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$\mathcal{K}$-PURITY AND ORTHOGONALITY

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ABSTRACT. Adámek and Sousa recently solved the problem of characterizing the subcategories $\mathcal{K}$ of a locally λ-presentable category $\mathcal{C}$ which are λ-orthogonal in $\mathcal{C}$, using their concept of $\mathcal{K}_\lambda$-pure morphism. We strenghten the latter definition, in order to obtain a characterization of the classes defined by orthogonality with respect to λ-presentable morphisms (where $f\colon A \longrightarrow B$ is called λ-presentable if it is a λ-presentable object of the comma category $(A \downarrow \mathcal{C})$). Those classes are natural examples of reflective subcategories defined by proper classes of morphisms. Adámek and Sousa's result follows from ours. We also prove that λ-presentable morphisms are precisely the pushouts of morphisms between λ-presentable objects of $\mathcal{C}$.

1. Introduction

We say that a morphism $f\colon A \longrightarrow B$ in a category $\mathcal{C}$ is λ-presentable if it is λ-presentable as an object of the comma category $(A \downarrow \mathcal{C})$. In [$H_1$, 98] and [H, 04], we demonstrated the usefulness of this concept to study purity. We start our paper by characterizing these morphisms as precisely the pushouts of the morphisms between λ-presentable objects (Theorem 2.3), a problem left open in [H, 04]. Then we compare subcategories defined by

(a) orthogonality with respect to morphisms between λ-presentable objects (called λ-orthogonality classes), to those defined by

(b) orthogonality with respect to λ-presentable morphisms (called λ_m-orthogonality classes).

In [AS, 04], Adámek and Sousa introduce the concept of $\mathcal{K}_\lambda$-pure morphism, and use it to characterize the λ-orthogonality classes as the ones closed under products, λ-directed colimits and $\mathcal{K}_\lambda$-pure subobjects. This completed the solution to a problem which arose from the works of Coste ([C, 79]) and Volger ([V, 79]). Using λ-presentable morphisms, we define *strongly* $\mathcal{K}_\lambda$-pure morphisms, and we obtain a characterization of the λ_m-orthogonality classes as those which are closed under products and strongly $\mathcal{K}_\lambda$-pure subobjects (Theorem 4.1). We derive the Adámek-Sousa's result from our characterization, by proving a lemma giving closure conditions on $\mathcal{K}$ under which all $\mathcal{K}_\lambda$-pure morphisms are strong.

This parallels the results and methods of [H_1, 98] and [H, 04]. In the former paper, we obtained a characterization of λ_m-injectivity classes as those closed under products and

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(strongly) λ-pure subobjects, and in the latter we showed how to derive from this the characterization of λ-injectivity classes of [RAB, 02].

On our way, we observe that all λ_m-orthogonality classes are reflective in $\mathcal{C}$. Note that, in contrast with the morphisms between λ-presentable objects, there is a proper class of λ-presentable morphisms. Similarly, λ_m-cone-orthogonality classes are multireflective, but despite partial results in this direction, we could not determine whether the "expected" characterization for them holds, namely as the classes closed under pullbacks and strongly $\mathcal{K}_\lambda$-pure subobjects. However, an example will show that closedness under pullbacks, $\mathcal{K}_\lambda$-pure subobjects and λ-directed colimits is not sufficient for being a λ-cone-orthogonality class.

Much of the usefulness of the λ-presentable morphisms rests on the fact that if $\mathcal{C}$ is a λ-accessible category with pushouts, then all the associated comma categories $(A \downarrow \mathcal{C})$ are also λ-accessible (Proposition 2.6). Whether this holds without the assumption that pushouts exist is left as an open problem.

Finally, we also consider general orthogonality classes in locally presentable categories. We find the orthogonality hull of any subcategory $\mathcal{K}$ to be the one obtained by adding to $\mathcal{K}$ all the objects which, for every $\alpha \geq \lambda$, is a strongly $\mathcal{K}_\alpha$-pure subobject of some product of objects in $\mathcal{K}$ (Corollary 4.5).

2. $\mathcal{K}_\lambda$-purity and strong $\mathcal{K}_\lambda$-purity

For basic definitions and results on locally presentable and accessible categories, we refer the reader to [AR, 94]. For convenience, we recall the following.

If λ is a regular infinite cardinal, an object A of a category $\mathcal{C}$ is λ-*presentable* if the hom-functor $\mathcal{C}(A,-): \mathcal{C} \longrightarrow \mathbf{Set}$ preserves λ-directed colimits. Then, $\mathcal{C}$ is λ-*accessible* if it has all λ-directed colimits, as well as a (small) set S of λ-presentable objects such that every object in $\mathcal{C}$ is the λ-directed colimit of a diagram with all its vertices in S. Finally, $\mathcal{C}$ is *locally* λ-*presentable* if it is λ-accessible and cocomplete (or, equivalently, complete). We write *finitely presentable* for ω-presentable.

Throughout the paper we will need to distinguish, for a given category $\mathcal{C}$, between the λ-presentable objects of the comma categories $(A \downarrow \mathcal{C})$ $(A \in \mathcal{C})$, and those of $\mathcal{C}^2$ (= the category of morphisms in $\mathcal{C}$). In order to do this, we denote by $\mathcal{C}_\lambda$ the full subcategory of $\mathcal{C}$ on the λ-presentable objects of $\mathcal{C}$, and we recall, from [H$_1$, 98], the following definition.

2.1. Definition. A morphism $f: A \longrightarrow B$ in a category $\mathcal{C}$ will be called λ-*presentable* if it is a λ-presentable object of the comma category $(A \downarrow \mathcal{C})$.

2.2. Remarks and Examples.

(a) In a λ-accessible category, being a λ-presentable morphism is easily seen to be strictly weaker than being λ-presentable as an object of $\mathcal{C}^2$: this is clear from the fact that the latter is the same than being a morphism in $\mathcal{C}_\lambda$ (see [AR, 94], Exercise 2.c), and from Theorem 2.3 below. For example, a function $f: X \longrightarrow Y$ is finitely presentable in $\mathbf{Set^2}$ just when X and Y are finite, but it is a finitely presentable morphism in $\mathbf{Set}$ if and only if the sets $Y - X$ and $\{(x, x') \mid x \neq x', f(x) = f(x')\}$ are finite.

(b) Let $\mathcal{C}$ be the category of all right R-modules, for some ring R. One can verify directly (or using Theorem 2.3 below) that the embedding $A \hookrightarrow B$ of a submodule is a finitely presentable morphism in $\mathcal{C}$ if and only if the quotient B/A is a finitely presentable module.

