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EQUIVALENCE OF 2D-MULTITOPIC CATEGORY AND ANA-BICATEGORY

Dedicated to Prof. V. S. Borkar for creating interest in mathematics

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Abstract. In this paper equivalence of the concepts of ana-bicategory and the 2D-multitopic category is proved. The equivalence is FOLDS equivalence of the FOLDS-Specifications of the two concepts. Two constructions for transforming one form of category to another are given and it is shown that we get a structure equivalent to the original one when we compose the two constructions.

1. Introduction

In category theory there is an emergence of higher dimensional categories. There are two distinct flavors of higher dimensional categories:

1. Pure algebraic: In these the composition of cells is defined by a composition function and the composition functions are constrained by huge coherence conditions. Examples include bicategory, tricategory, 2-category etc.

2. Virtual: In these the composition of cells is defined by the universal property of certain special cells called "universals". Examples are multitopic category, opetopic category etc.

Even for the case $n = 3$ the pure algebraic version becomes intractable with lots of isomorphisms and coherence diagrams. Thr geometric nature of the coherence conditions suggests a deep underlying truth which when revealed would generate them automatically. The virtual version does not have the same problem and in a certain sense is "scalable". Another point to be remembered is that the virtual version defines the composition "up to isomorphism" in a true categorical spirit. In view of these advantages it is tempting to propose the virtual definitions for categories. The first step towards accepting the virtual definitions of categories is to show that for the case of $n = 1, 2$, the virtual definition

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reduces to the ordinary definitions of category and bicategory respectively. In this thesis, multitopic category is considered as the basic virtual definition for higher dimensional categories and it is shown that this reduces to concept of bicategory for $n = 2$. Let the multitopic category for the case $n = 2$, which is the case being dealt here, be called 2D-multitopic category.

In parallel there is another point to be made about category theory. Though it is emphasized in category theory that concepts should be defined "up to isomorphism", this does not go beyond structures internal to the category like limits, colimits etc. For example, while defining functor we do not say that it takes certain values up to isomorphism. In [Makkai (1996)], Makkai has proposed a version of category theory in which external concepts like functors and natural transformations are also defined up to isomorphism. The functor there called anafunctor is defined up to isomorphism. This was extended to define bicategory as ana-bicategory. Another concept introduced in the same paper was saturation, which more or less means that anafunctor can take any one of the isomorphic copies of an object as its value.

In light of what has been said, I feel that the "correct" "algebraic" concept of bicategory is ana-bicategory, and it is actually found to be the case that the 2D-multitopic category is equivalent to an ana-bicategory with saturation. This should not be surprising because the horizontal composition internal to 2D-multitopic category is defined by the universal property hence defined up to isomorphism. Once the equivalence of 2D-multitopic category and ana-bicategory is shown, from the fact that bicategory is a special case of ana-bicategory, it follows that the definition of 2D-multitopic category reduces to bicategory.

In section 2, the formal definitions that will be used in this paper are listed. In section 3, the construction of ana-bicategory from 2D-multitopic category is described. In section 4, the construction of 2D-multitopic category from ana-bicategory is described. In section 5, the equivalence of these two definitions is shown.

2. Preliminaries

In this section the mathematical definitions that are required for subsequent sections are given.

2.1. ANA-BICATEGORY: The concept of ana versions of categorical definitions were introduced in [Makkai (1996)]. First the concept of anafunctor and natural anatransformation have to be given.

Anafunctor: An anafunctor F between categories $\mathcal{C}$ and $\mathcal{D}$ is given by the following data and conditions:

Datum-1. A class $|F|$, with two maps $\sigma : |F| \longrightarrow \mathcal{O}(\mathcal{C})$ (source) and $\tau : |F| \longrightarrow \mathcal{O}(\mathcal{D})$ (target). For $X \in \mathcal{O}(\mathcal{C})$ we denote $|F|(X) = \{s \in |F| : \sigma(s) = X\}$, and for $s \in |F|(X)$ we denote $\tau(s)$ by $F_s(X)$. $|F|$ is called class of specifications.

Datum-2. For each $X, Y \in \mathcal{O}(\mathcal{C})$, $s \in |F|(X)$, $t \in |F|(Y)$ and $f : X \longrightarrow Y$, an arrow $F_{s,t}(f) : F_s(X) \longrightarrow F_t(Y)$ in $\mathcal{D}$ is given.

Condition-1. For every $X \in \mathcal{O}(\mathcal{C})$, $|F|(X)$ is non-empty.

Condition-2. For all $X \in \mathcal{O}(\mathcal{C})$ and $s \in |F|(X)$, $F_{s,s}(\mathrm{Id}_X) = \mathrm{Id}_{F_s(X)}$.

Condition-3. For all $X, Y, Z \in \mathcal{O}(\mathcal{C})$, $s \in |F|(X)$, $t \in |F|(Y)$, $u \in |F|(Z)$, $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$, we have $F_{s,u}(f \cdot g) = F_{s,t}(f) \cdot F_{t,u}(g)$.

Saturated anafunctor: Given an anafunctor F, F is said to be saturated if $F_s(X) = A$ and $i : A \cong B$ in $\mathcal{D}$ then there is an unique $t \in |F|(X)$ such that $F_t(X) = B$ and $F_{s,t}(\mathrm{Id}_X) = i$.

Saturation is an external condition on anafunctor, but usually anafunctors that arise naturally have this saturation property, for example product anafunctor.

Natural anatransformation: A natural anatransformation ϕ between anafunctors F and G is given by the following datum and condition:

Datum-1. A family $\langle \phi_{X,s,t} : F_s(X) \longrightarrow G_t(X) \rangle_{X \in \mathcal{O}(\mathcal{C}), s \in |F|(X), t \in |G|(X)}$.

Condition-1. For every $f : X \longrightarrow Y$ in $\mathcal{C}(X, Y)$, and for every $s \in |F|(X)$, $t \in |G|(X)$, $u \in |F|(Y)$, $v \in |G|(Y)$, the following diagram commutes

$$\begin{tikzcd} F_s(X) \arrow[r, "F_{s,u}(f)"] \arrow[d, "\phi_{X,s,t}"'] & F_u(Y) \arrow[d, "\phi_{Y,u,v}"] \\ G_t(X) \arrow[r, "G_{t,v}(f)"'] & G_v(Y) \end{tikzcd}$$

Natural anaisomorphism: Natural anaisomorphism is a natural anatransformation in which each member of the family $\langle \phi_{X,s,t} : F_s(X) \longrightarrow G_t(X) \rangle_{X \in \mathcal{O}(\mathcal{C}), s \in |F|(X), t \in |G|(X)}$ is an isomorphism.

