^t \times \mathbb{R}^l \arrow[r, "\Phi \times 1_{\mathbb{R}^l}"] \arrow[ddr, "\Psi"'] & \mathbb{R}^n \times \mathbb{R}^s \times \mathbb{R}^t \times \mathbb{R}^l \arrow[d, "(\widetilde{\Phi}, \widetilde{\phi})"] \\
& \mathbb{R}^n \times \mathbb{R}^s \times \mathbb{R}^t \times \mathbb{R}^k \arrow[d, "\Phi^{-1} \times 1_{\mathbb{R}^k}"] \\
& \mathbb{R}^n \times \mathbb{R}^s \times \mathbb{R}^t \times \mathbb{R}^k
\end{tikzcd}
$$

where $(\widetilde{\Phi}, \widetilde{\phi})(x', \lambda', \mu', \beta) = (\widetilde{X}(x', \lambda', \mu', \beta), \widetilde{\Lambda}(\lambda', \mu', \beta), \widetilde{M}(\mu', \beta), \widetilde{\phi}(\beta))$. It is not difficult to see that we can write Ψ in the form

$$\Psi(x, \lambda, \mu, \beta) = (X_1(x, \lambda, \mu, \beta), \Lambda_1(\lambda, \mu, \beta), M_1(\mu, \beta), \psi_1(\beta)),$$

and clearly $\psi_1(\beta) = \widetilde{\phi}(\beta)$. Therefore we can rewrite (16) as

$$K(x, \lambda, \mu, \beta) = S_1(x, \lambda, \mu, \beta) H(X_1(x, \lambda, \mu, \beta), \Lambda_1(\lambda, \mu, \beta), M_1(\mu, \beta), \psi_1(\beta)).$$

Thus to prove K factors through H we only need to verify that $S_1, X_1, \Lambda_1, M_1, \psi_1$ satisfy the conditions corresponding to (6)-(8). Because S, $\Phi = (X, \Lambda, M)$ are determined by u, S, X, X', S^{-1} and $\widetilde{S}, \widetilde{X}$ satisfy the equivariant conditions corresponding to (8) hence so does S_1. Since $\widetilde{S}, \widetilde{X}, \widetilde{\Lambda}, \widetilde{M}$ satisfy (6),(7), $S_1(x, \lambda, \mu, 0) = I_n$. By

$$
\begin{aligned}
& X_1(x, \lambda, \mu, \beta) = \\
& X'(\widetilde{X}(X(x, \lambda, \mu), \Lambda(\lambda, \mu), M(\mu), \beta), \widetilde{\Lambda}(\Lambda(\lambda, \mu), M(\mu), \beta), \widetilde{M}(M(\mu), \beta)), \\
& \Lambda_1(\lambda, \mu, \beta) = \Lambda'(\widetilde{\Lambda}(\Lambda(\lambda, \mu), M(\mu), \beta), \widetilde{M}(M(\mu), \beta)), \\
& M_1(\mu, \beta) = M'(\widetilde{M}(M(\mu), \beta)),
\end{aligned}
$$

we have

$$
\begin{aligned}
& X_1(\gamma x, \lambda, \mu, \beta) = \gamma X_1(x, \lambda, \mu, \beta), \forall \gamma \in \Gamma, \\
& X_1(x, \lambda, \mu, 0) = X'(X(x, \lambda, \mu), \Lambda(\lambda, \mu), M(\mu)) = x, \\
& \Lambda_1(\lambda, \mu, 0) = \Lambda'(\Lambda(\lambda, \mu), M(\mu)) = \lambda, \\
& M_1(\mu, 0) = M'(M(\mu)) = \mu
\end{aligned}
$$

and Fact 1 is proved.

Fact 2. H and G are equivalent.

Suppose $l = dim_{\mathbb{R}} \frac{\mathbf{E}_{x,\lambda,\mu}(\Gamma)}{T(g, \mathfrak{U})}$, it is clear by Theorem 5 in §3 that $l \leqslant k$. By Lemma 1 we can take a l-parameter versal unfolding K of g, $K \in \mathbf{E}_{x,\lambda,\mu,\beta}(\Gamma), \beta \in \mathbb{R}^l$. Hence there are $S_i, X_i, \Lambda_i, M_i, \psi_i (i = 1, 2)$ which satisfy the conditions corresponding to (6)-(8) shch that

$$G(x, \lambda, \mu, \alpha) = S_1(x, \lambda, \mu, \alpha) K(X_1(x, \lambda, \mu, \alpha), \Lambda_1(\lambda, \mu, \alpha), M_1(\mu, \alpha), \psi_1(\alpha)), \tag{17}$$

$$H(x, \lambda, \mu, \alpha) = S_2(x, \lambda, \mu, \alpha) K(X_2(x, \lambda, \mu, \alpha), \Lambda_2(\lambda, \mu, \alpha), M_2(\mu, \alpha), \psi_2(\alpha)). \tag{18}$$

We claim: There is a diffeomorphism germ $\phi : (\mathbb{R}^k, 0) \to (\mathbb{R}^k, 0)$ such that

$$\psi_2 = \psi_1 \circ \phi. \tag{19}$$

In fact, since G and H are both versal unfoldings of g, theorem 5 implies that the real linear space $\frac{\mathbf{E}_{x,\lambda,\mu}(\Gamma)}{T(g,\mathfrak{U})}$ is generated by $\{ \frac{\partial G}{\partial \alpha_1}|_{\alpha=0}, \cdots, \frac{\partial G}{\partial \alpha_k}|_{\alpha=0} \}$ and also by $\{ \frac{\partial H}{\partial \alpha_1}|_{\alpha=0}, \cdots, \frac{\partial H}{\partial \alpha_k}|_{\alpha=0} \}$. Hence by differentiating (17) and (18) we have

$$\frac{\partial G}{\partial \alpha_j}|_{\alpha=0} \equiv \frac{\partial K}{\partial \beta} \cdot \frac{\partial \psi_1}{\partial \alpha_j}|_{\alpha=0}(mod\, T(g,\mathfrak{U})), j = 1, \cdots, k;$$

$$\frac{\partial H}{\partial \alpha_j}|_{\alpha=0} \equiv \frac{\partial K}{\partial \beta} \cdot \frac{\partial \psi_2}{\partial \alpha_j}|_{\alpha=0}(mod\, T(g,\mathfrak{U})), j = 1, \cdots, k.$$

It follows that ψ_1 and ψ_2 are both submersion germs. So there is a diffeomorphism $\phi : (\mathbb{R}^k, 0) \to (\mathbb{R}^k, 0)$ such that $\psi_2 = \psi_1 \circ \phi$. Now let $K_1(x, \lambda, \mu, \alpha) = K(x, \lambda, \mu, \psi_1(\alpha))$, then

$$G(x, \lambda, \mu, \alpha) = S_1(x, \lambda, \mu, \alpha)K_1(X_1(x, \lambda, \mu, \alpha), \Lambda_1(\lambda, \mu, \alpha), M_1(\mu, \alpha), \alpha).$$

Let $\Psi(x, \lambda, \mu, \alpha) = (X_1(x, \lambda, \mu, \alpha), \Lambda_1(\lambda, \mu, \alpha), M_1(\mu, \alpha), \alpha)$, then Ψ is a Γ-equivariant diffeomorphism germ, and it's inverse can be written in the form

$$\Psi^{-1}(x', \lambda', \mu', \alpha) = (X_1'(x', \lambda', \mu', \alpha), \Lambda_1'(\lambda', \mu', \alpha), M_1'(\mu', \alpha), \alpha).$$

Thus by (17) we have

$$K_1(x', \lambda', \mu', \alpha) = S_1^{-1}(\Psi^{-1}(x', \lambda', \mu', \alpha))G(\Psi^{-1}(x', \lambda', \mu', \alpha)). \tag{20}$$

From (18),(19),(20) we get

$$\begin{aligned} H(x, \lambda, \mu, \alpha) &= S_2(x, \lambda, \mu, \alpha)K(X_2(x, \lambda, \mu, \alpha), \Lambda_2(\lambda, \mu, \alpha), M_2(\mu, \alpha), \psi_1(\phi(\alpha))) \\ &= S_2(x, \lambda, \mu, \alpha)K_1(X_2(x, \lambda, \mu, \alpha), \Lambda_2(\lambda, \mu, \alpha), M_2(\mu, \alpha), \phi(\alpha)) \\ &= S_2(x, \lambda, \mu, \alpha)S_1^{-1}(\Psi^{-1}(X_2(x, \lambda, \mu, \alpha), \Lambda_2(\lambda, \mu, \alpha), M_2(\mu, \alpha), \phi(\alpha))) \\ &\quad G(\Psi^{-1}(X_2(x, \lambda, \mu, \alpha), \Lambda_2(\lambda, \mu, \alpha), M_2(\mu, \alpha), \phi(\alpha))). \end{aligned} \tag{21}$$

Let

$$\begin{aligned} &\widetilde{S}(x, \lambda, \mu, \alpha) = S_2(x, \lambda, \mu, \alpha)S_1^{-1}(\Psi^{-1}(X_2(x, \lambda, \mu, \alpha), \Lambda_2(\lambda, \mu, \alpha), M_2(\mu, \alpha), \phi(\alpha))), \\ &\widetilde{X}(x, \lambda, \mu, \alpha) = X_1'(X_2(x, \lambda, \mu, \alpha), \Lambda_2(\lambda, \mu, \alpha), M_2(\mu, \alpha), \phi(\alpha)), \\ &\widetilde{\Lambda}(\lambda, \mu, \alpha) = \Lambda_1'(\Lambda_2(\lambda, \mu, \alpha), M_2(\mu, \alpha), \phi(\alpha)), \\ &\widetilde{M}(\mu, \alpha) = M_1'(M_2(\mu, \alpha), \phi(\alpha)),\ \widetilde{\phi} = \phi, \end{aligned}$$

then (21) can be written in the form of (9). Note the conditions which S_i,X_i,Λ_i,M_i and ψ_i $(i = 1, 2)$ satisfy, $\Psi|_{\alpha=0}$ is the identity germ on $(\mathbb{R}^n \times \mathbb{R}^s \times \mathbb{R}^t, 0)$, thus

$$\widetilde{S}(x, \lambda, \mu, 0) = I_n, \widetilde{X}(x, \lambda, \mu, 0) = x, \widetilde{\Lambda}(\lambda, \mu, 0) = \lambda, \widetilde{M}(\mu, 0) = \mu, \tag{22}$$

and $\widetilde{S}, \widetilde{X}$ also satisfy the equivariant condintions (8). This completes the proof of Fact 2.

Finally, we prove F and G are equivalent. Interchange G and H in the proof above, then G can be written in the form:

$$G(x,\lambda,\mu,\alpha) = \widetilde{S}(x,\lambda,\mu,\alpha)H(\widetilde{X}(x,\lambda,\mu,\alpha),\widetilde{\Lambda}(\lambda,\mu,\alpha),\widetilde{M}(\mu,\alpha),\alpha),$$

where $\widetilde{S}, \widetilde{X}$ satisfy (8) and (22). By (12), we have

$$G(x,\lambda,\mu,\alpha) = \widetilde{S}(x,\lambda,\mu,\alpha)S(\widetilde{X},\widetilde{\Lambda},\widetilde{M})F(X(\widetilde{X},\widetilde{\Lambda},\widetilde{M}),\Lambda(\widetilde{\Lambda},\widetilde{M}),M(\widetilde{M}),\widetilde{\phi}(\alpha)). \tag{23}$$

Therefore F and G are equivalent, this completes the proof of the theorem.

Remark. The Theorem 2 is fundamental in classification of two-parameter bifurcation problems, we will discuss classification in another paper. In addition, if $f = g$, then theorem 2 implies that F and G are Γ-isomorphic.Thus by the following corollary, theorem 2 is a generalization of Theorem 7.4 in [8].

Corollary 2 *Suppose F and G are versal k-parameter, l-parameter unfoldings of f respectively. Then F and G are Γ-isomorphic if and only if $l = k$.*

proof. We need only prove the "if" part. At the begining of the proof of theorem 2, let u be the unit of the group $\mathfrak{U}$, so H=F there. By the proof of Fact 2, see (21), H namely F can be written in the form

$$F(x,\lambda,\mu,\alpha) = \hat{S}(x,\lambda,\mu,\alpha)G(\hat{X}(x,\lambda,\mu,\alpha),\hat{\Lambda}(\lambda,\mu,\alpha),\hat{M}(\mu,\alpha),\phi(\alpha)),$$

where $\hat{S},\hat{X},\hat{\Lambda},\hat{M}$ satisfy (22) and $\hat{S},\hat{X}$ satisfy (8). Since ϕ is a diffeomorphism here, so F and G are Γ-isomorphic.

3 Stability and Versal Unfolding Theorem

Definition 2 *(1) Let F be an unfolding of a bifurcation problem f, the unfolding parameter is $\omega \in \mathbb{R}^m$. Suppose H is a k-parameter unfolding of F, the unfolding parameter is $\alpha \in \mathbb{R}^k$ (where H=F when k=0), and G is a l-parameter unfolding of H, the unfolding parameter is $\beta \in \mathbb{R}^l$. If G can be written in the form*

$$\begin{aligned} G(x,\lambda,\mu,\omega,\alpha,\beta) = S(x,\lambda,\mu,\omega,\alpha,\beta)H(X(x,\lambda,\mu,\omega,\alpha,\beta),\\ \Lambda(\lambda,\mu,\omega,\alpha,\beta),M(\mu,\omega,\alpha,\beta),\phi(\omega,\alpha,\beta)),\end{aligned} \tag{24}$$

where $\phi : (\mathbb{R}^m \times \mathbb{R}^k \times \mathbb{R}^l, 0) \to (\mathbb{R}^m \times \mathbb{R}^k, 0)$,and $\forall \gamma \in \Gamma$,

$$S(\gamma x,\cdots) = \gamma S(x,\cdots)\gamma^{-1}, X(\gamma x,\cdots) = \gamma X(x,\cdots). \tag{25}$$

$$S(\cdots,0) = I_n, X(\cdots,0) = x, \Lambda(\cdots,0) = \lambda, M(\cdots,0) = \mu, \phi(\omega,\alpha,0) = (\omega,\alpha), \tag{26}$$

then we say G is **isomorphic** *to the l-parameter trivial unfolding of H.*
(2) If any l-parameter unfolding of F is isomorphic to the l-parameter trivial unfolding of F, $\forall l \in \mathbb{N}$, then F is called **stable***. Where F is also understood as the 0-parameter unfolding of itself.*

Notation 1 *Suppose G is a unfolding of H, H is a unfolding of K. If G is isomorphic to a l_1-parameter trivial unfolding of H, H is isomorphic to a l_2-parameter trivial unfolding of K, then G is isomorphic to a $(l_1 + l_2)$-parameter trivial unfolding of K.*

Notation 2 *Suppose $G(x, \lambda, \mu, \omega, \beta)$ is isomorphic to the 1-parameter trivial unfolding of $F(x, \lambda, \mu, \omega)$, $\beta \in \mathbb{R}$. Then G factor through F.*

Proof. By Definition 2, G can be written in the form

$$G(x, \lambda, \mu, \omega, \beta) = S(x, \lambda, \mu, \omega, \beta)F(X(x, \lambda, \mu, \omega, \beta), \Lambda(\lambda, \mu, \omega, \beta), M(\mu, \omega, \beta), \phi(\omega, \beta)),$$

where ϕ, S, X, Λ, M satisfy (25),(26) thus they satisfy (6)-(8). So G factor through F.

We suppose F is of finite codimension in this section, namely $dim_{\mathbb{R}}\mathbf{E}_{x,\lambda,\mu,\omega}(\mathit{\Gamma})/\tilde{T}(F) < +\infty$, where

$$\tilde{T}(F) = \vec{\mathbf{E}}_{x,\lambda,\mu,\omega}(\mathit{\Gamma}) \cdot F + (D_x F)\mathbf{E}_{x,\lambda,\mu,\omega}(\mathit{\Gamma}),$$

and the definitions of $\vec{\mathbf{E}}_{x,\lambda,\mu,\omega}(\mathit{\Gamma}), \mathbf{E}_{x,\lambda,\mu,\omega}(\mathit{\Gamma})$ are similar to $\vec{\mathbf{E}}_{x,\lambda,\mu}(\mathit{\Gamma})$ and $\mathbf{E}_{x,\lambda,\mu}(\mathit{\Gamma})$ in §1, they are also finitely generated modules over the ring $\varepsilon_{x,\lambda,\mu,\omega}(\mathit{\Gamma})$ of $\mathit{\Gamma}$-invariant function germs. $\tilde{T}(F)$ is a submodule of $\mathbf{E}_{x,\lambda,\mu,\omega}(\mathit{\Gamma})$.

Theorem 3 *(Stability Theorem) Suppose $F \in \mathbf{E}_{x,\lambda,\mu,\omega}(\mathit{\Gamma})$ is a m-parameter unfolding of bifurcation problem $f \in \mathbf{E}_{x,\lambda,\mu}(\mathit{\Gamma})$. Then F is stable if and only if*

$$\begin{aligned} \mathbf{E}_{x,\lambda,\mu,\omega}(\mathit{\Gamma}) &= \vec{\mathbf{E}}_{x,\lambda,\mu,\omega}(\mathit{\Gamma}) \cdot F + (D_x F)\mathbf{E}_{x,\lambda,\mu,\omega}(\mathit{\Gamma}) \\ &+ (D_\lambda F)\mathbf{E}_{\lambda,\mu,\omega} + (D_\mu F)\mathbf{E}_{\mu,\omega} + (D_\omega F)\mathbf{E}_\omega \end{aligned} \tag{27}$$

Proof. Necessity is obvious: for any $h \in \mathbf{E}_{x,\lambda,\mu,\omega}(\mathit{\Gamma})$, consider the 1-parameter unfolding G of F, $G(x, \lambda, \mu, \omega, \beta) = F + \beta h$, $\beta \in \mathbb{R}$. Thus $F + \beta h = G$ isomorphic to the 1-parameter trivial unfolding of F implies $h \in$ the right side of (27).

Remark. Before the proof of sufficiency, we give an importent lemma which will be proved at the end of this section, it will makes the proofs of the sufficiency in this theorem and in versal unfolding theorem very simple.

Lemma 1 *Suppose G is a k-parameter unfolding of F, the unfolding parameter is $\alpha \in \mathbb{R}^k$. If (27) holds and G_0 satisfies*

$$G_0 \in \mathbf{M}_{x,\lambda,\mu,\omega,\alpha,\tau}(\mathit{\Gamma}), G_0(x, \lambda, \mu, \omega, \alpha, 0) \equiv 0, (\tau \in \mathbb{R}), \tag{28}$$

then $\hat{G} \equiv G+G_0$ is isomorphic to the 1-parameter trivial unfolding of G.

Now we assume the lemma 1 is proved. Let H be a l-parameter unfolding of F, the unfolding parameter is $\beta \in \mathbb{R}^l$. Define

$$
\begin{aligned}
&H_0(x,\lambda,\mu,\omega) = F(x,\lambda,\mu,\omega),\\
&H_i(x,\lambda,\mu,\omega,\beta_1,\cdots,\beta_i) = H(x,\lambda,\mu,\omega,\beta_1,\cdots,\beta_i,0,\cdots,0), i=1,\cdots,l;\\
&\hat{G}_0(x,\lambda,\mu,\omega) = F = H_0,\\
&\hat{G}_1(x,\lambda,\mu,\omega,\beta_1) = \hat{G}_0 + [H_1 - H_0] = H_1,\\
&\cdots\\
&\hat{G}_l(x,\lambda,\mu,\omega,\beta_1,\cdots,\beta_l) = \hat{G}_{l-1} + [H_l - H_{l-1}] = H_l = H.
\end{aligned}
$$

Let $\hat{G} = \hat{G}_1(x,\lambda,\mu,\omega,\beta_1)$, $G = \hat{G}_0(x,\lambda,\mu,\omega) = F$, and $G_0(x,\lambda,\mu,\omega,\alpha,\beta_1) = H_1 - H_0$. For F, (27) holds; and for $G_0 = H_1 - H_0$ (28) holds. Thus it follows from Lemma 1 that $\hat{G}_1$ is isomorphic to the 1-parameter unfolding of $\hat{G}_0$. By induction principle we can prove that $\hat{G}_i$ is isomorphic to the 1-parameter trivial unfolding of $\hat{G}_{i-1}$ because of Lemma 1, where $i = 1,2,\cdots,l$. Thus $H = \hat{G}_l$ is isomorphic to the l-parameter trivial unfolding of $\hat{G}_0 = F$ by Notation 1. Therefore the theorem is proved.

Theorem 4 *Suppose f is a bifurcation problem of finite codimension, F is a k-parameter unfolding of f, the unfolding parameter is $\alpha = (\alpha_1,\cdots,\alpha_k) \in \mathbb{R}^k$. Then F is stable if and only if*

$$
T(g,\mathfrak{U}) + \mathbb{R}\{\frac{\partial F}{\partial \alpha_1}|_{\alpha=0},\cdots,\frac{\partial F}{\partial \alpha_k}|_{\alpha=0}\} = \mathbf{E}_{\mathbf{x},\lambda,\mu}(\mathit{\Gamma}). \tag{29}
$$

Proof. Necessity. Let $T(F) = \tilde{T}(F) + (D_\lambda F)(\mathbf{E}_{\lambda,\mu,\alpha}) + (D_\mu F)(\mathbf{E}_{\mu,\alpha}) + (D_\alpha F)(\mathbf{E}_\alpha)$, where $\tilde{T}(F) = \vec{\mathbf{E}}_{x,\lambda,\mu,\alpha}(\mathit{\Gamma})\cdot F + (D_x F)\mathbf{E}_{x,\lambda,\mu,\alpha}(\mathit{\Gamma})$. By Theorem 3, $T(F) = \mathbf{E}_{x,\lambda,\mu,\alpha}(\mathit{\Gamma})$. Restricting $\alpha = 0$ we can get a map π from $\mathbf{E}_{x,\lambda,\mu,\alpha}(\mathit{\Gamma})$ onto $\mathbf{E}_{x,\lambda,\mu}(\mathit{\Gamma})$. But $\pi(T(F)) \subset T(g,\mathfrak{U}) + \mathbb{R}\{\frac{\partial F}{\partial \alpha_1}|_{\alpha=0},\cdots,\frac{\partial F}{\partial \alpha_k}|_{\alpha=0}\}$, this implies (29). Sufficiency. By Theorem 3, it is enough to prove $\mathbf{E}_{x,\lambda,\mu,\alpha}(\mathit{\Gamma}) \subset T(F)$. The proof for this is just like that in the first part of the proof of Lemma 1 hence we omit the details here.

Remark. The following is a simple form of Lemma 1, it will be used to prove versal unfolding theorem next.

Corollary 3 *Suppose G is a k-parameter unfolding of bifurcation problem g, the unfolding parameter is $\alpha \in \mathbb{R}^k$. If $\mathbf{E}_{x,\lambda,\mu}(\mathit{\Gamma}) = T(g,\mathfrak{U}) + \mathbb{R}\{\frac{\partial G}{\partial \alpha_1}|_{\alpha=0},\cdots,\frac{\partial G}{\partial \alpha_k}|_{\alpha=0}\}$. and G_0 satisfies*

$$
G_0 \in \mathbf{M}_{x,\lambda,\mu,\alpha,\tau}(\mathit{\Gamma}), G_0(x,\lambda,\mu,\alpha,0) \equiv 0, (\tau \in \mathbb{R}), \tag{30}
$$

then $\hat{G} \equiv G + G_0$ factor through G.

Proof. In fact, kill ω (or let $\omega \equiv 0$ at all time) in (27) and Lemma 1, then Lemma 1 implies that $\hat{G}$ is isomorphic to the 1-parameter trivial unfolding of G, thus Notation 2 implies $\hat{G}$ factor through G.

Theorem 5 *(Versal Unfolding Theorem,cf.[2][8][5]) Let bifurcation problem g be of finite codimension, G be a k-parameter unfolding of g. Then G is a versal unfolding of g if and only if* $\mathbf{E}_{x,\lambda,\mu}(\Gamma) = T(g,\mathfrak{U}) + \mathbb{R}\{\frac{\partial G}{\partial \alpha_1}|_{\alpha=0}, \cdots, \frac{\partial G}{\partial \alpha_k}|_{\alpha=0}\}$.

Proof. Necessity. Consider the 1-parameter unfolding of g $H = g + \beta p$ for $\forall p \in \mathbf{E}_{x,\lambda,\mu}(\Gamma)$.It is easy to see $p \in T(g,\mathfrak{U}) + \mathbb{R}\{\frac{\partial G}{\partial \alpha_1}|_{\alpha=0}, \cdots, \frac{\partial G}{\partial \alpha_k}|_{\alpha=0}\}$. Sufficiency. By Corollary 3, it is not difficult to see that the proof is similar to that in Theorem 3.

Corollary 4 *Suppose f is a bifurcation problem of finite codimension, F is a k-parameter unfolding of f. Then F is stable iff F is a versal unfolding.*

*Proof.*Use Theorem 4 and Theorem 5 directly.

At last we prove Lemma 1.

Proof of Lemma 1. Since F is of finite codimension, so from condition (27) there exists

$$Y_1, \cdots, Y_a \in \mathbf{E}_{\lambda,\mu,\omega}; Z_1, \cdots, Z_b \in \mathbf{E}_{\mu,\omega}; W_1, \cdots, W_c \in \mathbf{E}_{\omega}$$

(where a,b,c are nonnegative integers) such that

$$\begin{aligned} &\mathbf{E}_{x,\lambda,\mu,\omega}(\Gamma) = \vec{\mathbf{E}}_{x,\lambda,\mu,\omega}(\Gamma) \cdot F + (D_x F)\mathbf{E}_{x,\lambda,\mu,\omega}(\Gamma) + \mathbb{R}\{(D_\lambda F)Y_1, \\ &\cdots, (D_\lambda F)Y_a, (D_\mu F)Z_1, \cdots, (D_\mu F)Z_b, (D_\omega F)W_1, \cdots, (D_\omega F)W_c\} \end{aligned} \tag{31}$$

Consider the finitely generated modules $\vec{\mathbf{E}}_{x,\lambda,\mu,\omega,\alpha,\tau}(\Gamma), \mathbf{E}_{x,\lambda,\mu,\omega,\alpha,\tau}(\Gamma)$ over the ring $\varepsilon_{x,\lambda,\mu,\omega,\alpha,\tau}(\Gamma)$ of Γ-invariant function germs (they are similar to $\vec{\mathbf{E}}_{x,\lambda,\mu}(\Gamma)$ and $\mathbf{E}_{x,\lambda,\mu}(\Gamma)$ in §1). Let $\hat{G} \in \mathbf{E}_{x,\lambda,\mu,\omega,\alpha,\tau}(\Gamma)$ and

$$\begin{aligned} \tilde{T}(\hat{G}) &= \vec{\mathbf{E}}_{x,\lambda,\mu,\omega,\alpha,\tau}(\Gamma) \cdot \hat{G} + (D_x \hat{G})\mathbf{E}_{x,\lambda,\mu,\omega,\alpha,\tau}(\Gamma), \\ T(\hat{G}) &= \tilde{T}(\hat{G}) + (D_\lambda \hat{G})\mathbf{E}_{\lambda,\mu,\omega,\alpha,\tau} + (D_\mu \hat{G})\mathbf{E}_{\mu,\omega,\alpha,\tau} + (D_\omega \hat{G})\mathbf{E}_{\omega,\alpha,\tau}, \end{aligned} \tag{32}$$

then $\tilde{T}(\hat{G})$ is a submodule of $\mathbf{E}_{x,\lambda,\mu,\omega,\alpha,\tau}(\Gamma)$. We claim:

$$\mathbf{E}_{x,\lambda,\mu,\omega,\alpha,\tau}(\Gamma) \subset T(\hat{G}). \tag{33}$$

In fact, let $A = \mathbf{E}_{x,\lambda,\mu,\omega,\alpha,\tau}(\Gamma)$, $A' = \tilde{T}(\hat{G})$, $B = \mathbf{E}_{x,\lambda,\mu,\omega}(\Gamma)$, $B' = \tilde{T}(F) = \vec{\mathbf{E}}_{x,\lambda,\mu,\omega}(\Gamma) \cdot F + (D_x F)\mathbf{E}_{x,\lambda,\mu,\omega}(\Gamma)$, $\pi : A \to B$ be defined by $\pi(p)(x,\lambda,\mu,\omega) = p(x,\lambda,\mu,\omega,0,0)$ and

$$N = \{(D_\lambda \hat{G})y_1, \cdots, (D_\lambda \hat{G})y_a; (D_\mu \hat{G})z_1, \cdots, (D_\mu \hat{G})z_b; (D_\omega \hat{G})w_1, \cdots, (D_\omega \hat{G})z_c\},$$

where $(D_\lambda \hat{G})y_i \in \pi^{-1}((D_\lambda F)Y_i), (D_\mu \hat{G})z_i \in \pi^{-1}((D_\mu F)Z_i), (D_\omega \hat{G})w_i \in \pi^{-1}((D_\omega F)W_i)$, thus (31) can be written in the form $B = B' + \mathbb{R} \cdot \pi(N)$. So the Theorem 2.5 in [9] implies that $A = A' + \varepsilon_{\alpha,\tau} \cdot N$, hence the claim follows $\varepsilon_{\alpha,\tau} \cdot N \subset T(\hat{G})$. Thus $-\frac{\partial \hat{G}}{\partial \tau}$ can be written in the form

$$\begin{aligned} -\tfrac{\partial \hat{G}}{\partial \tau} &= -\hat{S}(x,\lambda,\mu,\omega,\alpha,\tau)\hat{G} + \sum_{i=1}^{n} X_i(x,\lambda,\mu,\omega,\alpha,\tau) \cdot \frac{\partial \hat{G}}{\partial x_i} + \sum_{i=1}^{s} \Lambda_i(\lambda,\mu,\omega,\alpha,\tau) \cdot \frac{\partial \hat{G}}{\partial \lambda_i} \\ &+ \sum_{i=1}^{t} M_i(\mu,\omega,\alpha,\tau) \cdot \frac{\partial \hat{G}}{\partial \mu_i} + \sum_{i=1}^{m} W_i(\omega,\alpha,\tau) \cdot \frac{\partial \hat{G}}{\partial \omega_i}, \end{aligned} \tag{34}$$

where $\hat{S}$ and $\hat{X} = (X_1(x,\lambda,\mu,\omega,\alpha,\tau),\cdots,X_n(x,\lambda,\mu,\omega,\alpha,\tau)$ satisfy appropriate equivariance conditions. Define a vector field germ on $(\mathbb{R}^n \times \mathbb{R}^s \times \mathbb{R}^t \times \mathbb{R}^m \times \mathbb{R}^k \times \mathbb{R}, 0)$:

$$V = (\hat{X}, \hat{\Lambda}, \hat{M}, \hat{W}, 0, 1), \tag{35}$$

where $\hat{X}$, $\hat{\Lambda} = (\Lambda_1(\lambda,\mu,\omega,\alpha,\tau),\cdots,\Lambda_s(\lambda,\mu,\omega,\alpha,\tau))$, $\hat{M} = (M_1(\mu,\omega,\alpha,\tau),\cdots,M_t(\mu,\omega,\alpha,\tau))$, $\hat{W} = (W_1(\omega,\alpha,\tau),\cdots,W_m(\omega,\alpha,\tau))$ are given by (34).
Further, by the argument similar to that in [8] (cf. p.239-241) or see the proof of Lemma 2.2 in [9] we can construct $S(x,\lambda,\mu,\omega,\alpha,\tau), X(x,\lambda,\mu,\omega,\alpha,\tau), \Lambda(\lambda,\mu,\omega,\alpha,\tau), M(\mu,\omega,\alpha,\tau)$ and $\phi(\omega,\alpha,\tau)$ such that

$$\begin{aligned} \hat{G}(x,\lambda,\mu,\omega,\alpha,\tau) = S(x,\lambda,\mu,\omega,\alpha,\tau)G(&X(x,\lambda,\mu,\omega,\alpha,\tau), \\ &\Lambda(\lambda,\mu,\omega,\alpha,\tau), M(\mu,\omega,\alpha,\tau), \phi(\omega,\alpha,\tau)), \end{aligned} \tag{36}$$

and (25)(26) hold. Therefore $\hat{G}$ is isomorphic to the 1-parameter trivial unfolding of G by Definition 2.

Acknowledgment. The main part of this work was done during the first author's stay at the Institute of Mathematics, Academia Sinica. He would like to thank Professor Peixin LI and Jianming YU and their colleagues for their kind encouragement and hospitality.

References

[1] Golubitsky M, Schaeffer D G. *A theory for imperfect bifurcation via singularity theory.* Commun. Pure Appl. Math., 1979, 32:21-98.

[2] Golubitsky M, Schaeffer D G. *Imperfect bifurcation in the presence of symmetry.* Commun. Math. Phys.,1979, 67:205-232.

[3] Peter M. *Classification of Two-Parameter Bifurcations.* Lecture Notes in Mathematics, Vol.1463:294-300, Berlin Heidelberg: Springer-Verlag, 1991.

[4] Li Yangcheng. *The recognition of equivariant bifurcation problems.* Science in China, Series A, 1996, 39(6):604-612.

[5] Izumiya S. *Generic Bifurcation of Varieties.* Manuscripta Math. 1984, 46:137-164.

[6] Mather J N. *stability of C^∞ mappings, IV. Classification of stable germs by R-algebras.* Publications Math. I.H.E.S., 1969, 37:223-248

[7] Martinet J. *Singularities of Smooth Functions and Maps.* London: Cambridge University Press, 1982

[8] Golubitsky M, Stewart I, Schaeffer D G. *Singularities and Groups in Bifurcation Theory,* Vol. II. New York: Springer-Verlag, 1988.

[9] Li Bing, Qian Xiangzheng *Some Finitely Generated Modules in Equivariant Bifurcation Theory* Journal of North China University of Technology, 1999,11(1):10-15

Bing LI
Department of Mathematics, Changsha Electric Power University,CHANGSHA 410077, P.R. CHINA
or
Department of Mathematics, Hunan University , CHANGSHA 410082, P.R.CHINA
e-mail: lib796@public.cs.hn.cn
Jiancheng ZOU
College of Fundamental Sciences, North China University of Technology, BEIJING 100041 P.R.CHINA

Electronic Journal: Southwest Journal of Pure and Applied Mathematics
Internet: http://rattler.cameron.edu/swjpam.html
ISBN 1083-0464
Issue 1 June 2001, pp. 13 – 21
Submitted: December 22, 2000. Published: June 30, 2001

A Modification of the Kantorovich Conditions for the Secant Method

M. A. Hernández and M. J. Rubio

Abstract

In this work we carry out a modification of the Kantorovich conditions for the semilocal convergence of the Secant method. This modification generalizes the Lipschtz and Hölder continuity conditions. Besides, we apply the result obtained to approximate the solution of a nonlinear boundary value problem.

A.M.S Classification Codes : 47H17, 65J15.
Keywords and Phrases: The Secant method, recurrence relations, boundary value problems.

1 Introduction

Let X, Y be two Banach spaces and $F : \Omega \subseteq X \to Y$ a nonlinear operator and consider the equation

$$F(x) = 0. \tag{1}$$

Let us denote by $\mathcal{L}(X, Y)$ the space of bounded linear operators from X to Y. An operator $[x, y; F] \in \mathcal{L}(X, Y)$ is called a divided difference of first order for the operator F on the points x and y $(x \neq y)$ if the following equality holds

$$[x, y; F](x - y) = F(x) - F(y).$$

Using this definition, the Secant method is described by the following algorithm:

$$x_{n+1} = x_n - [x_{n-1}, x_n; F]^{-1}F(x_n), \quad x_0,\ x_{-1} \text{ given.} \tag{2}$$

The convergence of (2) to a solution of (1) has been studied by other authors ([1], [2], [3], [4], [5], [6]). The basic assumption made is that the divided difference of F is Lipschitz or Hölder continuous in some ball around the initial iterate. We relax this requierement and we only assume that the divided difference $[x, y; F]$ satisfies

$$\|[x, y; F] - [v, w; F]\| \leq \omega(\|x - v\|, \|y - w\|); \quad x, y, v, w \in \Omega$$

Department of Mathematics and Computation. University of La Rioja, Spain
E-mail: mahernan@dmc.unirioja.es, mjesus.rubio@dmc.unirioja.es

where $\omega : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous funtion nondecreasing in both components and $\omega(0,0) = 0$. It is evident that this condition generalizes the conditions previously indicaded, by only considering $\omega(u_1, u_2) = k(u_1 + u_2)$ for the Lipschitz continuous case and $\omega(u_1, u_2) = k(u_1^p + u_2^p)$ for the (k,p)-Hölder continuous case.

In this paper, we analyse the semilocal convergence of (2) using a technique that consists of a new system of recurrence relations. Finally, we apply our results to solve a nonlinear boundary value problem of second order.

2 Convergence study

In this paper, we consider Ω as a convex open subset of X and we assume that F is once Fréchet differentiable in Ω. It is known ([3],[4]) that, if for any $x, y \in \Omega$ there is a divided difference operator $[x, y; F]$ which satisfies a Lipschitz or Hölder condition, then the Fréchet derivative of F exists in Ω and verifies

$$[x, x; F] = F'(x), \quad x \in \Omega. \tag{3}$$

Note that, by means of continuity, it is easy to prove that the following result is also verified for our conditions.

Lemma 2.1 *Let Ω be a convex open subset of X and suppose that, for each pair of points $x, y \in \Omega$, there exists a first order divided difference $[x, y; F] \in \mathcal{L}(X, Y)$ such that*

$$\|[x, y; F] - [v, w; F]\| \leq \omega(\|x - v\|, \|y - w\|); \quad x, y, v, w \in \Omega,$$

where $\omega : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a continuos funtion nondecreasing in both components and $\omega(0,0) = 0$. Then, (3) holds.

Now, we are going to study the semilocal convergence of the Secant method.
Let $x_0, x_{-1} \in \Omega$ and assume that

- (I) $\|x_0 - x_{-1}\| \leq \alpha$,
- (II) there exists $L_0{}^{-1} = [x_{-1}, x_0; F]^{-1}$ such that $\|L_0{}^{-1}\| \leq \beta$,
- (III) $\|L_0{}^{-1} F(x_0)\| \leq \eta$,
- (IV) $\|[x, y; F] - [v, w; F]\| \leq \omega(\|x - v\|, \|y - w\|); \quad x, y, v, w \in \Omega$, where $\omega : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a continuos funtion nondecreasing in both components and $\omega(0,0) = 0$,
- (V) there exists $h : [0,1] \to \mathbb{R}$ such that if $t \in [0,1]$, $\omega(tu_1, tu_2) \leq h(t)\omega(u_1, u_2)$.

Notice that the last condition is not restrictive, since ω is non-decreciente in its two components, and we obtain that $\omega(tu_1, tu_2) \leq \omega(u_1, tu_2) \leq \omega(u_1, u_2)$, and therefore, in particular, it is always possible to consider $h(t) = 1$. This condition is included in order to can be more precise in certain bounds we shall carry out later on.

In the following result, we use a new technique to analyse the convergence of the Secant method. Taking into account certain recurrence relations and forcing the sequence to be within a particular ball, we obtain a semilocal convergence result under mild conditions.

Before getting the semilocal convergence result for the Secant method under these new conditions, we will introduce some notations that allow us to relax the expressions appearing in its calculation. Thus, we will take into account the following auxiliary functions:

$$a(u) = \frac{\beta\left[\omega(\alpha,0) + M\omega(u,u)\right]}{1-\beta\omega(\alpha,u)}, \quad b(u) = \frac{\beta\left[\omega(u,0) + M\omega(2u,2u)\right]}{1-\beta\omega(\alpha+u,u)},$$
$$c(u) = \frac{\beta\left[\omega(2u,0) + M\omega(2u,2u)\right]}{1-\beta\omega(\alpha+u,u)}, \quad \text{where } M = \int_0^1 h(t)\,dt.$$

Theorem 2.2 *Under conditions* (I)*-*(V)*, we assume that the equation*

$$u = \left(\frac{b(u)a(u)}{1-c(u)} + a(u) + 1\right)\eta \tag{4}$$

has a minimum positive zero R*. If* $\beta\omega(\alpha+R,R)<1$*,* $c(R)<1$ *and* $\overline{B(x_0,R)}\subset\Omega$*, then the sequence* $\{x_n\}$ *given by (2) is well defined, remain in* $\overline{B(x_0,R)}$ *and converges to a solution* x^* *of equation (1) in* $\overline{B(x_0,R)}$*.*

Proof. To simplify the notation we denote $a(R)=a$, $b(R)=b$, $c(R)=c$ and $[x_{n-1},x_n;F]=L_n$. Firstly, we prove, by mathematical induction, that the sequence given in (2) is well defined, namely iterative procedure (2) makes sense if, at each step, the operator $[x_{n-1},x_n;F]$ is invertible and the point x_{n+1} lies in Ω.