(c) The intuition might suggest that a morphism $f\colon A \longrightarrow B$ is λ-presentable when f provides a way to "present B" in the usual sense of classical algebra, by adding less than λ generators and relations to some presentation of A. This is indeed correct in the category of all structures of some λ-ary signature (see [H, 04]). In categorical terms, there is a translation of this idea in terms of pushouts that seems reasonable, and that the following theorem shows to be valid in a wider context.

2.3. Theorem. *Let $\mathcal{C}$ be a locally λ-presentable category. Then $f\colon A \longrightarrow B$ is λ-presentable if and only if there exists a pushout diagram*

$$\begin{tikzcd} C \arrow[r] \arrow[d] & D \arrow[d] \\ A \arrow[r, "f"'] \arrow[ur, phantom, "\ulcorner"] & B \end{tikzcd}$$

where C and D are λ-presentable

Proof. Proposition 1.2 of [H, 04] already shows that this is true up to a retraction in $(A \downarrow \mathcal{C})$:

2.4. Lemma. *([H, 04]) In a λ-accessible category with pushouts, $f\colon A \longrightarrow B$ is λ-presentable if and only if there exists a commutative diagram*

$$\begin{tikzcd} C \arrow[r, "h"] \arrow[d, "m"'] & D \arrow[d, "n"] \\ A \arrow[r, "f'"] \arrow[dr, "f"'] \arrow[ur, phantom, "\ulcorner"] & B' \arrow[d, shift right, "r"'] \\ & B \arrow[u, shift right, "s"'] \end{tikzcd}$$

where C and D are λ-presentable, the square is a pushout, and $rs = 1_B$.

We need another, technical lemma, which will be used again later:

2.5. Lemma. *Any pushout square*

$$\begin{tikzcd} C \arrow[r, "h"] \arrow[d, "m"'] & D \arrow[d, "n"] \\ A \arrow[r, "f'"'] & B' \end{tikzcd}$$

with C and D λ-presentable can be completed as follows:

$$\begin{tikzcd} C \arrow[r, "h"] \arrow[d, "m_{o,i}"] \arrow[ddd, bend right=60, "m"'] & D \arrow[d, "n_{o,i}"'] \arrow[ddd, bend left=60, "n"] \\ A_i \arrow[r, "f_i"'] \arrow[d, "m_{i,j}"] & B_i \arrow[d, "n_{i,j}"'] \\ A_j \arrow[r, "f_j"'] \arrow[d, "m_j"] & B_j \arrow[d, "n_j"'] \\ A \arrow[r, "f'"'] & B' \end{tikzcd}$$

where all A_i's and B_i's are λ-presentable, all squares are pushouts, and $(m_i\colon A_i \longrightarrow A)_I$ and $(n_i\colon B_i \longrightarrow B')_I$ are λ-directed colimit diagrams.

Proof. Let $(m_{i,j}\colon A_i \longrightarrow A_j)_I$ be a λ-directed diagram with the A_i's λ-presentable, and with colimit cocone $(m_i\colon A_i \longrightarrow A)_I$. m must factorize $m = m_i m_{o,i}$ for some $i \in I$, and then A is the colimit of $(m_{j,k}\colon A_j \longrightarrow A_k)_{k\geq j\geq i}$. For each $j \geq i$, form the pushout $(f_j, n_{o,j})$ of $(h, m_{o,j})$, and let the n_j, $(j \geq i)$ and the $n_{j,k}$ $(j \leq k)$ be the induced morphisms, as in the diagram above. Then all the B_i's are λ-presentable, and the squares in the diagram are pushouts. The fact that $(n_j\colon B_j \longrightarrow B')_{j\geq i}$ is the colimit diagram of $(n_{j,k}\colon B_j \longrightarrow B_k)_{k\geq j\geq i}$ is a straighforward computation. ∎

End of the proof of Theorem 2.3:

Given $f\colon A \longrightarrow B$ λ-presentable, there exist diagrams as in Lemmas 2.4 and 2.5. If $srn = n$ (in the diagram of Lemma 2.4), it follows easily that r is an isomorphism, and we are done. Otherwise, srn must factorize through some n_j, $srn = n_j n'$. Let $g\colon B_j \twoheadrightarrow E$ be the coequalizer of n' and $n_{o,j}$, and $(g'\colon B' \longrightarrow F, n'_j\colon E \longrightarrow F)$ be the pushout of (g, n_j), as in the diagram below. Then E is λ-presentable, and $(g'f', n'_j)$ is the pushout of (gf_j, m_j).

We have $rn_j n_{o,j} = rn = rsrn = rn_j n'$, so that there exists a unique $q\colon E \longrightarrow B$ such that $qg = rn_j$. r and q then induce a unique $t\colon F \longrightarrow B$ such that $tn'_j = q$ and $tg' = r$:

$$\begin{tikzcd}
C \arrow[r, "h"] \arrow[d, "m_{o,j}"'] & D \arrow[d, shift right, "n_{o,j}"'] \arrow[d, shift left, "n'"] & \\
A_j \arrow[r, "f_j"'] \arrow[d, "m_j"'] & B_j \arrow[r, two heads, "g"] \arrow[d, "n_j"] & E \arrow[d, "n'_j"] \arrow[dd, dotted, bend left, "q"] \\
A \arrow[r, "f'"] \arrow[drr, "f"'] & B' \arrow[r, "g'"] \arrow[dr, shift right, "r"'] & F \arrow[d, dotted, "t"] \\
& & B \arrow[ul, shift right, "s"']
\end{tikzcd}$$

To complete the proof, we need only to show that t is an isomorphism: the required pushout will then be $(f)(m_j) = (q)(gf_j)$.

We have $t(g's) = rs = 1_B$, and we need to show that $(g's)t = 1_F$. The lower right hand square being a pushout, this amounts to show that

(1) $g'stg' = g'$, and

(2) $g'stn'_j = n'_j$.

As for (1), we have $g'stg' = g'sr$, and showing that $g'sr = g'$ amounts to show that (a) $g'srf' = g'f'$ and (b) $g'srn = g'n$, because the (vertical) left rectangle $nh = f'm$ is a pushout. Now, (a) is clear since $srf' = sf = f'$; for (b), we have $g'srn = g'n_jn' = n'_jgn' = n'_jgn_{o,j} = g'n_jn_{o,j} = g'n$.

For the second equality (2), the top left square being a pushout and g being epi, it will be sufficient to show that (i) $g'stn'_jgn_{o,j} = n'_jgn_{o,j}$ and (ii) $g'stn'_jgf_j = n'_jgf_j$. For (i), we have $g'stn'_jgn_{o,j} = g'stg'n_jn_{o,j} = g'n_jn_{o,j} = n'_jgn_{o,j}$ (using (1)). Finally, (ii) follows from $g'stn'_jgf_j = g'stg'f'm_j = g'f'm_j = n'_jgf_j$ (using (1) again). ∎

Theorem 2.3 solves a problem left open in [H, 04].

