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FIXED POINT THEOREM AND APPLICATION IN MODULAR SPACE

B. Marzouki

Abstract. In this paper, by analogy with Banach contraction principle, we establish a result concerning fixed point in modular space, an application of the result is given in Orlicz space for the integral equation of Voltera type.

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1 Introduction and Preliminaries

In [1], the author has proved the following result using the fact that the modular ρ satisfies the Δ_2 condition and the Fatou property, the proof was based on the theorem of Cantor which he showed in a modular space. Also, let us note that the author used implicitly (without indicating it) the fact that B is ρ-bounded (for the notations and definitions, see the preliminaries below).
In this paper and by a simplified method, we show that the result remains true without supposing that the modular ρ verifies neither the Δ_2 condition nor the Fatou property.

Theorem 1.1. *Let B be a ρ-closed subset of a ρ-complete modular space X_ρ and $T : B \longrightarrow B$ satisfying*

$$\rho(2(Tx - Ty)) \leq k\rho(x - y) \quad \forall x, y \in B, \quad \textit{where} \quad 0 < k < 2,$$

we suppose also that ρ is convex and satisfies the Δ_2 condition and the Fatou property, then T has a unique fixed point.

Now let us recall some basic definitions in the theory of modular spaces. For more details we refer the reader to the book of Musielak [6].

Département de Mathématiques, Faculté des Sciences, Oujda, Maroc
E-mail : marzoukib@yahoo.fr

Definition 1.1. *Let X be a vector space over K ($K = \mathbb{R}$ or C), a functional $\rho : X \longrightarrow [0, +\infty]$ is called a modular if for all x and y in X, there holds*

a) $\rho(x) = 0$ *if and only if* $x = 0$,

b) $\rho(\alpha x) = \rho(x)$ *for every* $\alpha \in K$ *with* $|\alpha| = 1$,

c) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ *for every* $\alpha, \beta \geq 0$, *and* $\alpha + \beta = 1$.

If we replace c) by $\rho(\alpha x + \beta y) \leq \alpha\rho(x) + \beta\rho(y)$, *then the modular ρ is called convex.*

A modular ρ defines a corresponding modular space X_ρ given by

$$X_\rho = \Big\{ x \in X \quad / \quad \lim_{\lambda \to 0} \rho(\lambda x) = 0 \Big\}.$$

Definition 1.2. *Let X_ρ be a modular space.*

a) *A sequence x_n is called ρ-convergent to x and write $x_n \to_\rho x$, if and only if $\rho(x_n - x) \to 0$ as $n \to +\infty$.*

b) *A sequence x_n is called ρ-Cauchy whenever $\rho(x_n - x_m) \to 0$ as $n, m \to +\infty$.*

c) *The modular space X_ρ is called ρ-complete if any ρ-Cauchy sequence is ρ-convergent.*

d) *The modular ρ is said to satisfy the Δ_2 condition if $\rho(2x_n) \to 0$ whenever $\rho(x_n) \to 0$ as $n \to +\infty$.*

e) *ρ is said to have the Fatou property if $\rho(x) \leq \liminf\limits_{n \to +\infty} \rho(x_n)$ whenever $x_n \to_\rho x$.*

f) *A subset $B \subset X_\rho$ is called ρ-closed if for any sequence $x_n \subset B$ ρ-convergent to $x \in X_\rho$, we have $x \in B$.*

g) *A subset $B \subset X_\rho$ is said to be ρ-bounded if $\sup\{\rho(x - y) : x, y \in B\} < +\infty$.*

2 Main result

Theorem 2.1. *Let B be a ρ-closed and ρ-bounded subset of a ρ-complete modular space X_ρ and $T : B \longrightarrow B$ satisfying*

$$\rho(2(Tx - Ty)) \leq k\rho(x - y) \quad \forall x, y \in B, \quad \textit{where} \quad 0 < k < 2,$$

we suppose also that ρ is convex, then T has a unique fixed point.

Proof. Select $x_0 \in B$ and put $x_{n+1} = Tx_n, (n \geq 0)$ and $\alpha_{n+1} = \rho(x_{n+1} - x_n)$, then

$$\begin{aligned}
\alpha_{n+1} &= \rho(x_{n+1} - x_n) \\
&= \rho(Tx_n - Tx_{n-1}) \\
&\leq \frac{1}{2}\rho(2(Tx_n - Tx_{n-1})) \\
&\leq \frac{k}{2}\rho(x_n - x_{n-1}) \\
&= \frac{k}{2}\alpha_n,
\end{aligned}$$

so $\alpha_{n+1} \leq (\frac{k}{2})^{n+1}\alpha_0$, hence $\lim\limits_{n \to +\infty} \rho(x_{n+1} - x_n) = 0$.