From the initial hypotheses, it follows that x_1 is well defined and $\|{L_0}^{-1}F(x_0)\| \leq \eta < R$. Therefore, $x_1 \in B(x_0,R)\subseteq\Omega$.

Using (IV) and assuming that ω is nondecreasing, we obtain

$$\|I-{L_0}^{-1}L_1\| \leq \|{L_0}^{-1}\|\;\|L_0-L_1\| = \|{L_0}^{-1}\|\;\|[x_{-1},x_0;F]-[x_0,x_1;F]\|$$
$$\leq \|{L_0}^{-1}\|\omega(\|x_{-1}-x_0\|,\|x_0-x_1\|) \leq \beta\omega(\alpha,R) \leq \beta\omega(\alpha+R,R) < 1,$$

and, by the Banach lemma, ${L_1}^{-1}$ exists and

$$\|{L_1}^{-1}\| \leq \frac{\beta}{1-\beta\omega(\alpha,R)}.$$

By (2) and Taylor's formula, we have

$$F(x_1) = (F'(x_0)-L_0)(x_1-x_0) + \int_0^1 \left(F'(x_0+t(x_1-x_0)) - F'(x_0)\right)(x_1-x_0)\,dt.$$

As $[x,x;F]=F'(x)$, then, by (IV),

$$\|F(x_1)\| \leq \|[x_0,x_0;F]-[x_{-1},x_0;F]\|\;\|x_1-x_0\|$$
$$+\int_0^1 \|[x_0+t(x_1-x_0),x_0+t(x_1-x_0);F]-[x_0,x_0;F]\|\;\|x_1-x_0\|\,dt$$
$$\leq \left[\omega(\|x_0-x_{-1}\|,0) + \int_0^1 \omega(t\|x_1-x_0\|,t\|x_1-x_0\|)\,dt\right]\|x_1-x_0\|$$
$$\leq \left[\omega(\alpha,0) + \int_0^1 h(t)\omega(\|x_1-x_0\|,\|x_1-x_0\|)\,dt\right]\|x_1-x_0\|$$

$$\leq [\omega(\alpha,0) + M\omega(\|x_1-x_0\|,\|x_1-x_0\|)]\,\|x_1-x_0\| \leq [\omega(\alpha,0)+M\omega(R,R)]\,\|x_1-x_0\|$$

The iterate x_2 is well defined since $F(x_1)$ and ${L_1}^{-1}$ are. Moreover,

$$\|x_2-x_1\| \leq \|{L_1}^{-1}\|\ \|F(x_1)\| \leq \frac{\beta\,[\omega(\alpha,0)+M\omega(R,R)]}{1-\beta\omega(\alpha,R)}\|x_1-x_0\| = a\|x_1-x_0\|. \quad (5)$$

On the other hand, if we take into account that R is a solution of (4),

$$\|x_2-x_0\| \leq \|x_2-x_1\|+\|x_1-x_0\| \leq \left[\frac{\beta\,[\omega(\alpha,0)+M\omega(R,R)]}{1-\beta\omega(\alpha,R)}+1\right]\|x_1-x_0\| \leq (a+1)\eta < R, \quad (6)$$

then, $x_2 \in B(x_0,R) \subseteq \Omega$.

We now assume the linear operators L_j are invertible and $x_{j+1} \in B(x_0,R) \subseteq \Omega$ for all $j = 1,\ldots,n$. Then,

$$\|I - {L_0}^{-1}L_{n+1}\| \leq \|{L_0}^{-1}\|\ \|L_0 - L_{n+1}\| = \|{L_0}^{-1}\|\ \|[x_{-1},x_0;F]-[x_n,x_{n+1};F]\|$$

$$\leq \beta\,\omega(\|x_{-1}-x_n\|,\|x_0-x_{n+1}\|) \leq \beta\omega(\|x_{-1}-x_0\|+\|x_0-x_n\|,\|x_0-x_{n+1}\|)) \leq \beta\omega(\alpha+R,R) < 1,$$

and, according to Banach lemma, ${L_{n+1}}^{-1}$ exists and

$$\|{L_{n+1}}^{-1}\| \leq \frac{\beta}{1-\beta\omega(\alpha+R,R)}.$$

Taking norms in

$$F(x_{n+1}) = (F'(x_n)-L_n)(x_{n+1}-x_n) + \int_{x_n}^{x_{n+1}} (F'(x)-F'(x_n))\,dx$$

we get

$$\begin{aligned}
&\|F(x_{n+1})\| \leq \|[x_n,x_n;F]-[x_{n-1},x_n;F]\|\ \|x_{n+1}-x_n\| \\
&+\int_0^1 \|[x_n+t(x_{n+1}-x_n),x_n+t(x_{n+1}-x_n);F]-[x_n,x_n;F]\|\ \|x_{n+1}-x_n\|\,dt \\
&\leq \left[\omega(\|x_n-x_{n-1}\|,0)+\int_0^1 \omega(t\|x_{n+1}-x_n\|,t\|x_{n+1}-x_n\|)\,dt\right]\|x_{n+1}-x_n\| \\
&\leq \left[\omega(\|x_n-x_{n-1}\|,0)+\int_0^1 h(t)\omega(\|x_{n+1}-x_n\|,\|x_{n+1}-x_n\|)\,dt\right]\|x_{n+1}-x_n\| \\
&= [\omega(\|x_n-x_{n-1}\|,0)+M\omega(\|x_{n+1}-x_n\|,\|x_{n+1}-x_n\|)]\,\|x_{n+1}-x_n\| \\
&\leq [\omega(\|x_n-x_{n-1}\|,0)+M\omega(\|x_{n+1}-x_0\|+\|x_0-x_n\|,\|x_{n+1}-x_0\|+\|x_0-x_n\|)] \\
&\|x_{n+1}-x_n\| \leq [\omega(\|x_n-x_{n-1}\|,0)+M\omega(2R,2R)]\,\|x_{n+1}-x_n\|.
\end{aligned}$$

For $n=1$,

$$\|F(x_2)\| \leq [\omega(R,0)+M\omega(2R,2R)]\,\|x_2-x_1\|,$$

and

$$\|x_3-x_2\| \leq \|{L_2}^{-1}\|\ \|F(x_2)\| \leq \frac{\beta\,[\omega(R,0)+M\omega(2R,2R)]}{1-\beta\omega(\alpha+R,R)}\|x_2-x_1\| = b\|x_2-x_1\|. \quad (7)$$

Then, by (4), (5), (6) and (7), we have

$$\|x_3-x_0\| \leq \|x_3-x_2\|+\|x_2-x_0\| \leq b\|x_2-x_1\|+\|x_2-x_0\| \leq (ba+a+1)\|x_1-x_0\|$$

$$\leq (ba + a + 1)\eta < R.$$

Therefore, $x_3 \in B(x_0, R) \subseteq \Omega$.

For $n > 1$:

$$\|F(x_{n+1})\| \leq [\omega(\|x_n - x_0\| + \|x_0 - x_{n-1}\|, 0) + M\omega(2R, 2R)]\, \|x_{n+1} - x_n\|$$

$$\leq [\omega(2R, 0) + M\omega(2R, 2R)]\, \|x_{n+1} - x_n\|.$$

Thus

$$\|x_{n+2} - x_{n+1}\| \leq \|{L_{n+1}}^{-1}\|\ \|F(x_{n+1})\| \leq \frac{\beta\,[\omega(2R, 0) + M\omega(2R, 2R)]}{1 - \beta\omega(\alpha + R, R)}\|x_{n+1} - x_n\|$$

$$= c\|x_{n+1} - x_n\|. \tag{8}$$

Consequently, from (4), (5), (6), (7) and (8), it follows

$$\|x_{n+2} - x_0\| \leq \|x_{n+2} - x_{n+1}\| + \|x_{n+1} - x_n\| + \cdots + \|x_3 - x_2\| + \|x_2 - x_0\|$$

$$\leq \left[c^{n-1} + c^{n-2} + \cdots + 1\right] \|x_3 - x_2\| + \|x_2 - x_0\| \leq \left[\frac{1 - c^n}{1 - c}ba + a + 1\right] \|x_1 - x_0\|$$

$$< \left[\frac{ba}{1 - c} + a + 1\right] \eta = R.$$

So, $x_{n+2} \in B(x_0, R) \subseteq \Omega$ and the induction is complete.

Secondly, we prove that $\{x_n\}$ is a Cauchy sequence. For $m \geq 1$ and $n \geq 2$ we obtain

$$\|x_{n+m} - x_n\| \leq \|x_{n+m} - x_{n+m-1}\| + \|x_{n+m-1} - x_{n+m-2}\| + \cdots + \|x_{n+1} - x_n\|$$

$$\leq \left[c^{m-1} + c^{m-2} + \cdots + 1\right] \|x_{n+1} - x_n\| \leq \frac{1 - c^m}{1 - c}\|x_{n+1} - x_n\| < \frac{1}{1 - c}c^{n-2}\|x_3 - x_2\|.$$

Therefore, $\{x_n\}$ is a Cauchy sequence and converges to $x^* \in \overline{B(x_0, R)}$.

Finally, we see that x^* is a zero of F. Since

$$\|F(x_n)\| \leq [\omega(2R, 0) + M\omega(2R, 2R)]\, \|x_n - x_{n-1}\|,$$

and $\|x_n - x_{n-1}\| \to 0$ when $n \to \infty$, we obtain $F(x^*) = 0$. ■

Theorem 2.3 *Under the assumption of theorem 2.2, if* $\beta\,[\omega(\alpha, 0) + \omega(2R, 2R)] < 1$, *the solution* x^* *of equation (1) is unique in* $\overline{B(x_0, R)}$.

Proof. Let us assume that there exists $y^* \in \overline{B(x_0, R)}$ such that $F(y^*) = 0$. Moreover,

$$L_0^{-1}[F(y^*) - F(x^*)] = \int_{x^*}^{y^*} L_0^{-1}F'(x)\, dx = \int_0^1 L_0^{-1}F'(x^* + t\ (y^* - x^*))\ (y^* - x^*)\, dt.$$

Denoting $A = \int_0^1 {L_0}^{-1}F'(x^* + t\ (z^* - x^*))\, dt$, we have

$$\|A - I\| \leq \|{L_0}^{-1}\| \int_0^1 \|F'(x^* + t\ (y^* - x^*)) - L_0\|\, dt$$

$$\leq \|L_0{}^{-1}\| \int_0^1 \left(\|F'(x^* + t\,(y^* - x^*)) - F'(x_0)\| + \|F'(x_0) - L_0\|\right)\, dt$$

$$\leq \beta\, \left(\int_0^1 \omega\left(\|x^* + t\,(y^* - x^*) - x_0\|, \|x^* + t\,(y^* - x^*) - x_0\|\right)\, dt + \|F'(x_0) - L_0\|\right)$$

$$\leq \beta \left(\int_0^1 \omega\left(\,(1-t)\,\|x^* - x_0\| + t\,\|y^* - x_0\|, (1-t)\,\|x^* - x_0\| + t\,\|y^* - x_0\|\,\right)\, dt + \omega(\alpha, 0)\right)$$

$$\leq \beta \left(\int_0^1 \omega\left(\,\|x^* - x_0\| + \|y^* - x_0\|, \|x^* - x_0\| + \|y^* - x_0\|\right)\, dt + \omega(\alpha, 0)\right)$$

$$\leq \beta\,(\omega(\alpha, 0) + \omega(2R, 2R)) < 1.$$

In consequence, A is invertible and $y^* = x^*$. ■

3 Numerical Example

Now we consider a situation where if divided differences are Lipschitz or (k,p)-Hölder continuous, then we cannot work. Now we apply the semilocal convergence result given above to the following boundary value problem:

$$\left\{ \begin{array}{l} x'' + x^{1+p} + x^2 = 0, \quad p \in [0,1] \\ x(0) = x(1) = 0. \end{array} \right. \tag{9}$$

To solve this problem by finite differences, we start drawing the usual grid line with the grid points $t_i = ih$, where $h = 1/n$ and n is an appropriate integer. Denote $x(t_i) = x_i$, notice that x_0 and x_n are given by the boundary conditions, then $x_0 = 0 = x_n$. We first approximate the second derivative $x''(t)$ by

$$\begin{aligned} x''(t) &\approx [x(t+h) - 2x(t) + x(t-h)]/h^2, \\ x''(t_i) &= (x_{i+1} - 2x_i + x_{i-1})/h^2, \quad i = 1, 2, \ldots, n-1. \end{aligned}$$

Substituting this expression into the differential equation, we have the following system of nonlinear equations

$$\left\{ \begin{array}{l} 2x_1 - h^2 x_1^{1+p} - h^2 x_1^2 - x_2 = 0, \\ -x_{i-1} + 2x_i - h^2 x_i^{1+p} - h^2 x_i^2 - x_{i+1} = 0, \quad i = 2, 3, \ldots, n-2, \\ -x_{n-2} + 2x_{n-1} - h^2 x_{n-1}^{1+p} - h^2 x_{n-1}^2 = 0. \end{array} \right. \tag{10}$$

We can then consider an operator $F : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ such that $F(x) = H(x) - h^2 g(x) - h^2 f(x)$, where

$$g(x) = \left(x_1^{1+p}, x_2^{1+p}, \ldots, x_{n-1}^{1+p}\right)^t, \quad f(x) = \left(x_1^2, x_2^2, \ldots, x_{n-1}^2\right)^t,$$

and

$$H = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 2 \end{pmatrix}.$$

Thus

$$F'(x) = H - h^2(1+p)\begin{pmatrix} x_1^p & 0 & \cdots & 0 \\ 0 & x_2^p & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_{n-1}^p \end{pmatrix} - 2h^2\begin{pmatrix} x_1 & 0 & \cdots & 0 \\ 0 & x_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_{n-1} \end{pmatrix}$$

Let $x \in \mathbb{R}^{n-1}$, and choose the norm $\|x\| = \max_{1\le i\le n-1} |x_i|$, the corresponding norm on $A \in \mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$ is

$$\|A\| = \max_{1\le i\le n-1} \sum_{j=1}^{n-1} |a_{ij}|.$$

It is known (see [5]) that F has a divided difference at the points $x, y \in \mathbb{R}^{n-1}$, which is defined by the matrix whose entries are

$$[x,y;F]_{ij} = \frac{1}{x_j - y_j}\,\left(F_i(x_1,\dots,x_j,y_{j+1},\dots,y_{n-1}) - F_i(x_1,\dots,x_{j-1},y_j,\dots,y_{n-1})\right).$$

Consequently,

$$[x,y;F] = H - h^2\begin{pmatrix} \frac{x_1^{1+p}-y_1^{1+p}+x_1^2-w_1^2}{x_1-y_1} & 0 & \cdots & 0 \\ 0 & \frac{x_2^{1+p}-y_2^{1+p}+x_2^2-y_2^2}{x_2-y_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{x_{n-1}^{1+p}-y_{n-1}^{1+p}+x_{n-1}^2-y_{n-1}^2}{x_{n-1}-y_{n-1}} \end{pmatrix}.$$

In this case, we have that $[x,y;F] = \int_0^1 F'(x+t(y-x))\,dt$. So we study the value $\|F'(x) - F'(v)\|$ to obtain a bound for $\|[x,y;F] - [v,w;F]\|$.

For all $x, v \in \mathbb{R}^{n-1}$ with $|x_i| > 0, |v_i| > 0,\ (i = 1,2,\dots,n-1)$, and taking into account the max-norm it follows

$$\|F'(x) - F'(v)\| = \|\mathrm{diag}\{h^2(1+p)\ (v_i^p - x_i^p) + 2h^2(v_i - x_i)\}\|$$

$$= \max_{1\le i\le n-1} |h^2(1+p)\ (v_i^p - x_i^p) + 2h^2(v_i - x_i)| \le (1+p)h^2 \max_{1\le i\le n-1} |v_i^p - x_i^p| + 2h^2 \max_{1\le i\le n-1} |v_i - x_i|$$

$$\le (1+p)h^2\ [\max_{1\le i\le n-1} |v_i - x_i|\ ]^p + 2h^2\|v - x\| = (1+p)h^2\|v-x\|^p + 2h^2\|v-x\|.$$

Therefore

$$\|[x,y;F] - [v,w;F]\| \le \int_0^1 \|F'\left(x + t(y-x)\right) - F'\left(u + t(w-v)\right)\| \, dt$$

$$\le h^2 \int_0^1 \left((1+p)\|(1-t)(x-v) + t(y-w)\|^p + 2\|(1-t)(x-v) + t(y-w)\|\right)\, dt$$

$$\le h^2(1+p)\int_0^1 \left((1-t)^p\|x-v\|^p + t^p\|y-w\|^p\right)\, dt + 2h^2\int_0^1 \left((1-t)\|x-v\| + t\|y-w\|\right)\, dt$$

$$= h^2\left(\|x-v\|^p + \|y-w\|^p + \|x-v\| + \|y-w\|\right)$$

From (IV), we consider the function $\omega(u_1,u_2) = h^2(u_1^p + u_2^p + u_1 + u_2)$ and $h(t) = t^p$. So, $M = \frac{1}{1+p}$.

Now we apply the Secant method to approximate the solution of $F(x) = 0$. We choose $p = 1/2$ and if $n = 10$, then (10) gives 9 equations. Since a solution of (9) would vanish at the end points and be positive in the interior, a reasonable choice of initial approximation seems to be $10\sin\pi t$. This approximate gives us the following vector:

$$y_{-1} = \begin{pmatrix} 3.090169943749474 \\ 5.877852522924731 \\ 8.090169943749475 \\ 9.51056516295136 \\ 10.0000000000000 \\ 9.51056516295136 \\ 8.090169943749475 \\ 5.877852522924731 \\ 3.090169943749474 \end{pmatrix}.$$

Choose y_0 by setting $y_0(t_i) = y_{-1}(t_i) - 10^{-5}, i = 1, 2, \ldots, 9$ and using iteration (2), after two iterations, we obtain:

$$y_1 = \begin{pmatrix} 2.453176290658909 \\ 4.812704101582601 \\ 6.8481873135861 \\ 8.252997367741953 \\ 8.75737771678512 \\ 8.252997367741953 \\ 6.8481873135861 \\ 4.812704101582601 \\ 2.453176290658909 \end{pmatrix} \quad \text{and} \quad y_2 = \begin{pmatrix} 2.404324055268407 \\ 4.713971539035271 \\ 6.7003394962933925 \\ 8.066765882171131 \\ 8.556329565792526 \\ 8.066765882171131 \\ 6.7003394962933924 \\ 4.713971539035271 \\ 2.404324055268407 \end{pmatrix}.$$

Now we apply (2), then we take $x_{-1} = y_1$ and $x_0 = y_2$. With the notation of teorem 2.2 we can easily obtain the following results: $\alpha = 0.201048$, $\beta = 15.319$, $\eta = 0.0346555$. Since $h^2 = 0.01$, in this particular case, the solution of equation (4) is $R = 0.0416182$. Besides, $\beta\omega(\alpha + R, R) = 0.150264 < 1$ and $c = 0.156374 < 1$. Therefore, the hypotheses of theorem 2.2 are fulfilled, what ensures that a solution of the equation (1) exists in $\overline{B(x_0, R)}$. This solution is also unique in $\overline{B(x_0, R)}$, since $\beta[\omega(\alpha, 0) + \omega(2R, 2R)] = 0.213381 < 1$.

We obtain the vector x^* as the solution of system (10), after eleven iterations:

$$x^* = \begin{pmatrix} 2.394640794786742 \\ 4.694882371216001 \\ 6.672977546934751 \\ 8.033409358893319 \\ 8.520791423704788 \\ 8.033409358893319 \\ 6.67297754693475 \\ 4.694882371216 \\ 2.394640794786742 \end{pmatrix}.$$

If x^* is now interpolated, its approximation $\bar{x}^*$ to the solution of (9) with $p = 1/2$ is that appearing in figure 1.

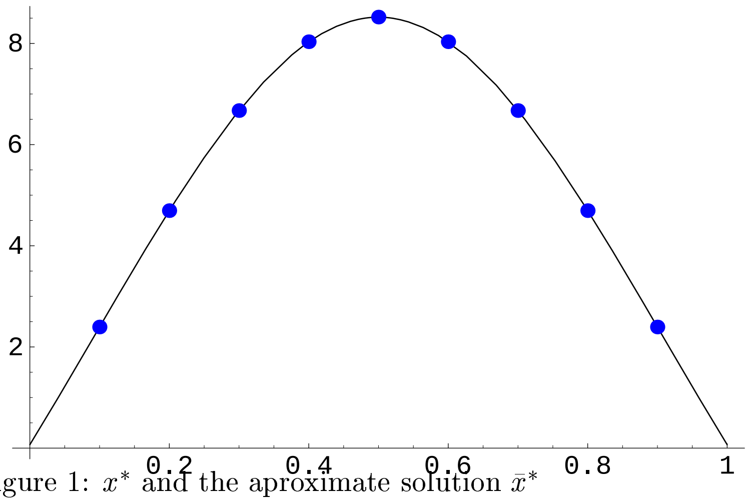

Figure 1: x^* and the aproximate solution $\bar{x}^*$

Note that in this example, we cannot guarantee the convergence to the solution through the classical Lipschitz or Hölder conditions for the divided differences.

References

[1] ARGYROS, I. K., The Secant Method and Fixed Points of Nonlinear Operators, *Mh. Math.*, **106**, (1988), pp. 85–94.

[2] ARGYROS, I. K., On an Application of a Variant of the Closed Graph Theorem and the Secant Method, *Tamkang J. Math.* **24, 3**, (1993), pp. 251–267.

[3] ARGYROS, I. K., On the Secant Method, *Publ. Math. Debrecen*, **43, 3-4**, (1993), pp. 223–238.

[4] DENNIS, J. R., Toward a Unified Convergence Theory for Newton-Like Methods. Article in *Nonlinear Funcional Analysis and Aplications*. Edited by L. B. Rall, *Academic Press*, New York, (1970), pp. 425-472.

[5] POTRA, F. A. AND PTAK, V., *Nondiscrete induction and iterative processes* (Pitman Publ. 1984).

[6] RHEINBOLDT, W. C., A Unified Convergence Theory for a Class of Iterative Processes, *SIAM J. Numer. Anal.*, **5, 1**, (1968), pp. 42–63.

Electronic Journal: Southwest Journal of Pure and Applied Mathematics
Internet: http://rattler.cameron.edu/swjpam.html
ISSN 1083-0464
Issue 1, June 2001, pp. 22–28.
Submitted: January 3, 2001. Published: June 30, 2001.

LOCAL AND SEMILOCAL CONVERGENCE THEOREMS FOR NEWTON'S METHOD BASED ON CONTINUOUSLY FRÉCHET DIFFERENTIABLE OPERATORS

Ioannis K. Argyros

Abstract. We approximate locally unique solutions of nonlinear equations in a Banach space setting using Newton's method. In contrast with earlier works which are based on the Lipschitz continuity of the operators involved, we only use the continuity of the first Fréchet derivative of the operator involved. Local and semilocal convergence theorems are given and the quadratic convergence of Newton's method under weaker than before conditions is established.

A.M.S. (MOS) Subject Classification Codes:. 65H10, 47H15, 49M17.

Key Words and Phrases:. Banach space, Newton's method, Fréchet-derivative, order of convergence, local-semilocal convergence.

1. Introduction

In this study we are concerned with the problem of approximating a locally unique solution x^* of equation

$$F(x) = 0, \tag{1}$$

where F is a continuously Fréchet-differentiable operator defined on an open convex subset D of a Banach space X with values in a Banach space Y.

The most popular method for approximating x^* is Newton's method

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n) \quad (n \geq 0) \quad (x_0 \in D). \tag{2}$$

Local and semilocal convergence theorems for Newton's method can be found in [1]–[3] and the references there. Most known results on the convergence of iteration (2) are based on the possibility of computing the Lipschitz constant of operator $F'(x_0)^{-1}F'(x)$ $(x \in D)$ (see Remark 2). Computing such a constant is a difficult or impossible task in general despite the fact that sufficient conditions based on this constant are considered to be the weakest known so far.

Department of Mathematics, Cameron University, Lawton, OK 73505, U.S.A.
E-mail Address: ioannisa@cameron.edu

There are many real-life problems whose solution reduces to finding x^* for a suitable operator F and on a proper space. However the computation of the Lipschitz constant is problematic. That is why we need local and semilocal results that avoid the computation of this constant and still guarantee the quadratic convergence of Newton's method. Such results are provided here, based on the continuity of the first Fréchet-derivative $F'(x)$ of F.

2. Convergence Analysis

We can show the following semilocal convergence theorem for Newton's method and continuously Fréchet-differentiable operators.

Theorem 1. *Let $F\colon D \subseteq X \to Y$ be a continuously Fréchet-differentiable operator defined on an open convex subset D of a Banach space X with values in a Banach space Y.*

Assume:

(a) *there exists $x_0 \in D$ such that $F(x_0) \neq 0$;*

(b) *$F'(x)^{-1} \in L(Y,X)$ $(x \in D)$, and there exists $b > 0$ such that*:

$$\|F'(x)^{-1}F'(x_0)\| \leq b \quad (x \in D); \tag{3}$$

(c) *for each fixed $p \in D$ with $F(p) \neq 0$, there exists*

$$\delta = \delta(p,t) \geq t\|F'(p)^{-1}F(p)\|, \quad t \in [0,1] \tag{4}$$

such that

$$\|F'(x_0)^{-1}[F'(p) - F'(y_t)]\| < t\|F'(p)^{-1}F(p)\| \tag{5}$$

for all

$$y_t \in U(p,\delta) = \{z \in X \mid \|z - p\| < \delta\} \subseteq D, \tag{6}$$

and y_t collinear to p;

(d) *there exists $\eta > 0$ such that*:

$$\|F'(x_0)^{-1}F(x_0)\| \leq \eta, \tag{7}$$

and

$$c = \frac{b\eta}{2} \in (0,1); \tag{8}$$

and

(e) *$U = U(x_0, \delta^*) \subseteq D$, where*

$$\delta^* \geq \frac{\eta}{1-c}. \tag{9}$$

Then, Newton's method $\{x_n\}$ $(n \geq 0)$ *generated by (2) is well defined, remains in* U *for all* $n \geq 0$ *and converges to a unique solution* $x^* \in \overline{U}$ *of equation* $F(x) = 0$. *Moreover the following error bounds hold*:

$$\|x_{n+1} - x_n\| < \frac{b}{2}\|x_n - x_{n-1}\|^2 \quad (n \geq 1), \tag{11}$$

and

$$\|x_n - x^*\| \leq a_n\|x_n - x_{n-1}\|^2 \quad (n \geq 1), \tag{12}$$

where,

$$a_n = \frac{b}{2}\sum_{j=0}^{\infty}\left(c^{2^n}\right)^{2^j-1} \leq \frac{b}{2(1-c^{2^n})}. \tag{13}$$

Proof. Choose $p = x_0$. It follows that $t\|F'(x_0)^{-1}F(x_0)\| > 0$, $t \in (0,1]$ since $F(x_0) \neq 0$. By (5) we obtain

$$\|F'(x_0)^{-1}[F'(x_0) - F'(x_0 + t(x_1 - x_0))]\| < t\|x_1 - x_0\| \tag{14}$$

since $x_1 \in U(x_0, \delta) \subseteq D$. Using the approximation

$$\begin{aligned} x_{k+2} - x_{k+1} = -F'(x_{k+1})^{-1}\int_0^1 \Big[&F'(x_k + t(x_{k+1} - x_k)) \\ &- F'(x_k)\Big](x_{k+1} - x_k)dt \end{aligned} \tag{15}$$

for $k = 0$, (15), and (2) we obtain in turn

$$\|x_2 - x_1\| < \frac{b}{2}\|x_1 - x_0\|^2 \leq c\|x_1 - x_0\|, \tag{16}$$

and

$$\begin{aligned} \|x_2 - x_0\| &\leq \|x_2 - x_1\| + \|x_1 - x_0\| \leq (1+c)\|x_1 - x_0\| \\ &< \frac{1-c^2}{1-c}\|x_1 - x_0\| \leq \delta^*. \end{aligned} \tag{17}$$

Hence (11) holds for $n = 1$ and $x_2 \in U$. Let us assume that (11) and $x_k \in U$ hold for all integer values k smaller or equal to $n-1$. Choose $p = x_{n-1}$. If $x_n = x_{n-1}$ then we found a solution $x^* = x_{n-1}$. If $x_n = x_{n-1}$ then we found a solution $x^* = x_{n-1}$ (by (2)). Otherwise we have $t\|x_n - x_{n-1}\| > 0$, $t \in (0,1]$. We obtain by (5)

$$\|F'(x_0)^{-1}[F'(x_{n-1}) - F'(x_{n-1} + t(x_n - x_{n-1}))]\| < t\|x_n - x_{n-1}\| \tag{18}$$

since $x_n \in U(x_{n-1}, \delta) \subseteq D$. Using (2), (3), (18) for $k = n-1$ we obtain (11). Hence (11) holds for all $n \geq 1$, and

$$\begin{aligned} \|x_{n+1} - x_0\| &\leq \|x_{n+1} - x_n\| + \|x_n - x_{n-1}\| + \cdots + \|x_1 - x_0\| \\ &< \frac{1-c^n}{1-c}\|x_1 - x_0\| \leq \delta^*. \end{aligned} \tag{19}$$

That is $x_n \in U$ for all $n \geq 1$. It follows from (11) and (9)

$$\|x_{n+1} - x_n\| < c^n \|x_1 - x_0\|.$$

Since $c \in (0,1)$ sequence $\{x_n\}$ $(n \geq 0)$ is Cauchy in a Banach space X and as such it converges to some $x^* \in \overline{U}$ (since $\overline{U}$ is a closed set). Letting $n \to \infty$ in (2) we obtain $F(x^*) = 0$. Estimate (12) is obtained by standard majorization techniques [1], [3].

The uniqueness of the solution x^* of $\overline{U}$ can be shown using the identity

$$F(y^*) - F(x^*) = \int_0^1 F'(x^* + t(y^* - x^*))(y^* - x^*)dt \tag{20}$$

where $F(x^*) = F(y^*) = 0$, $y^* \in \overline{U}$. Since

$$\begin{aligned} \|x^* + t(y^* - x^*) - x_0\| &= \|(1-t)(x^* - x_0) + t(y^* - x_0)\| \\ &\leq (1-t)\|x^* - x_0\| + t\|y^* - x_0\| \\ &\leq (1-t)\delta^* + t\delta^* = \delta^*, \end{aligned}$$

it follows that $x^* + t(y^* - x^*) \in \overline{U}$. That is, by (20) $\int_0^1 F'(x^* + t(y^* - x^*))dt$ is an invertible operator. Hence, (20) gives $x^* = y^*$.

That completes the proof of Theorem 1.

Remark 1. We can replace (3) as follows: if $x_n \in U(x_0, \delta)$ then

$$\|F'(x_0)^{-1}[F'(x_0) - F'(x_n)]\| < \delta^*. \tag{21}$$

Hence, if we choose $\delta^* \in (0,1)$, then by the Banach Lemma on invertible operators [2] $F'(x_n)^{-1} \in L(Y, X)$ and

$$\|F'(x_n)^{-1}F'(x_0)\| \leq \frac{1}{1-\delta^*} = b.$$

It is simple algebra to show that estimates (8) and (10) hold provided that:

$$\delta^* \in \left[\frac{2+\eta-\sqrt{\eta^2-12\eta+4}}{4}, \frac{2+\eta+\sqrt{\eta^2-12\eta+4}}{4}\right] \tag{22}$$

and

$$\eta \in (0, 6-4\sqrt{2}]. \tag{23}$$

Hence, we showed:

Theorem 2. *Let F, (a), (c) be as in Theorem 1. Moreover, assume*:

(a) *parameters δ^*, η satisfy (22) and (23) respectively with $b = \frac{1}{1-\delta^*}$*;

(b) $\overline{U} = \overline{U}(x_0, \delta^*) \subseteq D$.

Then, Newton's method $\{x_n\}$ $(n \geq 0)$ *generated by (2) is well defined, remains in* U *for all* $n \geq 0$ *and converges to a unique solution* $x^* \in \overline{U}$ *of equation* $F(x) = 0$, *so that estimates (11), (12) and (13) hold for all* $n \geq 1$.

Condition (5) can be weakened as follows:

Theorem 3. *Let* $F\colon D \subseteq X \to Y$ *be continuously differentiable at* $x_0 \in D$. *Assume*:

(a) *there exists* $x_0 \in D$ *such that* $F(x_0) \neq 0$ *and* $F'(x_0)^{-1} \in L(Y, X)$;

(b) *for all* $\varepsilon > 0$ *there exists* $\delta \geq \varepsilon$ *such that*

$$\|F'(x_0)^{-1}[F'(x_0) - F'(y_t)]\| < \varepsilon, \quad t \in (0, 1]$$

for all $y_t \in U(x_0, \delta) \subseteq D$ *with* y_t *collinear to* x_0;

(c) $\eta \in (0, 5 - 2\sqrt{6})$ (*given by (7)*) *and*

$$\delta^* \in \left[\frac{1 + \eta - \sqrt{\eta^2 - 10\eta + 1}}{6}, \frac{1 + \eta + \sqrt{\eta^2 - 10\eta + 1}}{6}\right];$$

and

(d) $U = U(x_0, \delta^*) \subseteq D$.

Then, Newton's method $\{x_n\}$ $(n \geq 0)$ *generated by (2) is well defined, remains in* U *for all* $n \geq 0$ *and converges to a unique solution* $x^* \in \overline{U}$ *of equation* $F(x) = 0$. *Moreover the following error bounds hold for all* $n \geq 0$

$$\|x_{n+2} - x_{n+1}\| \leq d\|x_{n+1} - x_n\|, \quad d = \frac{2\delta^*}{1 - \delta^*},$$

and

$$\|x_n - x^*\| \leq \frac{d^\eta}{1 - d}\eta.$$

Proof. The proof follows along the lines of Theorem 1 and is based on the approximation

$$x_{n+2} - x_{n+1} = -[F'(x_{n+1})^{-1}F'(x_0)]\bigg\{F'(x_0)^{-1}\int_0^1 [F'(x_n + t(x_{n+1} - x_n)) \\ - F'(x_0)](x_{n+1} - x_n)dt + F'(x_0)^{-1}[F'(x_0) - F'(x_n)](x_{n+1} - x_n)\bigg\}.$$

We complete this study with the following local result for Newton's method and continuously Fréchet-differentiable operators:

Theorem 4. *Let* $F\colon D \subseteq X \to Y$ *be a continuously Fréchet-differentiable operator defined on an open convex subset* D *of a Banach space* X *with values in a Banach space* Y.

Assume:

(a) *there exists a simple zero* $x^* \in D$ *of* F *in the sense that* $F'(x^*)^{-1} \in L(Y, X)$;

(b) $F'(x)^{-1} \in L(Y, X)$ $(x \in D)$ *and there exists* $q > 0$ *such that*:

$$\|F'(x)^{-1}F'(x^*)\| \leq q \quad (x \in D);$$

(c) *for each fixed* $p \in D$ *with* $F(p) \neq 0$ *there exists* $\delta \geq t\|p - x^*\|$, $t \in (0,1]$ *such that*:

$$\|F'(x^*)^{-1}[F'(p) - F'(p_t)]\| < t\|p - x^*\|$$

for all $p_t = x + t(x^* - x) \in U(p, \delta) \subseteq D$;

(d) $U^* = U(x^*, r^*) \subseteq D$, *where*

$$r^* > \frac{2}{q}.$$

Then, Newton's method $\{x_n\}$ $(n \geq 0)$ *generated by (2) is well defined, remains in* U^* *for all* $n \geq 0$ *and converges to* x^* *provided that* $x_0 \in U^*$. *Moreover the following error bounds hold for all* $n \geq 0$:

$$\|x_{n+1} - x^*\| \leq \frac{q}{2}\|x_n - x^*\|^2.$$

Proof. The proof follows along the lines of Theorem 1 and is based on the approximation:

$$\begin{aligned} x_{n+1} - x^* = [F'(x_n)^{-1}F(x^*)]\Big\{F'(x^*)^{-1}\int_0^1 \Big[F'(x_n + t(x^* - x_n)) \\ - F'(x_n)](x^* - x_n)dt\Big\}. \end{aligned}$$

Remark 2. The results obtained here do not make use of the Lipschitz constant appearing in the famous Newton–Kantorovich Theorem [1], [3]. It is stated there that if

$$\|F'(x_0)^{-1}[F'(x) - F'(y)]\| \leq \ell\|x - y\| x, y \in D$$

and

$$2\ell\eta \leq 1, \tag{24}$$

then equation (1) has a unique solution x^* inside a ball centered at x_0 and included in D. Our results are weaker than the above and promising especially if one compares condition (8) with (24) (if, e.g., $b < 2\ell$ and (c) is violated). Moreover condition (5) is a consequence of the continuity of the operator $F'(x_0)^{-1}F'(x)$ on D. Condition (4) may not hold in general. However since

$$F'(x_0)^{-1}[F'(p) - F'(y_t)] = F'(x_0)^{-1}\int_0^1 F''[y_t + t(p - y_t)](p - y_t)dt \tag{25}$$

it follows from (25) that (4) and (5) hold if, e.g.,

$$\overline{\ell} = \left\|F'(x_0)^{-1}\int_0^1 F''[y_t + t(p - y_t)]dt\right\| \leq 1, \quad t \in [0,1]$$

but not only then. Let us provide an example. For simplicity, let $X = Y = \mathbf{R}$, $D = [1.5, 2.5]$, $x_0 = 1.9$ and $F(x) = x^2 - 4$. Then we get $b = 1.2\overline{6}$, $\eta = .102631579$, $c = .065$ and $\delta^* = .109766395$. That is, all hypotheses of Theorem 1 are satisfied. Moreover note that $\overline{\ell} = \frac{2}{2(1.9)} = .526315789 < 1$.

REFERENCES

[1] Argyros, I.K. *Advances in the Efficiency of Computational Methods and Applications*, World Scientific Publ. Co., River Edge, NJ, 2000.

[2] Kantorovich, L.V. and Akilov, G.P. *Functional Analysis*, Pergamon Press, Oxford, 1982.

[3] Zabrejko, P.P. and Nguen, D.F., The majorant method in the theory of Newton–Kantorovich approximations and the Ptak error estimates, *Numer. Funct. Anal. and Optmiz.* **9**, (5 and 6) (1987), 671–684.

Electronic Journal: Southwest Journal of Pure and Applied Mathematics
Internet: http://rattler.cameron.edu/swjpam.html
ISSN 1083-0464
Issue 1, June 2001, pp. 29–37.
Submitted: January 15, 2001. Published: June 30, 2001.

GEOMETRY OF STATIONARY SETS FOR THE WAVE EQUATION IN $\mathbb{R}^n$. THE CASE OF FINITELY SUPPORTED INITIAL DATA: AN ANNOUNCEMENT

Mark L. Agranovsky* and Eric Todd Quinto**

Abstract We consider the Cauchy problem for the wave equation in the whole space $\mathbb{R}^n$, with initial data which are distributions supported on finite sets. The main result is a precise description of the geometry of the sets of stationary points of the solutions to the wave equation. Complete proofs are given in [6].

A.M.S. Subject Classification Codes:. 35L05, 44A12, 35B05, 35S30

Keywords and Phrases:. Wave Equation, Nodal sets, spherical Radon transform, Coxeter systems

§1. Introduction.

Our goal is to understand the structure of stationary (nodal) sets of solutions to the wave equation $\mathbb{R}^n$, $\quad n \geq 2$:

$$(1.1) \qquad \begin{aligned} &u_{tt} = \Delta u,\ u = u(x,t),\ x \in \mathbb{R}^n,\ t > 0, \\ &u\big|_{t=+0} = 0, \quad u_t\big|_{t=+0} = f\,. \end{aligned}$$

We will consider the case $f \in \mathcal{E}'_{\text{fin}}(\mathbb{R}^n)$, the set of distributions supported on a finite set of points. Stationary sets are sets of points $x \in \mathbb{R}^n$ for which the solution to the wave equation is always zero. This article is an announcement and proof sketches for the results in [6]; complete proofs will be given there.