Recall that, since finitely accessible categories have filtered colimits, finitely accessible categories with finite colimits are locally finitely presentable. In the following, we will often work with the slightly weaker assumption of the existence of pushouts. The next result gives the main reason:

2.6. Proposition. *Let A be any object of a λ-accessible category $\mathcal{C}$ with pushouts. Then $(A \downarrow \mathcal{C})$ is λ-accessible.*

Proof. From Lemma 2.4, one sees easily that there is essentially only a set of λ-presentable objects in $(A \downarrow \mathcal{C})$. To obtain $f\colon A \longrightarrow B$ as the colimit of a λ-directed diagram of λ-presentable morphisms, one proceeds as in the proof of 2.4 (see [H, 04]): $\mathcal{C}^2$ being locally λ-presentable, consider any λ-directed diagram $((a_{ij}, b_{ij})\colon f_i \longrightarrow f_j)_I$ of λ-presentable objects of $\mathcal{C}^2$ with colimit $((a_i, b_i)\colon f_i \longrightarrow f)_I$; let then (f_i^*, a_i^*) be the pushout (in $\mathcal{C}$) of (f_i, a_i) $(i \in I)$, and $f^*_{ij}\colon f^*_i \longrightarrow f^*_j$ and $f'_i\colon f^*_i \longrightarrow f$ be the induced morphisms. One can verify that $\operatorname{colim}_I(f^*_{ij}\colon f^*_i \longrightarrow f^*_j)_I = (f'_i\colon f^*_i \longrightarrow f)$ in $(A \downarrow \mathcal{C})$. ■

2.7. Remarks.

(a) One can compare Proposition 2.6 to Corollary 2.44 (and its proof) in [AR, 94], which shows that if $\mathcal{C}$ is λ-accessible and A is α-presentable, then $(A \downarrow \mathcal{C})$ is β-accessible for any β sharply greater than λ and α.

(b) As opposed to what we erroneously claimed in [H_1, 98], we do not know if Proposition 2.6 holds in every λ-accessible category. As a consequence, all categories in [H_1, 98] should be assumed to have pushouts.

We now consider some generalizations of the concept of purity.

2.8. Definition. Let $\mathcal{C}$ be a λ-accessible category $\mathcal{C}$ with pushouts, $\mathcal{K}$ a full subcategory of $\mathcal{C}$. Let $f\colon A \longrightarrow B$ be a morphism in $\mathcal{C}$.

(a) f is a *$\mathcal{K}$-epi* if for all pairs $B \overset{g}{\underset{h}{\rightrightarrows}} K$, $K \in \mathcal{K}$ and $gf = hf$ imply $g = h$.

(b) ([AS, 04]) f is *$\mathcal{K}_\lambda$-pure* if for every commutative square

$$\begin{tikzcd} C \arrow[r, "h"] \arrow[d, "m"'] & D \arrow[d, "n"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

where h is a $\mathcal{K}$-epi in $\mathcal{C}_\lambda$, there exists $d\colon D \longrightarrow A$ such that $dh = m$.

(c) f is *strongly $\mathcal{K}_\lambda$-pure* if for every factorization $A \xrightarrow{h} D \xrightarrow{n} B$ of f, h is a split mono whenever it is a λ-presentable $\mathcal{K}$-epi.

2.9. Remarks.

(a) If one replaces, in 2.8(b), the condition "h is a $\mathcal{K}$-epi in $\mathcal{C}_\lambda$" by "h is a λ-presentable $\mathcal{K}$-epi", the definition obtained is equivalent to the one in (c). This is completely analogous to the equivalence of strong monos and extremal monos in categories with pushouts (take $\mathcal{K} = \mathcal{C}$).

(b) For $\mathcal{K}$ empty, every morphism in $\mathcal{C}$ is $\mathcal{K}$-epi, and the (strongly) $\mathcal{K}_\lambda$-pure morphisms are just the (strongly) λ-pure morphisms of ([H_1, 98]) [AR, 94]. Recall from [H_1, 98] (but see Remark 2.7(b) above) that in the context of our definition, $f\colon A \longrightarrow B$ is strongly λ-pure if and only if it is the λ-directed colimit in $(A \downarrow \mathcal{C})$ of split monos. As a consequence, they are just the same than the λ-pure morphisms. For the context in which $\mathcal{K}$-purity and strong $\mathcal{K}$-purity are equivalent, see Proposition 2.11 below.

(c) In general λ-accessible categories, strong purity is not equivalent to purity (as defined in [AR, 94]): [AHT, 96] provides an example of an ω-pure morphism in a finitely accessible category, which is not a strong mono. From the description of the category, the morphism is easily seen to be finitely presentable. Hence it is not strongly ω-pure, otherwise it would be a split mono.

2.10. Examples.

(a) In the category of abelian groups, where λ-purity was originally defined, L. Fuchs ([Fu, 70]) calls a subgroup G of A λ-pure if G is a direct summand of any group B such that $G \leq B \leq A$ and $|B/G| < \lambda$. Note first that the condition $|B/G| < \lambda$ can be replaced by B/G being λ-generated: this is clear for $\lambda > \omega$, and follows from standard results in abelian group theory for $\lambda = \omega$. Also known are the facts that an abelian group G is a direct summand of its extension B if and only if the embedding is a split mono, and that λ-generated abelian groups are necessarily λ-presentable. Using 2.2(b), we see that Fuchs' definition corresponds to our formulation of strong purity (i.e., 2.8(c) with $\mathcal{K}$ empty).

(b) Let $\mathcal{C}$ be the category $\mathbf{Str}\Sigma$ of all Σ-structures and Σ-homomorphisms, where Σ is a (multi-sorted) λ-ary signature. It is well-known that the λ-pure morphisms in $\mathcal{C}$ are the homomorphisms reflecting the λ-ary *positive primitive* (or *pp*-) formulas, i.e., the existentially quantified conjunctions of less than λ atomic formulas. One can compare this with the *embeddings* (reflecting the atomic formulas), and the λ-*elementary embeddings* (reflecting all the λ-ary formulas).
More generally, using the usual syntactic translation of presentability in this context (see [AR, 94] or [H, 04]), as well as the syntactic characterization of epimorphisms in locally λ-presentable categories (see [H_2, 98]), one can prove that for a reflective subcategory $\mathcal{K}$ of $\mathcal{C}$ closed under λ-directed colimits, a morphism $f\colon A \longrightarrow B$ is $\mathcal{K}_\lambda$-pure (or, equivalently, strongly $\mathcal{K}_\lambda$-pure: see the next proposition) iff for every set ϕ of less than λ atomic formulas in Σ such that $\mathcal{K} \models \forall \boldsymbol{x} \exists^{\leq 1} \boldsymbol{y} \exists \boldsymbol{z} (\wedge \phi(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}))$, and for every string $\boldsymbol{a}$ in A, the condition