Ana-bicategory: An ana-bicategory $\mathcal{A}$ consists of the following data and conditions:

Datum-1. Collection $\mathcal{O}(\mathcal{A})$ of objects (0-cells).

Datum-2. For any pair of objects $A, B \in \mathcal{O}(\mathcal{A})$, a category $\mathcal{A}(A, B)$ (1-cells as its objects and 2-cells as its arrows).

Datum-3. For any object $A \in \mathcal{O}(\mathcal{A})$, an identity anaobject in $\mathcal{A}(A, A)$, determined by anafunctor

$$1_A : \mathbf{1} \longrightarrow \mathcal{A}(A, A)$$

The elements of class $|1_{\_}|$ are called 0-specifications.

Datum-4. For any three objects $A, B, C \in \mathcal{O}(\mathcal{A})$, composition anafunctor

$$\circ_{A,B,C} : \mathcal{A}(A,B) \times \mathcal{A}(B,C) \longrightarrow \mathcal{A}(A,C)$$

The elements of class $| \circ_{-,-,-} |$ are called 2-specifications.

Datum-5. Associativity natural anaisomorphism

$$\alpha_{A,B,C,D} : ((-) \circ (-)) \circ (-) \overset{\cong}{\Longrightarrow} (-) \circ ((-) \circ (-))$$

where $((-)\circ(-))\circ(-) = (\circ_{A,B,C}, \mathrm{Id}_{\mathcal{A}(C,D)}) \cdot \circ_{A,C,D}$ and $(-)\circ((-)\circ(-)) = (\mathrm{Id}_{\mathcal{A}(A,B)}, \circ_{B,C,D}) \cdot \circ_{A,B,D}$. Thus,

$$\alpha_{A,B,C,D,s,t,u,v} : (f \circ_s g) \circ_t h \longrightarrow f \circ_v (g \circ_u h)$$

Datum-6. Left identity natural anaisomorphism

$$\lambda_{A,B} : (-) \circ 1_B \overset{\cong}{\Longrightarrow} \mathrm{Id}_{\mathcal{A}(A,B)}$$

where $\mathrm{Id}_{\mathcal{A}(A,B)}$ is an identity functor and $(-) \circ 1_B = (\mathrm{Id}_{\mathcal{A}(A,B)}, ! \cdot 1_B) \cdot \circ_{A,B,B} : \mathcal{A}(A,B) \longrightarrow \mathcal{A}(A,B)$. Thus,

$$\lambda_{A,B,s,p} : f \circ_s 1_{B,p} \longrightarrow f$$

Datum-7. Right identity natural anaisomorphism

$$\rho_{A,B} : 1_A \circ (-) \overset{\cong}{\Longrightarrow} \mathrm{Id}_{\mathcal{A}(A,B)}$$

where $\mathrm{Id}_{\mathcal{A}(A,B)}$ is identity functor and $1_A \circ (-) = (! \cdot 1_A, \mathrm{Id}_{\mathcal{A}(A,B)}) \cdot \circ_{A,A,B} : \mathcal{A}(A,B) \longrightarrow \mathcal{A}(A,B)$. Thus,

$$\rho_{A,B,s,p} : 1_{A,p} \circ_s f \longrightarrow f$$

Condition-1. For any five objects $A, B, C, D, E \in \mathcal{O}(\mathcal{A})$, and four 1-cells $f \in \mathcal{O}(\mathcal{A}(A,B))$, $g \in \mathcal{O}(\mathcal{A}(B,C))$, $h \in \mathcal{O}(\mathcal{A}(C,D))$, $i \in \mathcal{O}(\mathcal{A}(D,E))$, the coherence pentagon (since there are too many specifications to be considered we use numbers to represent them rather than letters.):

$$\begin{tikzcd}
((f \circ_1 g) \circ_2 h) \circ_3 i \arrow[r, Rightarrow, "\alpha_{1,2,4,5} \circ_{3,6} \mathrm{Id}_i"] \arrow[d, Rightarrow, "\alpha_{2,3,10,12}"] & (f \circ_5 (g \circ_4 h)) \circ_6 i \arrow[r, Rightarrow, "\alpha_{5,6,7,8}"] & f \circ_8 ((g \circ_4 h) \circ_7 i) \arrow[d, Rightarrow, "\mathrm{Id}_f \circ_{8,9} \alpha_{4,7,10,11}"'] \\
(f \circ_1 g) \circ_{12} (h \circ_{10} i) \arrow[rr, Rightarrow, "\alpha_{1,12,11,9}"'] & & f \circ_9 (g \circ_{11} (h \circ_{10} i))
\end{tikzcd}$$

Condition-2. For any two objects $A, B \in \mathcal{O}(\mathcal{A})$, and two 1-cells $f \in \mathcal{O}(\mathcal{A}(A,B))$, $g \in \mathcal{O}(\mathcal{A}(B,C))$, the coherence triangle:

$$\begin{tikzcd}
(f \circ_s 1_{B,p}) \circ_t g \arrow[dd, Rightarrow, "\alpha_{s,t,u,v}"'] \arrow[dr, Rightarrow, "\lambda_{s,p} \circ_{t,w} \mathrm{Id}_g"] & \\
& f \circ_w g \\
f \circ_v (1_{B,p} \circ_u g) \arrow[ur, Rightarrow, "\mathrm{Id}_f \circ_{v,w} \rho_{u,p}"'] &
\end{tikzcd}$$

Saturated ana-bicategory: Ana-bicategory is said to be saturated if the anafunctors 1_A and $\circ_{A,B,C}$ are:

2.2. Lemma. *Given a saturated anafunctor $F : \mathcal{C} \longrightarrow \mathcal{D}$, an anafunctor $G : \mathcal{C} \longrightarrow \mathcal{D}$ and natural anaisomorphism $\phi : F \longrightarrow G$, then*

$$(\forall X \in \mathcal{O}(\mathcal{C}))(\forall t \in |G|(X))(\exists! s \in |F|(X))(\phi_{X,s,t} = \mathrm{Id}_{G_t(X)})$$