We use standard notation for function spaces; for example, $\mathcal{D}(\mathbb{R}^n)$ is the space of all C^∞-functions with compact support; $\mathcal{E}'(\mathbb{R}^n)$ is the space of compactly supported distributions; $C^\infty_{\text{rad}}(\mathbb{R}^n)$ and $\mathcal{D}_{\text{rad}}(\mathbb{R}^n)$ are the subspaces of corresponding spaces, consisting of radial functions $f, i.e., f(x) = f(|x|)$. Finally, let $R_+ = (0, \infty)$.

For classical solutions to (1.1), we define stationary sets as follows.

Mark L. Agranovsky, Bar Ilan University, Israel
E-mail: agranovs@macs.cs.biu.ac.il
Eric Todd Quinto, Tufts University, U.S.A
E-mail: equinto@math.tufts.edu

Partially supported by Grant 408/97-3 from the Academy of Sciences of Israel.
Partially supported by US National Science Foundation grants 9622947 and 9877155.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Definition 1. Let $f \in C^\infty(\mathbb{R}^n)$ and u be the (classical) solution for (1.1). Define the stationary set $S(f)$ as the set of time-invariant zeros of the solution u :

$$S(f) = \{x \in \mathbb{R}^n : u(x,t) = 0,\ t > 0\}. \tag{1.2}$$

We use regularization to extend Definition 1 to distributional solutions. Namely, if $f \in \mathcal{D}'(\mathbb{R}^n)$ and $\varphi \in \mathcal{D}_{\text{rad}}(\mathbb{R}^n)$, the convolution $f * \varphi$ is smooth and $u * \varphi$ (convolution with respect to x) is in C^∞ and solves (1.1) for the data $f * \varphi$.

Definition 2. For $f \in \mathcal{D}'(\mathbb{R}^n)$ define

$$S(f) = \bigcap_{\varphi \in \mathcal{D}_{\text{rad}}(\mathbb{R}^n)} S(f * \varphi), \tag{1.3}$$

where $S(f * \varphi)$ is defined by (1.2).

For continuous f, one can use Theorem 2 below to show Definition 2 coincides with Definition 1. In fact, the set of all $\varphi \in \mathcal{D}_{\text{rad}}(\mathbb{R}^n)$ in (1.3) can be replaced by an δ-sequence φ_n, $\varphi_n \in \mathcal{D}_{\text{rad}}(\mathbb{R}^n)$.

The main question under consideration is the following.

Problem. *Which sets $S \subset \mathbb{R}^n$ are stationary sets, $S = S(f)$, for some $f \in \mathcal{D}'(\mathbb{R}^n)$ ($f \in \mathcal{E}'(\mathbb{R}^n)$)?*

The problem has been solved in the plane [3], [4] if f is an arbitrary distribution of compact support, but not much is known in general. For distributions in the plane, the stationary sets have very restrictive structure; they must be the union of a finite set and a Coxeter system of lines (lines through one point generated by a finite Coxeter group of reflections). This Coxeter set is contained in a translate of the zero set of a homogeneous harmonic polynomial, and it is conical about the point of intersection. Loosely speaking, we will prove that a similar pattern occurs for finitely supported distributions in $\mathbb{R}^n$.

Characterizing stationary sets in $\mathbb{R}^n$ for $n > 2$ is more difficult and only partial results are known. It is known for compactly supported initial data in $\mathbb{R}^n$ that stationary sets are contained in zero sets of harmonic polynomials union algebraic varieties of lower dimension, and it is conjectured that the harmonic polynomial can be assumed to be a translate of a homogeneous polynomial [3], [4]. It is shown in [2] for f sufficiently integrable at infinity that stationary sets cannot have bounded closed components. In [1] and [7] more precise analyses are given in $\mathbb{R}^n$ for stationary sets of lower dimension and conical stationary sets for f with arbitrary growth.

In this article we describe stationary sets for the case of the initial data with finite support for arbitrary dimension n. We prove that, up to a low-dimensional component, the stationary sets are affine algebraic cones with a special geometry.

A *cone* is understood to be a union of straight lines with a common point which is the vertex of the cone. We will call a cone $K \subset \mathbb{R}^n$ *$k-$flat* with edge L, where L is a k-dimensional plane in $\mathbb{R}^n$, if K is a union of $(k+1)$-planes containing L.

A union $\Sigma = H_1 \cup \cdots \cup H_q$ of hyperplanes $H_i \subset \mathbb{R}^n$ is called a *Coxeter system* of hyperplanes if Σ is invariant with respect to any reflection σ_i around the hyperplane H_i, $i = 1, \ldots, q$. The *Coxeter group* generated by the reflection $\sigma_1, \ldots, \sigma_q$ will be denoted by $W(\Sigma)$.

We will call a polynomial P in $\mathbb{R}^n$, with real coefficients *a harmonic divisor* if P divides a nonzero harmonic polynomial. Zero sets of homogeneous harmonic divisors will be called *harmonic cones.*

For any set $F \subset \mathbb{R}^n$, the affine subspace spanned by F will be denoted by span F.

Our main result is the following.

Theorem 1. *Let f be a distributions supported at a finite number of points, $f \in \mathcal{E}'_{\text{fin}}(\mathbb{R}^n)$. Assume $f \neq 0$. If $S(f) \neq \emptyset$, then*

(a) *$S(f)$ is an algebraic variety in $\mathbb{R}^n$, contained in the zero set of a nonzero harmonic polynomial.*

(b) *After a suitable translation, the set $S(f)$ can be represented in the form*

$$S(f) = S_0 \cup V,$$

where V is an algebraic variety of $\operatorname{codim} V > 1$ and S_0, assuming it is nonempty, is a harmonic cone, which is a $(n-1)-$dimensional real algebraic variety.

In addition, the following is true:

(c) *The conical component S_0, in general, has the two components*

$$S_0 = \Sigma \cup K,$$

where K contains $\operatorname{supp} f$ but Σ does not, $\Sigma \cap K \neq \emptyset$ provided both Σ and K are nonempty, Σ is a Coxeter system and K is a k-flat harmonic cone with the edge $L = \text{span }(\operatorname{supp} f)$, $k = \dim L < n$. If $\operatorname{supp} f$ is a generic set, i.e., $k = n$, then $K = \emptyset$. If $k = n-1$, then K is a hyperplane and $\Sigma \cup K$ is a Coxeter system.

(d) *If $\tilde{\Sigma}$ is the union of all hyperplanes contained in S_0, then $\tilde{\Sigma}$ is again a Coxeter system; the distribution f is odd with respect to any reflection $\sigma \in W(\tilde{\Sigma})$, i.e. $f \circ \sigma = -f$; the sets S_0, V, $S(f)$ and $\operatorname{supp} f$ are $W(\tilde{\Sigma})$-invariant.*

Theorem 1 says that finite sets of point sources generate stationary sets which are necessarily algebraic varieties and which are either small (empty or low-dimensional) or up to a low-dimensional component, are $(n-1)$-dimensional cones which suitably translated are determined by zeros of spatial harmonics.

The geometry of the essential, conical part is as follows. If the set of points in $\operatorname{supp} f$ is generic then the cone is a Coxeter system of hyperplanes. These stationary sets may appear only as a result of a Coxeter skew-symmetry of the initial data. If $\operatorname{supp} f$ lies in a proper affine subspace in $\mathbb{R}^n$, then another component may appear which is a cone containing $\operatorname{supp} f$. In the plane, for any compactly supported f, $S(f)$ is, up to a finite set, a Coxeter system of lines [3]. However, in the plane the set K in Theorem 1 would be a collection of lines and therefore a Coxeter system by (d).

An important problem in studying of the wave equation is characterizing *nodal* sets (see [12], I, Ch.5, S.5), that is zero sets of eigenfunctions of the Laplace operator, or, equivalently, zero sets of time-harmonic solutions of the wave equation. This problem has been studied by many authors. Results on this subject mainly say that nodal sets are hypermanifolds with singularities and the eigenfunctions cannot vanish to high order on the nodal sets (see, *e.g.,* [13], [14], [11], [9] and others).

The problem under consideration is directly related to describing nodal sets. Indeed, extending the solution $u(x,t)$ of (1.1) for $t<0$ by $u(x,t)=-u(x,-t)$ and applying Fourier transform in t to the both sides of (1.1) yields

$$-\lambda^2 v(x,\lambda) = \Delta v(x,\lambda),$$

where $v(x,\lambda)$ is the Fourier transform evaluated at arbitrary $\lambda \in \mathbb{R}$.

Thus the stationary set $S(f)$ (1.2) is just the intersection of nodal sets of all the eigenfunctions $v(\cdot,\lambda)$ which are, since the initial data f has compact support, nonzero for an infinite number of λ. Thus, while for a single eigenfunction or a finite linear combination of eigenfunctions, the available information is the general analytic structure of nodal sets, our result shows that joint nodal hypermanifolds of one-parameter families of eigenfunctions are strongly determined geometrically. It is worth noting in this connection the result of [10] which describes geometry of nodal lines of bounded membrane under the assumption that these lines contain a open piece of a straight line.

This announcement continues a series of works [3], [4], [5], [2], [1], [7], [8], started by [3] and devoted to the description of injectivity sets for the spherical transform, stationary sets for the wave and heat equations, and related problems. Our initial interest in the problem was motivated by a problem in approximation theory posed in [18] (cf. [3], [4]). Complete proofs of our results are given in [6].

§2. Proof outline for Theorem 1.

We will relate our main theorem to a spherical Radon transform and prove a support theorem for this transform.

Let $f \in C(\mathbb{R}^n)$. Define the spherical transform $C(\mathbb{R}^n) \ni f \to \hat{f} \in C(\mathbb{R}^n \times \mathbb{R}_+)$ by

$$\hat{f}(x,r) = Rf(x,r) = \int_{|\theta|=1} f(x+r\theta)\,dA(\theta), \tag{2.1}$$

where dA is the normalized surface measure on the unit sphere. Because this spherical transform is a Fourier integral operator, it can be defined on distributions.

Theorem 2. *Let $f \in C(\mathbb{R}^n)$ and let $S(f)$ be defined by* (1.2). *Then*

$$S(f) = \{x \in \mathbb{R}^n : \hat{f}(x,r) = 0 \quad \text{for all} \quad r>0\} \tag{2.2}$$

and also

$$S(f) = \{x \in \mathbb{R}^n : (f * u)(x) = 0 \quad \text{for all} \quad u \in \mathcal{D}_{\mathrm{rad}}(\mathbb{R}^n)\}. \tag{2.3}$$

The second equality is a key to proving the consistency of Definitions 1 and 2 for smooth functions. The proof of Theorem 2 follows from the Poisson-Kirchoff formula and integral equations techniques (e.g., [19]).

For $f \in \mathcal{E}'(\mathbb{R}^n)$, one can show that $S(f)$ satisfies equality (2.2) distributionally. One needs to observe that, for each $x \in \mathbb{R}^n$ one can define $\hat{f}(x,\cdot)$ as a distribution on even functions in $\mathcal{D}(\mathbb{R})$ using the natural map between this set of functions and the set of smooth, compactly supported radial functions centered at x. This is related to (2.3).

If Q is a polynomial, we let $N(Q)$ denote the zero set of Q in $\mathbb{R}^n$. Using ideas related to those in [3], [4], we prove

Theorem 3. *Let $f \in \mathcal{E}'(\mathbb{R}^n)$ and $f \neq 0$. Then*

a) *The set $S(f)$ is an algebraic variety in $\mathbb{R}^n$ contained in the zero sets of a nonzero harmonic polynomial.*
b) *$S(f) = S_0 \cup V$, where V is an algebraic variety of* $\operatorname{codim} V > 1$, *$S_0 = N(Q)$ and Q is a harmonic divisor.*

This gives us the key to understand the algebraic structure of $S(f)$.

We now need to understand the geometric structure of $S(f)$. We introduce some new notation. Let $r \in \mathbb{R}_+$, $S \subset \mathbb{R}^n$, and $x \in S$. The point x is called a *regular point* of S if and only if there is a connected real-analytic hypersurface, A, (an $(n-1)-$dimensional real-analytic submanifold of $\mathbb{R}^n$) such that $x \in A \subset S$. Let x be a regular point of S, and let A be such an associated hypersurface ($x \in A \subset S$). Then, we let T_x denote the hyperplane tangent to A at x. The points y and y' in $\mathbb{R}^n$ are said to be *T_x-mirror* if and only if they are reflections about T_x. If $y \in T_x$, then y is its own mirror point, and we say y is *self-mirror*. Our next theorem is a microlocal version of a reflection principle of Courant and Hilbert.

Theorem 4. *Let $f \in \mathcal{E}'_{\text{fin}}(\mathbb{R}^n)$, $f \neq 0$. We let $x_0 \in S = S(f)$ be a regular point. Let $A \subset S$ be a connected real-analytic hypersurface containing x_0, and let T_{x_0} be the hyperplane tangent to A at x_0. Let $y_0 \in \operatorname{supp} f \setminus x_0$. Then, the $T_{x_0}-$mirror point to y_0 must also be in* $\operatorname{supp} f$.

Courant and Hilbert [12 II, pp. 699 *ff.*] proved the *reflection principle* that if $A = T_{x_0}$ is a hyperplane and f has zero integrals over all spheres centered on T_{x_0}, then f has to be an odd function about T_{x_0}. Therefore, if f has zero integrals over such spheres, and f is zero at one mirror point, it must be zero at the other. Our theorem is a microlocal version of that fact where the tangent plane provides the microlocal symmetry. It is proven by calculating the microlocal properties of the spherical transform and using a theorem of Kawai, Kashiwara and Hörmander about analytic wavefront at boundary points of supp f [17].

Proof sketch for Theorem 1. Let $f \in \mathcal{E}'_{\text{fin}}(\mathbb{R}^n)$. Let $x_0 \in A \subset S$ be a regular point, let A be a real-analytic hypersurface such that $x_0 \in A \subset S$. Let T_{x_0} be the plane tangent to A at x_0. Using the result of Theorem 3, we have $S(f) = N(Q) \cup V$ where Q is a harmonic divisor; $S_0 = N(Q)$.

The geometric part of the proof has two cases.

First, assume $\operatorname{supp} f \not\subset T_{x_0}$. Then, there is an $r_0 > 0$ and a point $a_0 \in S(x_0, r_0)$ that is in $\operatorname{supp} f \setminus T_{x_0}$. By the support theorem, the mirror point, a_m, to a_0 must also be in $\operatorname{supp} f$. By definition, T_{x_0} is the perpendicular bisector of the segment $\overline{a_0 a_m}$. For $x \in A$ near x_0, the sphere $S(x, |x - a_0|)$ must meet $\operatorname{supp} f$ at a_0 and at its T_x mirror point by Theorem 4. Because there are only a finite number of points in $\operatorname{supp} f$ and x is close to x_0, this mirror point must be a_m. So, T_x is the perpendicular bisector of $\overline{a_0 a_m}$ for all $x \in A$ sufficiently close to x_0. Therefore, $T_x = T_{x_0}$. This shows locally near x_0, A is flat. Because $S(f)$ is an algebraic variety, this shows that $T_{x_0} \subset S(f)$. Now, we use the reflection principle [12 II, pp. 699 *ff.*] to conclude f is odd about T_{x_0}.

We do this construction for each regular point in $x \in S$ for which $\operatorname{supp} f \notin T_x$. This gives us a collection of hyperplanes contained in $S(f)$ such that f is odd about each one. We know this collection is finite since $S(f)$ is an algebraic variety. It also gives us a Coxeter group of reflections about these hyperplanes, and the set of all such hyperplanes generated by the Coxeter group is Σ. Using properties of Coxeter groups [15, Ch. 8, S.10, Th. 8], [16, Prop. 4.1.3], we show the Coxeter group is finite and the intersection $\cap\Sigma$ is nonempty. We use Theorem 3 and the irreducible factors of the harmonic divisor Q corresponding to Σ to show these factors are all homogeneous about the points in $\cap\Sigma$.

Now, for the second case, assume $\operatorname{supp} f \subset T_{x_0}$. This implies $\dim L < n$. Let $a_0 \in \operatorname{supp} f$. In this case, a_0 is a $T_{x_0}-$self-mirror point. If there were an $x \in A$ near x_0 for which a_0 was not T_x-self-mirror, then this would contradict Theorem 4 because by finiteness of $\operatorname{supp} f$, there would be no T_x-mirror point to a_0 in $\operatorname{supp} f$. This shows that locally A generates a subset of S_0 that is conical about a_0. By a linearity argument, A generates a subset of S_0 conical about any point in $L = \text{span } \operatorname{supp} f$. We use Theorem 3 and the irreducible factors of the harmonic divisor Q corresponding to K to show these factors are all homogeneous about points in $L = \text{span } \operatorname{supp} f$.

This finishes the geometric part of the proof. If $K \neq \emptyset$ and $\Sigma \neq \emptyset$, we show, using Kakutani' Theorem [15] and invariance of L under $\Sigma-$ reflections, that L and Σ have common points and therefore $S_0 = K \cup \Sigma$ is a cone about any such a point. This finishes the proof sketch. $\square$

§3. Sufficient Conditions for Stationary Sets.

Theorem 1 says that the stationary sets $S(f)$ for $f \in \mathcal{E}'_{\text{fin}}(\mathbb{R}^n)$ may consist of three parts: a low-dimensional variety V, a Coxeter system Σ, and a cone K having all points in span $(\operatorname{supp} f)$ as vertices. In addition, the union $S_0 = \Sigma \cup K$ must be a cone containing the zero set of some shifted harmonic homogeneous polynomial and the entire stationary set $S(f) = \Sigma \cup K \cup V$ must belong to the zero set of some nonzero harmonic (not necessarily homogeneous, if V is not a cone with the common vertex with S_0).

Now the question is whether all the possibilities are realizable. Namely, whether each of the sets Σ, K, V and any unions of the sets of these three types are the stationary set $S(f)$ for some $f \in \mathcal{E}'_{\text{fin}}(\mathbb{R}^n)$ or, more generally, are contained in a stationary set?

Below we give positive answers for the sets Σ, K, V and $\Sigma \cup V$. The case of $\Sigma \cup K \cup V$, where each of the three sets are nonempty, remains unsolved.

Given a polynomial $G \in \mathbb{R}[x_1, \dots, x_n]$, denote by T_G the distribution $\langle T_G, \varphi\rangle = G(\partial)\varphi(0)$, $\varphi \in \mathcal{D}(\mathbb{R}^n)$. Here $G(\partial) = G\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)$.

The following theorem shows that the stationary set generated by a homogeneous distribution (of finite order) supported at a single point coincides with common zeros of iterated Laplacians of the symbol of the corresponding differential operator:

Theorem 5. *For any homogeneous polynomial $G \in \mathbb{R}[x_1 \dots x_n]$, we have*

$$S(T_G) = \bigcap_{j \geq 0} N(\Delta^j(G)).$$

The following two corollaries prove that zero sets of harmonics are stationary sets of a homogeneous distribution supported at a single point and describe all such distributions:

Corollary 6. *If Ψ is a homogeneous harmonic polynomial, then $N(\Psi) = S(T_\Psi)$.*

Corollary 7. *Let Ψ be a homogeneous harmonic polynomial and G a polynomial in $\mathbb{R}^n$. Then $N(\Psi) \subset S(T_G)$ if and only if Ψ divides all the polynomials G, ΔG, $\Delta^2 G, \dots$.*

Proof. Since Ψ is homogeneous, then it is easy to check that $N(\Psi) \subset S(T_G)$ is equivalent to $N(\Psi) \subset S(T_{G_m})$, where G_m is any homogeneous term of G. In turn, by Theorem 5, this is equivalent to $G_m, \Delta G_m, \Delta^2 G_m, \dots$ vanishing on $N(\Psi)$. It can be proven that vanishing on zeros of a real harmonic polynomial is equivalent to divisibility, therefore all the homogeneous terms G_m, along with their iterated Laplacians, are divisible by Ψ. This proves the corollary. □

Finally, any low-dimensional real algebraic variety can be stationary for some solution of the wave equation with point supported initial data and, moreover, any Coxeter system can be added:

Theorem 8. [1] *Let V be an algebraic variety in $\mathbb{R}^n$, $\operatorname{codim} V > 1$. Let Σ be either empty or a Coxeter system of hyperplanes. Then there exists a nontrivial polynomial $G \in \mathbb{R}[x_1, \dots, x_n]$ such that $\Sigma \cup V \subset S(T_G)$.*

Remark. In order to prove that $N(\Psi) \cup V$, Ψ is a homogeneous harmonic polynomial, $\operatorname{codim} V > 1$, can be realized as a stationary set, it would be sufficient to prove, according to Corollary 7, that the set of homogeneous polynomials G such that Ψ divides all $\Delta^s G$, $s = 0, 1, \dots$ is big enough to satisfy the additional condition on the low-dimensional part: $\Delta^s G\big|_V = 0$, $s = 0, 1, \dots$. Due to what has been proven in Theorem 5, this means that all the harmonic homogeneous polynomials h_{k-2j} in the decomposition

$$G(x) = h_k(x) + |x|^2 h_{k-2}(x) + \dots, \quad k = \deg G$$

are divisible by Ψ and vanish on V. However, the question about whether the space of harmonic homogeneous polynomials h divisible by a given harmonic Ψ is big enough turned out to be very nontrivial in $\mathbb{R}^n$ for $n > 2$ (*cf.* [1]). We even do not know whether this space is always infinite dimensional or not.

§4. The Case of Balls.

Similar arguments can be used to prove a theorem similar to Theorem 1 if $\operatorname{supp} f$ is (roughly) the union of disjoint balls. Let $\mathcal{E}'_D(\mathbb{R}^n)$ be the set of distributions whose support is contained in the union of a finite number of disjoint closed balls and whose support contains the boundaries of these balls. Distributions in $\mathcal{E}'_D(\mathbb{R}^n)$ can be arbitrary inside each closed ball, but their support must contain the entire boundary of each ball. Here is the geometric analogue of Theorem 1 for $\mathcal{E}'_D(\mathbb{R}^n)$.

Theorem 9 (Support Theorem). *Let $f \in \mathcal{E}'_D(\mathbb{R}^n)$, $f \neq 0$. Assume $S = S(f) \neq \emptyset$. Assume there are regular points in S, and let $x_0 \in S$ be a regular point and $x_0 \notin \operatorname{supp} f$. Let A be a connected real-analytic hypersurface in $\mathbb{R}^n$ such that $x_0 \in A \subset S$. Let T_{x_0} be the hyperplane tangent to A at x_0. Let C be the set of centers of the disks making up $\operatorname{supp} f$. There are two possibilities.*

(a) *For some $c_0 \in C$, $c_0 \notin T_{x_0}$. In this case $A \subset T_{x_0} \subset S$ and $\operatorname{supp} f$ is symmetric about T_{x_0}. Furthermore, f is odd about T_{x_0}.*

(b) *Or, $C \subset T_{x_0}$. In this case, near x_0, S is conical about $L = \operatorname{span} C$. Precisely, A generates a subset of S that is conical with edge L. In this case, $k = \dim L < n$.*

§5. Concluding Remarks.

Theorem 1 asserts that for an initial distributions with finite support, the essential $(n-1)-$dimensional part of the stationary set is a cone. From Section 3, we learn this cone appears as the set of common zeros of spatial harmonics in the Fourier decomposition of the initial distribution. Correspondingly, this happens only when these harmonics have a large set of common zeros (are coherent). More specifically, the cone may contain a system of Coxeter mirrors, if the initial data (sources) admit a corresponding symmetry. In this case vanishing of the solution of the wave equation on the mirrors is the result of cancelling of waves propagated by symmetric sources.

We expect that the stationary sets have a similar geometry for compactly supported initial data and, more generally, for distributions vanishing sufficiently fast at infinity. The main difficulty in proving that is obtaining the conical structure of the essential part of stationary sets. This was done in [3], [4] for $n = 2$. There, the simple structure of zero sets of harmonic polynomials of two variables and the support theorem, Theorem 4, play important roles. Lack of information about zero sets of harmonic polynomials of more than two variables was our main obstacle in extending our approach to $n > 2$. Nevertheless, we hope to succeed using a deeper analysis of the algebraic and geometric structure of stationary sets and by refining the microlocal results that go into the proof of Theorem 1 to be valid more generally, such as for rapidly decreasing functions.

The authors would like to thank Carlos Berenstein and Eric Grinberg for useful comments on the article. Conversations with Valery Volchkov at the beginning of this research were stimulating.

References

1. Agranovsky, M., *On some injectivity problems for the Radon transform on a paraboloid*, Contemporary Math **251** (2000), 1-14.
2. Agranovsky, M., Berenstein, C., and Kuchment, P., *Approximation by spherical waves in L^p spaces*, J. Geom. Analysis **6** (1996), 365-383.
3. Agranovsky, M.L., and Quinto, E.T., *Injectivity sets for a Radon transform and complete systems of radial functions, an announcement*, International Mathematical Research Notes (Duke J.) **11** (1994), 467-473.
4. ______, *Injectivity sets for the Radon transform over circles and complete systems of radial functions*, Journal of Functional Analysis **139** (1996), 383-414.
5. ______, *Injectivity of the spherical mean operator and related problems*, Complex Analysis, Harmonic Analysis and Applications (R. Deville, J. Esterle, V. Petkov, A. Sebbar, A. Yger, eds.), Addison Wesley, London, 1996.
6. ______, *Geometry of Stationary Sets for the Wave Equation in $\mathbb{R}^n$. The Case of Finitely Supported Initial Data*, Duke Math. J. (to appear).
7. Agranovsky, M.L., Volchkov, V.V., and Zalcman, L.A., *Conical injectivity sets for the spherical Radon transform*, Bull. London Math. Soc. **31** (1999), 231-236.
8. Agranovsky, M.L., and Rawat, R., *Injectivity sets for spherical means on the Heisenberg group*, Journal of Fourier Analysis and Appl. **5** (1999), 367-376.
9. Brunning, J., *Über Knoten von Eigenfunktionen des Laplace-Beltrami operators*, Math. Z. **158** (1978), 15-21.
10. ______, *Über Membranen mit speziellen Knotenlinien*, Comm. Math. Helv. **55** (1980), 13-19.
11. Cheng, S.Y., *Eigenfunctions and nodal sets*, Comm. Math. Helv. **57** (1976), 43-55.
12. Courant, R., and Hilbert, D., *Methods of Mathematical Physics, I, II*, Wiley-Interscience, New York, 1962.
13. Donelly, H., and Fefferman, C., *Nodal sets of eigenfunction on Riemannian manifolds*, Invent. Math. **93, 1** (1988), 161-183.

14. ______, *Nodal sets of eigenfunction on surfaces*, J. Amer. Math. Soc. **3** (1990), 333-353.
15. Dunford, N., and Schwartz, J.T., *Linear Operators, I*, Interscience Publ., New York-London, 1958.
16. L.C. Grove and C.T. Benson, *Finite Reflection Groups*, Springer Verlag, 1977.
17. Hörmander, L., *The Analysis of Linear Partial Differential Operators I*, Springer, New York, 1983.
18. Lin, V., and Pinkus, A., *Approximation of multivariate functions*, Advances in Computational Math. (H.P. Dikshit and C.A. Micchelli, eds.), World Scientific Publishers, 1994.
19. Quinto, E.T., *The invertibility of rotation invariant Radon transforms*, J. Math. Anal. Appl. **94** (1983), 602-603.

Electronic Journal: Southwest Journal of Pure and Applied Mathematics
Internet: http://rattler.cameron.edu/swjpam.html
ISSN 1083-0464
Issue 1, June 2001, pp. 38–53.
Submitted: February 3, 2001. Published: June 30, 2001.

INTERPOLATION AND THE LAGUERRE–PÓLYA CLASS

Andrew Bakan, Thomas Craven and George Csordas

Abstract A long-standing open problem in the theory of distribution of zeros of real entire functions requires the characterization of all real sequences $T = \{\gamma_k\}_{k=0}^{\infty}$ such that for any real polynomial $p(x) := \sum_0^n a_k x^k$, the polynomial $\sum_0^n \gamma_k a_k x^k$ has no more nonreal zeros than $p(x)$ has. The main result of this paper shows that under a mild growth restriction, an entire function of exponential type $f(z)$ for which the sequence $T = \{f(k)\}_{k=0}^{\infty}$ satisfies the above condition must have only real zeros. The paper concludes with some applications to the Riemann Hypothesis.

1991 A.M.S. Subject Classification Codes:. 41A05, 26C10, 30D15

Keywords and Phrases:. Laguerre-Pólya class, multiplier sequences, distribution of zeros of entire functions, interpolation, Riemann Hypothesis

1. Introduction.

In the vast literature dealing with the distribution of zeros of real polynomials and real entire functions, an important role is played by linear transformations T which possess the following property:

$$Z_c(T[p(x)]) \leq Z_c(p(x)), \tag{1.1}$$

where $p(x)$ is a polynomial and $Z_c(p(x))$ denotes the number of nonreal zeros of $p(x)$, counting multiplicities. If $T = D = \frac{d}{dx}$, then (1.1) is a consequence of Rolle's

Andrew Bakan, Institute of Mathematics, National Academy of Sciences of Ukraine
E-mail: bakan@imat.gluk.apc.org
Thomas Cravan, Department of Mathematics, University of Hawaii, Honolulu, HI
E-mail: tom@math.hawaii.edu
George Csordas, Department of Mathematics, University of Hawaii, Honolulu, HI
E-mail: george@math.hawaii.edu

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX theorem. If $q(x)$ is a real polynomial with only real zeros and $T = q(D)$, then (1.1) follows from the classical Hermite-Poulain theorem [14, p. 4]. There are many other linear transformations T which satisfy inequality (1.1). Indeed, set $T = \{\gamma_k\}_{k=0}^{\infty}$, $\gamma_k \in \mathbb{R}$, and for an arbitrary real polynomial $p(x) = \sum_0^n a_k x^k$, define

$$T[p(x)] := \sum_{k=0}^{n} \gamma_k a_k x^k. \tag{1.2}$$

If $Q(x)$ is a real polynomial with only real negative zeros and if $T = \{Q(k)\}_{k=0}^{\infty}$, then by Laguerre's theorem (cf. Theorem 2.4 below)

$$Z_c\left(\sum_{k=0}^{n} Q(k) a_k x^k\right) \leq Z_c(p(x)),$$

where $p(x) = \sum_0^n a_k x^k$ is an arbitrary real polynomial.

A long-standing open problem, suggested by the work of Laguerre in the 19th century [11, p. 116] (see also [10, p. 379]) is to characterize all real sequences $T = \{\gamma_k\}_{k=0}^{\infty}$ such that (1.1) holds, where $T[p(x)]$ is defined in (1.2).

On account of the work of Laguerre and Karlin cited above, it may be appropriate to refer to this problem as the *Karlin-Laguerre problem.* While the Karlin-Laguerre problem has remained open, recently there has been significant progress made in a series of investigations (see, for example [2], [1], [6] and [8]). In particular, in [1] (see Theorem 3.2 below) it was proved that *if* this sequence does not decrease "too rapidly", in the sense that $\overline{\lim}_{k\to\infty} \gamma_k^{1/k} > 0$, then the γ_k's can be interpolated by a function $\phi(x)$ in the Laguerre-Pólya class (Definition 2.1), that is $\gamma_k = \phi(k)$, $k = 0, 1, 2, \ldots$.

In Section 2 we begin with a brief review of some requisite terminology and we present some examples in order to motivate our main result which answers the following question raised in [7, Problem 9]. Suppose that $f(z)$ is a real entire function of exponential type such that the sequence $T = \{f(k)\}_{k=0}^{\infty}$ satisfies (1.1). Under what supplementary conditions does it follow that $f(z)$ has only real zeros? In Section 3, we make use of the result from [1, Theorem 2] as well as some classical uniqueness theorems and show that if the growth of $f(z)$ is suitably restricted along the imaginary axis, then $f(z)$ must possess only real zeros (Theorem 3.6). In fact, all the zeros of $f(z)$ are shown to be of one sign and thus Theorem 3.6 provides a converse, under additional assumptions, to Laguerre's theorem (Theorem 2.4). We also prove a slight extension of Laguerre's theorem to finite intervals (Theorem 3.7). By way of applications of the aforementioned results, we conclude the paper with two new necessary and sufficient conditions for the validity of the Riemann Hypothesis (Corollaries 3.10 and 3.11).

2. Definitions and background information.

We commence here with some pertinent definitions involving the Laguerre–Pólya class, multiplier sequences and complex zero decreasing sequences. Also we shall state Laguerre's theorem, mentioned in the introduction, and provide some examples to elucidate and motivate the main result of this paper.

Definition 2.1. A real entire function $\phi(x) := \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k$ is said to be in the *Laguerre-Pólya class*, written $\phi(x) \in \mathcal{L}\text{-}\mathcal{P}$, if $\phi(x)$ can be expressed in the form

$$\phi(x) = cx^n e^{-\alpha x^2+\beta x} \prod_{k=1}^{\infty} \left(1+\frac{x}{x_k}\right) e^{-\frac{x}{x_k}}, \tag{2.1}$$

where $c, \beta, x_k \in \mathbb{R}$, $c \neq 0$, $\alpha \geq 0$, n is a nonnegative integer and $\sum_{k=1}^{\infty} 1/x_k^2 < \infty$. We allow, by the usual convention, the possibility that the product is finite or empty. If $-\infty \leq a < b \leq \infty$ and if $\phi(x) \in \mathcal{L}\text{-}\mathcal{P}$ has all its zeros in (a,b) (or $[a,b]$), then we will use the notation $\phi \in \mathcal{L}\text{-}\mathcal{P}(a,b)$ (or $\phi \in \mathcal{L}\text{-}\mathcal{P}[a,b]$). If $\gamma_k \geq 0$ (or $(-1)^k\gamma_k \geq 0$ or $-\gamma_k \geq 0$) for all $k = 0, 1, 2\ldots$, then $\phi \in \mathcal{L}\text{-}\mathcal{P}$ is said to be of *type I in the Laguerre-Pólya class*, and we will write $\phi \in \mathcal{L}\text{-}\mathcal{P}I$. We will also write $\phi \in \mathcal{L}\text{-}\mathcal{P}I^+$, if $\phi \in \mathcal{L}\text{-}\mathcal{P}I$ and if $\gamma_k \geq 0$ for all $k = 0, 1, 2\ldots.$

In order to clarify the terminology above, we remark that if $\phi \in \mathcal{L}\text{-}\mathcal{P}I$, then $\phi \in \mathcal{L}\text{-}\mathcal{P}(-\infty, 0]$ or $\phi \in \mathcal{L}\text{-}\mathcal{P}[0,\infty)$, but that an entire function in $\mathcal{L}\text{-}\mathcal{P}(-\infty,0]$ need not belong to $\mathcal{L}\text{-}\mathcal{P}I$. Indeed, if $\phi(x) = \frac{1}{\Gamma(x)}$, where $\Gamma(x)$ denotes the gamma function, then $\phi(x) \in \mathcal{L}\text{-}\mathcal{P}(-\infty, 0]$, but $\phi(x) \notin \mathcal{L}\text{-}\mathcal{P}I$. This can be seen, for example, by noting that functions in $\mathcal{L}\text{-}\mathcal{P}I$ are of exponential type (see, for example, [1, Proof of Corollary 2]), whereas $\frac{1}{\Gamma(x)}$ is an entire function of order one of maximal type (see [3, p. 8] or [12, Chapter 2] for the definition of the "type" of an entire function). Also, it is easy to see that $\mathcal{L}\text{-}\mathcal{P}I^+ = \mathcal{L}\text{-}\mathcal{P}I(-\infty, 0]$.

Definition 2.2. A sequence $T = \{\gamma_k\}_{k=0}^{\infty}$ of real numbers is called a *multiplier sequence* if, whenever the real polynomial $p(x) = \sum_{k=0}^{n} a_k x^k$ has *only* real zeros, the polynomial $T[p(x)] = \sum_{k=0}^{n} \gamma_k a_k x^k$ also has *only* real zeros.

The following are well-known characterizations of multiplier sequences (cf. [18], [16, pp. 100–124] or [14, pp. 29–47]). A sequence $T = \{\gamma_k\}_{k=0}^{\infty}$ is a multiplier sequence if and only if

$$\phi(x) = T[e^x] := \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k \in \mathcal{L}\text{-}\mathcal{P}I. \tag{2.2}$$

Moreover, the algebraic characterization of multiplier sequences asserts that a sequence $T = \{\gamma_k\}_{k=0}^{\infty}$ is a multiplier sequence if and only if

$$g_n(x) := \sum_{j=0}^{n} \binom{n}{j} \gamma_j x^j \in \mathcal{L}\text{-}\mathcal{P}I \quad \text{for all } n = 1, 2, 3 \ldots. \tag{2.3}$$

A particularly important collection of multiplier sequences were named in [6] as *complex zero decreasing sequences.* These sequences are defined as follows.

Definition 2.3. We say that a sequence $\{\gamma_k\}_{k=0}^{\infty}$ is a *complex zero decreasing sequence (CZDS)*, if for any real polynomial $\sum_{k=0}^{n} a_k x^k$,

$$Z_c\left(\sum_{k=0}^{n} \gamma_k a_k x^k\right) \le Z_c\left(\sum_{k=0}^{n} a_k x^k\right), \tag{2.4}$$

where $Z_c(p(x))$ denotes the number of nonreal zeros of the polynomial $p(x)$, counting multiplicities. (The acronym CZDS will also be used in the plural.)

Now it follows from (2.4) that any complex zero decreasing sequence is also a multiplier sequence. If $T = \{\gamma_k\}_{k=0}^{\infty}$ is a CZDS of *nonzero* real numbers, then inequality (2.4) is equivalent to the statement that for any polynomial p, $T[p]$ has at least as many real zeros as p has. There are, however, CZDS which have zero terms (cf. [6, Section 3]) and consequently it may happen that $\deg T[p] < \deg p$. When counting the real zeros of p, the number generally increases with the application of T, but may in fact decrease due to a decrease in the degree of the polynomial. For this reason, we count nonreal zeros rather than real ones.

For the sake of clarification of the terminology used in the literature, we remark that if $T = \{\gamma_k\}_{k=0}^{\infty}$ is a CZDS of nonzero real numbers, then the sequence of reciprocals, $\Lambda = \{\frac{1}{\gamma_k}\}_{k=0}^{\infty}$ is sometimes called a *zero decreasing sequence* (cf. [1]). In the present setting it is more convenient for us to work with CZDS rather than with the sequences of reciprocals.

The existence of a *nontrivial* CZDS is a consequence of the following theorem proved by Laguerre and extended by Pólya (see Pólya [15] or [16, pp. 314-321]).

Theorem 2.4. (Laguerre [14, Satz 3.2])

(1) *Let* $f(x) = \sum_{k=0}^{n} a_k x^k$ *be an arbitrary real polynomial of degree* n *and let* $h(x)$ *be a polynomial with only real zeros, none of which lie in the interval* $(0, n)$*. Then* $Z_c(\sum_{k=0}^{n} h(k) a_k x^k) \leq Z_c(f(x))$.