$$B \models \exists \boldsymbol{y}\boldsymbol{z} (\wedge \phi[f(\boldsymbol{a}), \boldsymbol{y}, \boldsymbol{z}]) \qquad \Longrightarrow \qquad A \models \exists \boldsymbol{y}\boldsymbol{z} (\wedge \phi[\boldsymbol{a}, \boldsymbol{y}, \boldsymbol{z}])$$

is satisfied. If $\mathcal{K}$ is empty, the condition $\mathcal{K} \models \forall \boldsymbol{x} \exists^{\leq 1} \boldsymbol{y} \exists \boldsymbol{z} (\wedge \phi(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}))$ is void, and we obtain the above characterization of λ-purity.
This actually applies if $\mathcal{C}$ is any variety of Σ-structures. For example, if $\mathcal{C}$ is the category of abelian groups and $\mathcal{K}$ is the subcategory of all groups with trivial 2-torsion, then a subgroup A of B is (strongly) $\mathcal{K}_\omega$-pure iff every element of A which has a 2^n-th root in B for some positive integer n, must have one in A itself. This follows from the fact that $\mathcal{K} \models \forall x \exists^{\leq 1} y (2^n \cdot y = x)$ for all n.

In 2.10(b), we stated in particular that for sufficiently nice subcategories $\mathcal{K}$ of varieties, the $\mathcal{K}_\lambda$-pure morphisms are always strongly so. The following result is more precise:

2.11. Proposition. *Let $\mathcal{K}$ be a full subcategory of a locally λ-presentable category closed under products and λ-directed colimits. Then the $\mathcal{K}_\lambda$-pure morphisms and the strongly $\mathcal{K}_\lambda$-pure morphisms coincide.*

Proof. That a strongly $\mathcal{K}_\lambda$-pure morphism is $\mathcal{K}_\lambda$-pure is true for every subcategory $\mathcal{K}$ of any λ-accessible category with pushouts. This follows easily from the fact that the pushout of a λ-presentable $\mathcal{K}$-epi is also a λ-presentable $\mathcal{K}$-epi, and because morphisms in $\mathcal{C}_\lambda$ are λ-presentable.

To prove the reverse implication, we first show that under the conditions of the proposition, any λ-presentable $\mathcal{K}$-epi is the pushout of a $\mathcal{K}$-epi in $\mathcal{C}_\lambda$.

Let $f\colon A \longrightarrow B$ be a λ-presentable $\mathcal{K}$-epi. By Theorem 2.3, there exists a diagram as in the statement of Lemma 2.5, with f' and B' replaced by f and B. We show that one of the f_i's must be a $\mathcal{K}$-epi.

Suppose not. Then for each $i \in I$, there exist distinct morphisms $u_i, v_i\colon B_i \rightrightarrows K_i$ with $K_i \in \mathcal{K}$, such that $u_i f_i = v_i f_i$. We will use the categorical version of the standard model-theoretic construction of reduced product (with respect to a λ-complete filter: see [CK, 90] or [H, 04]). For each $i \in I$, let $i^* = \{j \in I \mid j \geq i\}$. Consider the set

$$F = \{D \subseteq I \mid D \supseteq i^* \text{ for some } i \in I\}.$$

The diagram $(\pi_{i,j}\colon \prod_{k\in i^*} K_k \longrightarrow \prod_{k\in j^*} K_k)_I$ of the canonical projections $(i \leq j)$ between the products is λ-directed, and we will denote by $(\pi_i\colon \prod_{k\in i^*} K_k \longrightarrow \prod_F K_k)_I$ its colimit diagram ($\prod_F K_k$ is just the reduced product with respect to the λ-complete filter F). For $i \in I$, the set of all $u_k n_{i,k}\colon B_i \longrightarrow B_k \longrightarrow K_k,\ k \geq i$, induces a morphism $u'_i\colon B_i \longrightarrow \prod_{k\in i^*} K_k$, and similarly for v'_i, defined from the $v_k n_{i,k}$'s. Let $t_i\colon A_i \longrightarrow \prod_{k\in i^*} K_k$ be the morphism $u'_i f_i = v'_i f_i$. The t_i's, u'_i's and v'_i's respectively induce morphisms t, u', v' between the colimits, as in the diagram:

$$\begin{tikzcd}
A_i \arrow[r, "f_i"] \arrow[d, "m_{i,k}"'] & B_i \arrow[r, shift left, "u_i"] \arrow[r, shift right, "v_i"'] \arrow[d, "n_{i,k}"] & K_i \\
A_k \arrow[r, "f_k"'] \arrow[d, "m_k"'] & B_k \arrow[r, shift left, "u_k"] \arrow[r, shift right, "v_k"'] \arrow[d, "n_k"] & K_k \\
A \arrow[r, "f"] & B &
\end{tikzcd}
\qquad
\begin{tikzcd}
A_i \arrow[rr, "f_i"] \arrow[dr, "t_i"'] \arrow[dd, "m_i"'] & & B_i \arrow[dl, shift left, "v'_i"] \arrow[dl, shift right, "u'_i"'] \arrow[dd, "n_i"] \\
& \prod\limits_{k\in i^*} K_k \arrow[d, "\pi_i"] & \\
A \arrow[r, dotted, "t"] \arrow[rr, bend right, "f"'] & \prod_F K_k & B \arrow[l, dotted, shift left, "u'"] \arrow[l, dotted, shift right, "v'"']
\end{tikzcd}$$

Clearly $u'_i \neq v'_i$, since $u_i \neq v_i$. But we actually have more: $u'_k n_{i,k} \neq v'_k n_{i,k}$ for all $k \geq i$, because of the top left pushout square. Now $u'f = t = v'f$ implies $u' = v'$, since $\prod_F K_k \in \mathcal{K}$ and f is $\mathcal{K}$-epi. Hence $\pi_i u'_i = \pi_i v'_i$. But B_i is λ-presentable, and $\prod_F K_k = \operatorname{colim}_{j\geq i}(\prod_{k\in j^*} K_k)$, so there must exist $j \geq i$ such that $\pi_{i,j}u'_i = \pi_{i,j}v'_i$. This means $u'_j n_{i,j} = v'_j n_{i,j}$, a contradiction.

The proposition now follows easily, using again the fact that λ-presentable $\mathcal{K}$-epis are preserved by pushouts. ■

The details of the proof show that the closure conditions in the statement of the proposition could actually be relaxed to the closure under λ-complete reduced products.