Proof. Fix $X \in \mathcal{O}(\mathcal{C})$ and $t \in |G|(X)$ and select any $s' \in |F|(X)$. Since $\phi_{X,s',t} : F_{s'}(X) \cong G_t(X)$, $\exists! s \in |F|(X)$ such that $F_s(X) = G_t(X)$ and $Fs', s(\mathrm{Id}_X) = \phi_{X,s',t}$. From naturality of ϕ,

$$\begin{tikzcd}
F_{s'}(X) \arrow[r, "F_{s',s}(\mathrm{Id}_X)"] \arrow[d, "\phi_{X,s',t}"', "\cong"] & F_s(X) \arrow[d, "\phi_{X,s,t}"] \\
G_t(X) \arrow[r, "G_{t,t}(\mathrm{Id}_X)"'] & G_t(X)
\end{tikzcd}$$

$$\phi_{X,s,t} = \mathrm{Id}_{G_t(X)}$$

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2.3. 2D-multitopic category: The concept of multitopic category was introduced in [Hermida, Makkai and Power (2000)], [Hermida, Makkai and Power (2001)] and [Hermida, Makkai and Power (2002)]. Since here only a 2 dimensional case is considered, the definition of multitopic category is simplified by removing the amalgamation mechanism that was built into its definition. First multicategory is defined and then based on it the definition of 2D-multitopic category is given.

Multicategory: Multicategory $\mathcal{M}$ consists of the following data and conditions:

Datum-1. A collection $\mathcal{O}(\mathcal{M})$ of objects and $\mathcal{O}(\mathcal{M})^*$ of tuples (strings) of objects.

Datum-2. A collection $\mathcal{A}(\mathcal{M})$ of arrows with domain in $\mathcal{O}(\mathcal{M})^*$ and codomain in $\mathcal{O}(\mathcal{M})$.

Datum-3. A natural number indexed, partially defined composition $\cdot_{\_}$ of arrows in $\mathcal{A}(\mathcal{M})$. Composition of $\alpha, \beta \in \mathcal{A}(\mathcal{M})$ is defined if and only if the codomain of α fits into domain of β. Formally, if the domain and codomain of α are $\bar{f}_\alpha$ and g_α and the domain and codomain of β are $\bar{f}_\beta$ and g_β such that $g_\alpha = \bar{f}_\beta(i)$, then composite $\alpha \cdot_i \beta$ is defined; it is an arrow in $\mathcal{A}(\mathcal{M})$ with domain $\bar{f}_\beta[\bar{f}_\alpha/(i, i+1)]$ and codomain g_β. (From here on $s[t/(i,j)]$ is the string formed by replacing i^{th} to $j-1^{th}$ substring of s by t. From here on for convenience the subscript for composition is removed. But it should be kept in mind that composition is placed.)

Condition-1. Composition is associative i.e. $\alpha \cdot_{i+j} (\beta \cdot_i \gamma) = (\alpha \cdot_j \beta) \cdot_i \gamma$

Condition-2. Composition is commutative i.e. $\alpha \cdot_i (\beta \cdot_j \gamma) = \beta \cdot_{j+|\bar{f}_\alpha|-1} (\alpha \cdot_i \gamma)$ where $i < j$.

Condition-3. For any $f \in \mathcal{O}(\mathcal{M})$, there is an identity in $\mathcal{A}(\mathcal{M})$ with domain the string $\langle f \rangle$ and codomain f denoted as Id_f such that, for any appropriate α, $\mathrm{Id}_f \cdot \alpha = \alpha = \alpha \cdot \mathrm{Id}_f$.

2D-multitopic category: A 2D-multitopic category consists of the following data and conditions:

Datum-1. A collection $\mathrm{Cell}_0(\mathcal{M})$ of 0-cells.

Datum-2. A collection $\mathrm{Cell}_1(\mathcal{M})$ of 1-cells with domain and codomain in $\mathrm{Cell}_0(\mathcal{M})$. We denote by $\mathrm{Cell}_1(\mathcal{M})^*$ a collection of all composable strings of 1-cells from $\mathrm{Cell}_1(\mathcal{M})$. $\leq$ denotes substring relation on $\mathrm{Cell}_1(\mathcal{M})^*$ and $\epsilon \in \mathrm{Cell}_1(\mathcal{M})^*$ denotes empty string. $\mathrm{Cell}_1(\mathcal{M})$ is referred to as (1-) pasting diagrams, also abbreviated as PD.

Datum-3. A collection $\mathrm{Cell}_2(\mathcal{M})$ of 2-cells with domain in $\mathrm{Cell}_1(\mathcal{M})^*$ and codomain in $\mathrm{Cell}_1(\mathcal{M})$, such that their initial and terminal 0-cells match.

Condition-1. The collections $\mathrm{Cell}_1(\mathcal{M})$, $\mathrm{Cell}_1(\mathcal{M})^*$ and $\mathrm{Cell}_2(\mathcal{M})$ form a multicategory with composition $\cdot$.

Condition-2. For every $\bar{f} \in \mathrm{Cell}_1(\mathcal{M})^*$, there exists a 2-cell say $\alpha \in \mathrm{Cell}_2(\mathcal{M})$, with domain $\bar{f}$, such that for every $\beta \in \mathrm{Cell}_2(\mathcal{M})$ with domain containing the string $\bar{f}$, there is a unique $\gamma \in \mathrm{Cell}_2(\mathcal{M})$, for which $\alpha \cdot \gamma = \beta$. Such an α is called an universal 2-cell (or simply universal).

2.4. Lemma. *If s, u are two universals in a multitopic category $\mathcal{M}$ such that* $\mathrm{dom}(s) \leq \mathrm{dom}(u)$, *then there is an universal t such that $u = s \cdot t$.*

PROOF. Existence of t satisfying $u = s \cdot t$ follows since s is an universal. Let α, be such that $\mathrm{dom}(t) \leq \mathrm{dom}(\alpha)$. We need to show the existence and uniqueness of β such that $\alpha = t \cdot \beta$.

Since $\mathrm{dom}(u) = \mathrm{dom}(s \cdot t) \leq \mathrm{dom}(s \cdot \alpha)$, there exists an unique β such that $s \cdot \alpha = u \cdot \beta$. Hence,

$$\begin{aligned} & s \cdot \alpha &&= u \cdot \beta \\ \Longrightarrow\ & s \cdot \alpha &&= s \cdot t \cdot \beta \\ \Longrightarrow\ & \alpha &&= t \cdot \beta \end{aligned}$$

■

2.5. FOLDS EQUIVALENCE: FOLDS stands for First Order Logic with Dependent Sorts. Here just a short overview is given. Details are in [Makkai (1995)] and [Makkai (1998)].