Now we have to show that x_n is ρ-Cauchy.

Let $\varepsilon > 0$, if we show that $\rho(x_{n+1} - x_n) \leq \varepsilon \implies \rho(x_{n+2} - x_n) \leq \varepsilon$, then by the same way we show that $\rho(x_{n+3} - x_n) \leq \varepsilon$ and so on we show that $\rho(x_{n+m} - x_n) \leq \varepsilon \quad \forall m = 1, 2, ...$

We have $\lim_{n\to+\infty} \rho(x_{n+1}-x_n)=0$, so for n large we have $\rho(x_{n+1}-x_n)\le\varepsilon$ and $\rho(x_n-x_{n-1})\le\varepsilon(1-\frac{k}{2})$, thus

$$\begin{aligned}\rho(x_{n+2}-x_n) &= \rho(Tx_{n+1}-Tx_{n-1})\\ &= \rho(Tx_{n+1}-Tx_n+Tx_n-Tx_{n-1})\\ &\le \frac{1}{2}\rho(2(Tx_{n+1}-Tx_n))+\frac{1}{2}\rho(2(Tx_n-Tx_{n-1}))\\ &\le \frac{k}{2}\rho(x_{n+1}-x_n)+\frac{k}{2}\rho(x_n-x_{n-1})\\ &\le \frac{k}{2}\rho(x_{n+1}-x_n)+\rho(x_n-x_{n-1})\\ &\le \frac{k}{2}\varepsilon+\varepsilon(1-\frac{k}{2})\\ &= \varepsilon,\end{aligned}$$

repeating the same argument, we have

$$\begin{aligned}\rho(x_{n+3}-x_n) &= \rho(Tx_{n+2}-Tx_n+Tx_n-Tx_{n-1})\\ &\le \frac{1}{2}\rho(2(Tx_{n+2}-Tx_n))+\frac{1}{2}\rho(2(Tx_n-Tx_{n-1}))\\ &\le \frac{k}{2}\rho(x_{n+2}-x_n)+\frac{k}{2}\rho(x_n-x_{n-1})\\ &\le \frac{k}{2}\rho(x_{n+2}-x_n)+\rho(x_n-x_{n-1})\\ &\le \frac{k}{2}\varepsilon+\varepsilon(1-\frac{k}{2})\\ &= \varepsilon.\end{aligned}$$

So x_n is ρ-Cauchy and consequently there exists $x\in B$ such that $x_n\to_\rho x$, that is $\lim_{n\to+\infty}\rho(x_n-x)=0$.

Let us show that x is a fixed point of T, we have

$$\begin{aligned}\rho(\frac{x-Tx}{2}) &= \rho(\frac{x-Tx_n}{2}+\frac{Tx_n-Tx}{2}))\\ &\le \rho(x-Tx_n)+\rho(Tx_n-Tx)\\ &\le \rho(x-x_{n+1})+\rho(2(Tx_n-Tx))\\ &\le \rho(x-x_{n+1})+k\rho(x_n-x),\end{aligned}$$

thus, letting $n\to+\infty$, we obtain

$$\rho(\frac{x-Tx}{2})=0,\quad \text{hence}\quad x=Tx.$$

Now assume that there is $y\in B$ such that $y=Ty$, then

$$\begin{aligned}\rho(x-y) &= \rho(Tx-Ty)\\ &\le \frac{1}{2}\rho(2(Tx-Ty))\\ &\le \frac{k}{2}\rho(x-y),\end{aligned}$$

since $k<2$ and B is ρ-bounded, we obtain a contradiction $\square$

Remarks 2.1.

1) *If we suppose in the previous theorem that $0 < k < 1$, then we can suppress the hypothesis : ρ is convex.*

2) *If we suppose that there is $x_0 \in B$ such that $\rho(x_0 - Tx_0) < +\infty$, then we can eliminate the hypothesis : B is ρ-bounded, but in this case, the fixed point of T is not necessarily unique. If in particular the modular ρ is with values in $[0, +\infty[$, then the fixed point is unique without assuming that B is ρ-bounded.*

From theorem 2.1 we deduce the following corollary,

Corollary 2.1. *Under the same hypothesis of theorem 2.1, suppose that T satisfies*

$$\rho(\theta(Tx - Ty)) \leq k\rho(x - y) \quad \textit{with} \quad \theta \geq 2 \quad \textit{and} \quad k < \theta,$$

then T has a unique fixed point

Proof. We have

$$\begin{aligned} \rho(2(Tx - Ty)) &= \rho(\frac{2\theta}{\theta}(Tx - Ty)) \\ &\leq \frac{2}{\theta}\rho(\theta(Tx - Ty)) \\ &\leq \frac{2k}{\theta}\rho(x - y) \\ &= k'\rho(x - y) \end{aligned}$$

where $\quad k' = \dfrac{2k}{\theta} < 2, \quad$ so from theorem 2.1, T has a unique fixed point. $\square$