2.12. Facts. For convenience, we collect a few facts about purity, some of them already mentioned. References are given for most. The others are obvious or left as straightforward exercises.
Let $\mathcal{C}$ be a λ-accessible category with pushouts, $\mathcal{K}$ any full subcategory of $\mathcal{C}$. Then

(1) the following implications hold:
$$\lambda\text{-pure} \quad\Rightarrow\quad \text{strongly } \mathcal{K}_\lambda\text{-pure} \quad\Rightarrow\quad \mathcal{K}_\lambda\text{-pure};$$
(2) if $\alpha \geq \lambda$, then (any sort of) α-purity implies (the same sort of) λ-purity;

(3) all three types of purity are closed under composition ([AR, 94] and [H, 04]);

(4) for any α, the (multiple) equalizer of any set of morphisms with codomain in $\mathcal{K}$ is strongly $\mathcal{K}_\alpha$-pure;

(5) λ-pure morphisms are regular monos ([AHT, 96]);

(6) the λ-pure morphisms with domain A are precisely the λ-directed colimits in $(A \downarrow \mathcal{C})$ of split monos ([H_1, 98]);

(7) λ-pure morphisms are stable under pushouts ([AHT, 96]);

(8) the class of all $\mathcal{K}_\lambda$-pure morphisms in $\mathcal{C}$ is closed under λ-directed colimits in $\mathcal{C}^2$ (adapt the proof of [AR, 94], 2.30);

(9) for each object A in $\mathcal{C}$, the class of all strongly $\mathcal{K}_\lambda$-pure morphisms from A in $\mathcal{C}$ is closed under λ-directed colimits in $(A \downarrow \mathcal{C})$ (adapt the proof of [H, 04], 2.3(b)).

(10) the right-hand implication in (1) is an equivalence if $\mathcal{C}$ has and $\mathcal{K}$ is closed under products and λ-directed colimits (Proposition 2.11 above);

(11) all three types of purity are closed under products in $\mathcal{C}^2$ (use pushouts);

(12) if $\mathcal{C}$ has finite copowers, $\mathcal{K}_\lambda$-pure morphisms are monos for every $\mathcal{K}$ ([AS, 04]).

Accordingly, in the right context, we will speak of ($\mathcal{K}_\lambda$-) pure *subobjects.* Note finally that λ-pure subobjects are sometimes called λ-*algebraically closed subobjects* (for example in [Fa, 75] and [H, 04]).

3. Injectivity, orthogonality, reflectivity: earlier results

In this section, we briefly survey a body of known facts, in order to provide the context for our main results (in Section 4).

Recall that an object K in $\mathcal{C}$ is *injective with respect to* (respectively *orthogonal to*) a morphism $f: A \longrightarrow B$ if

$$\mathcal{C}(f, K): \mathcal{C}(B, K) \longrightarrow \mathcal{C}(A, K)$$

is surjective (respectively bijective). We write $A \perp f$ to mean that A is orthogonal to f. For a class $\mathcal{M}$ of morphisms in $\mathcal{C}$, $\mathrm{Inj}(\mathcal{M})$ (resp. $\mathcal{M}^{\perp}$) denotes the class of all the objects which are injective with respect to (resp. orthogonal to) all the morphisms in $\mathcal{M}$. A class of objects of the form $\mathrm{Inj}(\mathcal{M})$ (resp. $\mathcal{M}^{\perp}$) is called an *injectivity class* (resp. an *orthogonality class*). Some of the following might be less familiar:

3.1. Definition. A class $\mathcal{K}$ of objects in $\mathcal{C}$ is a *λ_m-injectivity class* (resp. a *λ-injectivity class*) if $\mathcal{K} = \mathrm{Inj}(\mathcal{M})$ for some class $\mathcal{M}$ of λ-presentable morphisms (resp. of morphisms in $\mathcal{C}_\lambda$). *λ_m-orthogonality classes* and *λ-orthogonality classes* are defined similarly, replacing $\mathrm{Inj}(\mathcal{M})$ by $\mathcal{M}^{\perp}$.

3.2. Related Characterizations. Our main Theorem (4.1) will complete the following array of known results:
Let $\mathcal{K}$ be a class of objects in a locally λ-presentable category $\mathcal{C}$. Then,

(1) ([RAB, 02]) $\mathcal{K}$ is a λ-injectivity class in $\mathcal{C}$ if and only if it is closed under (i) products, (ii) λ-pure subobjects, and (iii) λ-directed colimits.

(2) ([H_1, 98]) $\mathcal{K}$ is a λ_m-injectivity class in $\mathcal{C}$ if and only if it is closed under (i) products, and (ii) λ-pure subobjects.

(3) ([AS, 04]) $\mathcal{K}$ is a λ-orthogonality class in $\mathcal{C}$ if and only if it is closed under (i) products, (ii) $\mathcal{K}_\lambda$-pure subobjects, and (iii) λ-directed colimits.

Note that in statement (3), "products" can be replaced by "limits", since by 2.12 (4), closure under $\mathcal{K}_\lambda$-pure subobjects and products implies closure under all limits.

3.3. Remarks.

(a) Lemma 3.6 in [H, 04] showed how a class $\mathcal{M}$ of λ_m-morphisms such that $\mathcal{K} = \mathrm{Inj}(\mathcal{M})$, can be replaced by a class (and hence, a set) of morphisms in $\mathcal{C}_\lambda$ when $\mathcal{K}$ is closed under λ-directed colimits. This lemma was interpreted as a compactness-type result, and gave a different proof of the Rosický-Adámek-Borceux characterization (3.2(1)) above.

(b) One may feel that closure under $\mathcal{K}_\lambda$-pure subobjects is not a very pleasant property, as the definition of $\mathcal{K}$-purity depends on $\mathcal{K}$ globally. But this unpleasant feature might be unavoidable, at least in the finitary case, if one recalls that there is no possible *uniform* syntactic characterization of subcategories of varieties of finitary structures closed under limits and directed colimits. The latter means the following: given a finitary signature Σ, H. Volger showed in [V, 79] that every class of Σ-structures, closed in $\mathbf{Str}\Sigma$ under limits and directed colimits, is finitarily axiomatizable, but that no set Δ of (first-order, finitary) sentences exists such that a class of structures is closed under these operators if and only if it can be axiomatized by a subset of Δ. [V, 79] does provide a syntactic characterization for these classes, but its formulation involves, as expected, a "global" property of the theory.

(c) Note however that there is a simple uniform syntactic characterization of orthogonality classes in $\mathbf{Str}\Sigma$. Those are the ones definable by classes of ($L_\infty(\Sigma)$-) sentences of the form

$$\forall \boldsymbol{x}(\wedge\phi(\boldsymbol{x}) \longrightarrow \exists! \boldsymbol{y}(\wedge\psi(\boldsymbol{x},\boldsymbol{y}))),$$

where ϕ and ψ are sets (of any cardinality) of atomic formulas. These subcategories correspond to the *prevarieties* of [AS, 04], except that here we do not allow Σ to be a proper class (in order for the ambient category to be locally presentable). If Σ is λ-ary, the λ_m-orthogonality classes and the λ-orthogonality classes are those among them for which the cardinality of ψ, respectively of ϕ and ψ, are smaller than λ. Only in the latter case the classes of sentences are actually *sets* (of $L_\lambda(\Sigma)$-sentences). More details can be found [AR, 94], [AS, 04] and [H, 04].