A FOLDS theory (L, Σ) consists of a signature L and set of axioms Σ. The FOLDS signature L is a one way category, where one way category is a category in which identity morphisms are the only morphisms with the same domain and codomain objects. The objects in this category are sorts. Each sort is dependent on all the sorts that are below it (an arrow to it). The axioms in Σ are first order sentences with a restriction. The restriction is that the equality is disallowed and all the statements are about the existence of certain elements in sorts. For example instead of saying $g \circ f = h$, we would say $\exists \tau \in T(X, Y, Z; f, g, h).T$. The advantage is that all axioms turn out to be asserting existence of certain elements that represent the truth of axiom.

Now FOLDS structure S is a functor from L to any category, that satisfies the axioms in Σ. Given two L structures S and T, a homomorphism p is a natural transformation from S to T.

Two FOLDS structures S and T with the same signature are said to be equivalent if there is a span

$$S \xleftarrow{p} Q \xrightarrow{q} T$$

where p, q are natural transformations and are fiberwise surjective. This is denoted as $S \simeq_L T$.

$p : S \longrightarrow T$ is fiberwise surjective if the following diagram is a weak pullback, for all objects K in L.

$$\begin{tikzcd} S(K) \arrow[r, "p_K"] \arrow[d, "\pi_{K,S}"'] & T(K) \arrow[d, "\pi_{K,T}"] \\ S(\dot{K}) \arrow[r, "p_{\dot{K}}"'] & T(\dot{K}) \end{tikzcd}$$

$\dot{K}$ is the context of sort K. Intuitively context is the sorts on which the sort K depends. $\pi_{K,T}$ is the projection of context values from the sort K.

Up to this point the equivalence of two structures with the same signature is considered. To compare two structures with different signatures something more is needed [Makkai (2001)].

Suppose we have two theories $T_1 = (L_1, \Sigma_1)$ and $T_2 = (L_2, \Sigma_2)$. To say that T_1 and T_2 are equivalent we need two constructions, one taking any T_1-model S_1 to a T_2-model S_1^* and another taking any T_2-model S_2 to T_1-model $S_2^{\#}$. T_1 and T_2 are equivalent whenever $S_1 \simeq_{L_1} S_1^{*\#}$ and $S_2 \simeq_{L_2} S_2^{\#*}$ for all S_1 and S_2.

The constructions $(-)^*$ and $(-)^{\#}$, given in later sections are canonical; in particular, they do not use the axiom of choice. Moreover, the data for the equivalences $S_1 \simeq_{L_1} S_1^{*\#}$ and $S_2 \simeq_{L_2} S_2^{\#*}$ are also canonically constructed from S_1 respectively S_2. In fact, the combined constructions add up to an equivalence of the two concepts: ana-bicategory and 2D-multitopic category, in the sense of [Makkai (2001), Section 6].

3. 2D-multitopic category to ana-bicategory

In this section, the construction of an ana-bicategory from a 2D-multitopic category is given. This construction will be denoted as $\mathcal{M} \xmapsto{(-)^*} \mathcal{M}^*$, where $\mathcal{M}$ is the given 2D-multitopic category. For simplicity, in this chapter $\mathcal{M}^*$ will be denoted by $\mathcal{A}$. The construction first involves extraction of ana-bicategory data from a 2D-multitopic category and proving the axioms of ana-bicategory.

3.1. Objects: $\mathcal{O}(\mathcal{A}) = \mathrm{Cell}_0(\mathcal{M})$.

3.2. Category $\mathcal{A}(A, B)$: For $A, B \in \mathcal{O}(\mathcal{A})$, category $\mathcal{A}(A, B)$

Data:

1. Objects: $\mathcal{O}(\mathcal{A}(A, B)) = \{f : f \text{ is 1 cell of the form } A \xrightarrow{f} B \in \mathrm{Cell}_1(\mathcal{M})\}$.

2. Arrows: $\mathcal{A}(A, B)[f, g] = \{\beta : \beta \text{ is 2 cell of the form } \begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\beta"] & B \end{tikzcd} \in \mathrm{Cell}_2(\mathcal{M})\}$.

3. Identity: $\mathrm{Id}_f = \begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "f"'] \arrow[r, phantom, "\Downarrow\mathrm{Id}_f"] & B \end{tikzcd} \in \mathrm{Cell}_2(\mathcal{M})$

4. Composition: Composition of arrows is defined as a composition of 2-cells in $\mathcal{M}$ restricted to $\mathcal{A}(A, B)$.

Axioms:

1. Associativity: Follows from associativity in multicategory $\mathcal{M}$.

2. Existence of identity: Follows from existence of identity in $\mathcal{M}$.

3. Left and right identity laws: $\mathrm{Id}_f \cdot \delta = \delta \cdot \mathrm{Id}_g = \delta$ (Identity law in multicategory), where $\delta : f \Longrightarrow g$.

3.3. Identity: Identity anaobject for A in $\mathcal{A}(A, A)$, anafunctor

$$1_A : \mathbf{1} \longrightarrow \mathcal{A}(A, A)$$

Data:

1. 0-Specifications: $|1_A|(1) = \{p : p \in \mathrm{Cell}_2(\mathcal{M})$ is an universal from empty pd $A\}$.
2. Anafunctor on objects $1_{A,p}(1) = co\mathrm{dom}(p)$ where $p \in |1_A|(1)$. Since 1 is the only object in $\mathbf{1}$, we denote $1_{A,p}(1)$ by $1_{A,p}$.
3. Anafunctor on arrows $1_{A,p,q}(\mathrm{Id}_1) = \delta$, where $p, q \in |1_A|(1)$, and $\delta \in \mathcal{A}(A, A)[1_{A,p}, 1_{A,q}]$ such that $p \cdot \delta = q$. Since Id_1 is only arrow in $\mathbf{1}$, we denote $1_{A,p,q}(\mathrm{Id}_1)$ by $1_{A,p,q}$.