(2) *Let* $f(x) = \sum_{k=0}^{n} a_k x^k$ *be an arbitrary real polynomial of degree* n*, let* $\phi \in \mathcal{L}\text{-}\mathcal{P}$ *and suppose that none of the zeros of* ϕ *lie in the interval* $(0, n)$*. Then the inequality* $Z_c(\sum_{k=0}^{n} \phi(k) a_k x^k) \leq Z_c(f(x))$ *holds.*

(3) *Let* $\phi \in \mathcal{L}\text{-}\mathcal{P}(-\infty, 0]$*, then the sequence* $\{\phi(k)\}_{k=0}^{\infty}$ *is a complex zero decreasing sequence.*

Partial converses of Laguerre's theorem may be found in [6, 8]. While the theory of multiplier sequences commenced with the work of Laguerre [11] in the 19th century, and while in the ensuing decades this theory gained prominence at the hands of such renowned mathematicians as Pólya, Schur, Schoenberg and Obreschkov among others (see [5] and the references therein), today the *only* known CZDS are those sequences $T = \{\gamma_k\}_{k=0}^{\infty}$ which can be interpolated by functions ϕ in $\mathcal{L}\text{-}\mathcal{P}$ (i.e. $\gamma_k = \phi(k)$). The vexing problem of the existence of other types of CZDS has eluded all previous investigations and so we highlight this problem as

Open Problem 2.5. *Do there exist complex zero decreasing sequences which cannot be interpolated by functions in the Laguerre-Pólya class? In particular, are the sequences* $T = \{\gamma_k\}_{k=0}^{\infty}$*, where* $\gamma_k = e^{-k^p}$*,* $p \geq 3$*, CZDS?*

Next, suppose that $\phi(x) \in \mathcal{L}\text{-}\mathcal{P}(-\infty, 0)$ and set $f(z) := \phi(z) + \sin(\pi z)$ for $z \in \mathbb{C}$. Then, since $f(k) = \phi(k)$ for $k = 0, 1, 2, \ldots$, the sequence $\{f(k)\}_{k=0}^{\infty}$ is a CZDS, but, in general, $f \notin \mathcal{L}\text{-}\mathcal{P}$ (cf. Example 2.8 below). Thus, in the absence of additional assumptions, the converse of Laguerre's theorem need not hold true. These considerations have led to the following problem.

Problem 2.6. ([7, Problem 9]) *Let $f(z)$ be a real entire function of exponential type. Suppose that all the zeros of $f(z)$ lie in the strip $|\Im z| < M$ for some $M > 0$. If $\{f(k)\}_{k=0}^{\infty}$ is a CZDS, when does it follow that $f(z) \in \mathcal{L}\text{-}\mathcal{P}$?*

Remark 2.7. We recall (cf. [3, p. 8] or [12, p. 84]) here that entire functions $f(z)$ of at most order one and of normal type are called *entire functions of exponential type* and the *exponential type* of $f(z)$ is

$$\sigma = \overline{\lim_{r\to\infty}} \frac{\log M(r,f)}{r} < \infty,$$

where $M(r,f) = \max_{|z|=r} |f(z)|$. Thus, entire functions of order less than one, or of order one and of minimal type, are said to be of exponential type zero. Equivalently, an entire function $f(z)$ is of exponential type if $f(z) = O(e^{c|z|})$ for some $c \in \mathbb{R}$ as $|z| \to \infty$. Moreover, if $f(z)$ is analytic in $\Re z \geq 0$ and $f(z) = O(e^{c|z|})$ (as $|z| \to \infty$), then we will say that $f(z)$ is of exponential type in $\Re z \geq 0$.

In order to motivate and elucidate the supplementary hypothesis which we will use to prove our main theorem (Theorem 3.6), we conclude this section with the following elementary example.

Example 2.8. Let $f(z) := \phi(z) - \sin(\pi z)$ for $z \in \mathbb{C}$, where $\phi(z) = 2$ for all $z \in \mathbb{C}$. Then $\phi \in \mathcal{L}\text{-}\mathcal{P}(-\infty, 0)$, the sequence $\{\phi(k)\}_{k=0}^{\infty} = \{2, 2, 2, \ldots\}$ is a CZDS and $f(z)$ is an entire function of exponential type. Moreover, a calculation shows (with $z = x + iy$, $x, y \in \mathbb{R}$) that

$$|f(z)| = |2 - \sin(\pi z)| \geq |\sinh(\pi y) - 2| > 0$$

for $y > A := \frac{1}{\pi}\log(2 + \sqrt{5})$. Thus the zeros of $f(z)$ lie in the strip $|\Im z| \leq A$. But $f \notin \mathcal{L}\text{-}\mathcal{P}$ as it has the nonreal zeros $(\frac{1}{2} + 2n) + \frac{i}{\pi}\log(2 + \sqrt{3})$, $n = 0, 1, 2, \ldots$.

These observations suggest that the assumption in Problem 2.6 that the zeros of f lie in a strip of the form $|\Im z| < M$ is not necessary. On the other hand, if we calculate the exponential type of $f(z)$ (see Remark 2.7), we find that

$$\sigma = \overline{\lim_{r\to\infty}} \frac{\log M(r,f)}{r} = \overline{\lim_{y\to\infty}} \frac{\log|f(\pm iy)|}{y} = \pi. \tag{2.5}$$

This, together with the familiar uniqueness theorems (see, for example, [3, Chapter 9] or [12, Chapter IV]) gives us the clue that a converse of Laguerre's theorem is valid for entire functions $f(z)$ of exponential type, provided we restrict the growth of $f(z)$ along the imaginary axis.

3. The main theorem and applications.

Let $f(z)$ be analytic in a domain D which contains a ray making an angle θ with the positive real axis. In order to measure the growth of $|f(z)|$ along this ray, we will make use of the *(Phragmén–Lindelöf) indicator function*, which is defined as

$$h_f(\theta) := h(\theta) := \overline{\lim_{r\to\infty}} \frac{\log|f(re^{i\theta})|}{r}. \tag{3.1}$$

The proof of our main theorem requires several results which we include here for the reader's convenience.

Theorem 3.1. ([3, Theorem 9.8.1, p. 166]) *Let $f(z)$ be analytic and of exponential type for $|\arg z| \le \alpha$, $\alpha > \pi/2$. Suppose that $h_f(\pm\pi/2) < \pi$. Let $\{\lambda_k\}_{k=0}^{\infty}$ be a sequence such that $\lim_{n\to\infty} \frac{n}{\lambda_n} = 1$ and $|\lambda_m - \lambda_n| \ge \delta|n-m|$ for some $\delta > 0$. If*

$$\lim_{n\to\infty} \frac{\log|f(\lambda_n)|}{\lambda_n} = -\infty,$$

then $f(z) \equiv 0$.

Using the terminology and the notation adopted in Section 2, we can formulate a special case of a recent result proved in [1] as follows.

Theorem 3.2. ([1, Theorem 2]) *Let $\{\gamma_k\}_{k=0}^{\infty}$, $\gamma_k > 0$, be a CZDS. If*

$$\overline{\lim_{k\to\infty}} \gamma_k^{1/k} > 0, \tag{3.2}$$

then there is a function $\phi(z) \in \mathcal{L}\text{-}\mathcal{P}$ of the form

$$\phi(z) := be^{az}\psi(z) := be^{az} \prod_{n=1}^{\infty} \left(1 + \frac{z}{x_n}\right), \tag{3.3}$$

where $a, b \in \mathbb{R}$, $b \neq 0$, $x_n > 0$ *and* $\sum_{n=1}^{\infty} 1/x_n < \infty$, *such that* $\phi(z)$ *interpolates the sequence* $\{\gamma_k\}_{k=0}^{\infty}$; *that is,* $\gamma_k = \phi(k)$ *for* $k = 0, 1, 2, \ldots$.

Remark 3.3. If $\{\gamma_k\}_{k=0}^{\infty}$ is a multiplier sequence (cf. Definition 2.2) with $\gamma_k \neq 0$, then it is known [18] that either $\gamma_k > 0$, $-\gamma_k > 0$ or $(-1)^k\gamma_k > 0$ for all k. Thus, Theorem 3.2 could be stated in a slightly more general form with the concomitant conclusion that $\gamma_k = \phi(k)$, $-\gamma_k = \phi(k)$ or $(-1)^k\gamma_k = \phi(k)$. We also note that if $a \geq 0$, then $\phi \in \mathcal{L}\text{-}\mathcal{P}I$.

Theorem 3.4. (Carlson's theorem [3, p. 153] or [12, p. 168]) *Let* $f(z)$ *be analytic and of exponential type in the half plane* $\Re z \geq 0$. *Suppose that* $h_f(-\pi/2) + h_f(\pi/2) < 2\pi$. *If* $f(k) = 0$ *for* $k = 0, 1, 2, \ldots$, *then* $f(z) \equiv 0$.

Remark 3.5. In Carlson's theorem, the assumption $f(k) = 0$ for $k = 0, 1, 2, \ldots$ can be weakened to $f(k) = 0$ for $k = m, m+1, m+2, \ldots$ for some positive integer m. In this case, let $p(z) = z(z-1)\cdots(z-m+1)$. We can apply the theorem to $g(z) = p(z)f(z)$ since $h_g(-\pi/2) + h_g(\pi/2) < 2\pi$ still holds.

Preliminaries aside, we can now prove the following theorem.

Theorem 3.6. *Let* $f(z)$ *be an entire function of exponential type. Suppose that* $\{f(k)\}_{k=0}^{\infty}$ *is a CZDS, where* $f(0) = 1$. *Let* $h_f(\theta)$ *denote the indicator function of* $f(z)$ *(cf. (3.1)), where* $\theta \in [-\pi, \pi]$. *If* $h_f(\pm\pi/2) < \pi$, *then* $f(z)$ *is in* $\mathcal{L}\text{-}\mathcal{P}$ *and* $f(z)$ *can be expressed in the form*

$$f(z) = e^{az} \prod_{n=1}^{\infty} \left(1 + \frac{z}{x_n}\right), \tag{3.4}$$

where $a \in \mathbb{R}$, $x_n > 0$ *and* $\sum_{n=1}^{\infty} 1/x_n < \infty$.

Proof. We first show that the hypotheses imply that $f(k) > 0$ for $k = 1, 2, \ldots$. If $f(m) = 0$ for any positive integer m, then $f(k) = 0$ for $k \geq m$ since any multiplier sequence (see Definition 2.2) has this property [4, Theorem 3.4(a)]. By Theorem 3.4 and Remark 3.5, this cannot happen for the function $f(z)$. Furthermore, by Remark 3.3, the values $f(k)$ are either all positive or alternate in sign. But it is known [3, Theorem 9.9.4, p. 169] that if $f(z)$ is an entire function of exponential type with $h_f(\pm\pi/2) < \pi$, then it is impossible for the numbers $f(k)$ to alternate in sign.

For convenience of notation, we set $\gamma_k := f(k)$, $k = 0, 1, 2, \dots$ and we begin by showing that the limit

$$L = \lim_{k\to\infty} (f(k))^{1/k} = \lim_{k\to\infty} \gamma_k^{1/k} \tag{3.5}$$

exists. Since $T = \{\gamma_k\}_{k=0}^{\infty}$ is a CZDS, it is known (see, for example, [5]) that T satisfies the *Turán inequalities*

$$\gamma_{k+1}^2 \geq \gamma_k\gamma_{k+2}, \qquad \text{or equivalently,} \qquad \frac{\gamma_{k+2}}{\gamma_{k+1}} \leq \frac{\gamma_{k+1}}{\gamma_k},$$

for $k = 0, 1, 2, \dots$, where we have used the assumption that $\gamma_k > 0$. Also, from the calculus we know that [19, p.68]

$$\varliminf_{k\to\infty} \frac{\gamma_{k+2}}{\gamma_{k+1}} \leq \varliminf_{k\to\infty} \gamma_k^{1/k} \leq \varlimsup_{k\to\infty} \gamma_k^{1/k} \leq \varlimsup_{k\to\infty} \frac{\gamma_{k+1}}{\gamma_k}. \tag{3.6}$$

But the sequence $\left\{\frac{\gamma_{k+1}}{\gamma_k}\right\}$ is monotone decreasing and so it follows from (3.6) that the limit in (3.5) exists.

Now, if in (3.5) the limit $L = 0$, then $\lim_{k\to\infty} \frac{\log f(k)}{k} = -\infty$ and by Theorem 3.1 (with $\lambda_k = k$) $f(z) \equiv 0$, which is impossible. Thus $L > 0$ and consequently, by Theorem 3.2, there is a function $\phi(z) \in \mathcal{L}$-$\mathcal{P}$ of the form (3.3) (with $b = 1$) such that

$$f(k) = \phi(k) \qquad (k = 0, 1, 2, \dots). \tag{3.7}$$

In order to estimate the indicator function of $f - \phi$ at $\theta = \pm\pi/2$, we recall that (see, for example [1, proof of Corollary 2]) the canonical product $\psi(z)$ in (3.3) is an entire function of order at most one and of minimal type, and hence $h_\psi(\theta) = 0$ for $\theta \in [-\pi, \pi]$. Setting $E(z) := e^{az}$, $a \in \mathbb{R}$, it is easy to verify that $h_E(\pm\pi/2) = 0$ and thus

$$h_\phi(\pm\pi/2) = 0. \tag{3.8}$$

Let $F(z) := f(z) - \phi(z)$ for $z \in \mathbb{C}$. Then $F(z)$ is an entire function of exponential type and by the well-known properties of the indicator function (cf. [12, p. 52]) and (3.8) we have

$$h_F(\pm\pi/2) \leq \max(h_f(\pm\pi/2), h_\phi(\pm\pi/2)) = \max(h_f(\pm\pi/2), 0). \tag{3.9}$$

Since by assumption $h_f(\pm\pi/2) < \pi$, (3.9) yields the estimate

$$h_F\left(-\frac{\pi}{2}\right) + h_F\left(\frac{\pi}{2}\right) < 2\pi. \tag{3.10}$$

Therefore, by (3.7) and (3.10), $F(z)$ satisfies the hypothesis of Carlson's theorem. Thus, $f(z) = \phi(z)$ for all $z \in \mathbb{C}$ and *a fortiori* $f(z)$ is in $\mathcal{L}$-$\mathcal{P}$ and $f(z)$ has a representation of the form (3.4). $\square$

In order to state some applications of Theorem 3.6, we provide here a slight generalization of Laguerre's theorem. While this result may be found, *mutatis mutandis*, in [1], we will sketch a different proof which may be of interest because of its elementary character.

Theorem 3.7. (Laguerre's theorem on intervals) *Let*

$$\phi(x) := \prod_{n=1}^{\infty} \left(1 + \frac{x}{x_n}\right), \tag{3.11}$$

where $x_n > 0$ *and* $\sum_{n=1}^{\infty} 1/x_n < \infty$. *Then, for any real polynomial* $p(x) = \sum_{k=0}^{n} a_k x^k$,

$$Z_{[0,1]}\left(\sum_{k=0}^{n} \phi(k) a_k x^k\right) \geq Z_{[0,1]}\left(\sum_{k=0}^{n} a_k x^k\right), \tag{3.12}$$

where $Z_{[0,1]}(p(x))$ *denotes the number of real zeros of* $p(x)$ *in the interval* $[0,1]$, *counting multiplicities.*

Proof. Only a sketch of the proof will be given since the main ideas of the proof may be found in [14, p. 6]. We first outline the proof under the simplifying assumptions that (a) $p(0) > 0$ and (b) the real zeros of $p(x)$ in $[0,1]$ are simple.

Consider the special case when $\phi_1(x) := x + \alpha_1$, $\alpha_1 > 0$, so that for the arbitrary polynomial $p(x)$, we have

$$\sum_{k=0}^{n} \phi_1(k) a_k x^k = \alpha_1 p(x) + x p'(x) =: f(x).$$

If $p(x)$ has no real zeros in $[0,1]$, then (3.12) is clear. Otherwise, let $0 < x_1 < x_2 < \cdots < x_m \leq 1$ denote the zeros of $p(x)$ in $[0,1]$. Then $f(x_j) = x_j p'(x_j)$, $j = 1, \ldots, m$, and it follows from Rolle's theorem that $f(x)$ has an odd number of zeros in (x_j, x_{j+1}), whence $f(x)$ has at least $m-1$ zeros in $[0,1]$. Since $p(0) > 0$ and x_1 is the first zero of $p(x)$ to the right of the origin, $p'(x_1) < 0$. Hence $f(x_1) = \alpha_1 p(x_1) + x_1 p'(x_1) < 0$, so that $f(x)$ has a zero in $(0, x_1)$ by the intermediate value theorem. Therefore, $f(x)$ has at least m zeros in $(0,1)$ and so this case of (3.12) is clear.

The argument above remains valid, *mutatis mutandis*, if $p(0) = 0$, since (i) $p(x)$ and $f(x)$ will then vanish at $x = 0$ with the same multiplicity and (ii) $p(x)$ and $f(x)$ have the same sign for sufficiently small $x > 0$. The case when $p(x)$ has a multiple zero in $[0,1]$ can be handled by standard continuity arguments.

Next, if $\phi(x) := \phi_N(x) := \prod_{n=1}^{N} (1 + x/x_n)$, $N \geq 1$, then repeated application of the foregoing argument shows that

$$Z_{[0,1]} \left(\sum_{k=0}^{n} \phi_N(k) a_k x^k \right) \geq Z_{[0,1]} \left(\sum_{k=0}^{n} a_k x^k \right).$$

Finally, taking the limit as $N \to \infty$, we obtain the desired conclusion (3.12). $\square$

Remark 3.8. Theorem 3.7 remains valid for functions of the form $\psi(x) := e^{ax}\phi(x)$, where $a \geq 0$ and $\phi(x)$ is given by (3.11). Moreover, in place of the interval $[0,1]$, we could have stated the theorem for any interval of the form $[-R, 0]$ or $[0, R]$, $0 \leq R \leq \infty$. We hasten to add that Theorem 3.7 is *not* valid for arbitrary functions $\phi \in \mathcal{L}\text{-}\mathcal{P}(-\infty, 0)$. Indeed, consider the simple examples $\phi_1(x) := e^{-x}$ and $\phi_2(x) := 1/\Gamma(x+1)$ and apply the sequences $\{\phi_1(k)\}_{k=0}^{\infty}$ or $\{\phi_2(k)\}_{k=0}^{\infty}$ to the polynomial $p(x) := 1 - 2x + x^2$.

The converse of Theorem 3.7, proved in [1], is a deeper theorem. The more precise result may be expressed as follows.

Theorem 3.9. ([1, Theorem 2]) *Let $\{\gamma_k\}_{k=0}^{\infty}$, $\gamma_0 = 1$, $\gamma_k > 0$, be a sequence of positive real numbers. Then, for any real polynomial $p(x) = \sum_{k=0}^{n} a_k x^k$,*

$$Z_{[0,1]}\left(\sum_{k=0}^{n} \gamma_k a_k x^k\right) \geq Z_{[0,1]}\left(\sum_{k=0}^{n} a_k x^k\right)$$

holds if and only if the sequence $\{\gamma_k\}_{k=0}^{\infty}$ can be interpolated by a function of the form (3.11).

For the reader's convenience, we remark that the sequences $\{\gamma_k\}_{k=0}^{\infty}$ which satisfy the inequality above are the reciprocals of sequences belonging to the class τ_1 as defined in [1, equation (7)].

In order to state our applications of the foregoing results, we need to recall some of the nomenclature pertaining to the Riemann Hypothesis. To begin with, Riemann's ξ-function can be defined (cf. [17] or [16]) by

$$\xi(iz) := \frac{1}{2}\left(z^2 - \frac{1}{4}\right)\pi^{-\frac{z}{2}-\frac{1}{4}}\Gamma\left(\frac{1}{4} + \frac{z}{2}\right)\zeta\left(\frac{1}{2} + z\right), \tag{3.13}$$

where ζ is the Riemann ζ-function. It is known that $\xi(z)$ is an entire function of order one (see [20, p. 29] and note that Titchmarsh uses the symbol Ξ for ξ) and of maximal type [9, p. 493]. Moreover, the Riemann ξ-function admits the Fourier integral representation ([17] or [20, p. 255])

$$H(x) := \frac{1}{8}\xi\left(\frac{x}{2}\right) = \int_0^{\infty} \Phi(t)\cos(xt)\,dt = \frac{1}{2}\int_{-\infty}^{\infty} \Phi(t)e^{ixt}\,dt, \tag{3.14}$$

where

$$\Phi(t) := \sum_{n=1}^{\infty}\left(2\pi^2 n^4 e^{9t} - 3\pi n^2 e^{5t}\right)\exp\left(-\pi n^2 e^{4t}\right).$$

The Riemann Hypothesis is the statement that all the zeros of $H(x)$ are real and so $H \in \mathcal{L}\text{-}\mathcal{P}$. Since

$$H(x) = \sum_{m=0}^{\infty} \frac{b_m}{(2m)!}(-x^2)^m, \tag{3.15}$$

where the b_m's are the moments defined by

$$b_m = \int_0^{\infty} t^{2m}\Phi(t)\,dt, \quad m = 0, 1, 2, \ldots,$$

the change of variables $u = -x^2$ in (3.15) yields

$$F(u) := \sum_{m=0}^{\infty} \frac{b_m}{(2m)!} u^m = \int_0^{\infty} \Phi(t) \cosh(t\sqrt{u})\, dt. \tag{3.16}$$

Thus F is an entire function of order $\frac{1}{2}$ and the Riemann Hypothesis is equivalent to the statement that F has only real negative zeros (so that $F \in \mathcal{L}\text{-}\mathcal{P}I$).

We can now state and prove the following consequence of Theorem 3.6 and Theorem 3.7.

Corollary 3.10. *Let $F(z)$ be the entire function defined by (3.16). Then the following statements are equivalent.*

(1) *The Riemann Hypothesis is true.*
(2) *The sequence $\{F(k)\}_{k=0}^{\infty}$ is a complex zero decreasing sequence.*
(3) *For every polynomial $p(x) = \sum_{k=0}^{n} a_k x^k$, $a_k \in \mathbb{R}$,*

$$Z_{[0,1]}\left(\sum_{k=0}^{n} F(k) a_k x^k\right) \geq Z_{[0,1]}(p(x)). \tag{3.17}$$

Proof. (1) $\iff$ (2) Suppose that the Riemann Hypothesis is true. Then $F \in \mathcal{L}\text{-}\mathcal{P}I$, whence by Laguerre's theorem (Theorem 2.4) the sequence $\{F(k)\}_{k=0}^{\infty}$ is a CZDS. Conversely, suppose that $\{F(k)\}_{k=0}^{\infty}$ is a CZDS. Then, from the discussion preceding this corollary, we know that the order of the entire function F is $\frac{1}{2}$ and consequently F is of exponential type zero. Also, the indicator function of F (cf. (3.1)) satisfies $h_F(\pm\frac{\pi}{2}) = 0$. Thus by Theorem 3.6, we have $F \in \mathcal{L}\text{-}\mathcal{P}I$.

(2) $\iff$ (3) Suppose that $\{F(k)\}_{k=0}^{\infty}$ is a CZDS. Then, as above, we conclude that $F \in \mathcal{L}\text{-}\mathcal{P}I$, whence F can be expressed in the form (3.11). Therefore by Theorem 3.7, (3.17) holds. Conversely, suppose that (3.17) holds. Then by Theorem 3.9 the sequence $\{F(k)\}_{k=0}^{\infty}$ can be interpolated by a function of the form (3.11) and hence by Theorem 2.4, $\{F(k)\}_{k=0}^{\infty}$ is a CZDS. $\square$

We remark that the proof of the implication (3) $\Longrightarrow$ (2) of Corollary 3.10 can be based on an elementary argument which does not use Theorem 3.9. Indeed, suppose that (3.17) holds. Now if the sequence $\{F(k)\}_{k=0}^{\infty}$ is not a CZDS, then there exists a polynomial $q(x) := \sum_0^n b_k x^k$, $b_k \in \mathbb{R}$ such that

$$Z_c\left(\sum_{k=0}^{n} F(k) b_k x^k\right) > Z_c(q(x)). \tag{3.18}$$

Let $Q(x) := \sum_0^n F(k) b_k x^k$ and choose $R > 0$ so large that all of the zeros of $Q(x)$ and $q(x)$ are contained in the disk $|z| < R$. Since $Z_c(Q(Rx)) = Z_c(Q(x))$, (3.18) implies that

$$Z_c(Q(Rx)) > Z_c(q(Rx)). \tag{3.19}$$

Using the fact that $F(k) > 0$ for $k = 0, 1, 2, \ldots$, we can express (3.19) in the equivalent form

$$Z_{[-1,1]}(Q(Rx)) < Z_{[-1,1]}(q(Rx)). \tag{3.20}$$

But then either $Z_{[-1,0]}(Q(Rx)) < Z_{[-1,0]}(q(Rx))$ or $Z_{[0,1]}(Q(Rx)) < Z_{[0,1]}(q(Rx))$. By considering $q(-x)$ if necessary, we may conclude that the latter holds. But this contradicts (3.17), and thus $\{F(k)\}_{k=0}^{\infty}$ is a CZDS.

In light of the importance of the theory of the zeta function, it is desirable to express the foregoing result (say the equivalence of (1) and (3) of Corollary 3.10) in terms of the ζ-function. To this end, we use (3.13), (3.14) and (3.16) and the fact that (at least for positive x)

$$8F(4x) = \frac{1}{2}\left(x - \frac{1}{4}\right) \pi^{-\frac{\sqrt{x}}{2} - \frac{1}{4}} \Gamma\left(\frac{1}{4} + \frac{\sqrt{x}}{2}\right) \zeta\left(\frac{1}{2} + \sqrt{x}\right). \tag{3.21}$$

Therefore, as a consequence of (3.21) and Corollary 3.10 and the fact that $F(x) \in \mathcal{L}\text{-}\mathcal{P}I$ if and only if $F(Rx) \in \mathcal{L}\text{-}\mathcal{P}I$ for $R > 0$, we have

Corollary 3.11. *Let $R > 0$. The Riemann Hypothesis is true if and only if*

$$\begin{aligned} Z_{[0,1]}\left(\sum_{k=0}^{n} a_k \left(4R^2 k - \frac{1}{4}\right) \pi^{-R\sqrt{k}} \Gamma\left(\frac{1}{4} + R\sqrt{k}\right) \zeta\left(\frac{1}{2} + 2R\sqrt{k}\right) x^k\right) \\ \geq Z_{[0,1]}\left(\sum_{k=0}^{n} a_k x^k\right), \end{aligned}$$

for every positive integer n *and for arbitrary* $a_0, \ldots, a_n \in \mathbb{R}$.

Remark 3.12. It may be of interest to note here that Corollary 3.11, in conjunction with some standard results from the theory of moments (see, for example, [13, p. 131 and p. 142]), yield several *necessary* conditions for the validity of the Riemann Hypothesis. We mention here two such necessary conditions. Let $R > 0$ and for nonnegative integers n and m set

$$\mu_{n,m}(R) := \frac{\pi^{R\sqrt{n+m}}}{(4R^2(n+m)-\frac{1}{4})\Gamma(\frac{1}{4}+R\sqrt{n+m})\zeta(\frac{1}{2}+2R\sqrt{n+m})}.$$

If the Riemann Hypothesis is true, then (a)

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\mu_{k,m}(R) > 0 \qquad (m, n = 0, 1, 2, \ldots)$$

and (b) there exists a bounded, nondecreasing function $\mu_R(t)$ on $[0, 1]$ such that

$$\mu_{n,0}(R) = \int_0^1 t^n \, d\mu_R(t) \qquad (n = 0, 1, 2, \ldots).$$

References

1. A. Bakan, T. Craven, G. Csordas and A. Golub, *Weakly increasing zero-diminishing sequences*, Serdica **22** (1997), 547–570.

2. A. G. Bakan and A. P. Golub, *On sequences that do not increase the number of real roots of polynomials*, Ukrainian Math. J. **45** (1993), 1481–1489.

3. R. P. Boas, Jr., *Entire Functions*, Academic Press, New York, 1954.

4. T. Craven and G. Csordas, *Multiplier sequences for fields*, Illinois J. Math. **21** (1977), 801–817.

5. T. Craven and G. Csordas, *Jensen polynomials and the Turán and Laguerre inequalities*, Pacific J. Math. **136** (1989), 241–260.

6. T. Craven and G. Csordas, *Complex zero decreasing sequences*, Methods Appl. Anal. **2** (1995), 420–441.

7. T. Craven and G. Csordas, *Problems and Theorems in the Theory of Multiplier Sequences*, Serdica **22** (1997), 515–524.

8. T. Craven and G. Csordas, *On a converse of Laguerre's theorem*, Electron. Trans. Numer. Anal. **5** (1997), 7–17.

9. G. Csordas, T. Norfolk and R. Varga, *A lower bound for the de Bruijn-Newman Constant* Λ, Numer. Math. **52** (1988), 483–497.

10. S. Karlin, *Total Positivity*, vol. 1, Stanford Univ. Press, Stanford, Calif., 1968.

11. E. Laguerre, *Sur quelques points de la théorie des équations numeriques*, Acta. Math. **4** (1884), 97–120.

12. B. Ja. Levin, *Distribution of Zeros of Entire Functions*, Transl. Math. Mono. vol. 5, Amer. Math. Soc., Providence, RI, 1964; revised ed. 1980.

13. I. P. Natanson, *Constructive Function Theory, vol. II, Approximation in Mean*, (transl. by J. R. Schulenberger), Fredrick Ungar Pub. Co., New York, 1965.

14. N. Obreschkoff, *Verteilung und Berechnung der Nullstellen reeller Polynome*, VEB Deutscher Verlag der Wissenschaften, Berlin, 1963.

15. G. Pólya, *Über einen Satz von Laguerre*, Jber. Deutsch. Math-Verein. **38** (1929), 161–168.

16. G. Pólya, *Collected Papers, Vol. II Location of Zeros*, (R. P. Boas, ed.), MIT Press, Cambridge, MA, 1974.

17. G. Pólya, *Über die algebraisch–funktionentheoretischen Untersuchungen von J. L. W. V. Jensen*, Klg. Danske Vid. Sel. Math.–Fys. Medd. **7** (1927), 3–33.

18. G. Pólya and J. Schur, *Über zwei Arten von Faktorenfolgen in der Theorie der algebraischen Gleichungen*, J. Reine Angew. Math. **144** (1914), 89–113.

19. W. Rudin, *Principles of Mathematical Analysis*, 3rd. ed., MacGraw–Hill, New York, 1976.

20. E. C. Titchmarsh, *The Theory of the Riemann Zeta–function*, 2nd. ed. (revised by D.R. Heath–Brown), Oxford University Press, Oxford, 1986.

Electronic Journal: Southwest Journal of Pure and Applied Mathematics
Internet: http://rattler.cameron.edu/swjpam.html
ISBN 1083-0464
Issue 1 June 2001, pp. 54 – 65
Submitted: February 22, 2001. Published: June 30, 2001

Universal Factorization Equalities over Generalized Quaternion Algebras

Yongge Tian

Abstract

A set of universal similarity factorization equalities are established over generalized quaternion algebras, their properties and some applications are also considered.

A.M.S. Subject Classification Codes: *15A06; 15A24; 15A33.*
Key words and phrases: *Generalized quaternion algebras, universal similarity factorizations, matrix representations*

1 Introduction

A generalized quaternion algebra over an arbitrary field $\mathbb{F}$ of characteristic not two is well known as follows(see e.g., [4], [6] and [9]).

Definition. *Let $\mathbb{F}$ be an arbitrary field of characteristic not two, u and v be two nonzero elements of $\mathbb{F}$. A generalized quaternion algebra $\left(\frac{u,v}{\mathbb{F}}\right)$ is a four dimensional vector space over $\mathbb{F}$ with the four basis 1, i, j and k satisfying the following multiplication laws*

$$i^2 = u, \qquad j^2 = v, \qquad k = ij = -ji, \tag{1.1}$$

and 1 acting as unity element.

Department of Mathematics and Statistics, Queen's University, Canada
E-mail: ytian@mast.queensu.ca

According to the above definition, the generalized quaternion algebra $\left(\frac{u,\,v}{\mathbb{F}}\right)$ *is a four-dimensional associative algebra with its center field* $\mathbb{F}$*. Three ordinary quaternion algebras over the real number field* $\mathbb{R}$ *and the complex number field* $\mathbb{C}$ *are well-known as follows:*

$$\mathbb{Q} := \left(\frac{-1,\,-1}{\mathbb{C}}\right), \qquad \text{the complex quaternion algebra,}$$

$$\mathbb{H} := \left(\frac{-1,\,-1}{\mathbb{R}}\right), \qquad \text{the real division quaternion algebra,}$$

$$\mathbb{S} := \left(\frac{-1,\,1}{\mathbb{R}}\right), \qquad \text{the split quaternion algebra.}$$

A well-known fact on generalized quaternions algebras is that $\left(\frac{u,\,v}{\mathbb{F}}\right)$ *is algebraically isomorphic to a subalgebra of the total matrix algebra* $M_4(\mathbb{F})$ *and each element* $a \in \left(\frac{u,\,v}{\mathbb{F}}\right)$ *has a matrix representation* $\phi(a)$ *over its central field* $\mathbb{F}$*.*

In this paper, we present some fundamental equalities for an element $a \in \left(\frac{u,\,v}{\mathbb{F}}\right)$ *and its matrix representation* $\phi(a)$*, and then derive from them some valuable consequences.*

Throughout, we use the following known definitions and notations. For any $a = a_0 + a_1 i + a_2 j + a_3 k \in \left(\frac{u,\,v}{\mathbb{F}}\right)$*,* $\overline{a} = a_0 - a_1 i - a_2 j - a_3 k$ *stands for the* conjugate *of* a*;* $n(a) = a\overline{a} = a_0^2 - ua_1^2 - va_2^2 + uva_3^2$ *stands for the* norm *of* a*;* $\overrightarrow{a} = [\,a_0,\ a_1,\ a_2,\ a_3\,]^T$ *stands for the* adjoint vector *of* a*. For a quaternion matrix* $A = (a_{st}) \in \left(\frac{u,\,v}{\mathbb{F}}\right)^{m\times n}$*,* $\overline{A} = (\overline{a_{st}})$ *denotes the* conjugate *of* A*; and* $A^* = (\overline{A})^T = (\overline{a_{ts}})$ *denotes the* conjugate transpose *of* A*. A square generalized quaternion matrix* A *of size* n *is said to be* invertible *if there is a matrix* B *over* $\left(\frac{u,\,v}{\mathbb{F}}\right)$ *such that* $AB = BA = I_n$*, the identity matrix.*

Some basic operation properties on generalized quaternions used in the sequel are listed below (see, e.g., [6] and [9]).

Lemma 1.1. Let $a = a_0 + a_1 i + a_2 j + a_3 k \in \left(\frac{u,\,v}{\mathbb{F}}\right)$ and $b \in \left(\frac{u,\,v}{\mathbb{F}}\right)$ be given. Then

(a) $\overline{\overline{a}} = a, \quad \overline{a+b} = \overline{a} + \overline{b}, \quad \overline{ab} = \overline{b}\,\overline{a}.$

(b) $a\overline{a} = \overline{a}a$, *i.e.*, $n(a) = n(\overline{a})$.

(c) $n(ab) = n(a)n(b)$.

(d) $a^2 - 2a_0 a + n(a) = 0$.

(e) if $n(a) \neq 0$, then a is invertible and $a^{-1} = n^{-1}(a)\,\overline{a}$.

Lemma 1.2. Suppose that $\left(\frac{u,\,v}{\mathbb{F}}\right)$ is a generalized quaternion algebra over an arbitrary field $\mathbb{F}$ of characteristic not two. Then

(a) $\left(\frac{u,\,v}{\mathbb{F}}\right)$ is a division algebra $\Longleftrightarrow n(a) \neq 0$, for any $0 \neq a \in \mathbb{F}(u,\,v)$.

(b) $\left(\frac{u,\,v}{\mathbb{F}}\right)$ is not a division algebra $\Longleftrightarrow$ $\mathbb{F}(u,\,v) \simeq M_2(\mathbb{F})$. In particular,

$$\left(\frac{u^2,\,v}{\mathbb{F}}\right) \simeq M_2(\mathbb{F}), \quad \textit{and} \quad \left(\frac{u,\,v^2}{\mathbb{F}}\right) \simeq M_2(\mathbb{F}). \tag{1.2}$$

2 A universal similarity factorrization equality over $\left(\frac{u,v}{\mathbb{F}}\right)$

In this section we first establish a universal similarity factorization equality over $\left(\frac{u,v}{\mathbb{F}}\right)$ *and then present some of its consequences.*

Theorem 2.1. Let $a = a_0 + a_1 i + a_2 j + a_3 k \in \left(\frac{u,v}{\mathbb{F}}\right)$ be given, where a_0—$a_3 \in \mathbb{F}$. Then the diagonal matrix $\mathrm{diag}(\, a, \ a, \ a, \ a\,)$ satisfies the following universal similarity factorization equality

$$P \begin{bmatrix} a & & & \\ & a & & \\ & & a & \\ & & & a \end{bmatrix} P^{-1} = \begin{bmatrix} a_0 & ua_1 & va_2 & -uva_3 \\ a_1 & a_0 & va_3 & -va_2 \\ a_2 & -ua_3 & a_0 & ua_1 \\ a_3 & -a_2 & a_1 & a_0 \end{bmatrix} := \phi(a) \tag{2.1}$$

where P has the following form

$$P = P^{-1} = \frac{1}{2} \begin{bmatrix} 1 & i & j & k \\ u^{-1}i & 1 & -u^{-1}k & -j \\ v^{-1}j & v^{-1}k & 1 & i \\ -(uv)^{-1}k & -v^{-1}j & u^{-1}i & 1 \end{bmatrix}. \tag{2.2}$$

Proof. *It is easy to verify that*

$$\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_0 & x_1 \\ x_1 & x_0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} x_0 + x_1 & 0 \\ 0 & x_0 - x_1 \end{bmatrix} \tag{2.3}$$

hold for any $x_0, \ x_1 \in \left(\frac{u,v}{\mathbb{F}}\right)$. *On the basis of (2.3), we further obtain*

$$\begin{bmatrix} x_0 & x_1 & x_2 & x_3 \\ x_1 & x_0 & x_3 & x_2 \\ x_2 & x_3 & x_0 & x_1 \\ x_3 & x_2 & x_1 & x_0 \end{bmatrix} = V \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & d_3 & \\ & & & d_4 \end{bmatrix} V, \tag{2.4}$$

where $x_0, \, x_1, \, x_2, \, x_3 \in \left(\frac{u,v}{\mathbb{F}}\right)$ *and*

$$d_1 = x_0 + x_1 + x_2 + x_3, \qquad d_2 = x_0 + x_1 - x_2 - x_3, \tag{2.5}$$

$$d_3 = x_0 - x_1 + x_2 - x_3, \qquad d_4 = x_0 - x_1 - x_2 + x_3, \tag{2.6}$$

$$V = V^T = V^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}. \tag{2.7}$$

Replacing x_0—x_3 *in (2.4) by* a_0, $a_1 i$, $a_2 j$ *and* $a_3 k$, *respectively, we get*

$$A := \begin{bmatrix} a_0 & a_1 i & a_2 j & a_3 k \\ a_1 i & a_0 & a_3 k & a_2 j \\ a_2 j & a_3 k & a_0 & a_1 i \\ a_3 k & a_2 j & a_1 i & a_0 \end{bmatrix} = V \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & d_3 & \\ & & & d_4 \end{bmatrix} V, \tag{2.8}$$

where

$$d_1 = a_0 + a_1 i + a_2 j + a_3 k = a, \qquad d_2 = a_0 + a_1 i - a_2 j - a_3 k,$$

$$d_3 = a_0 - a_1 i + a_2 j - a_3 k, \qquad d_4 = a_0 - a_1 i - a_2 j + a_3 k.$$

It is easy to verify that $d_2 = iai^{-1}$, $d_3 = jaj^{-1}$ *and* $d_4 = kak^{-1}$. *Thus*

$$\mathrm{diag}(\, d_1,\ d_2,\ d_3,\ d_4\,) = J\mathrm{diag}(\, a,\ a,\ a,\ a\,)J^{-1}, \tag{2.9}$$

where $J = \mathrm{diag}(\,1,\ i,\ j,\ k\,)$. *On the other hand, it is easy to verify that*

$$J^{-1}AJ = \begin{bmatrix} a_0 & a_1 ii & a_2 jj & a_3 kk \\ a_1 i^{-1} i & a_0 i^{-1} i & a_3 i^{-1} kj & a_2 i^{-1} jk \\ a_2 j^{-1} j & a_3 j^{-1} ki & a_0 j^{-1} j & a_1 j^{-1} ik \\ a_3 k^{-1} k & a_2 k^{-1} ji & a_1 k^{-1} ij & a_0 k^{-1} k \end{bmatrix} = \begin{bmatrix} a_0 & ua_1 & va_2 & -uva_3 \\ a_1 & a_0 & va_3 & -va_2 \\ a_2 & -ua_3 & a_0 & ua_1 \\ a_3 & -a_2 & a_1 & a_0 \end{bmatrix} = \phi(a). \tag{2.10}$$

Putting (2.8) and (2.9) in (2.10) yield

$$\phi(a) = J^{-1}AJ = J^{-1}V\mathrm{diag}(\, d_1,\ d_2,\ d_3,\ d_4\,)VJ = J^{-1}VJ\mathrm{diag}(\, a,\ a,\ a,\ a\,)J^{-1}VJ.$$

Let $P = J^{-1}VJ$. *Then we have (2.1) and (2.2).* $\square$

Observe that the matrix P *in (2.1) is composed just by* u, v *and the basis of* $\left(\frac{u,\,v}{\mathbb{F}}\right)$, *which has no relation with expression of* a. *We can easily derive from (2.1) various operation properties on generalized quaternions and their matrix representations* $\phi(a)$ *over* $\mathbb{F}$.