3.4. Orthogonality vs Reflectivity.

(a) One sees easily that for any subcategory of any category, we have

$$\text{reflective} \Rightarrow \text{orthogonal} \Rightarrow \text{limit-closed.}$$

Orthogonality classes have generated a fair amount of literature in the last 30 years or so, in particular concerning the conditions under which the arrows above can be reversed. In the context of locally presentable categories, note first that under the Weak Vopenka's Principle (a large-cardinal axiom), limit-closed subcategories are automatically α-orthogonal for some α (and hence reflective). In particular, both arrows above are equivalences under this assumption. Note also that this Principle is actually *equivalent* to the implication on the right being reversible (see [AR, 94]).

(b) The so-called "Orthogonal Subcategory Problem" is about the left arrow in (a): characterize the classes $\mathcal{M}$ of morphisms such that (the corresponding full subcategory) $\mathcal{M}^\perp$ is reflective. In the seminal paper [FK, 72] of Freyd and Kelly, one of the main theorems implies that, in a locally presentable category $\mathcal{C}$, if $\mathcal{M}$ is the union of a (small) set and of a class of epis from some proper factorization system in $\mathcal{C}$, then $\mathcal{M}^\perp$ is reflective. In particular, λ-orthogonality classes are reflective.

(c) The characterizations in 3.2(3) and Theorem 4.1 are about the right arrow in (a): what closure conditions should be added to reverse this implication? First note that adding closure under α-directed colimits for some α implies reflectivity (see [AR, 94]), and hence orthogonality. However, from the result of Volger already mentioned in 3.3(b), there exist reflective subcategories of locally finitely presentable categories which are closed under directed colimits, and not ω-orthogonal (but note that the proof of the main result in [HR, 01] shows that they must be ω_1-orthogonal). The main result in [HR, 01] states that no such counterexample exists for higher cardinalities. The remaining problem of characterizing the ω-orthogonality classes was finally solved in [AS, 04] (see 3.2(3)). We can summarize all this as follows (for a subcategory $\mathcal{K}$ of a locally λ-presentable category):

$$\omega\text{-orthogonal} \Leftrightarrow [\text{limits, } \mathcal{K}_\omega\text{-pure subobjects, directed colimits}]\text{-closed}$$

$$\max(\lambda, \omega_1)\text{-orthogonal} \Leftrightarrow [\text{limits, } \lambda\text{-directed colimits}]\text{-closed}$$

Note that closure under limits and directed colimits alone does not imply closure under $\mathcal{K}_\omega$-pure subobjects, since it does not imply ω-orthogonality, a crucial fact proved by H. Volger from the same example in [V, 79] (p. 37) used to prove the result mentioned in 3.3(b). However, we did not find an example showing that closure under limits and $\mathcal{K}_\omega$-pure subobjects does not imply closure under directed colimits. (Closure under limits and *strongly* $\mathcal{K}_\omega$-pure subobjects does *not* imply closure under directed colimits: see 4.3(c) below).

4. Main results

4.1. Theorem. *Let $\mathcal{K}$ be a class of objects in a locally λ-presentable category $\mathcal{C}$. Then $\mathcal{K}$ is a λ_m-orthogonality class in $\mathcal{C}$ if and only if it is closed under*

(i) products, and

(ii) strongly $\mathcal{K}_\lambda$-pure subobjects.

Proof. That the closure conditions are necessary is a straightforward exercise. To show that they are sufficient, we follow the same path than the one in [AS, 04] for the λ-orthogonality case. The proof will flow in a more natural way here, but we must show first that the conditions still imply that $\mathcal{K}$ is reflective. This is a consequence of the following lemma, which we will use again later:

4.2. Lemma. *Let $\mathcal{C}$ be a λ-accessible category with pushouts, and $\mathcal{K}$ be a subcategory of $\mathcal{C}$ closed under α-pure subobjects, for some $\alpha \geq \lambda$. Then the inclusion functor of $\mathcal{K}$ in $\mathcal{C}$ satisfies the Solution Set Condition.*

Proof. Let $A \in \mathcal{C}$. Using Theorem 2.33 of [AR, 94], one can find γ large enough so that A is γ-presentable, $\mathcal{C}$ is γ-accessible, $\mathcal{K}$ is closed under γ-pure subobjects, and any morphism $f: A \longrightarrow B$ factorizes through a γ-pure morphism $f': A' \longrightarrow B$ with A' γ-presentable (see [AR, 94]). Then the set

$$S_A = \{f: A \longrightarrow K \mid K \in \mathcal{K},\ K \text{ is } \gamma\text{-presentable in } \mathcal{C}\}$$

is easily seen to be a solution set for A. ∎

End of the proof of Theorem 4.1:

From 2.12 (4), $\mathcal{K}$ is closed under limits, and hence is reflective in $\mathcal{C}$, by the lemma. Let $\mathcal{M}$ be the class of all λ-presentable $\mathcal{K}$-epis f such that $K \perp f$ for all K in $\mathcal{K}$. We prove that for $A \in \mathcal{M}^{\perp}$, the reflection $\eta_A: A \longrightarrow rA$ in $\mathcal{K}$ is strongly $\mathcal{K}_\lambda$-pure. This will show that $\mathcal{M}^{\perp} \subseteq \mathcal{K}$, and hence that $\mathcal{M}^{\perp} = \mathcal{K}$.

Consider then any factorization $A \xrightarrow{h} D \xrightarrow{n} rA$ of η_A, with h a λ-presentable $\mathcal{K}$-epi. Then $h \in \mathcal{M}$. Indeed, any $g: A \longrightarrow K \in \mathcal{K}$ extends (uniquely) to some $g': rA \longrightarrow K$ such that $g'\eta_A = g$, and $g'n$ is then unique such that $(g'n)h = g$, since h is a $\mathcal{K}$-epi. But then 1_A factorizes through h (since $A \in \mathcal{M}$), i.e., h is a split mono. ∎

4.3. Remarks.

(a) The Adámek-Sousa characterization of λ-orthogonality classes (3.2(3) above) can be obtained from Theorem 4.1. The same Lemma 3.6 of [H, 04] used in the injectivity case (see 3.3(a)) actually holds if all the morphisms in its statement are replaced by $\mathcal{K}$-epis, with essentially the same proof. As a consequence, the replacement can be done in the same way if $\mathcal{K} = \mathcal{M}^{\perp}$ (instead of $\mathrm{Inj}(\mathcal{M})$). The proof of the lemma is rather technical, so this certainly does not provide a simpler proof of 3.2(3). However, it may help organizing a body of related results, and clarifying the role of each closure operator.