Axioms:

1. Well defined: $1_{A,p,q} \in \mathcal{A}(A, A)[1_{A,p}, 1_{A,q}]$, where $p, q \in |1_A|(1)$ is well defined since by universality of p there is an unique $1_{A,p,q}$ such that $p \cdot 1_{A,p,q} = q$.
2. Inhabitedness: $|1_A|(1)$ is nonempty because of the existence of universals from every PD, in particular from PD A.
3. Identity: $1_{A,p,p}(\mathrm{Id}_1) = \mathrm{Id}_{1_{A,p}}$, since $p \cdot \mathrm{Id}_{1_{A,p}} = p$, where $p \in |1_A|(1)$.
4. Composition: Need to show $1_{A,p,q} \cdot 1_{A,q,r} = 1_{A,p,r}$, where $p, q, r \in |1_A|(1)$. We have $p \cdot 1_{A,p,q} = q$ and $q \cdot 1_{A,q,r} = r$, $p \cdot 1_{A,p,r} = r$ from definition.

$$\begin{aligned} p \cdot (1_{A,p,q} \cdot 1_{A,q,r}) &= (p \cdot 1_{A,p,q}) \cdot 1_{A,q,r} \\ &= q \cdot 1_{A,q,r} \\ &= r \\ &= p \cdot 1_{A,p,r} \end{aligned}$$

   Since universals are left cancellable, we have $1_{A,p,q} \cdot 1_{A,q,r} = 1_{A,p,r}$.

3.4. Horizontal composition: Composition anafunctor,

$$\circ_{A,B,C} : \mathcal{A}(A, B) \times \mathcal{A}(B, C) \longrightarrow \mathcal{A}(A, C)$$

Since A, B, C will be clear from the context, $\circ_{A,B,C,-}$ and $\circ_{A,B,C,-,-}$ will be referred to as $\circ_-$ and $\circ_{-,-}$ respectively. Furthermore, $\circ_-(f, g)$ and $\circ_{-,-}(\beta, \gamma)$ will be denoted in the *infix* form as $f \circ_- g$ and $\beta \circ_{-,-} \gamma$ respectively.

Data:

1. 2-Specifications: $|\circ_{A,B,C}|(f, g) = \{s : s \in \mathrm{Cell}_2(\mathcal{M})$ is an universal from $A \xrightarrow{f} B \xrightarrow{g} C\}$.
2. Anafunctor on objects: $f \circ_s g = co\mathrm{dom}(s)$ where $s \in |\circ_{A,B,C}|(f, g)$.
3. Anafunctor on arrows: $\beta \circ_{s,t} \gamma = \delta$, where $s \in |\circ_{A,B,C}|(f_1, g_1)$, $t \in |\circ_{A,B,C}|(f_2, g_2)$, $(f_1, g_1), (f_2, g_2) \in \mathcal{A}(A, B) \times \mathcal{A}(B, C)$, $(\beta, \gamma) : (f_1, g_1) \Longrightarrow (f_2, g_2)$, and $\delta \in \mathcal{A}(A, C)[f_1 \circ_s g_1, f_2 \circ_t g_2]$ such that $\beta \cdot (\gamma \cdot t) = s \cdot \delta$.

Axioms:

1. Well defined: $\beta \circ_{s,t} \gamma \in \mathcal{A}(A, C)[f_1 \circ_s g_1, f_2 \circ_t g_2]$, where $s \in |\circ_{A,B,C}|(f_1, g_1)$, $t \in |\circ_{A,B,C}|(f_2, g_2)$, $\beta \in \mathcal{A}(A, B)[f_1, f_2]$ and $\gamma \in \mathcal{A}(B, C)[g_1, g_2]$ is well defined since by universality of s, there is an unique $\beta \circ_{s,t} \gamma$ such that $\beta \cdot (\gamma \cdot t) = s \cdot \beta \circ_{s,t} \gamma$.

2. Inhabitedness: $|\circ_{A,B,C}|(f, g)$ is non empty because of the existence of universals from every PD, in particular from PD $A \xrightarrow{f} B \xrightarrow{g} C$

3. Identity: $\mathrm{Id}_f \circ_{s,s} \mathrm{Id}_g = \mathrm{Id}_{f \circ_s g}$, since $\mathrm{Id}_f \cdot (\mathrm{Id}_g \cdot s) = \mathrm{Id}_f \cdot s = s = s \cdot \mathrm{Id}_{f \circ_s g}$.

4. Composition: Need to show $(\beta_1 \circ_{s,t} \gamma_1) \cdot (\beta_2 \circ_{t,u} \gamma_2) = (\beta_1 \cdot \beta_2) \circ_{s,u} (\gamma_1 \cdot \gamma_2)$, where $s \in |\circ_{A,B,C}|(f_1, g_1)$, $t \in |\circ_{A,B,C}|(f_2, g_2)$, $u \in |\circ_{A,B,C}|(f_3, g_3)$, $\beta_i \in \mathcal{A}(A, B)[f_i, f_{i+1}]$ and $\gamma_i \in \mathcal{A}(B, C)[g_i, g_{i+1}]$. We have $\beta_1 \cdot (\gamma_1 \cdot t) = s \cdot \beta_1 \circ_{s,t} \gamma_1$, $\beta_2 \cdot (\gamma_2 \cdot u) = t \cdot \beta_2 \circ_{t,u} \gamma_2$, $(\beta_1 \cdot \beta_2) \cdot ((\gamma_1 \cdot \gamma_2) \cdot u) = s \cdot (\beta_1 \cdot \beta_2) \circ_{s,u} (\gamma_1 \cdot \gamma_2)$.

$$\begin{aligned} s \cdot ((\beta_1 \circ_{s,t} \gamma_1) \cdot (\beta_2 \circ_{t,u} \gamma_2)) &= (s \cdot (\beta_1 \circ_{s,t} \gamma_1)) \cdot (\beta_2 \circ_{t,u} \gamma_2) \\ &= (\beta_1 \cdot (\gamma_1 \cdot t)) \cdot (\beta_2 \circ_{t,u} \gamma_2) \\ &= \beta_1 \cdot ((\gamma_1 \cdot t) \cdot (\beta_2 \circ_{t,u} \gamma_2)) \\ &= \beta_1 \cdot (\gamma_1 \cdot (t \cdot (\beta_2 \circ_{t,u} \gamma_2))) \\ &= \beta_1 \cdot (\gamma_1 \cdot (\beta_2 \cdot (\gamma_2 \cdot u))) \\ &= \beta_1 \cdot (\beta_2 \cdot (\gamma_1 \cdot (\gamma_2 \cdot u))) \\ &= \beta_1 \cdot (\beta_2 \cdot ((\gamma_1 \cdot \gamma_2) \cdot u)) \\ &= (\beta_1 \cdot \beta_2) \cdot ((\gamma_1 \cdot \gamma_2) \cdot u) \\ &= s \cdot (\beta_1 \cdot \beta_2) \circ_{s,u} (\gamma_1 \cdot \gamma_2) \end{aligned}$$

Since universals are left cancellable, we have $(\beta_1 \circ_{s,t} \gamma_1) \cdot (\beta_2 \circ_{t,u} \gamma_2) = (\beta_1 \cdot \beta_2) \circ_{s,u} (\gamma_1 \cdot \gamma_2)$.