3 Application to the integral equation of Voltera type

We look for a solution of the integral equation

$$x(t) = x_0(t) + \mu \int_0^t K(t, s)Rx(s)ds \qquad (*)$$

in Orlicz space L^φ, where $x_0(.)$ and $K(.,.)$ and R are given. We consider the vector space X defined by

$$X = \Big\{f : [0, 1] \longrightarrow \mathbb{R} \quad \text{measurable}\Big\},$$

we define in X the following modular : for $f \in X$, we put

$$\rho(f) = \int_0^1 \varphi(t, |f(t)|)dt,$$

where $\varphi : [0, 1] \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ satisfies :

i) φ is continuous and increasing and convex with respect to the second variable.

ii) $\lim\limits_{x \to +\infty} \varphi(t, x) = +\infty, \quad \varphi(t, 0) = 0, \quad \varphi(t, x) > 0 \quad \forall x > 0.$

iii) φ is measurable with respect to the first variable.

Then it is known (see for example [6]) that ρ is a convex modular and :

1) $L^\varphi = \Big\{f \in X \quad / \quad \lim\limits_{\lambda \to 0} \rho(\lambda f) = 0\Big\}$ is a modular space.

2) L^φ is ρ-complete.
3) L^φ is a Banach space equipped with the Luxembourg norm defined by

$$\|x\|_\rho = \inf\Big\{\alpha > 0 \quad / \quad \rho(\frac{x}{\alpha}) \le 1\Big\}.$$

4) ρ satisfies the Δ_2 condition and the Fatou property.
5) $\lim\limits_{n\to+\infty} \|x_n - x\|_\rho = 0 \iff \lim\limits_{n\to+\infty} \rho(x_n - x) = 0,$ note that for an arbitrary modular ρ which does not satisfy the Δ_2 condition, we have

$$\lim_{n\to+\infty} \|x_n - x\|_\rho = 0 \quad \Longleftrightarrow \quad \lim_{n\to+\infty} \rho(\alpha(x_n - x)) = 0 \quad \forall \alpha > 0.$$

To resolve the integral equation (*), we construct another complete modular space on which the operator T defined by

$$(Tx)(t) = x_0(t) + \mu \int_0^t K(t,s)Rx(s)ds$$

admits a fixed point which corresponds to a solution of (*).
We consider the vector space Y defined by

$$Y = \Big\{x : [0,1] \longrightarrow L^\varphi \quad \text{continuous with respect the norm} \quad \|.\|_\rho\Big\},$$

we define on Y the functional ρ_∞ by $\rho_\infty(x) = \sup\limits_{t\in[0,1]} \rho(x(t)),$ where ρ is the modular of L^φ, we can verify easily that $\rho_\infty : Y \longrightarrow [0,+\infty]$ satisfies all the properties of a convex modular, we obtain then the modular space

$$Y_{\rho_\infty} = \Big\{x \in Y \quad / \quad \lim_{\lambda\to 0} \rho_\infty(\lambda x) = 0\Big\}.$$

lemma 3.1. *Y_{ρ_∞} is ρ_∞-complete.*

Proof. Let x_n be a ρ_∞-Cauchy sequence in Y_{ρ_∞}, then

$$\forall \varepsilon > 0 \quad \exists n_0, \quad \forall n,p \ge n_0, \quad \forall t \in [0,1] \quad \rho(x_n(t) - x_p(t)) \le \varepsilon, \qquad (**)$$

so $\forall t \in [0,1]$ $x_n(t)$ is ρ-Cauchy in L^φ, hence $x_n(t) \to_\rho x(t)$ and $x(t) \in L^\varphi$.
In the relation (**), we fix $n \ge n_0$, we obtain $\liminf\limits_{p\to+\infty} \rho(x_n(t) - x_p(t)) \le \varepsilon$, but ρ satisfies the Fatou property, thus

$$\forall t \in [0,1] \quad \rho(x_n(t) - x(t)) \le \liminf_{p\to+\infty} \rho(x_n(t) - x_p(t)) \le \varepsilon,$$

so $\rho_\infty(x_n - x) \le \varepsilon$, hence $x_n \to_{\rho_\infty} x$.
Now we have to show that $x \in Y_{\rho_\infty}$, that is, x is continuous and $\lim\limits_{\lambda\to 0} \rho_\infty(\lambda x) = 0$.
i) $x : [0,1] \longrightarrow L^\varphi$ is continuous, indeed