Theorem 2.2. Let $a,\, b \in \left(\frac{u,\,v}{\mathbb{F}}\right)$ be given and $\lambda \in \mathbb{F}$. Then

(a) $a = b \Longleftrightarrow \phi(a) = \phi(b)$.

(b) $\phi(a + b) = \phi(a) + \phi(b)$, $\quad \phi(ab) = \phi(a)\phi(b)$, $\quad \phi(\lambda a) = \phi(a\lambda) = \lambda\phi(a)$, $\quad \phi(1) = I_4$.

The results in Theorem 2.2(a) and (b) together with (2.1) manifest that through the bijective map

$$\phi : a \in \left(\frac{u,\,v}{\mathbb{F}}\right) \ \longrightarrow\ \phi(a) \in M_4(\mathbb{F}),$$

the generalized quaternion algebra $\left(\frac{u,\,v}{\mathbb{F}}\right)$ *is algebraically isomorphic to the matrix algebra constructed by all matrices with the form as* $\phi(a)$. *Another remarkable fact implied by (2.1) is that any* $a \in \left(\frac{u,\,v}{\mathbb{F}}\right)$ *is in fact an eigenvalue of its representation matrix* $\phi(a)$.

Theorem 2.3. Let $a \in \left(\frac{u,\,v}{\mathbb{F}}\right)$ be given. Then $\phi(\overline{a})$ can be expressed as $\phi(\overline{a}) = K^{-1}\phi^T(a)K$, where $K = \mathrm{diag}(\,1,\ -u,\ -v,\ uv\,)$.

Proof. *Follows from a direct verification.* $\square$

Theorem 2.4. Let $a = a_0 + a_1 i + a_2 j + a_3 k \in \left(\frac{u,\,v}{\mathbb{F}}\right)$ be given. Then the determinant of matrix representation $\phi(a)$ of a is $|\phi(a)| = n^2(a) = (\, a_0^2 - ua_1^2 - va_2^2 + uva_3^2\,)^2$.

Proof. *From* $\phi(a\overline{a}) = \phi(a)\phi(\overline{a})$ *and* $n(a) = a\overline{a}$, *we obtain* $|\phi(a)|\,|\phi(\overline{a})| = |\phi(a\overline{a})| = |n(a)I_4| = n^4(a)$. *On the other hand, by Theorem 2.3, it follows that* $|\phi(\overline{a})| = |K^{-1}|\,|\phi^T(a)|\,|K| = |\phi(a)|$. *Hence* $|\phi(a)|^2 = n^4(a)$. *Note that the term* a_0^4 *occurs in* $|\phi(a)|$. *Thus* $|\phi(a)| = n^2(a)$. $\square$

Theorem 2.5. Let $a \in \left(\frac{u,\,v}{\mathbb{F}}\right)$ be given. Then

$$a = \frac{1}{4}[\,1,\ i,\ j,\ k\,]\phi(a)[\,1,\ i^{-1},\ j^{-1},\ k^{-1}\,]^T = \frac{1}{4}E_4\phi(a)F_4^T, \tag{2.11}$$

where $E_4 = [\,1,\ i,\ j,\ k\,]$ and $F_4 = [\,1,\ i^{-1},\ j^{-1},\ k^{-1}\,]$.

Proof. *The equality (2.1) can equivalently be written as* $\mathrm{diag}(\,a,\ a,\ a,\ a\,) = P\phi(a)P^{-1}$. *Hence a can be expressed as* $a = [\,1,\ 0,\ 0,\ 0\,]P\phi(a)P^{-1}[\,1,\ 0,\ 0,\ 0\,]^T$. *Written in an explicit form, it is (2.11).* $\square$

Theorem 2.6. Let $a \in \left(\frac{u,\,v}{\mathbb{F}}\right)$ be given. Then a is invertible if and only if $\phi(a)$ is invertible, in which case,

$$\phi(a^{-1}) = \phi^{-1}(a) \quad \textit{and} \quad a^{-1} = \frac{1}{4}E_4\phi^{-1}(a)F_4^T, \tag{2.12}$$

where E_4 and F_4 are as in *(2.11)*.

Proof. *Follows from (2.1) and (2.11).* $\square$

Another matrix representation of $a \in \left(\frac{u,\,v}{\mathbb{F}}\right)$ *can be introduced as follows*

$$\tau(a) := G^{-1}\phi^T(a)G = \begin{bmatrix} a_0 & ua_1 & va_2 & -uva_3 \\ a_1 & a_0 & -va_3 & va_2 \\ a_2 & ua_3 & a_0 & -ua_1 \\ a_3 & a_2 & -a_1 & a_0 \end{bmatrix}, \tag{2.13}$$

where $G = \mathrm{diag}(\,1,\ u,\ v,\ -uv\,)$. *This representation together with (2.1) will play key roles for solving various linear equations over* $\left(\frac{u,\,v}{\mathbb{F}}\right)$.

Combining them with Theorem 2.3, we have the following.

Theorem 2.7. Let $a \in \left(\frac{u,\,v}{\mathbb{F}}\right)$ be given. Then

$$\tau(a) = L\phi(\overline{a})L \quad \textit{and} \quad \phi(a) = L\tau(\overline{a})L,$$

where $L = \mathrm{diag}(\,1,\ -1,\ -1,\ -1\,)$.

Proof. *Follows from a direct verification.* $\square$

Theorem 2.8. Let $a,\ b \in \left(\frac{u,\,v}{\mathbb{F}}\right)$ be given and $\lambda \in \mathbb{F}$. Then

(a) $a = b \Longleftrightarrow \tau(a) = \tau(b)$.

(b) $\tau(a+b) = \tau(a) + \tau(b)$, $\quad \tau(ab) = \tau(b)\tau(a)$, $\quad \tau(\lambda a) = \tau(a\lambda) = \lambda\tau(a)$.

(c) If $a = a_0 + a_1i + a_2j + a_3k$, then $|\tau(a)| = n^2(a) = (\,a_0^2 - ua_1^2 - va_2^2 + uva_3^2\,)^2$.

(d) a is invertible if and only if $\tau(a)$ is invertible, in which case, $\tau(a^{-1}) = \tau^{-1}(a)$.

The results in Theorem 2.8(a) and (b) clearly show that through the bijective map

$$\tau : a \in \left(\frac{u,\, v}{\mathbb{F}}\right) \longrightarrow \tau(a) \in M_4(\mathbb{F}),$$

the generalized quaternion algebra $\left(\frac{u,v}{\mathbb{F}}\right)$ is algebraically anti-isomorphic to the matrix algebra constructed by all matrices with the same form as $\tau(a)$ in (2.13).

Combining two matrix representations with adjoint vectors of generalized quaternions, we can derive the following basic results.

Theorem 2.9. Let $a,\, b,\, x \in \left(\frac{u,v}{\mathbb{F}}\right)$ be given. Then

$$\overrightarrow{ax} = \phi(a)\overrightarrow{x}, \qquad \overrightarrow{xb} = \tau(b)\overrightarrow{x}, \qquad \overrightarrow{axb} = \phi(a)\tau(b)\overrightarrow{x} = \tau(b)\phi(a)\overrightarrow{x}. \tag{2.14}$$

and

$$\phi(a)\tau(b) = \tau(b)\phi(a). \tag{2.15}$$

Proof. *It is easily seen from the structure of $\phi(a)$ and $\tau(a)$ that $\overrightarrow{a} = \phi(a)\alpha_4^T$ and $\overrightarrow{a} = \tau(a)\alpha_4^T$, where $\alpha_4 = [\,1,\ 0,\ 0,\ 0\,]$. From both of them and Theorems 2.2(b) and 2.8(b), we easily find*

$$\overrightarrow{ax} = \phi(ax)\alpha_4^T = \phi(a)\phi(x)\alpha_4^T = \phi(a)\overrightarrow{x}, \qquad \overrightarrow{xb} = \tau(xb)\alpha_4^T = \tau(b)\tau(x)\alpha_4^T = \tau(b)\overrightarrow{x},$$

$$\overrightarrow{axb} = \overrightarrow{a(xb)} = \phi(a)\overrightarrow{xb} = \phi(a)\tau(b)\overrightarrow{x}, \qquad \overrightarrow{axb} = \overrightarrow{(ax)b} = \tau(b)\overrightarrow{ax} = \tau(b)\phi(a)\overrightarrow{x},$$

as required for (2.14). Observe that $\overrightarrow{x}$ is arbitrarily given. Thus we get (2.15) from the third equality in (2.14). $\square$

In term of the three formulas in (2.14), all two-sided linear equations over $\mathbb{F}(u,\, v)$ can equivalently be transformed into ordinary linear equation systems over $\mathbb{F}$, such as

$$axb = c \Longleftrightarrow \phi(a)\tau(b)\overrightarrow{x} = \overrightarrow{c},$$

$$ax - xb = c \Longleftrightarrow [\,\phi(a) - \tau(b)\,]\overrightarrow{x} = \overrightarrow{c},$$

$$axb + cxd = e \Longleftrightarrow [\,\phi(a)\tau(b) + \phi(c)\tau(d)\,]\overrightarrow{x} = \overrightarrow{e}.$$

3 Some particular universal similarity factorization equalities over $\left(\frac{u,v}{\mathbb{F}}\right)$

As pointed out in Lemma 1.2(b), if $\left(\frac{u,v}{\mathbb{F}}\right)$ is not a division algebra, then it is algebraically isomorphic to the matrix algebra $M_2(\mathbb{F})$. Motivated by this result and the equality in (2.1), a question can naturally be asked: Does there exist an universal similarity factorization equality between elements in $\left(\frac{u,v}{\mathbb{F}}\right)$ and matrices over $M_2(\mathbb{F})$? For some particular choices of u and v in $\left(\frac{u,v}{\mathbb{F}}\right)$, or some *specified fields $\mathbb{F}$, the answer to this question is positive. To do so, we first present a simple but vital result.*

Lemma 3.1. Let $\mathbb{F}$ be an arbitrary field, and the $\mathcal{A}$ is a four dimensional algebra over $\mathbb{F}$ with the four bases e_{11}, e_{12}, e_{21} and e_{22} in it satisfying the following multiplication rules

$$e_{st}e_{pq} = \begin{cases} e_{sq}, & t = p, \\ 0, & t \neq p, \end{cases} \qquad s,\, t,\, p,\, q = 1,\, 2. \tag{3.1}$$

Then for any $a = a_{11}e_{11} + a_{12}e_{12} + a_{21}e_{21} + a_{22}e_{22} \in \mathcal{A}$, where $a_{st} \in \mathbb{F}$, the corresponding diagonal matrix $\mathrm{diag}(\, a,\ a\,)$ satisfies the following universal similarity factorization equality

$$Q \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} Q^{-1} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \tag{3.2}$$

where Q has the independent form

$$Q = Q^{-1} = \begin{bmatrix} e_{11} & e_{21} \\ e_{12} & e_{22} \end{bmatrix}. \tag{3.3}$$

Proof. *According to (3.1), it is easy to verify that the unity element in $\mathcal{A}$ is $e = e_{11} + e_{22}$. In that case, the matrix Q in (3.3) satisfies*

$$Q^2 = \begin{bmatrix} e_{11} & e_{21} \\ e_{12} & e_{22} \end{bmatrix} = \begin{bmatrix} e_{11}^2 + e_{21}e_{12} & e_{11}e_{21} + e_{21}e_{22} \\ e_{12}e_{11} + e_{22}e_{12} & e_{12}e_{21} + e_{22}^2 \end{bmatrix} = \begin{bmatrix} e_{11} + e_{22} & 0 \\ 0 & e_{11} + e_{22} \end{bmatrix} = eI_2,$$

which implies that Q is invertible over $\mathcal{A}$ and $Q = Q^{-1}$. As to (3.2), it comes from a direct calculation of the matrix product in the left-hand side of (3.2). $\square$

We can easily establish from (3.2) some universal similarity factorization equalities between elements in $\left(\frac{u,\, v}{\mathbb{F}}\right)$ and matrices in $M_2(\mathbb{F})$, when u and v in $\left(\frac{u,\, v}{\mathbb{F}}\right)$ are chosen some particular values, or the field $\mathbb{F}$ has some particular structure.

Theorem 3.2. Let $a = a_0 + a_1 i + a_2 j + a_3 k \in \left(\frac{u^2,\, v}{\mathbb{F}}\right)$ be given, where u and v are two nonzero elements of $\mathbb{F}$, a_0—$a_3 \in \mathbb{F}$, and

$$i^2 = u^2, \qquad j^2 = v, \qquad k = ij = -ji. \tag{3.4}$$

Then a satisfies the following universal similarity factorization equality

$$Q_1 \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} Q_1^{-1} = \begin{bmatrix} a_0 + ua_1 & v(\, a_2 + ua_3\,) \\ a_2 - ua_3 & a_0 - ua_1 \end{bmatrix} := \psi_1(a), \tag{3.5}$$

where Q_1 has the universal form

$$Q_1 = Q_1^{-1} = \frac{1}{2} \begin{bmatrix} 1 + u^{-1}i & j - u^{-1}k \\ v^{-1}j + (uv)^{-1}k & 1 - u^{-1}i \end{bmatrix}. \tag{3.6}$$

Proof. *According to (3.4), we choose a group of new basis for* $\left(\frac{u^2,\, v}{\mathbb{F}}\right)$ *as follows*

$$e_{11} = \frac{1}{2}\left(1 + \frac{1}{u}i\right), \quad e_{12} = \frac{1}{2}\left(\frac{1}{v}j + \frac{1}{uv}k\right), \quad e_{21} = \frac{1}{2}\left(j - \frac{1}{u}k\right), \quad e_{22} = \frac{1}{2}\left(1 - \frac{1}{u}i\right). \tag{3.7}$$

Then it is not difficult to verify that the basis (3.7) satisfies the multiplication rules in (3.1). Under this group of new basis, any element a of $\left(\frac{u^2,\, v}{\mathbb{F}}\right)$ *can be expressed as*

$$a = (\, a_0 + ua_1\,)e_{11} + (\, va_2 + uva_3\,)e_{12} + (\, a_2 - ua_3\,)e_{21} + (\, a_0 - ua_1\,)e_{22}. \tag{3.8}$$

Substituting (3.7) and (3.8) into (3.2), we obtain (3.5). $\square$

We easily find from (3.5) the following result.

Theorem 3.3. Let $a,\ b \in \left(\frac{u^2,\, v}{\mathbb{F}}\right)$, and $\lambda \in \mathbb{F}$. Then

(a) $a = b \Longleftrightarrow \psi_1(a) = \psi_1(b)$.

(b) $\psi_1(a + b) = \psi_1(a) + \psi_1(b)$, $\psi_1(ab) = \psi_1(a)\psi_1(b)$, $\psi_1(\lambda a) = \psi_1(a\lambda) = \lambda\psi_1(a)$, $\psi_1(1) = I_2$.

(c) $|\psi_1(a)| = n(a)$.

(d) $\psi_1(\overline{a}) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \psi_1^T(a) \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

(e) $a = \frac{1}{4}[\, 1 + u^{-1}i,\ j - u^{-1}k\,]\psi_1(a) \begin{bmatrix} 1 + u^{-1}i \\ v^{-1}j + (uv)^{-1}k \end{bmatrix}$.

The equality (3.5) and the results in Theorem 3.3(a) and (b) manifest that through the bijective map

$$\psi_1 : a \in \left(\frac{u^2,\, v}{\mathbb{F}}\right) \ \longrightarrow\ \psi_1(a) \in M_2(\mathbb{F}),$$

the generalized quaternion algebra $\left(\frac{u^2,\, v}{\mathbb{F}}\right)$ *is algebraically isomorphic to the 2-by-2 total matrix algebra* $M_2(\mathbb{F})$. *Another important fact implied by (3.5) is that any 2-by-2 matrix over the field* $\mathbb{F}$ *can be diagonalized over a generalized quaternion algebra* $\left(\frac{u^2,\, v}{\mathbb{F}}\right)$, *where u and v are any two nonzero elements in* $\mathbb{F}$.

An analogous result is given below.

Theorem 3.4. Let $a = a_0 + a_1i + a_2j + a_3k \in \left(\frac{u,\, v^2}{\mathbb{F}}\right)$ be given, where u and v are two nonzero elements of $\mathbb{F}$, a_0— $a_3 \in \mathbb{F}$, and

$$i^2 = u, \qquad j^2 = v^2, \qquad k = ij = -ji. \tag{3.9}$$

Then a satisfies the following universal similarity factorization equality

$$Q_2 \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} Q_2^{-1} = \begin{bmatrix} a_0 + va_2 & u(\, a_1 - va_3\,) \\ a_1 + va_3 & a_0 - va_2 \end{bmatrix} := \psi_2(a), \tag{3.10}$$

where Q_2 has the universal form

$$Q_2 = Q_2^{-1} = \frac{1}{2}\begin{bmatrix} 1+v^{-1}j & i+v^{-1}k \\ u^{-1}i-(uv)^{-1}k & 1-v^{-1}j \end{bmatrix}. \tag{3.11}$$

Proof. *Setting a group of new basis for* $\left(\frac{u,\, v^2}{\mathbb{F}}\right)$ *as follows*

$$e_{11} = \frac{1}{2}\left(1+\frac{1}{v}j\right), \ e_{12} = \frac{1}{2}\left(\frac{1}{u}i-\frac{1}{uv}k\right), \ e_{21} = \frac{1}{2}\left(i+\frac{1}{v}k\right), \ e_{22} = \frac{1}{2}\left(1-\frac{1}{v}j\right). \tag{3.12}$$

Then it is not difficult to verify that the basis (3.12) satisfies the multiplication laws in (3.1). Under this group of new basis, any element a of $\left(\frac{u,\, v^2}{\mathbb{F}}\right)$ *can be expressed as*

$$a = (\,a_0+va_2\,)e_{11} + (\,ua_1-uva_3\,)e_{12} + (\,a_1+va_3\,)e_{21} + (\,a_0-va_2\,)e_{22}. \tag{3.13}$$

Substituting (3.12) and (3.13) into (3.2), we obtain (3.10). □

The operation properties on $\psi_2(a)$ in (3.10) are much like to those in Theorem 3.3, hence we omit them here.

Some direct consequences of the above several theorems are given as follows.

Theorem 3.5. Let the field $\mathbb{F}$ be algebraically closed, and $\left(\frac{u,v}{\mathbb{F}}\right)$ be a generalized quaternion algebra over $\mathbb{F}$ with u and v nonzero. Then any $a = a_0 + a_1i + a_2j + a_3k \in \left(\frac{u,v}{\mathbb{F}}\right)$ satisfies the following two universal similarity factorization equalities

$$Q_1\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}Q_1^{-1} = \begin{bmatrix} a_0+u_0a_1 & v(\,a_2+u_0a_3\,) \\ a_2-u_0a_3 & a_0-u_0a_1 \end{bmatrix}, \tag{3.14}$$

$$Q_2\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}Q_2^{-1} = \begin{bmatrix} a_0+v_0a_2 & u(\,a_1-v_0a_3\,) \\ a_1+v_0a_3 & a_0-v_0a_2 \end{bmatrix}, \tag{3.15}$$

where u_0, v_0 and w are the elements in $\mathbb{F}$ satisfying $u_0^2 = u$, $v_0^2 = v$ and $w^2 = -uv$, and Q_1 and Q_2 have the independent forms

$$Q_1 = Q_1^{-1} = \frac{1}{2}\begin{bmatrix} 1+u_0^{-1}i & j-u_0^{-1}k \\ v^{-1}j+(u_0v)^{-1}k & 1-u_0^{-1}i \end{bmatrix},$$

$$Q_2 = Q_2^{-1} = \frac{1}{2}\begin{bmatrix} 1+v_0^{-1}j & i+v_0^{-1}k \\ u^{-1}i-(uv_0)^{-1}k & 1-v_0^{-1}j \end{bmatrix}.$$

4 Universal factorization equalities for matrices over $\left(\frac{u,v}{\mathbb{F}}\right)$

Observe that the matrix P in (2.1) has a universal form. Hence we can easily extend (2.1) to any $m \times n$ matrix over $\left(\frac{u,v}{\mathbb{F}}\right)$.

Theorem 4.1. Let $A = A_0 + A_1 i + A_2 j + A_3 k$ be an $m \times n$ matrix over $\left(\frac{u,\,v}{\mathbb{F}}\right)$, where A_0—A_3 are four $m \times n$ matrices over $\mathbb{F}$. Then the diagonal block matrix $\mathrm{diag}(\, A,\ A,\ A,\ A\,)$ satisfies the following universal factorization equality

$$J_{4m} \begin{bmatrix} A & & & \\ & A & & \\ & & A & \\ & & & A \end{bmatrix} J_{4n} = \begin{bmatrix} A_0 & uA_1 & vA_2 & -uvA_3 \\ A_1 & A_0 & vA_3 & -vA_2 \\ A_2 & -uA_3 & A_0 & uA_1 \\ A_3 & -A_2 & A_1 & A_0 \end{bmatrix} := \Phi(A), \quad (4.1)$$

where

$$J_{4t} = J_{4t}^{-1} = \frac{1}{2} \begin{bmatrix} I_t & iI_t & jI_t & kI_t \\ u^{-1}I_t & I_t & -u^{-1}kI_t & -jI_t \\ v^{-1}jI_t & v^{-1}kI_t & I_t & iI_t \\ -(uv)^{-1}kI_t & -v^{-1}jI_t & u^{-1}iI_t & I_t \end{bmatrix}, \qquad t = m,\ n.$$

In particular, when $m = n$, (4.1) becomes a universal similarity factorization equality over $\left(\frac{u,\,v}{\mathbb{F}}\right)$.

The proof of (4.1) is much like that of (2.1), hence omitted. We call the $4m \times 4n$ *matrix* $\Phi(A)$ *in (4.1) the* central representation matrix *of* A.

We can also derive from the equality (4.1) some operation properties on generalized quaternion matrices and their central representation matrices.

Theorem 4.2. Let $A,\ B \in \left(\frac{u,\,v}{\mathbb{F}}\right)^{m\times n}$, $C \in \left(\frac{u,\,v}{\mathbb{F}}\right)^{n\times p}$ be given, and $\lambda \in \mathbb{F}$. Then

(a) $A = B \Longleftrightarrow \Phi(A) = \Phi(B)$.

(b) $\Phi(A+B) = \Phi(A)+\Phi(B)$, $\Phi(AC) = \Phi(A)\Phi(C)$, $\Phi(\lambda A) = \lambda\Phi(A)$, $\Phi(I_m) = I_{4m}$.

(c) Cayley-Hamilton theorem: $p(A) = 0$, where A is square and $p(A) = |\,\lambda I - \Phi(A)\,|$, called the characteristic polynomial of A.

Theorem 4.3. Let $A = A_0 + A_1 i + A_2 j + A_3 k \in \left(\frac{u,\,v}{\mathbb{F}}\right)^{m\times n}$. Then $\Phi(A^*)$ can be expressed as $\Phi(A^*) = K_{4n}^{-1}\,\Phi^T(A)\,K_{4m}$, where $K_{4t} = \mathrm{diag}(I_{4t},\ -uI_{4t},\ -vI_{4t},\ uvI_{4t})$, $t = m,\ n$.

Proof. *According to (4.1),* $\Phi(A^*)$ *and* $\Phi^T(A)$ *has the following forms*

$$\Phi(A^*) = \begin{bmatrix} A_0^T & -uA_1^T & -vA_2^T & uvA_3^T \\ -A_1^T & A_0^T & -vA_3^T & vA_2^T \\ -A_2^T & uA_3^T & A_0^T & -uA_1^T \\ -A_3^T & A_2^T & -A_1^T & A_0^T \end{bmatrix},$$

and

$$\Phi^T(A) = \begin{bmatrix} A_0^T & A_1^T & A_2^T & A_3^T \\ uA_1^T & A_0^T & -uA_3^T & -A_2^T \\ vA_2^T & vA_3^T & A_0^T & A_1^T \\ -uvA_3^T & -vA_2^T & uA_1^T & A_0^T \end{bmatrix}.$$

Hence we have the desired result. $\square$

Theorem 4.4. Let $A \in \left(\frac{u,\,v}{\mathbb{F}}\right)^{m\times n}$ be given. Then A and its central representation matrix $\Phi(A)$ satisfies the following equality

$$A = \frac{1}{4}[\,I,\ iI_m,\ jI_m,\ kI_m\,]\Phi(A)[\,I_n,\ i^{-1}I_n,\ j^{-1}I_n,\ k^{-1}I_n\,]^T = \frac{1}{4}E_{4m}\Phi(A)F_{4n}^T, \tag{4.2}$$

where $E_{4m} = [\,I_m,\ iI_m,\ jI_m,\ kI_m\,]$ and $F_{4n} = [\,I_n,\ i^{-1}I_n,\ j^{-1}I_n,\ k^{-1}I_n\,]$.

Proof. *The equality (4.1) can also be written as* $\mathrm{diag}(\,A,\ A,\ A,\ A\,) = J_{4m}\phi(A)J_{4n}^{-1}$. *Hence A can be expressed as* $A = [\,I_m,\ 0,\ 0,\ 0\,]J_{4m}\Phi(A)J_{4n}^{-1}[\,I_n,\ 0,\ 0,\ 0\,]^T$. *Written in an explicit form, it is (4.2).* □

Theorem 4.5. Let $A \in \left(\frac{u,\,v}{\mathbb{F}}\right)^{m\times m}$ be given. Then A is invertible if and only if its central representation matrix $\Phi(A)$ is invertible, in which case,

$$\Phi(A^{-1}) = \Phi^{-1}(A), \qquad A^{-1} = \frac{1}{4}E_{4m}\Phi^{-1}(A)F_{4m}^T, \tag{4.3}$$

where E_{4m} and F_{4m} are as in (4.2).

Proof. *Follows directly from (4.1) and (4.2).* □

In addition to the universal factorization equality in (4.1), we can also extend all the universal similarity factorization equalities in Section 3 to $m\times n$ matrices over $\left(\frac{u,\,v}{\mathbb{F}}\right)$ and derive a variety of consequences analogous to those in Theorem 3.2—3.5. Next we only list two of them.

Theorem 4.6. Let $A = A_0 + A_1i + A_2j + A_3k \in \left(\frac{u^2,\,v}{\mathbb{F}}\right)^{m\times n}$ be given, where u and v are two nonzero elements of $\mathbb{F}$, A_0—$A_3 \in \mathbb{F}^{m\times n}$, and $i^2 = u^2,\ j^2 = v,\ k = ij = -ji$. Then A satisfies the following universal factorization equality

$$G_{2m}\begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}G_{2n} = \begin{bmatrix} A_0 + uA_1 & v(\,A_2 + uA_3\,) \\ A_2 - uA_3 & A_0 - uA_1 \end{bmatrix}, \tag{4.4}$$

where G_{2t} has the independent form

$$G_{2t} = G_{2t}^{-1} = \frac{1}{2}\begin{bmatrix} (\,1 + u^{-1}i\,)I_t & (\,j - u^{-1}k\,)I_t \\ [\,v^{-1}j + (uv)^{-1}k\,]\,I_t & (\,1 - u^{-1}i\,)I_t \end{bmatrix}, \qquad t = m,\ n. \tag{4.5}$$

In particular, when $m = n$, (4.4) becomes a universal similarity factorization equality over $\left(\frac{u^2,\,v}{\mathbb{F}}\right)$.

Theorem 4.7. Let $A = A_0 + A_1i + A_2j + A_3k \in \left(\frac{u,\,v^2}{\mathbb{F}}\right)^{m\times n}$ be given, where u and v are two nonzero elements of $\mathbb{F}$, A_0—$A_3 \in \mathbb{F}^{m\times n}$, and $i^2 = u,\ j^2 = v^2,\ k = ij = -ji$. Then A satisfies the following universal factorization equality

$$L_{2m}\begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}L_{2n} = \begin{bmatrix} A_0 + vA_2 & u(\,A_1 - vA_3\,) \\ A_1 + vA_3 & A_0 - vA_2 \end{bmatrix}, \tag{4.6}$$

where L_{2t} has the independent form

$$L_{2t} = L_{2t}^{-1} = \frac{1}{2}\left[\begin{array}{cc} (\,1 + v^{-1}j\,)I_t & (\,i + v^{-1}k\,)I_t \\ \left[\,u^{-1}i - (uv)^{-1}k\,\right] I_t & (\,1 - v^{-1}j\,)I_t \end{array}\right], \qquad t = m,\ n. \tag{4.7}$$

In particular, when $m = n$, Eq.(4.6) becomes a universal similarity factorization equality over $\left(\frac{u,\, v^2}{\mathbb{F}}\right)$.

The results given in the preceding sections are all applicable to matrices over the real quaternion algebra $\mathbb{H}$, the real split quaternion algebra $\mathbb{S}$, and the complex quaternion algebra $\mathbb{Q}$. Using those results, one can easily develop a perfect theory for matrices analysis over these algebras.

References

[1] *J. L. Brenner, Matrices of quaternions, Pacific J. Math. 1(1951), 329-335.*

[2] *P. M. Cohn, The range of the derivation and the equation $ax - xb = c$, J. Indian Math. Soc.37(1973), 1–9.*

[3] *P. M. Cohn, Skew Field Constructions, Cambridge U. P., London, 1977.*

[4] *B. F. Farb, and R. K. Dennis, Noncommutative Algebra, Springer-Verlag, New York, 1993.*

[5] *L. Huang, The matrix equation $AXB - CXD = E$ over the quaternion field, Linear Algebra Appl. 234(1996), 197–208.*

[6] *T. Y. Lam, The Algebraic Theory of Quadratic Forms, W. A. Benjamin, Reading, Mass., 1973.*

[7] *M. L. Mehta, Matrix Theory, Selected Topics and Useful Results, Hindustan P. Co., India, 1989.*

[8] *P. R. Michael, Quaternionic linear and quadratic equations,* J. Natur. Geom. *11(1997), 101–106.*

[9] *R. S. Pierce, Associative Algebras, Springer-Verlag, New York, 1982.*

[10] *L. A. Wolf, Similarity of matrices in which the elements are real quaternions, Bull. Amer. Math. Soc. 42:737–743(1936).*

[11] *N. A. Wiegmann, Some theorems on matrices with real quaternion elements, Canad. J. Math. 7:191–201(1955).*

[12] *F. Zhang, Quaternions and matrices of quaternions, Linear Algebra Appl. 251(1997), 21–57.*

Electronic Journal: Southwest Journal of Pure and Applied Mathematics
Internet: http://rattler.cameron.edu/swjpam.html
ISBN 1083-0464
Issue 1 June 2001, pp. 66 – 74
Submitted: March 22, 2001. Published: June 30, 2001

Submanifolds of F-Structure Manifold Satisfying $F^K + (-)^{K+1}F = 0$

LoveJoy S. Das

Abstract

Invariant submanifolds have been studied by Blair, Ludden, Yano [1], Kubo [2], Yano and Okumura [3], [4] and among others. Yano and Ishihara [5] have studied and shown that any invariant submanifold of codimension 2 in a contact Riemannian manifold is also a contact Riemannian manifold. The purpose of this paper is to study invariant submanifolds of an n-dimensional manifold M endowed with a F-structure satisfying $F^K+(-)^{K+1}F = 0$ and $F^W+(-)^{W+1}F \neq 0$ for $1 < W < K$, where K is a fixed positive integer greater than 2. The case when K is odd (≥ 3) has been considered in this paper. In the first part of the paper we have shown that an invariant submanifold $\widetilde{M}$ embedded in an F-structure manifold M in such a way that the complementary distribution D_m is never tangential to the invariant submanifold $\Psi(\tilde{M})$, is an almost complex manifold with the induced $\widetilde{F}$-structure. In the later part of the paper some theorems regarding the integrability conditions of induced $\widetilde{F}$-structure have been proved.

A.M.S. Subject Classification Codes: 53C15, 53C60, 53B40
Key Words and Phrases: Invariant submanifold, integrability, F-structure

1. F-STRUCTURE

Let F be a nonzero tensor field of the type (1,1) and of class C^∞ on an n-dimensional manifold M such that [6]

$$F^K+(-)^{K+1}F = 0 \text{ and } F^W+(-)^{W+1}F \neq 0 \text{ for } 1 < W < K \quad (1.1)$$

Department of Mathematics and Computer Science, Kent State University, USA

where K is a fixed positive integer greater than 2. Such a structure on M is called an F-structure of rank r and of degree K. If the rank of F is constant and $r = r(F)$, then M is called an F-structure manifold of degree $K(\geq 3)$.

Let the operator on M be defined as follows [6]

$$\ell = (-)^K F^{K-1} \quad \text{and} \quad m = I + (-)^{K+1} F^{K-1} \tag{1.2}$$

where I denotes the identity operator on M. For the operators defined by (1.2) we have

$$\ell + m = I, \quad \ell^2 = \ell; \quad \text{and} \quad m^2 = m. \tag{1.3}$$

For F-satisfying (1.1), there exist complementary distribution D_ℓ and D_m corresponding to the projection operators ℓ and m respectively. If rank (F) = constant on M then dim $D_\ell = r$ and dim $D_m = (n - r)$. We have the following results [6].

$$F\ell = \ell F = F \quad \text{and} \quad Fm = mF = 0 \tag{1.4a}$$

$$F^{K-1} = (-)^K \ell, \quad F^{K-1}\ell = -\ell \quad F^{K-1}m = 0. \tag{1.4b}$$

Thus F^{K-1} acts on D_ℓ as an almost complex structure and on D_m as a null operator.

2. INVARIANT SUBMANIFOLDS OF F-STRUCTURE MANIFOLD

Let $\widetilde{M}$ be a differentiable manifold embedded differentially as a submanifold in an n-dimensional C^∞ Riemannian manifold M with an F-structure and let us denote its embedding by $\Psi : \widetilde{M} \to M$. Denote by $B : T(\widetilde{M}) \longrightarrow T(M)$ the differential mapping of Ψ, where $d\Psi = B$ is the Jacobson map of Ψ. $T(\widetilde{M})$ and $T(M)$ are tangent bundles of $\widetilde{M}$ and M respectively. Let us call $T(\widetilde{M}, M)$ as the set of all vectors tangent to the submanifold $\Psi(\widetilde{M})$. It is known that $B : T(\widetilde{M}) \longrightarrow T(\widetilde{M}, M)$ is an isomorphism [7].

Let $\widetilde{X}$ and $\widetilde{Y}$ be two C^∞ vector fields defined along $\Psi(\widetilde{M})$ and tangent to $\Psi(\widetilde{M})$. Let X and Y be the local extensions of $\widetilde{X}$ and $\widetilde{Y}$. The restriction of $[X, Y]_{\widetilde{M}}$ is determined independently of the choice of these local extensions X and Y. Therefore we can define

$$[\widetilde{X}, \widetilde{Y}] = [X, Y]_{\widetilde{M}}.$$

Since B is an isomorphism it is easy to see that $[B\widetilde{X}, B\widetilde{Y}] = B[\widetilde{X}, \widetilde{Y}]$ for all $\widetilde{X}, \widetilde{Y} \in T(\widetilde{M})$.

We denote by G, the Riemannian metric tensor of M and put

$$\tilde{g}(\widetilde{X}, \widetilde{Y}) = g(B\widetilde{X}, B\widetilde{Y}) \tag{2.1}$$

for all $\widetilde{X}, \widetilde{Y}$ in $T(\widetilde{M})$, where g is the Riemannian metric in M and $\tilde{g}$ is the induced metric of $\widetilde{M}$.

Definition(2.1): We say that $\widetilde{M}$ is an invariant submanifold of M if

(i) the tangent space $T_p(\Psi(\widetilde{M}))$ of the submanifold $\Psi(\widetilde{M})$ is invariant by the linear mapping F at each point p of $\Psi(\widetilde{M})$.

(ii) For each $\widetilde{X} \in T(\widetilde{M})$, we have

$$F^{\frac{K-1}{2}}(B\widetilde{X}) = B\widetilde{X}'.$$

Definition (2.2): Let $\widetilde{F}$ be a (1,1) tensor field defined in $\widetilde{M}$ such that $\widetilde{F}(\widetilde{X}) = \widetilde{X}'$ and M is an invariant submanifold then we have

$$F(B\widetilde{X}) = B(\widetilde{F}\widetilde{X}) \tag{2.2a}$$

and

$$F^{\frac{K-1}{2}}(B\widetilde{X}) = B(\widetilde{F}^{\frac{K-1}{2}}\widetilde{X}). \tag{2.2b}$$

We see that there are two cases for any invariant submanifold $\widetilde{M}$. We assume the following: Case I: The distribution D_m is never tangential to $\Psi(\widetilde{M})$.

Case II: The distribution D_m is always tangential to $\Psi(\widetilde{M})$.

We will consider Case I and assume that no vector field of the type mX where $X \in T(\Psi(\widetilde{M}))$ is tangential to $\Psi(\widetilde{M})$.

Theorem (2.1): *An invariant submanifold $\widetilde{M}$ is an almost complex manifold if the following two conditions are satisfied:*

(i) *the distribution D_m is never tangential to $\Psi(\widetilde{M})$ and*

(ii) $\widetilde{F}$ in $\widetilde{M}$ *defines an induced almost complex structure satisfying* $\widetilde{F}^{K-1} = (-)^K I$.

Proof: Applying $F^{\frac{K-1}{2}}$ in (2.2) we obtain

$$F^{\frac{K-1}{2}}, \quad \left(F^{\frac{K-1}{2}}(B\widetilde{X})\right) = F^{\frac{K-1}{2}}, \left(B(\tilde{F}^{\frac{K-1}{2}}, \widetilde{X})\right) \tag{2.3}$$

Making use of equation (2.2)a in (2.3) we get

$$F^{K-1}(B\widetilde{X}) = B(\tilde{F}^{K-1}\widetilde{X}) \tag{2.4}$$

In order to show that vector fields of the type $B\widetilde{X}$ belong to the distribution D_ℓ we suppose that $m(B\widetilde{X}) \neq 0$, then using (1.2) we have

$$\begin{aligned} m(B\widetilde{X}) &= (I + (-)^{K+1}F^{K-1})B\widetilde{X} \\ &= B\widetilde{X} + (-)^{K+1}F^{K-1}(B\widetilde{X}), \end{aligned}$$

which in view of (2.4) becomes

$$\begin{aligned} &= B\widetilde{X} + (-)^{K+1}B(\tilde{F}^{K-1}\widetilde{X}) \\ &= B[\widetilde{X} + (-)^{K+1}\tilde{F}^{K-1}\widetilde{X}] \end{aligned}$$

which contrary to our assumption shows that in $m(B\widetilde{X})$ is tangential to $\Psi(\widetilde{M})$. Thus $m(B\widetilde{X}) = 0$.

Also, in view of (1.4)b, (1.3) and (2.4) we obtain

$$\begin{aligned} B(\tilde{F}^{K-1}\widetilde{X}) &= F^{K-1}(B\widetilde{X}) \\ &= (-)^K \ell(B\widetilde{X}) \\ &= (-)^K (I - m)B\widetilde{X} \\ &= (-)^K B\widetilde{X} - (-)^K mB\widetilde{X} \\ B(\tilde{F}^{K-1}\widetilde{X}) &= (-)^K B\widetilde{X}. \end{aligned}$$

Since B is an isomorphism, we get

$$\tilde{F}^{K-1} = (-)^K I$$

Let $\mathcal{F}(M)$ be the ring of real valued differentiable functions on M and $\mathcal{X}(M)$ be the module of derivatives of $\mathcal{F}(M)$. Then $\mathcal{X}(M)$ is Lie algebra over the real numbers and the elements of $\mathcal{X}(M)$ are called vector fields. Then M is equipped with (1,1) tensor field F which is a linear map such that

$$F : \mathcal{X}(M) \to \mathcal{X}(M).$$

Let M be of degree K and let K be a positive odd integer greater than 2. Then we consider a positive definite Riemannian metric w,r,t which D_ℓ and D_m are orthogonal so that

$$g(X,Y) = g(HX,HY) + g(mX,Y), \tag{2.5}$$

where $H = F^{\frac{K-1}{2}}$ for all $X,Y \in \mathcal{X}(M)$.