3.5. Associativity isomorphisms: Natural anaisomorphism

$$\alpha_{A,B,C,D} : ((-) \circ (-)) \circ (-) \overset{\cong}{\Longrightarrow} (-) \circ ((-) \circ (-))$$

where $((-) \circ (-)) \circ (-) = (\circ_{A,B,C}, \mathrm{Id}_{\mathcal{A}(C,D)}) \cdot \circ_{A,C,D}$ and $(-) \circ ((-) \circ (-)) = (\mathrm{Id}_{\mathcal{A}(A,B)}, \circ_{B,C,D}) \cdot \circ_{A,B,D}$. Since A, B, C, D will be clear from the context $\alpha_{A,B,C,D}$ will be denoted by α.

Data: Define $\alpha_{s,t,u,v} = \delta$, where $s \in |\circ_{A,B,C}|(f, g)$, $t \in |\circ_{A,C,D}|(f \circ_s g, h)$, $u \in |\circ_{B,C,D}|(g, h)$, $v \in |\circ_{A,B,D}|(f, g \circ_u h)$ and $\delta \in \mathcal{A}(A, D)[(f \circ_s g) \circ_t h, f \circ_v (g \circ_u h)]$ such that $(s \cdot t) \cdot \delta = u \cdot v$.

Axioms:

1. Well defined: $\alpha_{s,t,u,v} \in \mathcal{A}(A, D)[(f \circ_s g) \circ_t h, f \circ_v (g \circ_u h)]$, where $s \in |\circ_{A,B,C}|(f, g)$, $t \in |\circ_{A,C,D}|(f \circ_s g, h)$, $u \in |\circ_{B,C,D}|(g, h)$, $v \in |\circ_{A,B,D}|(f, g \circ_u h)$ is well defined, since $s \cdot t$ is a composite of universals and hence is a universal. So, there is an unique $\alpha_{s,t,u,v}$ such that $(s \cdot t) \cdot \alpha_{s,t,u,v} = u \cdot v$.

2. Isomorphism: $\alpha_{s,t,u,v}$ is invertible. Its inverse $\phi_{s,t,u,v}$ is such that $s \cdot t = (u \cdot v) \cdot \phi_{s,t,v,u})$. $\phi_{s,t,u,v}$ is well defined, since $u \cdot v$ is a composite of universals, so there is an unique $\phi_{s,t,u,v}$ such that $(u \cdot v) \cdot \phi_{s,t,u,v} = s \cdot t$.

   Need to show $\alpha_{s,t,u,v} \cdot \phi_{s,t,u,v} = \mathrm{Id}_{(f \circ_s g) \circ_t h}$ and $\phi_{s,t,u,v} \cdot \alpha_{s,t,u,v} = \mathrm{Id}_{f \circ_v (g \circ_u h)}$. We use universality of $s \cdot t$ and $u \cdot v$ and following calculations.

$$\begin{aligned}(s \cdot t) \cdot (\alpha_{s,t,u,v} \cdot \phi_{s,t,u,v}) &= ((s \cdot t) \cdot \alpha_{s,t,u,v}) \cdot \phi_{s,t,u,v} \\ &= (u \cdot v) \cdot \phi_{s,t,u,v} \\ &= (s \cdot t) \\ &= (s \cdot t) \cdot \mathrm{Id}_{(f \circ_s g) \circ_t h}\end{aligned}$$

$$\begin{aligned}(u \cdot v) \cdot (\phi_{s,t,u,v} \cdot \alpha_{s,t,u,v}) &= ((u \cdot v) \cdot \phi_{s,t,u,v}) \cdot \alpha_{s,t,u,v} \\ &= (s \cdot t) \cdot \alpha_{s,t,u,v} \\ &= (u \cdot v) \\ &= (u \cdot v) \cdot \mathrm{Id}_{f \circ_v (g \circ_u h)}\end{aligned}$$

3. Naturality: Need to show that following diagram commutes.

$$\begin{tikzcd}(f_1 \circ_{s_1} g_1) \circ_{t_1} h_1 \arrow[r, Rightarrow, "\alpha_{s_1,t_1,u_1,v_1}"] \arrow[d, Rightarrow, "(\beta \circ_{s_1,s_2} \gamma) \circ_{t_1,t_2} \delta"'] & f_1 \circ_{v_1} (g_1 \circ_{u_1} h_1) \arrow[d, Rightarrow, "\beta \circ_{v_1,v_2} (\gamma \circ_{u_1,u_2} \delta)"] \\ (f_2 \circ_{s_2} g_2) \circ_{t_2} h_2 \arrow[r, Rightarrow, "\alpha_{s_2,t_2,u_2,v_2}"'] & f_2 \circ_{v_2} (g_2 \circ_{u_2} h_2)\end{tikzcd}$$

   From universality of $s_1 \cdot t_1$, it is sufficient to show $(s_1 \cdot t_1) \cdot (\alpha_{s_1,t_1,u_1,v_1} \cdot \beta \circ_{v_1,v_2} (\gamma \circ_{u_1,u_2} \delta)) = (s_1 \cdot t_1) \cdot ((\beta \circ_{s_1,s_2} \gamma) \circ_{t_1,t_2} \delta \cdot \alpha_{s_2,t_2,u_2,v_2})$.

$$\begin{aligned}(s_1 \cdot t_1) \cdot (\alpha_{s_1,t_1,u_1,v_1} \cdot \beta \circ_{v_1,v_2} (\gamma \circ_{u_1,u_2} \delta)) &= ((s_1 \cdot t_1) \cdot \alpha_{s_1,t_1,u_1,v_1}) \cdot \beta \circ_{v_1,v_2} (\gamma \circ_{u_1,u_2} \delta) \\ &= (u_1 \cdot v_1) \cdot \beta \circ_{v_1,v_2} (\gamma \circ_{u_1,u_2} \delta) \\ &= u_1 \cdot (v_1 \cdot \beta \circ_{v_1,v_2} (\gamma \circ_{u_1,u_2} \delta)) \\ &= u_1 \cdot (\beta \cdot (\gamma \circ_{u_1,u_2} \delta \cdot v_2)) \\ &= \beta \cdot (u_1 \cdot (\gamma \circ_{u_1,u_2} \delta \cdot v_2)) \\ &= \beta \cdot ((u_1 \cdot \gamma \circ_{u_1,u_2} \delta) \cdot v_2) \\ &= \beta \cdot ((\gamma \cdot (\delta \cdot u_2)) \cdot v_2) \\ &= \beta \cdot (\gamma \cdot ((\delta \cdot u_2) \cdot v_2)) \\ &= \beta \cdot (\gamma \cdot (\delta \cdot (u_2 \cdot v_2)))\end{aligned}$$