$$\lim_{n\to+\infty} \rho_\infty(x_n - x) = 0 \quad \Longrightarrow \quad \forall t \in [0,1] \quad \lim_{n\to+\infty} \rho(x_n(t) - x(t)) = 0,$$

since ρ satisfies the Δ_2 condition, we deduce that $\lim\limits_{n\to+\infty}\|x_n(t)-x(t)\|_\rho = 0$, hence

$$\forall\varepsilon>0 \quad \exists n_0, \quad \forall n\geq n_0 \quad \forall t\in[0,1], \quad \|x_n(t)-x(t)\|_\rho\leq\varepsilon.$$

Let $t_0\in[0,1]$, then $\exists\eta>0$ such that $|t-t_0|\leq\eta \implies \|x_{n_0}((t)-x_{n_0}(t_0)\|_\rho\leq\varepsilon$, so

$$\begin{aligned}\|x(t)-x(t_0)\|_\rho &\leq \|x(t)-x_{n_0}(t)\|_\rho+\|x_{n_0}(t)-x_{n_0}(t_0)\|_\rho+\|x_{n_0}(t_0)-x(t_0)\|_\rho\\ &\leq 3\varepsilon.\end{aligned}$$

ii) $\lim\limits_{\lambda\to0}\rho_\infty(\lambda x)=0$, indeed

$$\begin{aligned}\rho_\infty(\lambda x) &= \rho_\infty(\lambda(x-x_n)+\lambda x_n)\\ &\leq \rho_\infty(2\lambda(x-x_n))+\rho_\infty(2\lambda x_n),\end{aligned}$$

we can suppose that $2\lambda<1$, so $\rho_\infty(\lambda x)\leq\rho_\infty(x-x_n)+\rho_\infty(2\lambda x_n)$, therefore $\limsup\limits_{\lambda\to0}\rho_\infty(\lambda x)\leq\rho_\infty(x-x_n)$, since this inequality is satisfied for all n, we deduce that $\lim\limits_{\lambda\to0}\rho_\infty(\lambda x)=0$. $\square$

Now we return to the integral equation (*) :

$$x(t)=x_0(t)+\mu\int_0^t K(t,s)Rx(s)ds$$

where $\mu>0$, $x_0\in Y_{\rho_\infty}$, $K:[0,1]\times[0,1]\longrightarrow\mathbb{R}^+$ is continuous and $R:L^\varphi\longrightarrow L^\varphi$ satisfies $\rho(Rx-Ry)\leq\theta\rho(x-y)$ $\forall x,y\in L^\varphi$ with $\theta\geq2$ and $\mu\theta<1$, then we have the following result

Theorem 3.1. *We suppose also that :*

a) $\sup\limits_{t\in[0,1]}\int_0^t K(t,s)ds\leq1$.

b) *There exists* $f_0\in Y_{\rho_\infty}$ *such that* $\rho_\infty(f_0-Tf_0)<+\infty$.

c) $\rho(R(0))<+\infty$ *and* $\rho(\varepsilon Rx)\leq\rho(R(\varepsilon x))$ $\forall x\in L^\varphi,\forall\varepsilon$ *with* ε *sufficiently small. Then we have :*

1) $\forall x\in Y_{\rho_\infty}$, $Tx\in Y_{\rho_\infty}$.

2) $\exists k<\theta$ *such that* $\rho_\infty(\theta(Tx-Ty))\leq k\rho_\infty(x-y)$.

We deduce that the operator $T:Y_{\rho_\infty}\longrightarrow Y_{\rho_\infty}$ *has a fixed point, and consequently the equation (*) has a solution.*

Proof. Without loss of generality, taking $\tilde{R}(x)=R(x)-R(0)$, we can suppose that $R(0)=0$.

1) **i)** Let $x\in Y_{\rho_\infty}$, then the map $(Tx)(.)$ is continuous on $[0,1]$, indeed, it suffices to show that the map $t\longmapsto Rx(t)$ is continuous.
Let $t_0\in[0,1]$, then since $x(.)$ is continuous, we have $\lim\limits_{t\to t_0}\|x(t)-x(t_0)\|_\rho=0$, but since ρ satisfies the Δ_2 condition, we deduce that $\lim\limits_{t\to t_0}\rho(x(t)-x(t_0))=0$, hence $\lim\limits_{t\to t_0}\rho(Rx(t)-Rx(t_0))=0$, and so $\lim\limits_{t\to t_0}\|Rx(t)-Rx(t_0)\|_\rho=0$.