(b) Theorem 4.1 shows in particular that $\mathcal{M}^{\perp}$ is reflective for every class $\mathcal{M}$ of λ-presentable morphisms. It would be interesting to see if the Freyd-Kelly theorem refered to in 3.4(b), or the related results in W. Tholen's [T, 83], could be generalized as to include this case. A related problem is the following: among the factorization systems (E,M) determining the λ_m-orthogonality class $\mathcal{M}^{\perp} = \mathcal{K}$ as a reflective subcategory, is there one for which E and M can be constructed in a natural way from the λ-presentable $\mathcal{K}$-epis and the strongly $\mathcal{K}_\lambda$-pure morphisms respectively?

(c) 3.3(c) provides an easy way to construct examples. In the category of groups, let $\mathcal{K}$ be the class of all groups in which any element which has a p-root for all odd primes p must have a unique square root. $\mathcal{K}$ is definable by the sentence

$$\forall x x_2 x_3 \ldots (\bigwedge_{i>1} x = x_i^{p_i} \longrightarrow \exists! y (x = y^2)),$$

where p_i is the i-th prime number. By 3.3(c), $\mathcal{K}$ is an ω_m- (and ω_1-) orthogonality class, hence closed under strongly $\mathcal{K}_\omega$-pure subobjects and under $\mathcal{K}_{\omega_1}$-pure subobjects. However it is not closed under directed colimits (and hence not ω-orthogonal):

the group of rational numbers with odd denominators in reduced form is not in $\mathcal{K}$, but it is the direct limit of its subgroups $A_i = \{a/p_2^{k_2}p_3^{k_3}...p_i^{k_i} \mid a, k_j \in \mathbb{Z}\}$, $i \geq 2$, which are all in $\mathcal{K}$.

Hoping to exhibit a λ_m-orthogonality class which is not a small-orthogonality class (i.e., not α-orthogonal for any α), one could try writing down a proper class of sentences as in 3.3(c). However, by 3.4(a), one could not prove, without negating the Vopenka's Principle, that this class is not equivalent to a set of such sentences.

If $\mathcal{K}$ and $\mathcal{N}$ are subcategories of $\mathcal{C}$, $\Pi(\mathcal{N})$ and $\mathcal{K}^s_\lambda(\mathcal{N})$ will denote respectively the class of all products and the class of all strongly $\mathcal{K}_\lambda$-pure subobjects of the members of $\mathcal{N}$.

4.4. Theorem. *Let $\mathcal{K}$ be a subcategory of a locally λ-presentable category $\mathcal{C}$. Then $\mathcal{K}^s_\lambda(\Pi(\mathcal{K}))$ is the λ_m-orthogonality hull of $\mathcal{K}$.*

Proof. Let $\mathcal{N} = \mathcal{K}^s_\lambda(\Pi(\mathcal{K}))$. Using Theorem 4.1, we need to prove that:

(a) $\Pi(\mathcal{N}) \subseteq \mathcal{N}$,

(b) $\mathcal{N}^s_\lambda(\mathcal{N})) \subseteq \mathcal{N}$, and

(c) $\mathcal{N} \subseteq \mathcal{S}$ for every λ_m-orthogonality class $\mathcal{S}$ containing $\mathcal{K}$.

(a) This follows from the fact that the product of a set of strongly $\mathcal{K}_\lambda$-pure morphisms is also strongly $\mathcal{K}_\lambda$-pure (Facts 2.12(11)).

(b) We first show that a morphism is $\mathcal{K}$-epi if and only if it is $\mathcal{N}$-epi. That $\mathcal{N}$-epis are $\mathcal{K}$-epis is trivial from $\mathcal{K} \subseteq \mathcal{N}$. Let $g\colon A \longrightarrow D$ be a $\mathcal{K}$-epi, and $u, v\colon D \rightrightarrows N \in \mathcal{N}$ be morphisms such that $ug = vg$. Then there exists a strongly $\mathcal{K}_\lambda$-pure morphism $h\colon N \longrightarrow \prod_{i\in I} K_i$ for some K_i's in $\mathcal{K}$. If $\pi_i\colon \prod_{i\in I} K_i \longrightarrow K_i$ is the canonical projection, then $\pi_i hug = \pi_i hvg$ implies $\pi_i hu = \pi_i hv$ for each $i \in I$ (since g is a $\mathcal{K}$-epi), which implies $hu = hv$, and hence $u = v$ since h is mono, by 2.12 (12).

Now, the fact that $\mathcal{K}$-epi $= \mathcal{N}$-epi implies that strongly $\mathcal{K}_\lambda$-pure $=$ strongly $\mathcal{N}_\lambda$-pure. Let then A be in $\mathcal{N}^s_\lambda(\mathcal{N})$. There exist $B \in \mathcal{N}$ and a diagram

$$A \xrightarrow{\quad f \quad} B \xrightarrow{\quad g \quad} \prod_{i\in I} K_i$$

with f strongly $\mathcal{N}_\lambda$-pure, g strongly $\mathcal{K}_\lambda$-pure, and $K_i \in \mathcal{K}$ for each i. Hence both morphisms are strongly $\mathcal{K}_\lambda$-pure, so that by the Fact 2.12 (3), gf is also strongly $\mathcal{K}_\lambda$-pure. We conclude that $A \in \mathcal{K}^s_\lambda(\Pi(\mathcal{K})) = \mathcal{N}$.

(c) If $\mathcal{S}$ is an λ_m-orthogonality class containing $\mathcal{K}$, then it contains $\Pi(\mathcal{K})$, and hence $\mathcal{S}^s_\lambda(\Pi(\mathcal{K}))$, by Theorem 4.1. But strongly $\mathcal{K}_\lambda$-pure morphisms are obviously strongly $\mathcal{S}_\lambda$-pure, and hence $\mathcal{S}$ contains $\mathcal{K}^s_\lambda(\Pi(\mathcal{K})) = \mathcal{N}$ ∎

4.5. Corollary. *Let $\mathcal{K}$ be a subcategory of a locally λ-presentable category $\mathcal{C}$. Then*

(1) $\mathcal{K}$ is an orthogonality class if and only if
(i) it is closed under products, and (ii) $\bigcap_{\alpha\geq\lambda} \mathcal{K}^s_\alpha(\mathcal{K}) = \mathcal{K}$.

(2) $\bigcap_{\alpha\geq\lambda} \mathcal{K}^s_\alpha(\Pi(\mathcal{K}))$ is the orthogonality hull of $\mathcal{K}$.