$$
\begin{aligned}
(s_1 \cdot t_1) \cdot ((\beta \circ_{s_1,s_2} \gamma) \circ_{t_1,t_2} \delta \cdot \alpha_{s_2,t_2,u_2,v_2}) &= ((s_1 \cdot t_1) \cdot (\beta \circ_{s_1,s_2} \gamma) \circ_{t_1,t_2} \delta) \cdot \alpha_{s_2,t_2,u_2,v_2} \\
&= (s_1 \cdot (t_1 \cdot (\beta \circ_{s_1,s_2} \gamma) \circ_{t_1,t_2} \delta)) \cdot \alpha_{s_2,t_2,u_2,v_2} \\
&= (s_1 \cdot (\beta \circ_{s_1,s_2} \gamma \cdot (\delta \cdot t_2))) \cdot \alpha_{s_2,t_2,u_2,v_2} \\
&= ((s_1 \cdot \beta \circ_{s_1,s_2} \gamma) \cdot (\delta \cdot t_2)) \cdot \alpha_{s_2,t_2,u_2,v_2} \\
&= ((\beta \cdot (\gamma \cdot s_2)) \cdot (\delta \cdot t_2)) \cdot \alpha_{s_2,t_2,u_2,v_2} \\
&= (\beta \cdot ((\gamma \cdot s_2) \cdot (\delta \cdot t_2))) \cdot \alpha_{s_2,t_2,u_2,v_2} \\
&= (\beta \cdot (\delta \cdot ((\gamma \cdot s_2) \cdot t_2))) \cdot \alpha_{s_2,t_2,u_2,v_2} \\
&= (\beta \cdot (\delta \cdot (\gamma \cdot (s_2 \cdot t_2)))) \cdot \alpha_{s_2,t_2,u_2,v_2} \\
&= (\beta \cdot (\gamma \cdot (\delta \cdot (s_2 \cdot t_2)))) \cdot \alpha_{s_2,t_2,u_2,v_2} \\
&= \beta \cdot ((\gamma \cdot (\delta \cdot (s_2 \cdot t_2))) \cdot \alpha_{s_2,t_2,u_2,v_2}) \\
&= \beta \cdot (\gamma \cdot ((\delta \cdot (s_2 \cdot t_2)) \cdot \alpha_{s_2,t_2,u_2,v_2})) \\
&= \beta \cdot (\gamma \cdot (\delta \cdot ((s_2 \cdot t_2) \cdot \alpha_{s_2,t_2,u_2,v_2}))) \\
&= \beta \cdot (\gamma \cdot (\delta \cdot (u_2 \cdot v_2)))
\end{aligned}
$$

3.6. Left identity isomorphisms. Natural anaisomorphism

$$\lambda_{A,B} : (-) \circ 1_B \overset{\cong}{\Longrightarrow} \mathrm{Id}_{\mathcal{A}(A,B)}$$

where $\mathrm{Id}_{\mathcal{A}(A,B)}$ is an identity functor and $(-)\circ 1_B = (\mathrm{Id}_{\mathcal{A}(A,B)}, ! \cdot 1_B) \cdot \circ_{A,B,B} : \mathcal{A}(A,B) \longrightarrow \mathcal{A}(A,B)$. Since A, B will be clear from the context $\lambda_{A,B}$ will be denoted by λ.

Data: Define $\lambda_{s,p} = \delta$, where $s \in |\circ_{A,B,B}|(f, 1_{B,p})$, $p \in |1_B|(1)$, and $\delta \in \mathcal{A}(A,B)[f \circ_s 1_{B,p}, f]$ such that $(p \cdot s) \cdot \delta = \mathrm{Id}_f$.

Axioms:

1. Well defined: $\lambda_{s,p} \in \mathcal{A}(A,B)[f \circ_s 1_{B,p}, f]$ where $s \in |\circ_{A,B,B}|(f, 1_{B,p})$, $p \in |1_B|(1)$ is well defined, since $s \cdot p$ is a composite of universals and hence is an universal. So, there is an unique $\lambda_{s,p}$ such that $(p \cdot s) \cdot \lambda_{s,p} = \mathrm{Id}_f$.

2. Isomorphism: $\lambda_{s,p}$ is invertible. Its inverse is $p \cdot s$. One side $(p \cdot s) \cdot \lambda_{s,p} = \mathrm{Id}_f$ was verified above. Now,

$$
\begin{aligned}
& (p \cdot s) \cdot \lambda_{s,p} &&= \mathrm{Id}_f \\
\Longrightarrow\ & ((p \cdot s) \cdot \lambda_{s,p}) \cdot (p \cdot s) &&= \mathrm{Id}_f \cdot (p \cdot s) \\
\Longrightarrow\ & (p \cdot s) \cdot (\lambda_{s,p} \cdot (p \cdot s)) &&= (p \cdot s) \\
\Longrightarrow\ & (p \cdot s) \cdot (\lambda_{s,p} \cdot (p \cdot s)) &&= (p \cdot s) \cdot \mathrm{Id}_f \\
\Longrightarrow\ & \lambda_{s,p} \cdot (p \cdot s) &&= \mathrm{Id}_f
\end{aligned}
$$

3. Naturality: Need to show that the following diagram commutes.

$$
\begin{tikzcd}
f \circ_s 1_{B,p} \arrow[r, Rightarrow, "\lambda_{s,p}"] \arrow[d, Rightarrow, "\beta \circ_{s,t} 1_{B,p,q}"'] & f \arrow[d, Rightarrow, "\beta"] \\
f \circ_t 1_{B,q} \arrow[r, Rightarrow, "\lambda_{t,q}"'] & g
\end{tikzcd}
$$

From universality of $p{\cdot}s$, it is sufficient to show $(p{\cdot}s){\cdot}(\lambda_{s,p}{\cdot}\beta) = (p{\cdot}s){\cdot}(\beta\circ_{s,t}1_{B,p,q}{\cdot}\lambda_{t,q})$.