ii) $\lim\limits_{\lambda\to0}\rho_\infty(\lambda Tx)=0$, indeed $\forall t\in[0,1]$, we have

$$\begin{aligned}\rho(\lambda(Tx)(t)) &= \rho\Big(\lambda x_0(t) + \lambda\mu\int_0^t K(t,s)Rx(s)ds\Big)\\ &\leq \rho(2\lambda x_0(t)) + \rho\Big(2\lambda\mu\int_0^t K(t,s)Rx(s)ds\Big)\\ &\leq \rho_\infty(2\lambda x_0) + \rho\Big(\lambda\int_0^t K(t,s)Rx(s)ds\Big).\end{aligned}$$

To conclude, it is enough to show that $\rho\Big(\lambda\int_0^t K(t,s)Rx(s)ds\Big) \leq \theta\rho_\infty(\lambda x)$.
We have $\rho\Big(\lambda\int_0^t K(t,s)Rx(s)ds\Big) = \rho\Big(\int_0^t K(t,s)(\lambda Rx(s))ds\Big)$,
On the other hand, from the mean value theorem,(see for example [5]), we have

$$\begin{aligned}\int_0^t K(t,s)(\lambda Rx(s))ds &\in \Big(\int_0^t K(t,s)ds\Big)\overline{\mathrm{Co}}\Big\{\lambda Rx(\tau), \quad \tau\in[0,t]\Big\}\\ &\subset \Big(\int_0^t K(t,s)ds\Big)\overline{\mathrm{Co}}\Big\{\lambda Rx(\tau), \quad \tau\in[0,1]\Big\},\end{aligned}$$

so $\int_0^t K(t,s)(\lambda Rx(s))ds = \Big(\int_0^t K(t,s)ds\Big)y(\tau)$ with
$y(\tau)\in\overline{\mathrm{Co}}\Big\{\lambda Rx(\tau), \quad \tau\in[0,1]\Big\}$, hence

$$y(\tau) = \lim_{p\to+\infty}\Big(\sum_{i=0}^{n_p}\lambda_i^p\lambda Rx(\tau_i^p)\Big) \quad\text{with}\quad \sum_{i=1}^{n_p}\lambda_i^p = 1,$$

now since ρ satisfies the Δ_2 condition and the Fatou property, we deduce that

$$\lim_{p\to+\infty}\rho\Big(y(\tau) - \sum_{i=0}^{n_p}\lambda_i^p\lambda Rx(\tau_i^p)\Big) = 0 \quad\text{and}\quad \rho(y(\tau)) \leq \liminf_{p\to+\infty}\rho\Big(\sum_{i=0}^{n_p}\lambda_i^p\lambda Rx(\tau_i^p)\Big),$$

but we have

$$\begin{aligned}\rho\Big(\sum_{i=0}^{n_p}\lambda_i^p\lambda Rx(\tau_i^p)\Big) &\leq \sum_{i=0}^{n_p}\lambda_i^p\rho(\lambda Rx(\tau_i^p))\\ &\leq \sum_{i=0}^{n_p}\lambda_i^p\rho(R(\lambda x(\tau_i^p)))\\ &\leq \sum_{i=0}^{n_p}\lambda_i^p\theta\rho(\lambda x(\tau_i^p))\\ &\leq \sum_{i=0}^{n_p}\lambda_i^p\theta\sup_{t\in[0,1]}\rho(\lambda x(t))\\ &= \theta\rho_\infty(\lambda x).\end{aligned}$$

2) We have $Tx(t) - Ty(t) = \mu\int_0^t K(t,s)Rx(s)ds$, so

$$\rho\Big(\theta(Tx(t) - Ty(t))\Big) = \rho\Big(\theta\mu\int_0^t K(t,s)(Rx(s) - Ry(s))ds\Big).$$

Put $L(t_0) = \sup_{t\in[0,1]} \int_0^t K(t,s)ds$, then using the mean value theorem and the same technique as the previous one, we obtain

$$\rho\Big(\theta(Tx(t) - Ty(t))\Big) \leq \theta^2 \mu L(t_0)\rho_\infty(x-y) \quad \forall t \in [0,1],$$

so

$$\rho_\infty\Big(\theta(Tx - Ty)\Big) \leq k\rho_\infty(x-y) \quad \text{with} \quad k = \theta^2\mu L(t_0) < \theta.$$

Hence we conclude from corollary 2.1 and remark 2.1 that T has a fixed point. □

Remark 3.1. *If R is reduced to the identity, then the solution of the integral equation (*) can be obtained directly using the Banach fixed point theorem.*

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