Definition (2.3): The induced metric $\widetilde{g}$ defined by (2.1) is Hermitian if the following is satisfied

$$\widetilde{g}(H\widetilde{X}, H\widetilde{Y}) = \widetilde{g}(\widetilde{X},\widetilde{Y}), \quad \text{where} \quad H = F^{\frac{K-1}{2}} \tag{2.6}$$

Theorem (2.2): *If F-structure manifold has the following two properties i.e.*

(a) $\widetilde{M}$ is an invariant submanifold of F-structure manifold M such that distribution D_m is never tangential to $\Psi(\widetilde{M})$.

(b) the Riemannian metric g on M is defined by (2.5), then the induced metric $\widetilde{g}$ of $\widetilde{M}$ defined by (2.1) is Hermitian.

Proof: In view of equations (2.1) and (2.6) we obtain

$$\widetilde{g}(\widetilde{F}^{\frac{K-1}{2}}\widetilde{X}, \widetilde{F}^{\frac{K-1}{2}}\widetilde{Y}) = g(B\widetilde{F}^{\frac{K-1}{2}}\widetilde{X}, B\widetilde{F}^{\frac{K-1}{2}}\widetilde{Y}).$$

Applying equation (2.2) and (2.5) in the above equation we get

$$\widetilde{g}(\widetilde{F}^{\frac{K-1}{2}}\widetilde{X}, \widetilde{F}^{\frac{K-1}{2}}\widetilde{Y}) = g(F^{\frac{K-1}{2}}B\widetilde{X}, F^{\frac{K-1}{2}}B\widetilde{Y}) = g(B\widetilde{X}, B\widetilde{Y}) - g(mB\widetilde{X}, B\widetilde{Y}) \tag{2.7}$$

Since the distribution D_m is never tangential to $\Psi(\widetilde{M})$ on using (2.1) we get:

$$\begin{aligned} &= g(B\widetilde{X}, B\widetilde{Y}) \\ &= \widetilde{g}(\widetilde{X}, \widetilde{Y}). \end{aligned}$$

Now we consider the second case and assume that the distribution D_m is always tangential to $\Psi(\widetilde{M})$. In view of the Case II, we have $m(B\widetilde{X}) = B\widetilde{X}^*$, where $\widetilde{X}^* \in T(\widetilde{M})$ for some $\widetilde{X}^* \in T(\widetilde{M})$.

Let us define (1,1) tensor fields $\widetilde{m}$ and $\widetilde{\ell}$ in $\widetilde{M}$ as follows

$$\widetilde{\ell} = (-)^K \widetilde{F}^{K-1} \text{ and } \widetilde{m} = \widetilde{I} + (-)^{K+1}\widetilde{F}^{K-1} \tag{2.7a}$$

$$\widetilde{m}\widetilde{X} = \widetilde{X}^* \text{ and } m(B\widetilde{X}) = B(\widetilde{m}\widetilde{X}) \tag{2.7b}$$

Theorem (2.4): *We have*

$$B(\widetilde{\ell}\,\widetilde{X}) = \ell(B\,\widetilde{X}) \tag{2.8}$$

Proof: In view of (2.7)a, (2.8) assumes the following form

$$\begin{aligned} B(\widetilde{\ell}\,\widetilde{X}) &= B((-)^K \widetilde{F}^{K-1}\widetilde{X}), \\ &= (-)^K B(\widetilde{F}^{K-1}\widetilde{X}), \end{aligned}$$

On making use of (2.4) and (2.7) in the above equation, we get

$$\begin{aligned} B(\widetilde{\ell}\,\widetilde{X}) &= (-)^K \widetilde{F}^{K-1}(B\widetilde{X}), \\ &= \widetilde{\ell}(B\widetilde{X})\ . \end{aligned}$$

Theorem (2.5): *For $\widetilde{\ell}$ and $\widetilde{m}$ satisfying (2.7)a, we have*

$$\widetilde{\ell} + \widetilde{m} = \widetilde{I}, \qquad \widetilde{\ell}^2 = \widetilde{\ell} \qquad \text{and} \qquad \widetilde{m}^2 = \widetilde{m} \tag{2.9}$$

Proof: From (1.3) we have $\ell + m = I$, which can be written as $(\ell + m)B\widetilde{X} = B\widetilde{X}$, thus we have

$$\ell B\widetilde{X} + mB\widetilde{X} = B\widetilde{X}\ .$$

which in view of (2.7)b and (2.8) becomes

$$\begin{aligned} B(\widetilde{\ell}\widetilde{X}) + B(\widetilde{m}\widetilde{X}) &= B(\widetilde{\ell} + \widetilde{m})\widetilde{X} \\ &= B\widetilde{X}. \end{aligned}$$

Therefore $\widetilde{\ell} + \widetilde{m} = I$ since B is an isomorphism. Proof of the other relations follows in a similar manner.

Theorem (2.5) shows that $\widetilde{\ell}$ and $\widetilde{m}$ defined by (2.7)a are complementary projection operators on $\widetilde{M}$.

Theorem (2.6): *If F-structure manifold has the following properties i.e., $\widetilde{M}$ is an invariant submanifold of F-structure manifold M such that distribution D_m is always tangential to $\Psi(\widetilde{M})$. Then there exists an induced $\widetilde{F}$-structure manifold which admits a similar Riemannian metric $\widetilde{g}$ satisfying*

$$\widetilde{g}(\widetilde{X}, \widetilde{Y}) = \widetilde{g}(\widetilde{H}\widetilde{X}, \widetilde{H}\widetilde{Y}) + \widetilde{g}(\widetilde{m}\widetilde{X}\widetilde{Y}) \tag{2.10}$$

Proof: From (2.2)b we get

$$B(\tilde{F}^{\frac{K-1}{2}}\widetilde{X}) = F^{\frac{K-1}{2}}(B\widetilde{X}).$$

Furthermore,

$$B(\tilde{F}^{K}\widetilde{X}) = F^{K}(B\widetilde{X}),$$

which in view of (1.1) and (2.2)a yields

$$= B(-(-)^{K+1}\tilde{F}\widetilde{X}),$$

which shows that $\tilde{F}$ defines an $\tilde{F}$-structure manifold which satisfies

$$\tilde{F}^{K} + (-)^{K+1}\tilde{F} = 0. \tag{2.10}$$

In consequence of equations (2.1), (2.2)b and (2.5) we obtain:

$$\tilde{g}(\widetilde{H}, \widetilde{X}, \widetilde{H}\tilde{Y}) + \tilde{g}(\widetilde{m}\widetilde{X}, \tilde{Y}), \quad \text{where} \quad \widetilde{H} = \tilde{F}^{\frac{K-1}{2}} \tag{2.11}$$

$$\begin{aligned} &= g(B\widetilde{H}\widetilde{X}, B\widetilde{H}\tilde{Y}) + g(B\widetilde{m}\widetilde{X}, B\tilde{Y}) \\ &= g(H\,B\widetilde{X}, H\,B\tilde{Y}) + g(m\,B\widetilde{X}, B\tilde{Y}) \\ &= g(B\widetilde{X}, B\tilde{Y}) \end{aligned}$$

which in view of the fact that B is an isomorphism gives

$$= \tilde{g}(\widetilde{X}, \tilde{Y}).$$

3. INTEGRABILITY CONDITIONS

The Nijenhuis tensor N of the type (1.2) of F satisfying (1.1) in M is given by [14]

$$N(X, Y) = [FX, FY] - F[FX, Y] - F[X, F, Y] + F^2[X, Y] \tag{3.1}$$

and the Nijenhuis tensor $\widetilde{N}$ of $\tilde{F}$ satisfying (2.10) in $\widetilde{M}$ is given by

$$N(\widetilde{X}, \tilde{Y}) = [\tilde{F}\widetilde{X}, \tilde{F}\tilde{Y}] - \tilde{F}[\tilde{F}\widetilde{X}, \tilde{Y}] - \tilde{F}[\widetilde{X}\tilde{F}\tilde{Y}] + \tilde{F}^2[\widetilde{X}, \tilde{Y}] \tag{3.2}$$

Theorem (3.1): *The Nijenhuis tensors N and $\widetilde{N}$ of M and $\widetilde{M}$ given by (3.1) and (3.2) satisfy the following relation*

$$N(B\widetilde{X}, B\tilde{Y}) = B\widetilde{N}(\widetilde{X}, \tilde{Y}) \tag{3.3}$$

Proof: We have

$$\begin{aligned} N(B\widetilde{X}, B\widetilde{Y}) &= [F(B\widetilde{X}), F(B\widetilde{Y})] - F[F(B\widetilde{X}), B\widetilde{Y}] \\ &\quad -F[B\widetilde{X}, F(B\widetilde{Y})] + F^2[B\widetilde{X}, B\widetilde{Y}], \end{aligned}$$

which in view of (2.2)a becomes

$$\begin{aligned} &= B[\tilde{F}\widetilde{X}, \tilde{F}\widetilde{Y}] - F[B(\tilde{F}\widetilde{X}), B\widetilde{Y}] - F[B\widetilde{X}, B\tilde{F}\widetilde{Y})] + F^2[B\widetilde{X}, B\widetilde{Y}] \\ &= B[\tilde{F}\widetilde{X}, \tilde{F}\widetilde{Y}] - FB[\tilde{F}\widetilde{X}, \widetilde{Y}] - FB[\widetilde{X}, \tilde{F}\widetilde{Y}] + BF^2[\widetilde{X}, \widetilde{Y}] \\ &= B[\tilde{F}\widetilde{X}, \tilde{F}\widetilde{Y}] - B\tilde{F}[\tilde{F}, \widetilde{X}, \widetilde{Y}] - B\tilde{F}[\widetilde{X}, \tilde{F}\widetilde{Y}] + B\tilde{F}^2[\widetilde{X}, \widetilde{Y}] \\ &= B\widetilde{N}(\widetilde{X}, \widetilde{Y}) \end{aligned}$$

Theorem (3.2): *The following identities hold*

$$B\widetilde{N}(\tilde{\ell}\widetilde{X}, \tilde{\ell}\widetilde{Y}) = N(\tilde{\ell}B\widetilde{X}, \tilde{\ell}B\widetilde{Y}) \tag{3.4}$$

$$B\widetilde{N}(\widetilde{m}\widetilde{X}, \widetilde{m}\widetilde{Y}) = N(\widetilde{m}B\widetilde{X}, \widetilde{m}B\widetilde{Y}) \tag{3.5}$$

$$B\{\widetilde{m}\tilde{n}(\widetilde{X}, \widetilde{Y})\} = m\,N(B\widetilde{X}, B\widetilde{Y}) \tag{3.6}$$

Proof: The proof of (3.4) to (3.6) follows by virtue of theorem (3.1), equation (1.4)a, (2.2)a, (2.7)a,b and (3.3).

For $\widetilde{F}$ satisfying (2.10), there exists complementary distribution $D_{\tilde{\ell}}$ and $D_{\widetilde{m}}$ corresponding to the projection operators $\tilde{\ell}$ and $\widetilde{m}$ in $\widetilde{M}$ given by (2.7)a. Then in view of the intergrability conditions of $\widetilde{F}$ structure we state the following theorem:

Theorem (3.3): *If D_ℓ is integrable in M, then $D_{\tilde{\ell}}$ is also integrable in $\widetilde{M}$. If D_m is integrable in M then $D_{\widetilde{m}}$ is also integrable in $\widetilde{M}$.*

Theorem (3.4): *If D_ℓ and D_m are both integrable in M, then $D_{\tilde{\ell}}$ and $D_{\widetilde{m}}$ are also integrable in $\widetilde{M}$.*

Theorem (3.5): *If F-structure is integrable in M then the induced structure $\widetilde{F}$ is also integrable in $\widetilde{M}$.*

References

1. Blair, D., Ludden, G. and Yano, K. *Semi-Invariant immersions*, Kodai Math., Sem,, Rep., Vol 25(1976), 313-319.

2. Kubo, Y., *Invariant submanifolds of codimension 2 of a manifold with* (F, G, u, v, λ)*-structure,* Kodai Math., Sem., Rep., Vol. 24(1972), 50-61.

3. Yano, K. and Okumura, M., *On* (f, g, u, v, λ)*-structures*, Kodai, Math., Sem., Rep., Vol 22(1970), 401-423.

4. Yano, K. and Okumura, M., *Invariant hypersurfaces of a manifold with* (f, g, u, v, λ)*-structure*, Kodai, Math., Sem., Rep., Vol 23(1971), 290-304.

5. Yano, K. and Isaihara, S., *Invariant submanifolds of almost contact manifolds*, Kodai, Math., Sem., Rep., Vol 21(1969), 350-364.

6. Kim, J.B., *Notes on* f*-manifold*, Tensor(N.S.), Vol. 29(1975), 299-302.

7. Nekogova, H., f*-structures induced on submanifolds in spaces*, almost Hermitian or Kählerian, Kodai, Math., Sem., Rep., Vol 18(1966), 161-183.

8. Das, Lovejoy S., *Prolongations of* F*-structure to the tangent bundle of order 2.* International J. Math. & Math. Sci. Vol 16 No.1(1993), 201-204.

9. Das, Lovejoy S. *Fiberings on almost* r*-contact manifolds.* Publications Mathematicae, Debrecen, Hongrie, Vol. 43(1993), 1-7.

10. Hicks, N.J., *Notes on Differential Geometry*, D. Van Nostrand Company, Inc., New York.

11. Das, Lovejoy S. and Upadhyay, M.D., F*-structure manifold*, Kyung Pook Mathematical Journal, Korea, 18(1978), 272-283.

12. Das, Lovejoy S., *Complete lift of* F*-structure manifold*, Kyung Pook Mathematical Journal, Korea, 20(1980), 231-237.

13. Helgason, S., *Differential Geometry, Lie groups and Symmetric Spaces*, Academic Press, New York 1998.

14. Yano, k. & Ishihara, S., *On integrability of a structure of satisfying* $f^3 + f = 0$, Quart. J. Math. Oxford, Vol. 25(1964), 217-222.

Electronic Journal: Southwest Journal of Pure and Applied Mathematics
Internet: http://rattler.cameron.edu/swjpam.html
ISBN 1083-0464
Issue 1 June 2001, pp. 75 – 78
Submitted: March 24, 2001. Published: June 30, 2001

On Self-Similar Centralizers of Circle Maps

Robert Cowen

Abstract

In this paper we introduce the notion of self-similar centralizers and give sufficient conditions for certain circle maps not to have a self-similar centralizer (we say f has a self-similar centralizer if $g \in Z(f) \Rightarrow Z(g) \cong Z(f)$).

A.M.S. Subject Classification Codes: 28D
Key Words and Phrases: Circle maps, Centralizer

Let $Imm^n(S^1)$ be the space of C^n immersions of the circle (i.e.maps of S^1 onto itself for which $f'(z) \neq 0$ for for all $z \in S^1$) endowed with the C^n topology where $1 \leq n \leq \infty$. For $f \in Imm^n(S^1)$ let $Z(f)$ denote the set of elements of $Imm^n(S^1)$ that commute with f. This semi-group is called the centralizer of f. We say that $Z(f)$ is trivial if it is just the itterates of f (i.e $Z(f) = \{f^m : m \in \mathbf{N}\}$).Wesaythat f $\in Imm^n(S^1)$ is expanding if $|f'(z)| > 1$ for all $z \in S^1$ and these maps are denoted by $Exp^n(S^1)$. The expanding map $z \to z^m$ will be denoted by $T_m(z)$. In this paper sufficient conditions for an element of $Exp^n(S^1)$ not to have a self-similar centralizer are given. We shall need the following theorem:

Department of Mathematics, University of Botswana, Botswana
E-mail: cowenr@mopipi.ub.bw

THEOREM 1 [4]

Let f_1, f_2 be expanding orientation-preserving maps of S^1 that commute, have degrees p and q respectively and are of class $C^{n-}(n \geq 2)$. Then if p and q are coprime, there is a diffeomorphism h of class C^{n-} satisfying $h(f_1(z)) = T_p(h(z))$ and $h(f_2(z)) = T_q(h(z))$ for all $z \in S^1$.

We now state a sufficient condition for a centralizer semi-group not to be self-similar:

THEOREM 2

Let $f \in Exp^n(S^1)$ be orientation preserving with degree p. If these exists $g \in Z(f)$ with degree q where p and q are coprime then $Z(f)$ is not self-similar.

Proof: From Theorem 1 there exists a diffeomorphism h of S^1 such that $h(f(z)) = T_p(h(z))$ and $h(g(z)) = T_q(h(z))$ for all $z \in S^1$. In [1] it was shown that if h is the homeomorphism conjugating f and T_p (whose existence is guarenteed by a Theorem of Shub in [5]) then $Z(f)$ is a subset of $\{h^{-1}\ o\ \theta T_m\ o\ h : m \in \mathbf{N}, \theta^{p-1} = 1\}$. From above, we have h is a diffeomorphism of the same class as f and so:

$$Z(f) = \{h^{-1}\ o\ \theta T_m\ o\ h : m \in \mathbf{N} \text{ and } \theta^{p-1} = 1\}.$$

Similarly we have that:

$$Z(g) = \{h^{-1}\ o\ \omega T_m\ o\ h : m \in \mathbf{N} \text{ and } \omega^{q-1} = 1\}.$$

We now show (by contradiction) that there is no bijective semi-group homomorhism between $Z(f)$ and $Z(g)$:

Suppose $Z(f) \cong Z(g)$ and $\Phi : Z(f) \to Z(g)$ is the isomorphism. Assume $p > q$ (otherwise consider a homomorhism from $Z(g)$ to $Z(f)$). Let $\{\theta_i : 1 \leq i \leq p{-}1\}$ be the $p{-}1$ roots of unity and let $\{\omega_i : 1 \leq i \leq q{-}1\}$ be the $q-1$ roots of unity. The identity element of both the semi-groups $Z(f), Z(g)$ is $h^{-1}\ o\ T_1\ o\ h$ and so $\Phi(h^{-1}\ o\ T_1\ o\ h) = h^{-1}\ o\ T_1\ o\ h$. Let $i \in \{1, ..., p-1\}$ be arbitrary and denote $\Phi(h^{-1}\ o\ \theta_i T_1\ o\ h)$ by $h^{-1}\ o\ \omega_{j(i)} T_{k(i)}\ o\ h$. Now:
$h^{-1}\ o\ T_1\ o\ h$
$= \Phi(h^{-1}\ o\ T_1\ o\ h)$

$= \Phi((h^{-1}\ o\ \theta_i T_1\ o\ h)\ o\ ...\ o\ (h^{-1}\ o\ \theta_i T_1\ o\ h))$
(where we take $p-1$ compositions of $h^{-1}\ o\ \theta_i T_1\ o\ h$)
$= \Phi(h^{-1}\ o\ \theta_i T_1\ o\ h)\ o\ ...\ o\ \Phi(h^{-1}\ o\ \theta_i T_1\ o\ h)$
$= h^{-1}\ o\ \omega_{j(i)} T_{k(i)}\ o\ h\ o\ ...\ o\ h^{-1}\ o\ \omega_{j(i)} T_{k(i)}\ o\ h$
$= h^{-1}\ o\ \omega_{j(i)}^{1+k(i)+k(i)^2+...+k(i)^{p-2}} T_{k(i)^{p-1}}\ o\ h.$
Thus $(k(i))^{p-1} = 1 \Rightarrow k(i) = 1$, and so $\Phi(h^{-1}\ o\ \theta_i T_1\ o\ h) = h^{-1}\ o\ \omega_{j(i)} T_1\ o\ h.$
By injectivity of Φ we must have that $\omega_{j(i)} \neq \omega_{j(s)}$ if $i \neq s$ for all $i, s \in \{1, ..., p-1\}$. But this is impossible since $p > q$ and so $Z(f)$ is not isomorphic to $Z(g)$ and the Theorem is proved.

A semi-group of **N** is called nonlacunary if it is not contained in a single generated semi-group. For example 2 and 6 form a nonlacunary semi-group but 4 and 8 do not. It was shown in [2] that if f and g have a common fixed point then the condition that p and q be coprime in Theorem 1 can be replaced by the weaker condition that p and q form a nonlacunary semi-group of **N**. Thus we have:

COROLLARY 1

Let $f \in Exp^n(S^1)$ be orientation preserving with degree p. If there exists $g \in Z(f)$ with degree q and:
(i) p and q form a nonlacunary semi-group of **N**
(ii) f and g have a common fixed point.
Then $Z(f)$ is not self-similar.

We now give an example of a map in $Exp^n(S^1)$ whose centralizer is not self-similar:

EXAMPLE

Let $f : S^1 \to S^1$ be given by $f(z) = \dfrac{(z - \frac{4}{21})(z + \frac{4}{13})}{(1 - \frac{4}{21}z)(1 + \frac{4}{13}z)}$. Then $f(z)$ is the restriction to S^1 of a Blaschke product defined on the closed unit disk so $f(z)$ is anlytic on S^1. The map $f(z)$ has degree 2 and as

$$\frac{1 - \frac{4}{21}}{1 + \frac{4}{21}} + \frac{1 - \frac{4}{13}}{1 + \frac{4}{13}} = \frac{17}{25} + \frac{11}{17} = \frac{564}{425} > 1$$

we have that $f(z)$ is expanding (c.f [3]). Let $h(z)$ be the Möbius map on S^1 defined by $h(z) = \dfrac{z + \frac{1}{16}}{1 + \frac{1}{16}z}$. As the measure theoretic entopy of $f(z)$ is equal to the topological entopy of f (where the measure is derived from using the Poisson kernal of the unique fixed point of $f(z)$ in the open unit disk) we have that $h \ o \ f \ o \ h^{-1} = cT_2$ for some $c \in S^1$ (c.f [3]). Evaluating this equation

REFERENCES

[1] C. Arteaga, Centralizers of immersions of the circle, Proc. Amer. Math. Soc 109 (1990), 849 - 853.

[2] A.S.A Johnson and D.J. Rudolph, commuting endormorphisms of the circle, Ergod. Th. and Dynam. Sys. 12 (1992), 743 - 748.

[3] N.F.G. Martin, On finite Blaschke products whose restrictions to the unit circle are exact endomorphisms, Bull. Lond. Math. Soc. 15 (1983), 343 - 348.

[4] R. Sacksteder, Abelian semi-groups of expanding maps. Springer Lecture Notes in Mathematics 318. Springer, New York 1972, 235 - 238.

[5] M. Shub, Endomorphisms of compact differentiable manifolds, Amer. Jour. Math. 91 (1969), 175 - 199.

Electronic Journal: Southwest Journal of Pure and Applied Mathematics
Internet: http://rattler.cameron.edu/swjpam.html
ISSN 1083-0464
Issue 1, June 2001, pp. 79–92.
Submitted: April 3, 2001. Published: June 30, 2001.

THE VANISHING OF THE COHOMOLOGY OF THE ORLIK-SOLOMON ALGEBRAS

Kelly Jeanne Pearson

Abstract The Orlik-Solomon algebra of a hyperplane arrangement first appeared from the Brieskorn and Orlik-Solomon theorems as the cohomology of the complement of this arrangement (if the ground field is complex). Later, it was discovered that this algebra plays an important role in many other problems. In particular, define the cohomology of an Orlik-Solomon algebra as that of the complex formed by its homogeneous components with the differential defined via multiplication by an element of degree one. In this paper, we give necessary and sufficient conditions for the vanishing of the cohomology of the Orlik-Solomon algebra.

1991 A.M.S. Subject Classification Codes:. 05B35, 52C35

Key Words and Phrases:. Hyperplane arrangements, Orlik-Solomon algebra

1 Introduction

Let V be an affine space of dimension ℓ over a field F, and let $\mathcal{A} = \{H_1, \dots , H_n\}$ be a nonempty arrangement of hyperplanes of V. Let $L(\mathcal{A})$ be the intersection poset, ordered opposite to inclusion, of $\mathcal{A}$. Consider the complement space $M = V \setminus \cup_{i=1}^{n} H_i$. The problem of expressing $H^*(M, \mathcal{K})$ in terms of generators and relations was first studied by Arnold [2] in the case $\mathcal{A}$ was the braid arrangement and $\mathcal{K} = \mathbb{C}$; that is, $\mathcal{A}$ was the collection of hyperplanes $\{x_i - x_j : 1 \leq i < j \leq \ell\}$. This problem was later studied by Brieskorn [3] for an arbitrary arrangement. For each $H_i \in \mathcal{A}$ fix an affine functional α_i with $\operatorname{Ker} \alpha_i = H_i$. The Orlik-Solomon algebra is the graded F-algebra $= A(\mathcal{A}) = \oplus_p^{\ell} A_p$ generated by the differential forms $\omega_i = d\alpha_i/\alpha_i \in A_1$. If $F = \mathbb{C}$, then according to Brieskorn's theorem, this algebra is isomorphic under the de Rham map to the cohomology algebra of $M = V \setminus \cup_{i=1}^{n} H_i$. Orlik and Solomon [9] found a purely algebraic characterization of $H^*(M, \mathcal{K})$. These results can be briefly summarized as follows. An algebra $A(\mathcal{A})$ (referred to as the Orlik-Solomon algebra)

Mathematics Department, Murray State University, Murray KY
kelly.pearson@murraystate.edu

over $\mathcal{K}$ is constructed in terms of generators and relations using only $L(\mathcal{A})$. This is a graded algebra with $A(\mathcal{A}) \cong H^*(M, \mathcal{K})$. Hence, $H^*(M, \mathcal{K})$ can be determined by $L(\mathcal{A})$.

The above are important results in that topological invariants of the complement space are expressed in terms of combinatorics. Indeed, a central question in the theory of hyperplane arrangements is the problem of expressing topological invariants of the complement space in terms of combinatorics. In this manner, it is a natural question to consider a generalization of $H^*(M, \mathcal{K})$ to cohomology with local coefficients.

For $a \in A_1(\mathcal{A})$, one can define a local coefficient system $\mathcal{L}(a)$. It turnsout that $H^*(M, \mathcal{L}(a))$ relates closely to the cohomology of the Orlik-Solomon algebra. The connection between $H^*(M, \mathcal{L}(a))$ and the cohomology of the Orlik-Solomon algebra has been studied in many papers, for instance [6].

We construct a cochain complex on the graded linear space $A(\mathcal{A})$ as follows. Let $a \in A_1(\mathcal{A})$ with $a = \sum_{i=1}^{n} \lambda_i a_i$ for $\lambda_i \in \mathcal{K}$. Multiplication by a giving the differential $d_k : A_k(\mathcal{A}) \xrightarrow{a\cdot} A_{k+1}(\mathcal{A})$ forms a complex $(A(\mathcal{A}), a)$. The cohomology of this complex is said to be the cohomology of the Orlik-Solomon algebra and is denoted $H^*(A(\mathcal{A}), a)$.

Recently, there have been many results concerning $\dim H^1(A(\mathcal{A}), a)$. In the case char $\mathcal{K} \neq 2$, it has been shown in [6] that $\dim H^1(A(\mathcal{A}), a)$ can be determined by a particular set of elements from $L(\mathcal{A})$. However, little is known about the higher dimensions $H^p(A(\mathcal{A}), a)$ for $p > 1$ [11], and this is what our work is devoted to.

In this paper, we let $\mathcal{K} = \mathbb{R}$ or $\mathbb{C}$ and establish a necessary and sufficient condition for $H^*(A(\mathcal{A}), a) = 0$. Here is an outline of the paper.

In §2, we employ tools from operator theory to prove the upper semicontinuity of the map $t \mapsto \dim H^p(A(\mathcal{A}), t)$ for any $p \geq 0$ and for any $t \in A_1(\mathcal{A})$. In §3, we analyze tensor products in the category of graded commutative algebras in order to express the cohomology of a reducible arrangement in terms of the cohomology of each factor of the arrangement. In §4, we apply results discussed in §2 and §3 and show $H^*(A(\mathcal{A}), a) = 0$ if and only if $H^\ell(A(\mathcal{A}), a) = 0$, where $\operatorname{rank}(\mathcal{A}) = \ell$. The following theorem, which is the main result of this paper, gives a necessary and sufficient condition for $H^\ell(A(\mathcal{A}), a) = 0$.

Theorem 4.11. *Let $\mathcal{A}$ be an affine ℓ-arrangement. We may write*

$$\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2 \times \cdots \times \mathcal{A}_k \times \mathcal{B},$$

where $\mathcal{A}_j$ are each central and $\mathcal{B}$ not central and they contain no proper central factors. Let $a \in A_1(\mathcal{A})$. We have $\dim H^\ell(A(\mathcal{A}), a) \neq 0$ if and only if $\sum_{H_i \in \mathcal{A}_j} \lambda_i = 0$ for all j.

The cohomology of the Orlik-Solomon algebras is a particular case of a more general problem of skew commutative algebras, i.e. studying modules over an exterior algebra E, see [1]. If M is such a module, then $a \in E_1$ is said to be regular on M if and only if

$$H^*(M, a) = \{x \in M;\ ax = 0\}/aM = 0;$$

otherwise, a is said to be singular. The set of all singular elements is called a singular variety of M, denoted $Sing(M)$. So we study $Sing(A(\mathcal{A}))$ as $E(\mathcal{A})$-modules.

2 Upper Semicontinuity

Let $\mathcal{A} = \{H_1, ..., H_n\}$ be a hyperplane arrangement in $V = F^\ell$ for some field F. We recall the definition of the Orlik-Solomon algebra $A(\mathcal{A})$ and the cohomology of the Orlik-Solomon algebra.

Let $\mathcal{K}$ be a commutative ring. Let E_1 be the linear space over $\mathcal{K}$ on n generators, $e_1, \ldots, e_n$. Let $E(\mathcal{A}) := \Lambda(E_1)$ be the exterior algebra on E_1. We have $E(\mathcal{A}) = \bigoplus_{p\geq 0} E_p$ is a graded algebra over $\mathcal{K}$. The standard $\mathcal{K}$-basis for E_p is given by

$$\{e_{i_1} \cdots e_{i_p} : \ 1 \leq i_1 < \ldots < i_p \leq p\}.$$

Any ordered subset $S = \{H_{i_1}, ..., H_{i_p}\}$ of $\mathcal{A}$ corresponds to an element $e_S := e_{i_1} \cdots e_{i_p}$ in $E(\mathcal{A})$.

We define the map $\partial : \ E(\mathcal{A}) \to E(\mathcal{A})$ via the usual differential. That is,

$$\begin{aligned} \partial(1) &:= 0, \\ \partial(e_i) &:= 1, \\ \text{and for } p \geq 2, \ \partial(e_{i_1} \cdots e_{i_p}) &:= \sum_{k=1}^{p} (-1)^{k-1} e_{i_1} \cdots \hat{e}_{i_k} \cdots e_{i_p}. \end{aligned}$$

Let $S = \{H_{i_1}, ..., H_{i_p}\}$ be a subset of $\mathcal{A}$. We say S is dependent if $\cap S \neq \emptyset$ and $\text{rank}(\cap S) < |S|$. Equivalently, S is dependent if polynomials $\alpha_{i_k} \in F[x_1, \ldots, x_\ell]$ defining the hyperplanes H_{i_k} are linearly dependent.

We define $I(\mathcal{A})$ to be the ideal of $E(\mathcal{A})$ which is generated by the elements

$$\{\partial(e_S) : \ S \text{ is dependent } \} \cup \{e_S : \ \cap_{H\in S} H = \emptyset\}.$$

The Orlik-Solomon algebra, $A(\mathcal{A})$, is defined as

$$A(\mathcal{A}) := E(\mathcal{A})/I(\mathcal{A}).$$

Let $\pi : \ E(\mathcal{A}) \to A(\mathcal{A})$ be the canonical projection. We write a_S to represent the image of e_S under π.

The grading on $E(\mathcal{A})$ induces a grading on $A(\mathcal{A})$; hence, $A(\mathcal{A}) = \oplus A_p(\mathcal{A})$. The standard basis used for $A(\mathcal{A})$ is referred to as the broken circuit basis.

We construct a cochain complex on the homogeneous components of $A(\mathcal{A})$ as follows. Let $a \in A_1(\mathcal{A})$ with $a = \sum_{i=1}^{n} \lambda_i a_i$ for $\lambda_i \in \mathcal{K}$. Multiplication by a giving the differential $d_k : A_k(\mathcal{A}) \xrightarrow{a\cdot} A_{k+1}(\mathcal{A})$ forms a complex $(A(\mathcal{A}), a)$. The cohomology of this complex is said to be the cohomology of the Orlik-Solomon algebra and is denoted $H^*(A(\mathcal{A}), a)$.

In this section, we show that in the case $\mathcal{K} = \mathbb{R}$ or $\mathbb{C}$, the function $A_1(\mathcal{A}) \to \mathbb{Z}$ given by $t \mapsto \dim H^p(A(\mathcal{A}), t)$ is upper semicontinuous in t for any p. We show this in the more general setting of finite dimensional vector spaces and hence begin by establishing some standard definitions and notational conventions, which can be found in [4].

Let V be an n-dimensional vector space over $\mathcal{K}$. Relative to a basis $\{b_1, \dots, b_n\}$ for V over $\mathcal{K}$, for $v \in V$ we express $v = (v_1, \dots, v_n)$ as $v = \sum_{i=1}^{n} v_i b_i$.

Since $\mathcal{K} = \mathbb{C}$ or $\mathbb{R}$, we define the standard Euclidean norm, $\|\cdot\|_V$, on V as

$$\|v\|_V = \|(v_1, \dots, v_n)\|_V := \sqrt{v_1\bar{v}_1 + \cdots + v_n\bar{v}_n},$$

relative to the standard orthonormal basis $\{e_1, \dots, e_n\}$ for V.

With respect to the norm $\|\cdot\|_V$, we define the unit sphere in V by

$$S(V) := \{v \in V : \ \|v\|_V = 1\}.$$

We also have the corresponding standard inner product, $(\cdot,\cdot)$, defined on V as

$$(x, y) := x_1\bar{y}_1 + \cdots + x_n\bar{y}_n \text{ for } x, y \in V.$$

We define the orthogonal complement relative to the inner product. Let $X \subseteq V$ be a linear subspace. Then

$$X^{\perp} := \{v \in V : \ (v, x) = 0 \text{ for all } x \in X\}.$$

Let $X \subseteq V$ be a linear subspace and $v \in V$. We define the distance from v to X to be

$$\text{dist } (v, X) := \inf\{\|v - x\|_V : \ x \in X\}.$$

Note there exists an unique $x_0 \in X$ for which dist $(v, X) =$ dist (v, x_0).

For the remainder of this section, we fix two finite dimensional vector spaces V and W over $\mathcal{K}$. Fix $0 \neq A \in \text{Hom}_{\mathcal{K}}(V, W)$.

Definition 2.1. Define

$$\gamma(A) := \inf\{\|Ah\|_W : \ h \in S(V) \cap (\text{Ker}\, A)^{\perp}\}.$$

The following technical lemmas are well known and can be found in [4]; we include their proofs for clarity.

Lemma 2.2. *If* $0 \neq A \in \text{Hom}_{\mathcal{K}}(V, W)$, *then* $\gamma(A) > 0$.

Proof. Clearly, $\gamma(A) \geq 0$. Suppose that $\gamma(A) = 0$. By definition of the infimum, there exists a sequence $\{h_j\} \subset S(V) \cap (\text{Ker}\, A)^{\perp}$ so that $\|Ah_j\|_W \to 0$. This implies $\lim_{j\to\infty} h_j \in \text{Ker}\, A \cap (\text{Ker}\, A)^{\perp} = \{0\}$. But $\|\cdot\|$ is continuous in the metric; hence, $\lim_{j\to\infty} \|h_j\|_V = 1$. □

Lemma 2.3. *If* $A \in \text{Hom}_{\mathcal{K}}(V, W)$ *and if* $h \in V$, *then we have*

$$\gamma(A) \cdot \text{dist } (h, \text{Ker}\, A) \leq \|Ah\|_W.$$

Proof. Let $\rho : V \to (\text{Ker}\, A)^{\perp}$ be orthogonal projection of V onto $(\text{Ker}\, A)^{\perp}$. We relate the norm to the distance by noticing $\|\rho h\|_V =$ dist $(h, \text{Ker}\, A)$. Hence,

$$\begin{aligned} \|Ah\|_W &= \|A\rho h\|_W \\ &\geq \gamma(A) \cdot \|\rho h\|_V \\ &= \gamma(A) \cdot \text{dist } (h, \text{Ker}\, A). \end{aligned}$$

The lemma now follows. □

Lemma 2.4. *If $V_1, V_2 \subseteq V$ are linear subspaces with $\dim V_1 > \dim V_2$, then there exists $0 \neq v_1 \in V_1$ so that $\|v_1\|_V = \text{dist } (v_1, V_2)$.*

Proof. Let ρ_1 be the orthogonal projection of V onto V_1. We have the inequality $\dim \rho_1(V_2) \leq \dim V_2 < \dim V_1$, so $\rho_1(V_2)$ is a proper linear subspace of V_1. Take $0 \neq v_1 \in V_1 \cap (\rho_1(V_2))^{\perp}$. Then for any $v_2 \in V_2$, we have

$$\begin{aligned} 0 =&(\rho_1(v_2), v_1) \\ =&(v_2, \rho_1(v_1)) \\ =&(v_2, v_1). \end{aligned}$$

Thus, $v_1 \in V_2^{\perp}$. Consequently, $\|v_1\|_V = \text{dist } (v_1, V_2)$. □

Definition 2.5. Let $B \in \text{Hom}_{\mathcal{K}}(V, W)$. The operator norm of B is defined to be

$$\|B\|_{\text{op}} := \sup\{\|Bh\|_W : \; h \in S(V)\}.$$

We note that for any $h \in V$, the inequality holds:

$$\|Bh\|_W \leq \|B\|_{\text{op}} \|h\|_V.$$

Proposition 2.6. *If $B \in \text{Hom}_{\mathcal{K}}(V, W)$ with $\|B\|_{\text{op}} < \gamma(A)$ then*

$$\dim \text{Ker}(A + B) \leq \dim \text{Ker}\, A.$$

Proof. If $0 \neq h \in \text{Ker}(A + B)$, then $Ah = -Bh$. By Lemma 2.3, we have

$$\begin{aligned} \gamma(A) \cdot \text{dist } (h, \text{Ker}\, A) \leq& \|Ah\|_W \\ =&\|Bh\|_W \\ \leq&\|B\|_{op} \cdot \|h\|_V \\ <&\gamma(A) \cdot \|h\|_V. \end{aligned}$$

Thus, $\text{dist } (h, \text{Ker}\, A) < \|h\|_V$, for all $0 \neq h \in \text{Ker}(A + B)$. By Lemma 2.4, we have $\dim \text{Ker}(A + B) \leq \dim \text{Ker}\, A$. □

Definition 2.7. Let $A \in \text{Hom}_{\mathcal{K}}(V, W)$. The adjoint of A, denoted by $A^* \in \text{Hom}_{\mathcal{K}}(W, V)$, is given by $(x, A^*y) := (Ax, y)$ for all $x \in V$ and for all $y \in W$.

Lemma 2.8. *If $A \in \text{Hom}_{\mathcal{K}}(V, W)$, then $\text{Ker}\, A^* = (\text{range } \; A)^{\perp}$.*

Proof. Let $y \in \text{Ker}\, A^*$. Then $y \in (Ax)^{\perp}$ for any $x \in V$. Thus $y \in (\text{range } \; A)^{\perp}$.

Let $y \in (\text{range } \; A)^{\perp}$. Then for any $x \in V$, we have $0 = (Ax, y) = (x, A^*y)$. This implies $A^*y = 0$; hence, $y \in \text{Ker}\, A^*$. □

Proposition 2.9. *Let $A, B \in \text{Hom}_{\mathcal{K}}(V, W)$. If $\|B^*\|_{\text{op}} < \gamma(A^*)$, then we have* $\text{rank}(A + B) \geq \text{rank}\, A$.