PROOF. (1)($\Rightarrow$) Let $\mathcal{K} = \mathcal{M}^{\perp}$. Orthogonality classes are closed under all limits, as mentioned above, and $\cap_{\alpha\geq\lambda}\mathcal{K}^s_\alpha(\mathcal{K}) \supseteq \mathcal{K}$ is clear. We need to show that $\cap_{\alpha\geq\lambda}\mathcal{K}^s_\alpha(\mathcal{K}) \subseteq \mathcal{K}$.

Note that the conclusions of the Theorems 4.4 and 4.1 clearly hold for any $\alpha \geq \lambda$. Let $A \in \cap_{\alpha\geq\lambda}\mathcal{K}^s_\alpha(\mathcal{K})$. Now, any given $m: C \longrightarrow D$ in $\mathcal{M}$ is α-presentable for α large enough. We show that $A \perp m$. There exists a strongly $\mathcal{K}_\alpha$-pure morphism $f: A \longrightarrow K$ for some $K \in \mathcal{K}$, so that for any $g: C \longrightarrow A$, fg factorizes through m, by a unique $h: D \longrightarrow K$. We form the pushout (g', m') of g and m, and let n be the induced morphism:

$$\begin{tikzcd}
C \arrow[r, "m"] \arrow[d, "g"'] & D \arrow[d, "g'"] \arrow[ddr, "h"] & \\
A \arrow[r, "m'"] \arrow[drr, "f"'] \arrow[ur, phantom, "\ulcorner"] & P \arrow[dr, dotted, "n"] & \\
& & K
\end{tikzcd}$$

m' being α-presentable, it has a left inverse r, and $rg'm = rm'g = g$. We only need to show that rg' is unique to factorize g through m. We use the fact that f is mono (see 2.12 (12)): if $hm = (rg')m$ for some $h: D \longrightarrow A$, then $fhm = f(rg')m$, which implies $fh = f(rg')$ because m is $\mathcal{K}$-epi. Hence $h = rg'$.

($\Leftarrow$) $\mathcal{K}$ being closed under products, we have $\mathcal{K} = \cap_{\alpha\geq\lambda}\mathcal{K}^s_\alpha(\mathcal{K}) = \cap_{\alpha\geq\lambda}\mathcal{K}^s_\alpha(\Pi(\mathcal{K}))$, i.e., by Theorem 4.4, an intersection of orthogonality classes, and hence an orthogonality class itself.

(2) This is proved from (1) in the same way that Theorem 4.4 is proved from Theorem 4.1 ■

4.6. REMARK. One can compare Corollary 4.5 to the characterization given in Exercise 2.o of [AR, 94]. Correctly restated (personal communication from the authors), it says that the orthogonality hull of a subcategory $\mathcal{K}$ of a locally presentable category $\mathcal{C}$ is $\bigcap_{\alpha\in Ord}\mathcal{K}(\alpha)$, where $\mathcal{K}(\alpha)$ is the smallest subcategory of $\mathcal{C}$ containing $\mathcal{K}$ and closed under limits and α-directed colimits in $\mathcal{C}$. This is proved using mainly the facts that each $\mathcal{K}(\alpha)$ is a small-orthogonality class, and that every orthogonality class is the intersection of small-orthogonality classes.

For the rest of the paper, we briefly consider cone-orthogonality in locally λ-presentable categories. An object K in $\mathcal{C}$ is *injective with respect to the cone* $c = (c_i: C \longrightarrow D_i)_I$ if every $g: C \longrightarrow K$ factorizes through c (i.e., through at least one of the c_i's). K is *orthogonal to* c if every $g: C \longrightarrow K$ factorizes through exactly one c_i, and furthermore this factorization is unique. A λ_m*-cone* (λ*-cone*) is a cone made of λ-presentable morphisms (resp. of morphisms in $\mathcal{C}_\lambda$). If $\mathcal{M}$ is a class of (λ_m-, λ-) cones, $\mathrm{Inj}(\mathcal{M})$ and $\mathcal{M}^{\perp}$ will be called (λ_m-, λ-) cone-injectivity and cone-orthogonality classes respectively.

Example 3.1 of [RAB, 02] shows that classes closed under λ-pure subobjects and λ-directed colimits are not necessarily λ-cone-injectivity classes. In contrast, Theorem 4 of

[H_1, 98] proves that λ_m-cone-injectivity classes are precisely the ones closed under λ-pure subobjects (the characterization is actually proved in every λ-accessible category with pushouts).

Using the same technique than in [H_1, 98], one can show that for every classes $\mathcal{K}$ and $\mathcal{N}$ of objects in $\mathcal{C}$, $\mathcal{K}^s_\lambda(\mathcal{N}) = \mathrm{Inj}(\mathcal{M})$ for a class $\mathcal{M}$ of cones made of λ-presentable $\mathcal{K}$-epis. (One can use this to give another proof of Theorem 4.1, without using the fact that $\mathcal{K}$ is reflective.) As a consequence, a class $\mathcal{K}$ closed under strongly $\mathcal{K}_\lambda$-pure subobjects comes close to be a λ_m-cone-orthogonality class: what is missing is that a given morphism may factorize through several branches of the same cone of $\mathcal{M}$. The expected cure would be to add closure under pullbacks. Note that Lemma 4.2 implies that such a $\mathcal{K}$, closed under strongly $\mathcal{K}_\lambda$-pure subobjects and pullbacks, will be multireflective in $\mathcal{C}$. However, we were not able to deduce (nor to disprove) that this was enough to insure that $\mathcal{K}$ is a λ_m-cone-orthogonality class.

As for λ-cone-orthogonality, a natural conjecture from the characterization of λ-orthogonality in [AS, 04] (3.2(3) above), would be that they are the classes closed under (i) pullbacks, (ii) $\mathcal{K}_\lambda$-pure subobjects, and (iii) λ-directed colimits. However this is not true, the same Example 3.1 of [RAB, 02] being actually a counterexample. First, the category $\mathcal{B}$ described there is easily found to be closed (in the category $\mathbf{Str}\Sigma$ of all Σ-structures) under pullbacks. Also, every morphism in $\mathbf{Str}\Sigma$ is a $\mathcal{B}$-epi, for the simple reason that there are no two distinct morphisms from any given Σ-structure to any object in $\mathcal{B}$. Hence purity and $\mathcal{B}$-purity are the same. Finally, the fact that $\mathcal{B}$ is not a ω-cone-injectivity class (proved there) implies that it is not a ω-cone-orthogonality class either (see [AR, 94], 4.23).

Hence the following three problems remain:

4.7. Open Problems. In locally λ-presentable categories, characterize, by closure properties:

(1) the λ_m-cone-orthogonality classes,

(2) the λ-cone-orthogonality classes, and

(3) the cone-orthogonality classes.

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