Proof. From Proposition 2.6, we have

$$\dim \text{Ker}(A + B)^* = \dim \text{Ker}(A^* + B^*) \leq \dim \text{Ker}\, A^*.$$

Since $\dim \text{Ker}\, A^* = \dim(\text{range } \; A)^{\perp} = \dim W - \text{rank}\, A$, it follows that

$$\dim W - \text{rank}(A + B) \leq \dim W - \text{rank}(A). \quad □$$

Definition 2.10. Let X be a topological space. Let $f : X \to \mathbb{R}$ be a real-valued function; f is said to be upper semicontinuous if for any real number α the set $\{x \in X : f(x) < \alpha\}$ is open. Alternatively, for X a metric space we may define f to be upper semicontinuous at $x_0 \in X$ if

$$\limsup_{x \to x_0} f(x) \leq f(x_0).$$

We now apply these results to the graded algebra $A(\mathcal{A}) = \oplus A_p(\mathcal{A})$. Since $\mathcal{K} = \mathbb{R}$ or $\mathbb{C}$, we have $A_p(\mathcal{A})$ is a finite dimensional vector space. For any $t \in A_1(\mathcal{A})$, let the map $t\cdot : A_p(\mathcal{A}) \to A_{p+1}(\mathcal{A})$ be given by multiplication by t. Let $Z_p(A(\mathcal{A}), t)$ denote the kernel of the map $t\cdot : A_p(\mathcal{A}) \to A_{p+1}(\mathcal{A})$; let $B_p(A(\mathcal{A}), t)$ denote the image of the map $t\cdot : A_p(\mathcal{A}) \to A_{p+1}(\mathcal{A})$.

Lemma 2.11. *If $\|t - t_0\|_{\mathcal{K}^n} \to 0$, then $\|t \cdot - t_0 \cdot\|_{\text{op}} \to 0$.*

Proof. We have $A_p(\mathcal{A})$ and $E_p(\mathcal{A})$ are finite dimensional vector spaces over $\mathcal{K}$. We use the standard basis for $E_p(\mathcal{A})$ given by $\{e_{i_1} \cdots e_{i_p} : 1 \leq i_1 < \ldots < i_p \leq p\}$. We use the broken circuit basis for $A_p(\mathcal{A})$. Then $\| \cdot \|_{E_p(\mathcal{A})}$ and $\| \cdot \|_{A_p(\mathcal{A})}$ are defined as previously.

It will suffice to show $\|(t - t_0)(v)\|_{A_p(\mathcal{A})} \to 0$ for any $v \in A_{p-1}(\mathcal{A})$. But the maps given by $(t - t_0)\cdot : E_{p-1}(\mathcal{A}) \to E_p(\mathcal{A})$ commute with the projection map $\pi : E_p(\mathcal{A}) \to A_p(\mathcal{A})$; that is,

$$\begin{array}{ccc} E_{p-1}(\mathcal{A}) & \xrightarrow{(t-t_0)\cdot} & E_p(\mathcal{A}) \\ \downarrow \pi & \circ & \downarrow \pi \\ A_{p-1}(\mathcal{A}) & \xrightarrow{(t-t_0)\cdot} & A_p(\mathcal{A}). \end{array}$$

Hence, it will suffice to show $\|(t - t_0) \cdot v\|_{E_p(\mathcal{A})} \to 0$ for any $v \in E_{p-1}(\mathcal{A})$.

We need only show that $\|(t - t_0) \cdot b\|_{E_p(\mathcal{A})} \to 0$ for any standard basis element $b \in E_{p-1}(\mathcal{A})$. We write $b = e_{i_1} \cdots e_{i_{p-1}}$, where $1 \leq i_1 < \ldots < i_{p-1} \leq n$. Then

$$\begin{aligned} \|(t - t_0) \cdot b\|^2_{E_p(\mathcal{A})} &= \|(t - t_0) \cdot e_{i_1} \cdots e_{i_p}\|^2_{E_p(\mathcal{A})} \\ &= \|\left(\sum_{i=1}^{n} (t^i - t_0^i) e_i\right) e_{i_1} \cdots e_{i_{p-1}}\|^2_{E_p} \\ &= \sum_{i \neq i_1, \ldots, i_{p-1}} (t^i - t_0^i)^2 \end{aligned}$$

As $\|t - t_0\|_{\mathcal{K}^n} \to 0$, we have $\|(t - t_0) \cdot b\|^2_{E_p(\mathcal{A})} \to 0$ as required. □

Theorem 2.12. *Let $\mathcal{A}$ be a hyperplane arrangement with n hyperplanes. Let $A(\mathcal{A})$ be the Orlik-Solomon algebra on $\mathcal{A}$ over the field $\mathcal{K}$, where $\mathcal{K}$ is either $\mathbb{C}$ or $\mathbb{R}$. The function $t \mapsto \dim H^p(A(\mathcal{A}), t)$ from $A_1(\mathcal{A})$ to $\mathbb{Z}$ is upper semicontinuous.*

Proof. We first identify $A_1(\mathcal{A})$ with $\mathcal{K}^n$.

The result clearly holds for $t_0 = 0 \in A_1(\mathcal{A})$. That is,

$$\limsup_{t \to 0} \dim H^p(A(\mathcal{A}), t) \leq \dim H^p(A(\mathcal{A}), 0) = \dim A_p(\mathcal{A}).$$

Fix $0 \neq t_0 \in \mathcal{K}^n$. Let $\epsilon = \min\{\gamma(t_0\cdot), \gamma(t_0\cdot^*)\}$. By Lemma 2.2, $\epsilon > 0$. As $t \to t_0$ in $\mathcal{K}^n$, by Lemma 2.11, we have $\|t \cdot - t_0 \cdot\|_{\mathrm{op}} \to 0$. Hence, there exists $\delta > 0$ so that $\|t \cdot - t_0 \cdot\|_{\mathrm{op}} < \epsilon$ whenever $\|t - t_0\|_{\mathcal{K}^n} < \delta$.

Consequently, we use Proposition 2.6, Proposition 2.9, and Lemma 2.11 to see $\dim \mathrm{Ker}(A(\mathcal{A}), t) \leq \dim \mathrm{Ker}(A(\mathcal{A}), t_0)$ and $\mathrm{rank}(A(\mathcal{A}), t) \geq \mathrm{rank}(A, t_0)$. Thus,

$$\begin{aligned} \dim H^p(A(\mathcal{A}), t) &= \dim Z_p(A(\mathcal{A}), t) - \dim B_{p-1}(A(\mathcal{A}), t) \\ &\leq \dim Z_p(A(\mathcal{A}), t_0) - \dim B_{p-1}(A(\mathcal{A}), t_0) \\ &= \dim H^p(A(\mathcal{A}), t_0). \end{aligned}$$

The assertion now follows. □

3 Tensor Products in the Category of Graded Commutative Algebras

Let $\mathcal{K}$ be a commutative ring. We introduce the following definitions and notational conventions, as can be found in [7].

Definition 3.1. (Tensor Product of Modules) Let M and N be $\mathcal{K}$-modules. The tensor product $M \otimes N$ is the abelian group with generators being all symbols $m \otimes n$ for $m \in M$ and $n \in N$ subject to the relations ($k \in \mathcal{K}$)
(i) $(m + m', n) - (m, n) - (m', n)$
(ii) $(m, n + n') - (m, n) - (m, n')$
(iii) $(km, n) - (m, kn)$.
There exists a bilinear map $\phi : M \times N \to M \otimes N$ so that $\phi(m, n) = m \otimes n$. We have the following universal property. Let A be a $\mathcal{K}$-module. For any bilinear homomorphism $f : M \times N \to A$, there exists an unique $\hat{f} : M \otimes N \to A$ so that $f(m, n) = \hat{f}(m \otimes n)$.

Definition 3.2. (Graded Module) We say M is a graded $\mathcal{K}$-module if there is a family of $\mathcal{K}$-modules $\{M_n\}_{n \geq 0}$ so that $M = \bigoplus_{n \geq 0} M_n$. For $m \in M_n$, we write $\deg(m) = n$.

Definition 3.3. (Tensor Product of Graded Modules) Let M and N be graded $\mathcal{K}$-modules. The tensor product $M \otimes N$ is the graded module given by

$$(M \otimes N)_n = \bigoplus_{p+q=n} M_p \otimes N_q. \tag{3.3.a}$$

Let A be a graded $\mathcal{K}$-module. Let $f : M \times N \to A$ be any bilinear graded homomorphism, there exists an unique graded homomorphism $\hat{f} : M \otimes N \to A$ so that $f(m, n) = \hat{f}(m \otimes n)$.

Definition 3.4. (Graded Commutative Algebra) M is said to be a graded commutative $\mathcal{K}$-algebra if the following are satisfied:
1. M is a graded $\mathcal{K}$-module.
2. There is an associative multiplication in M so that $M_p M_q \subseteq M_{p+q}$.
3. (Commutative) For homogeneous elements $a, b \in M$ we have

$$ab = (-1)^{\deg(a) \cdot \deg(b)}\, ba.$$

Definition 3.5. (Tensor Product of Graded Commutative Algebras) Let M and N be graded commutative $\mathcal{K}$-algebras. The tensor product $M \otimes N$ is the graded commutative $\mathcal{K}$-algebra given by

1. $M \otimes N$ is a graded $\mathcal{K}$-module defined in (3.3.a).
2. Multiplication is defined by $(m \otimes n)(m' \otimes n') := (-1)^{\deg(n)\deg(m')} mm' \otimes nn'$.
   Note: One can check that this multiplication is commutative.

We have the following universality description of $M \otimes N$. Let A be graded commutative $\mathcal{K}$-algebra. Let $f: M \times N \to A$ be a bilinear graded homomorphism with

$$f((m,n)(m',n')) = (-1)^{\deg(n)\deg(m')} f(m,n)f(m',n').$$

There exists an unique $\hat{f}: M \otimes N \to A$ so that $f(m,n) = \hat{f}(m \otimes n)$.

Example 3.6. Let $\mathcal{A}_i$ be arrangements. Let $A(\mathcal{A}_i)$ denote the Orlik-Solomon algebra on the arrangement $\mathcal{A}_i$ over the commutative ring $\mathcal{K}$. Then $A(\mathcal{A}_i)$ is a graded commutative algebra over $\mathcal{K}$. Hence, we have defined $\bigotimes_i A(\mathcal{A}_i)$.

We recall the product arrangement. Let $\mathcal{A}_1$ be an arrangement in V_1, and let $\mathcal{A}_2$ be an arrangement in V_2. If $V = V_1 \oplus V_2$, then we put

$$\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2 = \{H \oplus V_2 : H \in \mathcal{A}_1\} \cup \{V_1 \oplus H : H \in \mathcal{A}_2\}.$$

We recall the broken circuit basis for the Orlik-Solomon algebra $A(\mathcal{A})$. Let $S = \{H_{i_1}, ..., H_{i_p}\}$ be an ordered subset of $\mathcal{A}$ with $i_1 < \cdots < i_p$. We say a_S is basic in $A_p(\mathcal{A})$ if

1. S is independent, and
2. For any $1 \le k \le p$, there does not exist a hyperplane $H \in \mathcal{A}$ so that $H < H_{i_k}$ with $\{H, H_{i_k}, H_{i_{k+1}}, ..., H_{i_p}\}$ dependent.

Lemma 3.7. *If $a_\gamma \in A_p(\mathcal{A}_1)$ and $a_\sigma \in A_q(\mathcal{A}_2)$ are basic, then $a_\gamma a_\sigma$ is basic in $A(\mathcal{A}_1 \times \mathcal{A}_2)$.*

Proof. Order the hyperplanes in $\mathcal{A}_1 \times \mathcal{A}_2$ via

1. $H \oplus V_2 < K \oplus V_2$ if $H < K$ in $\mathcal{A}_1$
2. $V_1 \oplus H < V_1 \oplus K$ if $H < K$ in $\mathcal{A}_2$
3. $H \oplus V_2 < V_1 \oplus K$ if $H \in \mathcal{A}_1$ and $K \in \mathcal{A}_2$

Let $a_\gamma \in A_p(\mathcal{A}_1)$ and $a_\sigma \in A_q(\mathcal{A}_2)$ be basic (i.e. in the broken circuit basis). Suppose $\mathcal{A}_1$ is an arrangement in V_1 and $\mathcal{A}_2$ is an arrangement in V_2. Suppose $a_\gamma a_\sigma$ is not basic in $A(\mathcal{A}_1 \times \mathcal{A}_2)$. By definition of the broken circuit basis, there are only two possibilities. Suppose $(\cap_{H\in\gamma}(H \oplus V_2)) \cap (\cap_{H\in\sigma}(V_1 \oplus H)) = \emptyset$. This happens only if $\cap_{H\in\gamma} H = \emptyset$ or $\cap_{H\in\sigma} H = \emptyset$. This is not possible since a_σ and a_γ are basic.

Suppose there exists a hyperplane $H \in \mathcal{A}_1 \times \mathcal{A}_2$ and a subset ρ of $\gamma \cup \sigma$ with $H < \rho$ so that $\{H, \rho\}$ is dependent. But this implies the linear functionals defining the hyperplanes are linearly dependent. Since $Q(\mathcal{A}_1)$ and $Q(\mathcal{A}_2)$ have no common variables, this implies H is dependent upon $\rho \cap \gamma$ or $\rho \cap \sigma$. This contradicts the fact a_γ and a_σ are basic. Our assertion now follows. □

Lemma 3.8. *For the product arrangement, we have $A(\mathcal{A}_1 \times \mathcal{A}_2) \cong A(\mathcal{A}_1) \otimes A(\mathcal{A}_2)$.*

Proof. We define the map $\phi: A(\mathcal{A}_1) \times A(\mathcal{A}_2) \to A(\mathcal{A}_1 \times \mathcal{A}_2)$ on the generators by $\phi(a_\gamma, a_\sigma) := a_\gamma a_\sigma$, and we extend the map ϕ bilinearly.

Moreover, we have

$$\begin{aligned}\phi((a_\gamma, a_\sigma)\cdot(a_{\gamma'}, a_{\sigma'})) &= \phi(a_\gamma a_{\gamma'}, a_\sigma a_{\sigma'})\\ &= a_\gamma a_{\gamma'} a_\sigma a_{\sigma'}\\ &= (-1)^{\deg(\sigma)\cdot\deg(\gamma')} a_\gamma a_\sigma a_{\gamma'} a_{\sigma'}\\ &= (-1)^{\deg(\sigma)\cdot\deg(\gamma')} \phi(a_\gamma, a_\sigma)\cdot\phi(a_{\gamma'}, a_{\sigma'}).\end{aligned}$$

By the universal mapping property, there exists

$$\hat{\phi}:\ A(\mathcal{A}_1)\otimes A(\mathcal{A}_2)\to A(\mathcal{A}_1\times A_2)$$

so that $\phi = \hat{\phi}\pi$, where $\pi : A(\mathcal{A}_1)\times A(\mathcal{A}_2)\to A(\mathcal{A}_1)\otimes A(\mathcal{A}_2)$ is the canonical projection. Now, $\hat{\phi}$ is clearly surjective. All that remains is to verify injectivity.

Let $a_\gamma \in A_p(\mathcal{A}_1)$ and $a_\sigma \in A_q(\mathcal{A}_2)$ be basic. Suppose $\hat{\phi}(\sum \alpha_i a_{\gamma_i}\otimes a_{\sigma_i}) = 0$. Then by the linearity of $\hat{\phi}$, we have $\sum \alpha_i a_{\gamma_i} a_{\sigma_i} = 0$. Since a_{γ_i} and a_{σ_i} are basic in $A(\mathcal{A}_1)$ and $A(\mathcal{A}_2)$, we have $a_{\gamma_i} a_{\sigma_i}$ is basic in $A(\mathcal{A}_1\times\mathcal{A}_2)$. Hence, we must have $\alpha_i = 0$ for each i. ▫

Suppose $\mathcal{A} = \mathcal{A}_1\times\mathcal{A}_2$. By Lemma 3.7, $A_1(\mathcal{A})$ can be identified with the linear space $A_1(\mathcal{A}_1)\oplus A_1(\mathcal{A}_2)$. Let $a\in A_1(\mathcal{A})$. We may express $a = a_1 + a_2$ uniquely for $a_1\in A_1(\mathcal{A})$ and $a_2 \in A_1(\mathcal{A}_2)$. For the chain complexes $(A(\mathcal{A}_1), a_1)$ and $(A(\mathcal{A}_2), a_2)$, we recall tensor products of chain complexes; see [8].

Let the differential (multiplication by a_i) for the complex $(A(\mathcal{A}_i), a_i)$ be denoted d_i for $i = 1, 2$. The differential for the chain complex $(A(\mathcal{A}_1), a_1)\otimes(A(\mathcal{A}_2), a_2)$, written $d_1\otimes d_2$, is defined on generators as

$$(d_1\otimes d_2)(a_\gamma\otimes a_\sigma) := a_1 a_\gamma\otimes a_\sigma + (-1)^{\deg(a_\gamma)} a_\gamma\otimes a_2 a_\sigma.$$

Lemma 3.9. *Let $\mathcal{A} = \mathcal{A}_1\times\mathcal{A}_2$. Let $a\in A_1(\mathcal{A})$ with $a = a_1 + a_2$ for $a_1\in A_1(\mathcal{A}_1)$ and for $a_2\in A_1(\mathcal{A}_2)$. As chain complexes, $(A(\mathcal{A}_1\times\mathcal{A}_2), a) = (A(\mathcal{A}_1)\otimes A(\mathcal{A}_2), d_1\otimes d_2)$*

Proof. From Lemma 3.8, we have $A(\mathcal{A}_1\times\mathcal{A}_2) = A(\mathcal{A}_1)\otimes A(\mathcal{A}_2)$. For a basic element $a_\rho\in A_p(\mathcal{A}_1\times\mathcal{A}_2)$, we may write $a_\rho = a_\gamma\cdot a_\sigma$ with $a_\gamma\in A_m(\mathcal{A}_1)$ and $a_\sigma\in A_n(\mathcal{A}_2)$ and $m+n = p$. Hence, multiplying by a in the chain complex $A(\mathcal{A}_1\times\mathcal{A}_2)$ gives the differential defined on generators as

$$\begin{aligned}a\cdot a_\rho =&(a_1+a_2)(a_\gamma\cdot a_\sigma)\\ =&a_1 a_\gamma\cdot a_\sigma + (-1)^{\deg(a_\gamma)} a_\gamma\cdot a_2 a_\sigma.\end{aligned}$$

The result follows immediately. ▫

Theorem 3.10. *Let $\mathcal{A} = \mathcal{A}_1\times\mathcal{A}_2$ be a product arrangement. Let $a\in A_1(\mathcal{A}_1\times\mathcal{A}_2)$. Write $a = a_1 + a_2$ for $a_1\in A_1(\mathcal{A}_1)$ and $a_2\in A_1(\mathcal{A}_2)$. Let $\mathcal{K}$ be a field. We have:*

$$H^m(A(\mathcal{A}_1\times\mathcal{A}_2), a) = \bigoplus_{p+q=m} H^p(A(\mathcal{A}_1), a_1)\otimes H^q(A(\mathcal{A}_2), a_2).$$

Proof. By Lemma 3.9, this is a direct application of the Künneth Formula (see [7]) to the cochain complex $(A(\mathcal{A}_1)\otimes A(\mathcal{A}_2), d_1\otimes d_2)$. ▫

4 $H^*(A(\mathcal{A}), a)$

In this section, we use the results of §2 and §3 to establish necessary and sufficient conditions for $H^*(A(\mathcal{A}), a) = 0$.

We recall an arrangement $\mathcal{A} = \{H_1, \ldots, H_n\}$ is central if $\cap_{i=1}^n H_i \neq \emptyset$. An affine arrangement $\mathcal{A}$ is an arrangement that may or may not be central.

Let $\mathcal{A}$ be an affine arrangement. We may write

$$\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2 \times \cdots \times \mathcal{A}_k \times \mathcal{B},$$

where $\mathcal{A}_i$ are each central and $\mathcal{B}$ is not central. Moreover, we may assume each $\mathcal{A}_i$ contains no proper central factors and $\mathcal{B}$ contains no central factors; otherwise, we would decompose the arrangement further. For $a \in A_1(\mathcal{A})$, write $a = a_1 + \cdots + a_k + a_{\mathcal{B}}$ for $a_j \in A_1(A_j)$ and $a_{\mathcal{B}} \in A_1(\mathcal{B})$.

Example 4.1. Let $Q(\mathcal{A}) = x(x-1)y(y-1)$. Although $\mathcal{A}$ is a product of affine arrangements, $\mathcal{A}$ contains no central factors. Hence, $\mathcal{A} = \mathcal{B}$ in this case.

We recall the deconing process. Suppose $\mathcal{A}$ is central. We form the deconed arrangement, $d\mathcal{A}$, of $\mathcal{A}$ as follows. Let α_i be the functional corresponding to H_i. Without loss of generality, we may assume $\alpha_1 = x_1$. Decone at $\alpha_1 = x_1$ by setting $x_1 = 1$.

We recall the coning process. Let $\mathcal{A}$ be an affine arrangement given by a polynomial $Q = \prod_{i=1}^n \alpha_i$ in the affine space $V = \{(x_1, \ldots, x_\ell)\}$. Put $cV = \{(x_0, x_1, \ldots, x_\ell)\}$ and regard V as the hyperplane of cV defined by $x_0 = 1$. Let $S = F[x_1, \ldots, x_\ell]$ and $S^c = F[x_0, x_1, \ldots, x_\ell]$ be the coordinate rings of V and cV, respectively. For each $g \in S$ define the homogenization of g as

$$g^h = x_0^{\deg g} g(x_1/x_0, \ldots, x_\ell/x_0)$$

where $\deg g$ is the degree of g. Then the coned arrangement $\mathcal{A}^c$ is the central arrangement in cV defined by the polynomial $Q^c = x_0 \prod_{i=1}^n \alpha_i^h$.

Lemma 4.2. *If $\mathcal{A}$ is a central hyperplane arrangement and $\mathcal{A}$ contains no proper central factor, then $d\mathcal{A}$ contains no central factor.*

Proof. Suppose $d\mathcal{A}$ contains a central factor. There exist subarrangements $\mathcal{C}_1$ and $\mathcal{C}_2$ of $d\mathcal{A}$ so that $d\mathcal{A} = \mathcal{C}_1 \times \mathcal{C}_2$. Moreover, wemay assume $\mathcal{C}_1$ is central. Since $\mathcal{C}_1$ is central, by taking a linear change of coordinates if necessary, we may assume the hyperplanes of $\mathcal{C}_1$ pass through the origin; i.e. we are assuming the defining equation $Q(\mathcal{C}_1)$ consists of linear functionals. Then by coning, we obtain $\mathcal{A}$. Since the defining equation of $\mathcal{C}_1$ is unaffected by coning the arrangement $d\mathcal{A}$, we have constructed a central factor of $\mathcal{A}$. This contradicts the assumption that $\mathcal{A}$ contains no central factors. □

Example 4.3. To demonstrate the proof of Lemma 4.2, we consider an arrangement $\mathcal{A}$ where both $\mathcal{A}$ and $d\mathcal{A}$ contain a central factor. Let $Q(\mathcal{A}) = z(x-z)(y-z)(w-z)w$. When deconing at $z = 1$, we have $Q(d\mathcal{A}) = (x-1)(y-1)(w-1)w$. Take $Q(\mathcal{C}_1) = (x-1)(y-1)$ and $Q(\mathcal{C}_2) = (w-1)w$. Let $\hat{x} = x - 1$ and $\hat{y} = y - 1$. We have $Q(d\mathcal{A}) = \hat{x}\hat{y}(w-1)w$. When coning, we have $Q((d\mathcal{A})^c) = z\hat{x}\hat{y}(w-z)w$. By taking the linear change of coordinate $\hat{x} = x - z$ and $\hat{y} = y - z$, we see that $\mathcal{A}$ and $(d\mathcal{A})^c$ are linearly isomorphic.

The rank of $\mathcal{A}$ is defined by

$$r(\mathcal{A}) = \max_{X \in L(\mathcal{A})} \operatorname{rank}(X).$$

Let $\operatorname{rank}(\mathcal{A}) = \ell$. The Euler characteristic is given by

$$\chi(\mathcal{A}) = \sum_{i=1}^{\ell} (-1)^i \dim A_i(\mathcal{A}).$$

We also note that $\chi(d\mathcal{A})$ depends only on $L(\mathcal{A})$, see [11].

Let $\mathcal{A}$ be an arrangement. Let $H_0 \in \mathcal{A}$. We recall the arrangements given by deletion and restriction

$$\mathcal{A}' = \{H : \ H \in \mathcal{A} \setminus H_0\}, \text{and}$$

$$\mathcal{A}'' = \{H_0 \cap H : \ H \in \mathcal{A} \text{ and } H \cap H_0 \neq \emptyset\}.$$

We need the following lemmas and proposition, established in [5].

Lemma 4.4. *Let $\mathcal{A}$ be an affine arrangement with $r(\mathcal{A}) > 1$. If $\mathcal{A}$ does not contain a central factor, then for any distinguished hyperplane $H_0 \in \mathcal{A}$ either $\mathcal{A}'$ or A'' does not contain a central factor.*

Proof. We refer to the proof given in Lemma to Theorem II in [5]. □

We define

$$\beta(\mathcal{A}) := (-1)^{r(\mathcal{A})}\chi(\mathcal{A}).$$

Lemma 4.5. *Let $\mathcal{A}$ be an arrangement with $|\mathcal{A}| > 1$. If $\mathcal{A}$ is not central, then there exists $H_0 \in \mathcal{A}$ so that* $\operatorname{rank}(\mathcal{A}') = \operatorname{rank}(\mathcal{A})$. *With respect to H_0, we have the equality $\beta(\mathcal{A}) = \beta(\mathcal{A}') + \beta(\mathcal{A}'')$. If $\mathcal{A}$ is central, then this inequality holds for any $H \in \mathcal{A}$.*

Proof. Suppose $\mathcal{A}$ is not central. Then there exists a maximal element $T \in L(\mathcal{A})$ and a hyperplane $H_0 \not< T$. Hence, T is a maximal element in $L(\mathcal{A}')$. Since $\chi(A) = \chi(\mathcal{A}') - \chi(\mathcal{A}'')$ by Theorem 2.56 in [10], we have

$$(-1)^{r(\mathcal{A})}\chi(\mathcal{A}) = (-1)^{r(\mathcal{A})}\chi(\mathcal{A}') - (-1)^{r(\mathcal{A})}\chi(\mathcal{A}'').$$

We have $r(\mathcal{A}'') = r(\mathcal{A}) - 1$ and $\operatorname{rank}(\mathcal{A}') = \operatorname{rank}(\mathcal{A})$, so

$$\beta(\mathcal{A}) = \beta(\mathcal{A}') + \beta(\mathcal{A}'').$$

If $\mathcal{A}$ is central, then $\mathcal{A}'$ and $\mathcal{A}''$ are both central, so $\chi(A) = \chi(\mathcal{A}') = \chi(\mathcal{A}'') = 0$. □

Lemma 4.6. *If $\mathcal{A}$ is an arrangement, then $\beta(\mathcal{A}) \geq 0$.*

Proof. We induct on $|\mathcal{A}|$.

If $\mathcal{A} = \emptyset$, then $\beta(\mathcal{A}) = 1$. If $|\mathcal{A}| = 1$, then $\beta(\mathcal{A}) = 0$.

Assume $\beta(\mathcal{B}) \geq 0$ for all arrangements $\mathcal{B}$ with $|\mathcal{B}| < k$. Suppose $|\mathcal{A}| = k > 1$. By Lemma 4.5, we have $\beta(\mathcal{A}) = \beta(\mathcal{A}') + \beta(\mathcal{A}'')$ for some hyperplane $H_0 \in \mathcal{A}$. By the induction hypothesis, we have $\beta(\mathcal{A}'), \beta(\mathcal{A}'') \geq 0$. We therefore have $\beta(\mathcal{A}) \geq 0$ as required. □

Proposition 4.7. *Let $\mathcal{A}$ be an affine arrangement. We have $\chi(\mathcal{A}) \neq 0$ if and only if $\mathcal{A}$ contains no central factors.*

Proof. Suppose $\mathcal{A}$ contains a central factor; that is, $\mathcal{A} = \mathcal{B} \times \mathcal{C}$, where $\mathcal{B}$ is central. Then $\chi(\mathcal{A}) = \chi(\mathcal{B})\chi(\mathcal{C})$; see Lemma 2.50 in [10]. Since $\mathcal{B}$ is central, we have $\chi(\mathcal{B}) = 0$; see Proposition 2.51 in [10]. Hence, $\chi(\mathcal{A}) = 0$.

Suppose $\mathcal{A}$ contains no central factors. We want to show $\chi(\mathcal{A}) \neq 0$. It will suffice to show $\beta(\mathcal{A}) \neq 0$. We proceed by induction. Suppose $|\mathcal{A}| = 2$, then $\mathcal{A}$ consists of two hyperplanes which don't intersect; hence, $\chi(\mathcal{A}) = -1$.

Suppose for any $\mathcal{B}$ with $|\mathcal{B}| < k$ ($k > 1$) for which $\mathcal{B}$ contains no central factors, we have $\beta(\mathcal{B}) \neq 0$ (hence, $\chi(\mathcal{B}) \neq 0$). Fix $H_0 \in \mathcal{A}$ so that $\beta(\mathcal{A}) = \beta(\mathcal{A}') + \beta(\mathcal{A}'')$. We apply Lemma 4.6 to see that $\mathcal{A}'$ or $\mathcal{A}''$ contains no central factors. By the induction hypothesis, $\beta(\mathcal{A}') > 0$ or $\beta(\mathcal{A}'') > 0$. Since $\beta(\mathcal{A}) = \beta(\mathcal{A}') + \beta(\mathcal{A}'')$ and $\beta(\mathcal{A}'), \beta(\mathcal{A}'') \geq 0$ with at least one positive, we have $\beta(\mathcal{A}) > 0$. Hence, $\chi(\mathcal{A}) \neq 0$ as required. □

Theorem 4.8. *Let $\mathcal{A}$ be an affine arrangement. We may write*

$$\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2 \times \cdots \times \mathcal{A}_k \times \mathcal{B},$$

where $\mathcal{A}_j$ are each central and $\mathcal{B}$ not central and they contain no proper central factors. For $a \in A_1(\mathcal{A})$, write $a = a_1 + \cdots + a_k + a_{\mathcal{B}}$ for $a_j \in A_1(A_j)$ and $a_{\mathcal{B}} \in A_1(\mathcal{B})$. We have $H^*(A(\mathcal{A}), a) \neq 0$ if and only if $\sum_{H_i \in \mathcal{A}_j} \lambda_i = 0$ for all $1 \leq j \leq k$.

Proof. We use the Künneth Formula from Theorem 3.10; that is,

$$H^m(A(\mathcal{A}), a) = \bigoplus_{p_1 + \cdots p_k + p_{k+1} = m} H^{p_1}(A(\mathcal{A}_1), a_1) \otimes \cdots \otimes H^{p_k}(A(\mathcal{A}_k), a_k) \otimes H^{p_{k+1}}(A(\mathcal{B}), a_{\mathcal{B}}).$$

Suppose $\sum_{A_1(\mathcal{A}_j)} \lambda_i \neq 0$ for some j. We have $H^*(\mathcal{A}_j, a_j) = 0$ since $\mathcal{A}_j$ is central; we refer to [11]. By the Künneth Formula, it follows that $H^*(\mathcal{A}, a) = 0$.

Suppose $\sum_{A_1(\mathcal{A}_j)} \lambda_i = 0$ for all j. By the Künneth Formula, it will suffice to show $H^*(A(\mathcal{A}_i), a_i) \neq 0$ and $H^*(A(\mathcal{B}), a_{\mathcal{B}}) \neq 0$. Since $\mathcal{B}$ contains no central factors, we have $\chi(\mathcal{B}) \neq 0$. Hence, $H^*(A(\mathcal{B}), a_{\mathcal{B}}) \neq 0$.

Take $\tilde{a} = \sum_{i=2}^{n} \lambda_i a_i$. We consider the chain complex formed by multiplication by $\tilde{a}$, $(d\mathcal{A}, \tilde{a})$; here, $d\mathcal{A}$ is $\mathcal{A}$ deconed at H_1. Since we have the short exact sequences, see [11]

$$0 \to H^{p-1}(A(d\mathcal{A}_i), \tilde{a}) \to H^p(A(\mathcal{A}_i), a) \to H^p(A(d\mathcal{A}_i), \tilde{a}) \to 0,$$

it will suffice to show $H^*(A(d\mathcal{A}_i), \tilde{a}) \neq 0$. But by Lemma 4.2, $d\mathcal{A}_i$ contains no central factors, so by Proposition 4.7 $\chi(d\mathcal{A}_i) \neq 0$; hence, $H^*(A(d\mathcal{A}_i), \tilde{a}) \neq 0$ as required. □

We recall the following theorem from [11].

Theorem 4.9. (*Yuzvinsky* [11]) *Let* $\mathcal{A} = \{H_1, \dots, H_n\}$ *be an affine arrangement with* $\operatorname{rank}(\mathcal{A}) = \ell$. *Let* $a = \sum_{i=1}^{n} \lambda_i a_i \in A_1(\mathcal{A})$ *and satisfy the condition* $\sum_{X \subset H_i} \lambda_i \neq 0$ *for all* $X(\mathcal{A})$ *such that* $\chi(dA(\mathcal{A}_X)) \neq 0$. *Then* $H^p(A(\mathcal{A}), a) = 0$ *for every* $p < \ell$.

We use Theorem 4.9 in conjunction with the upper semicontinuity of the map $t \mapsto \dim H^p(A(\mathcal{A}), t)$ discussed in §2 in order to establish conditions under which $\dim H^\ell(A(\mathcal{A}), a) \neq 0$ for an affine ℓ-arrangement $\mathcal{A}$.

Theorem 4.10. *Let* $\mathcal{A}$ *be an arbitrary* ℓ*-arrangement with* $|\mathcal{A}| = n$. *Suppose* $\mathcal{A}$ *contains no central factors. Let* $a = \sum_{i=1}^{n} \lambda_i a_i \in A_1(\mathcal{A})$. *We have:*

$$H^\ell(A(\mathcal{A}), a) \neq 0.$$

Proof. If $a = 0$, then $H^\ell(A(\mathcal{A}), a) = Z_\ell(A(\mathcal{A}), a) = A_\ell(\mathcal{A}) \neq 0$. Hence, we may assume $a \neq 0$ for the remainder of the proof.

Let

$$S = \{X \in L(\mathcal{A}) : \sum_{X \subset H_j} \lambda_j \neq 0\}.$$

Since $a \neq 0$, we have $S \neq \emptyset$. We define

$$\kappa := \min \{|\sum_{X \subset H_j} \lambda_j| : \ X \in S\}.$$

Since $S \neq \emptyset$, $\kappa > 0$.

We now construct a sequence $\alpha_i \in F^n$ so that $(\alpha_i)_j \to \lambda_j$ as $i \to \infty$. For $i \in \mathbb{N}$, define

$$(\alpha_i)_j := \lambda_j + \frac{\kappa}{n \cdot 2^i}.$$

It is clear that $(\alpha_i)_j \to \lambda_j$ as $i \to \infty$. Moreover, we now show $\sum_{X \subset H_j} (\alpha_i)_j \neq 0$ for any $X \in L(\mathcal{A})$ and any $i \in \mathbb{N}$.

Fix $X \in L(\mathcal{A})$. If $\sum_{X \subset H_j} \lambda_j = 0$, then $\sum_{X \subset H_j} (\alpha_i)_j \neq 0$ since $\kappa > 0$.

Suppose $\sum_{X \subset H_j} \lambda_j \neq 0$. If $\sum_{X \subset H_j} \lambda_j > 0$, then $\sum_{X \subset H_j} (\alpha_i)_j > 0$ since $\dfrac{\kappa}{n \cdot 2^i} > 0$.

Suppose $\sum_{X \subset H_j} \lambda_j < 0$. Then

$$\begin{aligned}
\sum_{X \subset H_j} (\alpha_i)_j = & \sum_{X \subset H_j} (\lambda_j + \frac{\kappa}{n \cdot 2^i}) \\
= & (\sum_{X \subset H_j} \lambda_j) + \frac{|X| \cdot \kappa}{n \cdot 2^i} \\
\leq & (\sum_{X \subset H_j} \lambda_j) + \frac{\kappa}{2^i} \\
< & 0,
\end{aligned}$$

where the last inequality is true because of the definition of κ.

Therefore, for any $i \in \mathbb{N}$, we have α_i satisfies the condition of Theorem 4.9 ensuring $\dim H^p(A(\mathcal{A}), \alpha_i) = 0$ for $p < \ell$. Since $\mathcal{A}$ contains no central factors, we have $H^*(A(\mathcal{A}), \alpha_i) \neq 0$; hence, $\dim H^\ell(A(\mathcal{A}), \alpha_i) \neq 0$. By Theorem 2.12, the function $t \mapsto \dim H^\ell(A(\mathcal{A}), t)$ is upper semicontinuous in t; therefore, $\dim H^\ell(A(\mathcal{A}), a) > 0$ as required. □

Theorem 4.11. *Let $\mathcal{A}$ be an affine ℓ-arrangement. We may write*

$$\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2 \times \cdots \times \mathcal{A}_k \times \mathcal{B},$$

where $\mathcal{A}_j$ are each central and $\mathcal{B}$ not central and they contain no proper central factors. Let $a \in A_1(\mathcal{A})$. We have $\dim H^\ell(A(\mathcal{A}), a) \neq 0$ if and only if $\sum_{H_i \in \mathcal{A}_j} \lambda_i = 0$ for all j.

Proof. Let $\mathcal{A}_j$ be an ℓ_j-arrangement, and let $\mathcal{B}$ be an $\ell_{\mathcal{B}}$-arrangement. Then $\mathcal{A}$ is an $(\ell_{\mathcal{B}} + \sum_{i=1}^{k} \ell_j)$-arrangement. Since we have the short exact sequences

$$0 \to H^{p-1}(A(d\mathcal{A}_i), \tilde{a}) \to H^p(A(\mathcal{A}_i), a) \to H^p(A(d\mathcal{A}_i), \tilde{a}) \to 0$$

and the Künneth Formula

$$H^m(A(\mathcal{A}), a) = \bigoplus_{p_1 + \cdots p_k + p_{k+1} = m} H^{p_1}(A(\mathcal{A}_1), a_1) \otimes \cdots \otimes H^{p_k}(A(\mathcal{A}_k), a_k) \otimes H^{p_{k+1}}(A(\mathcal{B}), a_{\mathcal{B}}),$$

it will suffice to show $H^{\ell_j - 1}(A(d\mathcal{A}_j), \tilde{a}_j) \neq 0$ and $H^{\ell_{\mathcal{B}}}(A(\mathcal{B}), a_{\mathcal{B}}) \neq 0$. This result was established in Theorem 4.10. □

Theorem 4.12. *Let $\mathcal{A}$ be an arrangement with $\ell = \mathrm{rank}(\mathcal{A})$. Fix $a \in A_1(\mathcal{A})$. Then $H^*(A(\mathcal{A}), a) = 0$ if and only if $H^\ell(A(\mathcal{A}), a) = 0$.*

Proof. This follows immediately from Theorem 4.11 and Theorem 4.8. □

References

[1] A. Aramova, L. Avramov, and J. Herzog, *Resolutions of monomial ideals and cohomology over exterior algebras*, Trans. Amer. Math. Soc. 352, no. 2 (2000), 579-594.

[2] V.I. Arnold, *The cohomology ring of the colored braid group*, Mat. Zametki 5 (1969), 138-140.

[3] E. Brieskorn, *Sur les groupes de tresses*, Seminaire Bourbaki 1971/72, Lecture Notes in Math. 317 (1973), Springer Verlag, 21-44.

[4] J. Conway, *A Course in Functional Analysis*, Graduate Texts in Mathematics 96, Springer-Verlag, Berlin, 1990.

[5] H. Crapo, *A Higher Invariant for Matroids*, Journal of Combinatorial Theory 2, (1967), 406-417.

[6] A. Libgober and S. Yuzvinsky, *Cohomology of the Brieskorn-Orlik-Solomon Algebras and local systems*, Compositio Math (to appear).

[7] S. Maclane, *Homology*, Die Grundlehren Der Mathematischen Wissenschaften in Einzeldarstellungen 114, Springer-Verlag, Berlin, 1963.

[8] J. Munkres, *Elements of Algebraic Topology*, Addison-Wesley Publishing Company, 1984.

[9] P. Orlik and L. Solomon, *Combinatorics and topology of complements of hyperplanes*, Invent. math. 56 (1980), 167-189.

[10] P. Orlik and H. Terao, *Arrangements of Hyperplanes*, Grundlehren der mathematischen Wissenschaften 300, Springer-Verlag, Berlin, 1991.

[11] S.Yuzvinsky, *Cohomology of Brieskorn-Orlik-Solomon Algebras*, Communications in Algebra **23(14)** (1995), 5339-5354.

Electronic Journal: Southwest Journal of Pure and Applied Mathematics
Internet: http://rattler.cameron.edu/swjpam.html
ISSN 1083-0464
Issue 1, June 2001, pp. 93–101.
Submitted: April 13, 2001. Published: June 30, 2001.

MAPS IN UNIFORM SPACES: TOWARD A UNIFIED APPROACH

Jaime Rodríguez–Montes And Jairo A. Charris

Abstract A simple condition on the functions $\varphi : \mathbb{R}^+ \to \mathbb{R}^+ = [0, +\infty)$ allows to establish results on fixed and coincidence points for contractive (when $\varphi(t) < t$ for all $t > 0$)or expansive (provided $\varphi(t) > t,\ t > 0$) maps with respect to a w-distance. These generalize recent results of different people, including the authors of the present paper. As in a previous work, an attempt is made toward a unification of the point of view in contractions and expansions.

1991 A.M.S. Subject Classification Codes:. 4H25, 47H10

Key words and phrases:. Metric (distance), w-distance, contractive and expansive maps for a distance or a w-distance, fixed and coincidence points of maps, Cauchy sequence, uniform and complete uniform spaces.

1.Introduction

A fundamental result in fixed point theory is the Banach contraction principle, which has been extended in many different directions ([2], [4], [9]). Of somewhat lesser importance is the expansion principle ([5], [8], [12], [13]). Recently, O. Kada , T. Suzuki and W. Takahashi [7] have introduced the concept of a w-distance on a metric space and have generalized important results in non-convex minimizations and in fixed point theory for both w-contractive or w-expansive maps.

In this paper we follow ideas in [4], [11], [12] to establish some new results on fixed and coincidence points of maps by means of appropriate w-contractive or w-expansive assumptions. These generalize results in [4], [6], [11], [12], [13] and in many other places. As in [12], a unified point of view for contractions and expansions is sought after by means of the same techniques in that paper but now adapted to w-distances.

In this section we review the basic definitions and the known results that are relevant to our purposes. To begin with, we slightly generalize the concept of a w-distance on

Universidad Nacional, Bogotá, Colombia
E-mail: jarodrig@matematicas.unal.edu.co

a metric space in [7] to that of a w-distance on a *uniform space*. For the description of the latter concept and related notions we refer to [1], Chapter II or to [3], Chapter IX. We call *uniform space* (X, ϑ) a nonempty set X endowed of a *uniformity* ϑ, the latter being a special kind of filter on $X \times X$, all whose elements contain the diagonal $\Delta = \{(x, x) | x \in X\}$.

If $V \in \vartheta$ and $(x, y) \in V$, $(y, x) \in V$, x and y are said to be V–close, and a sequence (x_n) in X is a *Cauchy sequence* for ϑ if for any $V \in \vartheta$, $N \geq 1$ exists such that x_m and x_n are V-close for m, $n \geq N$. A uniformity ϑ defines a unique topology $\tau(\vartheta)$ on X for which the neighborhoods of $x \in X$ are the sets $V(x) = \{y \in X | (x, y) \in V\}$ when V runs over ϑ, and a subspace Y of X is called *complete* if any Cauchy sequence of Y for ϑ converges to a point of Y for $\tau(\vartheta)$ (as a matter of fact, this is the definition of *sequential completeness*, the more general concept of *completeness* requiring the notion of a *Cauchy filter* ([1] pp. 197, 201), but it is all we need for our purposes). A uniform space (X, ϑ) is said to be *Hausdorff, compact, connected,* etc., if the topological space $(X, \tau(\vartheta))$ has the corresponding property.
We will assume all the relevant uniform spaces (X, ϑ) *to be Hausdorff*, which ocurrs if and only if the intersection of all the $V \in \vartheta$ reduces to the diagonal Δ of X (([1], p. 187); i.e., if $(x, y) \in V$ for all $V \in \vartheta$ implies $x = y$). This guarantees the uniqueness of limits of sequences.

A metric or distance d on X defines a Hausdorff uniformity, called the *uniformity of* d. This is the set ϑ_d of all subsets of $X \times X$ containing a "*band*" $B_\epsilon = \{(x, y) | d(x, y) < \epsilon\}$ for some $\epsilon > 0$. The topology $\tau(\vartheta_d)$ on X is the usual topology τ_d of d. Hence, results on general uniform spaces apply to metric spaces, but many other mathematical objects carry natural uniformities. Such is the case of topological groups, topological vector spaces, general (not necessarily metrizable) compact spaces, and also of subspaces of all these spaces. Thus, results on general uniform spaces apply equally well to these objects, which was the motivation for A. Weil [14] introduction of the concept. This also widens the scope of results on w-contractions or w-expansions.

Remark 1.1. If (X, ϑ) is a uniform space and $V \in \vartheta$, also $V^{-1} = \{(y, x) | (x, y) \in V\} \in \vartheta$ ([1], p. 181). If $V \in \vartheta$ and $V^{-1} = V$, V is called *symmetrical.* Observe that if V is symmetrical and $(x, y) \in V$ then $(y, x) \in V$, and x and y are automatically V-close. Since $V \cap V^{-1} \subseteq V$, any $V \in \vartheta$ contains a $W \in \vartheta$, symmetrical, and if $(x, y) \in W$ then x and y are both W and V-close. For many purposes it is then enough to assume $V \in \vartheta$ symmetrical.

Remark 1.2. When topological concepts are mentioned in the context of a uniform space (X, ϑ) (continuity or semi-continuity of maps on X, for example), they always refer to the topological space $(X, \tau(\vartheta))$.

Definition 1.1. *Let* (X, ϑ) *be a uniform space. A function* $\mathfrak{p} : X \times X \longrightarrow \mathbb{R}^+ = [0, \infty)$ *is called a w-distance on* X *(or more precisely, on* (X, ϑ)*), if the following conditions hold for all* x, y, z *in* X*:*

(1) $\mathfrak{p}(x, z) \leq \mathfrak{p}(x, y) + \mathfrak{p}(y, z)$.
(2) $\mathfrak{p}(x, \cdot) : X \longrightarrow \mathbb{R}^+$ *is lower semi-continuous.*
(3) *For any* $V \in \vartheta$ *there is* $\delta > 0$ *such that if* $\mathfrak{p}(z, x) \leq \delta$ *and* $\mathfrak{p}(z, y) \leq \delta$ *then* $(x, y) \in V$.

For example, if $\mathfrak{p}$ is a distance on X satisfying (2) above and such that $\vartheta \subseteq \vartheta_{\mathfrak{p}}$, then $\mathfrak{p}$ is a w-distance on (X, ϑ). Definition 1.1 is in [7] within the frame of metric spaces.

The following lemma contains some useful properties of w-distances. It is stated in [7] for metric spaces. The proof is straightforward.

Lemma 1.1. *Let (X, ϑ) be a uniform space and $\mathfrak{p}$ be a w-distance on X. Let (x_n), (y_n) be arbitrary sequences in X and (α_n), (β_n) be sequences in $\mathbb{R}^+$ converging to 0. Then, for $x, y, z \in X$, the following holds:*

(a) *If $\mathfrak{p}(x_n, y) \leq \alpha_n$ and $\mathfrak{p}(x_n, z) \leq \beta_n$ for all $n \in \mathbb{N}$, then $y = z$. In particular, if $\mathfrak{p}(x, y) = 0$ and $\mathfrak{p}(x, z) = 0$, then $y = z$.*
(b) *If $\mathfrak{p}(x_n, y_n) \leq \alpha_n$ and $\mathfrak{p}(x_n, z) \leq \beta_n$ for all $n \in \mathbb{N}$, then (y_n) converges to z.*
(c) *If $\mathfrak{p}(x_n, x_m) \leq \alpha_n$ for all $m, n \in \mathbb{N}$ with $m > n$, then (x_n) is a Cauchy sequence in (X, ϑ).*
(d) *If $\mathfrak{p}(y, x_n) \leq \alpha_n$ for all $n \in \mathbb{N}$, then (x_n) is a Cauchy sequence in (X, ϑ).*

Throughout this paper, Φ and Ψ will respectively denote the sets of maps ϕ and ψ of $\mathbb{R}^+$ into itself such that $\phi(t) < t$ and $\psi(t) > t$ for all $t > 0$, and satisfying the following condition:

Condition(A). *For any $\varphi \in \Phi \cup \Psi$ and any decreasing sequence (t_n) in $\mathbb{R}^+$ such that*

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} \varphi(t_n) = t,$$

it follows that $t = 0$.

Remark 1.3. Condition (A) above is the simple condition mentioned in the abstract. In [12], it proved a versatile and valuable instrument in establishing results on contractions and expansions with respect to the metric d in a metric space (X, d). We will see it is also useful in the context of w-distances. Condition (A) is generally satisfied by functions appearing in the framework of contractions or expansions. It is obviously equivalent to the apparently stronger condition below:

Condition (B). *For any $\varphi \in \Phi \cup \Psi$ and any non-increasing sequence (t_n) in $(0, +\infty)$, if*

$$\lim t_n = \lim \varphi(t_n) = t,$$

then $t = 0$.

Remark 1.4. For $\phi \in \Phi$, it is obvious that $\phi(0+) = \lim_{t\to 0+} \phi(t) = 0$. For $\psi \in \Psi$, it may happen that $\psi(0+)$ either does not exist or $\psi(0+) \neq 0$. The assumption $\psi(0+) = 0$ for $\psi \in \Psi$ is, however, frequently needed.

For $\psi \in \Psi$, we define $\widehat{\psi} : (0, +\infty) \longrightarrow \mathbb{R}^+$ by $\widehat{\psi}(t) = \sup\{x \mid \psi(x) < t\}$ if $\{x \mid \psi(x) < t\} \neq \emptyset$ and $\widehat{\psi}(t) = 0$ otherwise. Clearly $\widehat{\psi}$ is non–decreasing and $0 \leq \widehat{\psi}(t) \leq t$ for any $t > 0$. For $\phi \in \Phi$, we also define $\widetilde{\phi} : (0, +\infty) \to \mathbb{R}^+$ by $\widetilde{\phi}(t) = \sup\{x \mid \phi(x) \leq t\}$. Clearly $\widetilde{\phi}$ is non-decreasing and $\phi(t) \leq \widetilde{\phi}(t) \leq t$ for all $t > 0$.

The next two lemmas appear in [12] and will be repeatedly used in the sequel. For completeness we also include their proofs here.

Lemma 1.2. *If $\psi \in \Psi$ and $\psi(0+) = 0$, then*

(a) $0 < \widehat{\psi}(t) \leq t$, *for all* $t > 0$ *;*
(b) *for any $\epsilon > 0$ there is $0 < t \leq \epsilon$ such that $\widehat{\psi}(t) < t$.*

Proof. From $\psi(0+) = 0$ it follows that $\widetilde{\psi}(t) > 0$ for $t > 0$. If (b) were not satisfied, $\epsilon > 0$ could be found such that $\widehat{\psi}(t) = t$ for all $0 < t \leq \epsilon$. Let $0 < \alpha < \epsilon$. Since $\widehat{\psi}(\epsilon) = \epsilon$ then, for some t_1, $\alpha < t_1 < \psi(t_1) < \epsilon$, and having selected t_n such that $\alpha < t_n < \psi(t_n)$, we could choose t_{n+1} such that $\alpha < t_{n+1} < \psi(t_{n+1}) < t_n$, from which it follows that $t \geq \alpha > 0$ exists such that $\lim t_n = \lim \psi(t_n) = t > 0$. This is absurd. $\square$

Lemma 1.3. *Let $\phi \in \Phi$. Then, for any $\epsilon > 0$ there is $t \leq \epsilon$ such that $\widetilde{\phi}(t) < t$.*

Proof. Otherwise, for some $\epsilon > 0$, $\widetilde{\phi}(t) = t$ for all $t \leq \epsilon$, and since $\phi(\epsilon) < \epsilon = \widetilde{\phi}(\epsilon)$, for some t_1 in $\mathbb{R}^+$, $\phi(\epsilon) < \phi(t_1) < t_1 < \epsilon$. Thus, having chosen t_n with $\phi(t_{n-1}) < \phi(t_n) < t_n < t_{n-1}$, from $\widetilde{\phi}(t_n) = t_n$ we could also choose t_{n+1} such that $\phi(t_n) < \phi(t_{n+1}) < t_{n+1} < t_n$. Thus, a sequence (t_n) would exist such that $\lim t_n = \lim \phi(t_n) = t > 0$. Since $\phi \in \Phi$, this is contradictory. $\square$

In the next two sections we establish our main results.

2. Existence theorems of the w-contractive type.

Our first lemma adapts Lemma 2.1 in [12] to contractions for a w-distance.

Lemma 2.1. *Let (Y, ϑ) be a uniform space and $X \subseteq Y$ be a complete subspace of Y. Let f be a map of X into Y and $\mathfrak{p}$ be a w-distance on Y. Further assume that:*

i) There is a sequence (x_n) in X such that $x_{n+1} = f(x_n)$ for all $n \geq 1$.
ii) For any $\delta > 0$ there exists $0 < \delta_1 \leq \delta$ such that $\mathfrak{p}(x, f(z)) < \delta$ whenever $\mathfrak{p}(x, f(x)) < \delta_1$ and $\mathfrak{p}(x, z) \leq \delta$.
iii) Given $\delta_1 > 0$, $N \geq 1$ exists such that $\mathfrak{p}(x_N, f(x_N)) < \delta_1$.

Then, there is $x_0 \in X$ such that $f(x_0) = x_0$ and

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} f(x_n) = x_0. \tag{2.1}$$

Proof. Let $V \in \vartheta$, arbitrary, and let $\delta > 0$ small be such that Condition (3) of Definition 1.1 is satisfied. Also let $0 < \delta_1 \leq \delta$ and $N \geq 1$ be such that assertions *(ii)* and *(iii)* above hold. From $\mathfrak{p}(x_N, f(x_N)) = \mathfrak{p}(x_N, x_{N+1}) < \delta_1 \leq \delta$ and *(ii)*, also $\mathfrak{p}(x_N, f(x_{N+1})) < \delta$. Thus, again from $x_{N+2} = f(x_{N+1})$ and *(ii)*, $\mathfrak{p}(x_N, f(x_{N+2})) < \delta$. Continuing this way we conclude that $\mathfrak{p}(x_N, x_{N+p}) < \delta$, $p = 1, 2, 3, ...$, which implies, in view of *(c)* of Lemma 1.1, that (x_n) is a Cauchy sequence. Completeness of X then ensures that (2.1) holds for some $x_0 \in X$.
Now, from $\mathfrak{p}(x_N, x_m) < \delta$ for $m > N$ and the lower semicontinuity of $\mathfrak{p}$ it follows that $\mathfrak{p}(x_N, x_0) \leq \delta$, and hence , by (ii) above, $\mathfrak{p}(x_N, f(x_0)) < \delta$. Condition (3) of Definition 1.1 then guarantees that $(x_0, f(x_0)) \in V$, so that, since V is arbitrary, $f(x_0) = x_0$. Thus, x_0 is a fixed point of f. $\square$

The next theorem, of the contractive type, adapts Theorem 2.1 in [12] to w-distances.

Theorem 2.1. *Let f be a selfmap of a uniform space (X, ϑ), and assume that either X or $f(X)$ is complete. Let $\mathfrak{p}$ be a w-distance on X such that for some $\phi \in \Phi$,*

$$\mathfrak{p}(f(x), f(y)) \leq \phi(\mathfrak{p}(x, y)), \quad x, y \in X. \tag{2.2}$$

Then, a unique fixed point x_0 of f exists such that (2.1) holds for any sequence (x_n) in X satisfying $x_{n+1} = f(x_n)$, $n \geq 1$.

Proof. Since $f(X) \subseteq X$, a sequence (x_n) in X such that $x_{n+1} = f(x_n)$, $n \geq 1$, clearly exists. From (2.2) we get, provided $\mathfrak{p}(x_n, x_{n+1}) > 0$ for all $n \geq 1,$ *that*

$$\mathfrak{p}(x_{n+1}, x_{n+2}) = \mathfrak{p}(f(x_n), f(x_{n+1})) \leq \phi(\mathfrak{p}(x_n, x_{n+1})) < \mathfrak{p}(x_n, x_{n+1}).$$

Hence, $t_n = \mathfrak{p}(x_n, x_{n+1})$, $n \geq 1$, is a decreasing sequence such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} \phi(t_n) = t \geq 0$, so that $t = 0$. Let $\delta > 0$ be such that $\widetilde{\phi}(\delta) < \delta$ (Lemma 1.3), and let $\delta_1 = \delta - \widetilde{\phi}(\delta)$. If $\mathfrak{p}(x, f(x)) < \delta_1$ and $\mathfrak{p}(x, z) < \delta$, it follows from $\mathfrak{p}(f(x), f(z)) \leq \widetilde{\phi}(\mathfrak{p}(x, z)) \leq \widetilde{\phi}(\delta)$ that $\mathfrak{p}(x, f(z)) \leq \mathfrak{p}(x, f(x)) + \mathfrak{p}(f(x), f(z)) < \delta_1 + \widetilde{\phi}(\delta) = \delta$. Hence, from Lemma 2.1, there is x_0 in X such that (2.1) holds, and then $f(x_0) = x_0$. Now assume that $\mathfrak{p}(x_N, x_{N+1}) = 0$ for some $N \geq 1$. Then, from (2.2), also $\mathfrak{p}(x_{N+1}, x_{N+2}) = 0$, and since $\mathfrak{p}(x_N, x_{N+2}) \leq \mathfrak{p}(x_N, x_{N+1}) + \mathfrak{p}(x_{N+1}, x_{N+2})$, then $\mathfrak{p}(x_N, x_{N+2}) = 0$. Hence, from Lemma 1.1(a), $x_{N+1} = x_{N+2} = f(x_{N+1})$, and x_{N+1} is a fixed point of f. Finally assume that also $f(\widetilde{x}_0) = \widetilde{x}_0$. From (2.2) we have, provided $\mathfrak{p}(x_0, \widetilde{x}_0) > 0$, that $\mathfrak{p}(x_0, \widetilde{x}_0) = \mathfrak{p}(f(x_0), f(\widetilde{x}_0)) \leq \phi(\mathfrak{p}(x_0, \widetilde{x}_0)) < \mathfrak{p}(x_0, \widetilde{x}_0)$. This is absurd. Thus, $\mathfrak{p}(x_0, \widetilde{x}_0) = 0$. Similarly, $\mathfrak{p}(\widetilde{x}_0, x_0) = 0$, so that $\mathfrak{p}(x_0, x_0) \leq \mathfrak{p}(x_0, \widetilde{x}_0) + \mathfrak{p}(\widetilde{x}_0, x_0) = 0$. Lemma 1.1(a) then guarantees that $\widetilde{x}_0 = x_0$, and thus x_0 is the unique fixed point of f. $\square$

The following result is a generalization of Lemma 2.1 above to a pair of maps.

Lemma 2.2. *Let f and g be maps of a set X into a uniform space (Y, ϑ), and assume that $f(X)$ is a complete subspace of Y. Let $\mathfrak{p}$ be a w-distance on Y, and further assume that:*

i) There is a sequence (x_n) in X with $g(x_n) = f(x_{n+1})$, $n \geq 1$.
ii) For any $\delta > 0$ and x, y, z in X, there is $0 < \delta_1 \leq \delta$ such that $\mathfrak{p}(f(x), g(z)) < \delta$ whenever $\mathfrak{p}(f(x), g(x)) < \delta_1$ and $\mathfrak{p}(f(x), f(z)) \leq \delta$.
iii) Given $\delta_1 > 0$, there is $N \geq 1$ such that $\mathfrak{p}(f(x_N), g(x_N)) < \delta_1$.

Then, there is x_0 in X such that

$$\lim f(x_n) = \lim g(x_n) = f(x_0) = g(x_0). \tag{2.3}$$

Proof. Let $V \in \vartheta$ and $\delta > 0$ be as in Definition 1.1(3). Let $0 < \delta_1 \leq \delta$ and $N \geq 1$ be such that assertions *(ii)* and *(iii)* above hold. From $\mathfrak{p}(f(x_N), g(x_N)) = \mathfrak{p}(f(x_N), f(x_{N+1})) < \delta_1 \leq \delta$ and *(ii)*, also $\mathfrak{p}(f(x_N), g(x_{N+1})) = \mathfrak{p}(f(x_N), f(x_{N+2})) < \delta$, and continuing this way, $\mathfrak{p}(f(x_N), f(x_{N+p})) < \delta$, $p = 1, 2, 3, ...$, which implies, in view of *(c)* of Lemma 1.1, that $(f(x_n))$ is a Cauchy sequence in $f(X)$, which is complete. Thus, for some $x_0 \in X$, $\lim f(x_n) = \lim g(x_n) = f(x_0)$, and from Definition 1.1(2), $\mathfrak{p}(f(x_N), f(x_0)) \leq \delta$, and thus, by assumptions *(ii)* and *(iii)* above, $\mathfrak{p}(f(x_N), g(x_0)) \leq \delta$. By Definition 1.1(3), $(f(x_0), g(x_0)) \in V$, so that $f(x_0) = g(x_0)$, and (2.3) holds. $\square$

Theorem 2.2 below extends Corollary 3.3 in [12] to w-distances.

Theorem 2.2. *Let f, g be selfmaps of a uniform space (X, ϑ) such that $g(X) \subseteq f(X)$. Assume either $g(X)$ or $f(X)$ is complete and let $\mathfrak{p}$ be a w-distance on X such that, for some $\phi \in \Phi$,*

$$\mathfrak{p}(g(x), g(y)) \leq \phi(\mathfrak{p}(f(x), f(y))), \quad x, y \in X. \tag{2.4}$$

Then, there is $x_0 \in X$ such that

$$f(x_0) = \lim f(x_n) = \lim g(x_n) = g(x_0) \tag{2.5}$$

for any sequence (x_n) in X such that $g(x_n) = f(x_{n+1})$ for all $n \geq 1$. Furthermore, if f and g commute at their coincidence points, $f(x_0)$ is a unique common fixed point of f and g.

Proof. Since $g(X) \subseteq f(X) \subseteq X$, there is a sequence (x_n) in X such that $g(x_n) = f(x_{n+1})$ for all $n \geq 1$. Then, from (2.4), $\mathfrak{p}(f(x_{n+1}), f(x_{n+2})) = \mathfrak{p}(g(x_n), g(x_{n+1})) \leq \phi(\mathfrak{p}(f(x_n), f(x_{n+1}))) < \mathfrak{p}(f(x_n), f(x_{n+1}))$, provided $t_n = \mathfrak{p}(f(x_n), f(x_{n+1})) > 0$ for all $n \geq 1$. Thus (t_n) is a decreasing sequence in $\mathbb{R}^+$ such that $\lim t_n = \lim \phi(t_n)$, so that $\lim t_n = 0$. Now arguing as in Theorem 2.1, (2.4) ensures that f and g satisfy Lemma 2.2 *(ii)*, which guarantees the existence of $x_0 \in X$ such that

$$f(x_0) = \lim f(x_n) = \lim g(x_n) = g(x_0).$$

The commutativity of f and g at x_0 then ensures that

$$f^2(x_0) = f \circ g \ (x_0) = g \circ f \ (x_0) = g^2(x_0). \tag{2.6}$$

Now, from (2.4) and (2.6),

$$\mathfrak{p}(g(x_0), g(x_0)) \leq \phi(\mathfrak{p}(f(x_0), f(x_0))) < \mathfrak{p}(f(x_0), f(x_0)) = \mathfrak{p}(g(x_0), g(x_0))$$

and

$$\mathfrak{p}(g(x_0), g^2(x_0)) \leq \phi(\mathfrak{p}(f(x_0), f^2(x_0))) < \mathfrak{p}(f(x_0), f^2(x_0)) = \mathfrak{p}(g(x_0), g^2(x_0)),$$

provided $\mathfrak{p}(g(x_0), g(x_0)) \neq 0$ and $\mathfrak{p}(g(x_0), g^2(x_0)) \neq 0$. This is a contradiction. Thus $\mathfrak{p}(g(x_0), g(x_0)) = 0 = \mathfrak{p}(g(x_0), g^2(x_0))$, and from Lemma 1.1(a) we conclude that $g^2(x_0) = g(x_0) = f(x_0) = f^2(x_0)$. Hence, $f(x_0)$ is a common fixed point of f and g. If we now assume that $\mathfrak{p}(f(x_N), f(x_{N+1})) = 0$ for some $N \geq 1$, then $\mathfrak{p}(f(x_{N+1}), f(x_{N+2})) = 0$ and, since $\mathfrak{p}(f(x_N), f(x_{N+2})) \leq \mathfrak{p}(f(x_N), f(x_{N+1})) + \mathfrak{p}(f(x_{N+1}), f(x_{N+2}))$, also $\mathfrak{p}(f(x_N), f(x_{N+2})) = 0$. Thus, as above, $f(x_{N+1}) = f(x_{N+2}) = g(x_{N+1})$ is a fixed point of f and g. Uniqueness follows from an argument entirely similar to that in Theorem 2.1. □

The following example shows that Theorems 2.1 and 2.2 generalize results in [4], [6], [11], [12].

Example 2.1. Let $X = [0, 1]$ and $d(x, y) = |x - y|$. Let $f(x) = x^2$ for $x \in X$, not rational, and $f(x) = 0$ for rational x. It is not difficult to check that no $\phi \in \Phi$ exists such that $d(f(x), f(y)) \leq \phi(d(x, y))$ for all $x, y \in X$. On the other hand, if $\phi(t) = t^2$ for $0 \le t < 1$ and $\phi(t) = 0$ for $t \ge 1$, then $\phi \in \Phi$. Let $\mathfrak{p}(x, y) = y$ for $x, y \in X$. Then $\mathfrak{p}$ is a w-distance on (X, ϑ_d) and $\mathfrak{p}(f(x), f(y)) \le \phi(\mathfrak{p}(x, y))$. Theorem 2.1 thus ensures the existence of a unique fixed point of f.

Example 2.2. Let $X, d, \mathfrak{p}$ and ϕ be as in Example 2.1; $f(x) = x^2$, $x \in X$; $g(x) = x^4$, $x \in X - \mathbb{Q}$, and $g(x) = 0$, $x \in X \cap \mathbb{Q}$. Then no $\phi \in \Phi$ exists such $d(g(x), g(y)) \le \phi(d(f(x), f(y)))$ for all $x, y \in X$, but $\mathfrak{p}(g(x), g(y)) \le \phi(\mathfrak{p}(f(x), f(y)))$ does hold. Theorem 2.2 thus ensures the existence of a unique common fixed point of f and g. The assertions in this and the above example could not be drawn from results in [12].

3. Fixed point theorems for w-expansive maps

Lemma 3.1. *Let X be a subset of a uniform space (Y, ϑ), f a map of X into Y such that $f(X)$ is complete, $\mathfrak{p}$ a w-distance on Y. Further assume that:*

i) There is a sequence (x_n) in X such that $x_n = f(x_{n+1})$, $\quad n \ge 1$

ii) For each $\delta > 0$ there is $0 < \delta_1 \le \delta$ such that $\mathfrak{p}(f(x), z) < \delta$ whenever $\mathfrak{p}(f(x), x) < \delta_1$ and $\mathfrak{p}(f(x), f(z)) \le \delta$.

iii) Given $\delta_1 > 0$, there is $N \ge 1$ such that $\mathfrak{p}(f(x_N), x_N) < \delta_1$.

Then, there is $x_0 \in X$ such that $f(x_0) = x_0$ and

$$\lim x_n = \lim f(x_n) = x_0 = f(x_0). \tag{3.1}$$

Proof. For $V \in \vartheta$, let $\delta > 0$ be as in Definition 1.1(3), and let $0 < \delta_1 \le \delta$, $N \ge 1$, be such that conditions *(ii)* and *(iii)* above hold. From $\mathfrak{p}(f(x_N), x_N) = \mathfrak{p}(f(x_N), f(x_{N+1})) < \delta_1 \le \delta$ and *(ii)*, we have that $\mathfrak{p}(f(x_N), x_{N+1}) = \mathfrak{p}(f(x_N), f(x_{N+2})) < \delta$. Also $\mathfrak{p}(f(x_N), x_{N+1}) = \mathfrak{p}(f(x_N), f(x_{N+2})) < \delta$. Continuing this way we conclude that $\mathfrak{p}(x_{N-1}, x_{N+p}) = \mathfrak{p}(f(x_N), x_{N+p}) < \delta$, $p = 0, 1, 2, 3, ...$, which implies, by Lemma 1.1 (c), that (x_n) is a Cauchy sequence. Completeness of $f(X)$ then ensures that

$$\lim x_n = \lim f(x_n) = f(x_0)$$

for some $x_0 \in X$. On the other hand, from $\mathfrak{p}(f(x_N), x_m) < \delta$, $m > N$, and the lower semicontinuity of $\mathfrak{p}$, it follows that $\mathfrak{p}(f(x_N), f(x_0)) \le \delta$, and hence, from assumption *(ii)* above, $\mathfrak{p}(f(x_N), x_0) < \delta$. Definition 1.1(3) thus ensures that $(f(x_0), x_0) \in V$, so that, since V is arbitrary, $f(x_0) = x_0$, and (3.1) holds. □

Lemma 3.1 above is Lemma 2.1 in [12] for the case of w-distances. Theorem 3.1, next, also adapts Theorem 2.1 in the same paper to this more general situation.

Theorem 3.1. *Let X be a subset of a uniform space (Y, ϑ) and let f be a map of X into Y such that $X \subseteq f(X)$. Also assume that either X or $f(X)$ is complete. Let $\mathfrak{p}$ be a w-distance on Y such that, for some $\psi \in \Psi$ with $\psi(0+) = 0$,*

$$\mathfrak{p}(f(x), f(y)) \ge \psi(\mathfrak{p}(x, y)) \quad x, y \in X. \tag{3.2}$$

Then, f has a unique fixed point x_0 satisfying (3.1) for any sequence (x_n) in X such that $x_n = f(x_{n+1})$ for all $n \ge 1$.

Proof. Since $X \subseteq f(X)$, there is a sequence (x_n) in X such that $x_n = f(x_{n+1})$ for all $n \ge 1$, and by (3.2),if we assume $\mathfrak{p}(x_n, x_{n+1}) > 0$ for all $n \ge 1$ then

$$\mathfrak{p}(x_{n+1}, x_{n+2}) < \psi(\mathfrak{p}(x_{n+1}, x_{n+2})) \le \mathfrak{p}(f(x_{n+1}), f(x_{n+2})) = \mathfrak{p}(x_n, x_{n+1}),$$

so that $t_n = \mathfrak{p}(x_n, x_{n+1})$, $n \geq 1$, is a decreasing sequence such that $\lim t_n = \lim \psi(t_n) = t$ for some $t \geq 0$. Since $\psi \in \Psi$, then $t = 0$.
Let $\delta > 0$ be such that $0 < \widehat{\psi}(\delta) < \delta$ (Lemma 1.2) and $\delta_1 = \delta - \widehat{\psi}(\delta)$. If $\mathfrak{p}(f(x), x) < \delta_1$ and $\mathfrak{p}(f(x), f(z)) < \delta$, it follows from $\widehat{\psi}(\mathfrak{p}(x,z)) \leq \mathfrak{p}(f(x), f(z)) < \delta$ that $\mathfrak{p}(x,z) \leq \widehat{\psi}(\delta)$, so that $\mathfrak{p}(f(x), z) \leq \mathfrak{p}(f(x), x) + \mathfrak{p}(x,z) < \delta_1 + \widehat{\psi}(\delta) = \delta$. Lemma 3.1 then ensures the existence of x_0 in X such that (3.1) holds. If $\mathfrak{p}(x_N, x_{N+1}) = 0$ for some $N \geq 1$ then, from (3.2), $\mathfrak{p}(x_{N+1}, x_{N+2}) = 0$, and, as in Theorem 2.1, $x_{N+1} = x_{N+2} = f(x_{N+2})$, so that x_{N+2} is a fixed point of f. Uniqueness follows easily from (3.2) and the argument in Theorem 2.1. □

Lemma 3.2. *Let f and g be maps of a set X into a uniform space (Y, ϑ). Assume that $g(X)$ is complete and let $\mathfrak{p}$ be a w-distance on Y. Also assume that:*

i) There is a sequence (x_n) in X with $f(x_n) = g(x_{n+1})$, $n \geq 1$.
ii) Given $\delta > 0$, there is $0 < \delta_1 \leq \delta$ such that $\mathfrak{p}(g(x), f(z)) < \delta$, provided $\mathfrak{p}(g(x), f(x)) < \delta_1$ and $\mathfrak{p}(g(x), g(z)) < \delta$.
iii) For any $\delta_1 > 0$ there is $N \geq 1$ such that $\mathfrak{p}(g(x_N), f(x_N)) < \delta_1$.

Then, there is x_0 in X such that

$$f(x_0) = \lim f(x_n) = \lim g(x_n) = g(x_0). \tag{3.3}$$

Proof. Now based on Lemma 3.1, the proof is entirely similar to that of Lemma 2.2. □

Lemma 3.2 extends Lemma 3.1 to a pair of maps. We finally have:

Theorem 3.2. *Let f and g be maps of a set X into a uniform space (Y, ϑ) such that $f(X) \subseteq g(X)$. Assume either $f(X)$ or $g(X)$ is complete and let $\mathfrak{p}$ be a w-distance on Y such that*

$$\mathfrak{p}(g(x), g(y)) \geq \psi(\mathfrak{p}(f(x), f(y))), \quad x, y \in X \tag{3.4}$$

for some $\psi \in \Psi$ with $\psi(0+) = 0$. Then, there is x_0 in X such that (3.3) holds for any sequence (x_n) in X such that $f(x_n) = g(x_{n+1})$ for all $n \geq 1$. Furthermore, if $g(X) \subseteq X$ and f and g commute at any of their coincidence points, then $f(x_0)$ is the unique common fixed point of f and g.

Proof. As in Theorem 2.2, from $f(X) \subseteq g(X)$ it follows that there is a sequence (x_n) in X such that $f(x_n) = g(x_{n+1})$, $n \geq 1$, and (3.4) then ensures that $(f(x_n))$ is a Cauchy sequence for ϑ. Therefore $x_0 \in X$ exists such that (3.3) holds. Commutativity of f and g at their coincidence points and (3.4) then guarantee that $f(x_0)$ is the unique common fixed point of f and g. □

The following examples show that Theorems 3.1 and 3.2 generalize results in [5], [12] and [13].

Example 3.1. Let $X = [0,1)$ and $d(x,y) = |x - y|$. Also let $f(x) = 2x$ for $0 \leq x \leq \frac{1}{2}$ and $f(x) = 1$ if $\frac{1}{2} < x < 1$. Then no $\psi \in \Psi$ exists such that $d(f(x), f(y)) \geq \psi(d(x,y))$ for all x, y in X. Let $\psi(t) = f(t)$ if $0 \leq t < 1$ and $\psi(t) = 2t$ for $t \geq 1$. If $\mathfrak{p}(x,y) = y$, then $\mathfrak{p}(f(x), f(y)) \geq \psi(\mathfrak{p}(x,y))$ for x, y in X. Theorem 3.1 thus ensures the existence of a unique fixed point for f.

Example 3.2. Let X, d and $\mathfrak{p}$ be as in Example 3.1. Also let $g(x) = 2x$ for $0 \leq x \leq \frac{1}{2}$ and $g(x) = 1$ for $\frac{1}{2} < x < 1$, and $f(x) = \frac{x}{2}$ for $x \in X$. Then, no $\psi \in \Psi$ exists such that $d(g(x), g(y)) \geq \psi(d(f(x), f(y)))$ for all x, y in X, but $\mathfrak{p}(g(x), g(y)) \geq \psi(\mathfrak{p}(f(x), f(y)))$ holds for ψ as in Example 3.1. Therefore, Theorem 3.2 ensures the existence of a unique fixed point for f and g.

Acknowledgments

The first author acknowledges partial support from the Facultad de Ciencias to pursue graduate (Ph.D) studies.

References

1. N. Bourbaki, *Topologie Générale*, Chaps. I, II, 4 ed., Hermann, Paris, 1965.
2. K.M. Das and V. Naik, *Common fixed points theorems for commuting maps on metric space*, Proc. Amer. Math Soc., **77** (1979), 269–373.
3. J. Dugundji, *Topology*, Allyn and Bacon, Boston, 1966.
4. J. Dugundji and A. Granas, *Fixed Point Theory*, vol. I, Polish. Sci. Pub., Warszawa, 1982.
5. J. Jachymski, *Fixed points theorems for expansive mappings*, Math Japon. **42** (1995), 131–136.
6. G. Jungck, *Commuting mappings and fixed points*, Amer. Math. Monthly. **83** (1976), 261–263.
7. O. Kada, T. Suzuki and W. Takahashi, *Non convex minimization theorems and fixed point theorems in complete metric spaces*, Math. Japon. **44** (1996), 381–391.
8. S. M. Kang, *Fixed points for expansions mappings*, Math. Japon. **38** (1993), 713–717.
9. B. E. Rhoades, *A comparison of various definitions of contractive mapping*, Trans. Amer. Math. Soc. **226** (1977), 257–290.
10. J. Rodríguez-Montes, *Aplicaciones no conmutativas y puntos fijos*, Rev. Colombiana. Mat. **21** (1987), 257–262.
11. J. Rodríguez-Montes, *Some results on fixed and coincidence points for pairs of maps in metric spaces*, Rev. Colombiana. Mat. **27** (1993), 249–252.
12. J. Rodríguez-Montes and J. A. Charris, *Fixed points for contractive and expansive maps in metric space: Toward a unified approach*, submitted, Reporte interno No.73, Departamento de Matemáticas y Estadística, Universidad Nacional, Bogotá, Colombia, 2000.
13. S. Z. Wang, B. Y. Li, Z. M. Gao and K. Iseki, *Some fixed point theorems on expansion mappings*, Math. Japon. **29** (1984), 631–636.
14. A. Weil, *Sur les espaces á structure uniforme et sur la topologie générale*, Hermann, Paris, 1937.

Information for Authors cont.

Copyright. Authors will be asked, upon acceptance of an article, to transfer copyright of the article to the publisher. This will ensure the widest possible dissemination of information under copyright laws. Preprints are available via web page at http://rattler.cameron.edu/SWJPAM. Authorization to copy is granted provided that the SWJPAM copyright symbol is not removed.

Tables. Tables must be numbered consecutively with roman numerals and have titles. They should be prepared in such a manner that no break is necessary.

Figures. All graphs and diagrams should be referred to as 'figure' (not abbreviated) in the text. A detailed caption should be provided for each figure, and figures should be numbered consecutively.

Footnotes. The use of footnotes should be minimized.

Notification. The author will be notified of the posting of his/her work and the time of release for SWJPAM.

References. References should be listed at the end of each paper. They should be numbered consecutively using arabic numbers between square brackets in the text. In the list of references, the numbers should appear without square brackets. A journal reference should comprise of name(s) and initial(s) of author(s), title of article, abbreviated name of journal (following the style of 'Abbreviations of Names of Serials reviewed in Mathematical Reviews'), volume number, year of publication (within parentheses), and full page numbers, e.g.

1. Argyros, I.K. and Szidarovszky, F.: On the Monotone Convergence of Algorithmic Models, Appl. Math. Com. 48, (1992), 167–176.

A book reference should comprise of name(s) and initial(s) of author(s), title of book, publisher, town, year of publication, e.g.

11. Ortega, J.M. and Rheinboldt, W.C.: Iterative Solution of Nonlinear Equations in Several Variables, Academic Press, New York, 1970

An edited volume reference should comprise of name(s) and initial(s) of author(s), title of paper, name(s) of editor(s), title of book, publisher, town, year, inclusive page numbers, e.g.

12. Prenter, P.: On Polynomial Operators and Equations, L.B. Rall (ed), Computational Solution of nonlinear operator equations, Johy Wiley Publisher, New York, 1971, pp. 361–398.

Indexing and Abstracting services. Articles published in SWJPAM are selectively indexed or abstracted in: Mathematical Reviews of A.M.S., Zentralblatt für Mathematik, European Mathematical Society (EMIS).