$$\begin{aligned}(p\cdot s)\cdot(\lambda_{s,p}\cdot\beta) &= ((p\cdot s)\cdot\lambda_{s,p})\cdot\beta\\ &= \mathrm{Id}_f\cdot\beta\\ &= \beta\end{aligned}$$

$$\begin{aligned}(p\cdot s)\cdot(\beta\circ_{s,t}1_{B,p,q}\cdot\lambda_{t,q}) &= ((p\cdot s)\cdot\beta\circ_{s,t}1_{B,p,q})\cdot\lambda_{t,q}\\ &= (p\cdot(s\cdot\beta\circ_{s,t}1_{B,p,q}))\cdot\lambda_{t,q}\\ &= (p\cdot(\beta\cdot(1_{B,p,q}\cdot t)))\cdot\lambda_{t,q}\\ &= (\beta\cdot(p\cdot(1_{B,p,q}\cdot t)))\cdot\lambda_{t,q}\\ &= (\beta\cdot((p\cdot 1_{B,p,q})\cdot t))\cdot\lambda_{t,q}\\ &= (\beta\cdot(q\cdot t))\cdot\lambda_{t,q}\\ &= \beta\cdot((q\cdot t)\cdot\lambda_{t,q})\\ &= \beta\cdot\mathrm{Id}_f\\ &= \beta\end{aligned}$$

3.7. Right identity isomorphisms: Natural anaisomorphism

$$\rho_{A,B} : 1_A\circ(-)\overset{\cong}{\Longrightarrow}\mathrm{Id}_{\mathcal{A}(A,B)}$$

where $\mathrm{Id}_{\mathcal{A}(A,B)}$ is an identity functor and $1_A\circ(-) = (!\cdot 1_A, \mathrm{Id}_{\mathcal{A}(A,B)})\cdot\circ_{A,A,B} : \mathcal{A}(A,B)\longrightarrow\mathcal{A}(A,B)$. Since A,B will be clear from the context $\rho_{A,B}$ will be denoted by ρ.

Data: Define $\rho_{s,p} = \delta$, where $s\in|\circ_{A,A,B}|(1_{A,p},f)$, $p\in|1_A|(1)$, and $\delta\in\mathcal{A}(A,B)[1_{A,p}\circ_s f, f]$ such that $(p\cdot s)\cdot\delta = \mathrm{Id}_f$.

Axioms:

1. Well defined: $\rho_{s,p}\in\mathcal{A}(A,B)[1_{A,p}\circ_s f, f]$ where $s\in|\circ_{A,A,B}|(1_{A,p},f)$, $p\in|1_A|(1)$ is well defined, since $s\cdot p$ is a composite of universals and hence is an universal. So, there is an unique $\rho_{s,p}$ such that $(p\cdot s)\cdot\rho_{s,p} = \mathrm{Id}_f$.

2. Isomorphism: $\rho_{s,p}$ is invertible. Its inverse is $p\cdot s$. One side $(p\cdot s)\cdot\rho_{s,p} = \mathrm{Id}_f$ was verified above. Now,

$$\begin{aligned} & (p\cdot s)\cdot\rho_{s,p} &&= \mathrm{Id}_f\\ \Longrightarrow\ & ((p\cdot s)\cdot\rho_{s,p})\cdot(p\cdot s) &&= \mathrm{Id}_f\cdot(p\cdot s)\\ \Longrightarrow\ & (p\cdot s)\cdot(\rho_{s,p}\cdot(p\cdot s)) &&= (p\cdot s)\\ \Longrightarrow\ & (p\cdot s)\cdot(\rho_{s,p}\cdot(p\cdot s)) &&= (p\cdot s)\cdot\mathrm{Id}_f\\ \Longrightarrow\ & \rho_{s,p}\cdot(p\cdot s) &&= \mathrm{Id}_f\end{aligned}$$

3. Naturality: Need to show that the following diagram commutes.

$$\begin{tikzcd} 1_{A,p} \circ_s f \arrow[r, Rightarrow, "\rho_{s,p}"] \arrow[d, Rightarrow, "1_{A,p,q} \circ_{s,t} \beta"'] & f \arrow[d, Rightarrow, "\beta"] \\ 1_{A,q} \circ_t f \arrow[r, Rightarrow, "\rho_{t,q}"'] & g \end{tikzcd}$$

From the universality of $p \cdot s$, it is sufficient to show $(p \cdot s) \cdot (\rho_{s,p} \cdot \beta) = (p \cdot s) \cdot (1_{A,p,q} \circ_{s,t} \beta \cdot \rho_{t,q})$.

$$\begin{aligned} (p \cdot s) \cdot (\rho_{s,p} \cdot \beta) &= ((p \cdot s) \cdot \rho_{s,p}) \cdot \beta \\ &= \mathrm{Id}_f \cdot \beta \\ &= \beta \end{aligned}$$

$$\begin{aligned} (p \cdot s) \cdot (1_{A,p,q} \circ_{s,t} \beta \cdot \rho_{t,q}) &= ((p \cdot s) \cdot 1_{A,p,q} \circ_{s,t} \beta) \cdot \rho_{t,q} \\ &= (p \cdot (s \cdot 1_{A,p,q} \circ_{s,t} \beta)) \cdot \rho_{t,q} \\ &= (p \cdot (1_{A,p,q} \cdot (\beta \cdot t))) \cdot \rho_{t,q} \\ &= (p \cdot (\beta \cdot (1_{A,p,q} \cdot t))) \cdot \rho_{t,q} \\ &= (\beta \cdot (p \cdot (1_{A,p,q} \cdot t))) \cdot \rho_{t,q} \\ &= (\beta \cdot ((p \cdot 1_{A,p,q}) \cdot t)) \cdot \rho_{t,q} \\ &= (\beta \cdot (q \cdot t)) \cdot \rho_{t,q} \\ &= \beta \cdot ((q \cdot t) \cdot \rho_{t,q}) \\ &= \beta \cdot \mathrm{Id}_f \\ &= \beta \end{aligned}$$

3.8. Coherence.

Pentagon Condition:

$$\begin{tikzcd} ((f \circ_1 g) \circ_2 h) \circ_3 i \arrow[r, Rightarrow, "\alpha_{1,2,4,5} \circ_{3,6} \mathrm{Id}_i"] \arrow[d, Rightarrow, "\alpha_{2,3,10,12}"] & (f \circ_5 (g \circ_4 h)) \circ_6 i \arrow[r, Rightarrow, "\alpha_{5,6,7,8}"] & f \circ_8 ((g \circ_4 h) \circ_7 i) \arrow[d, Rightarrow, "\mathrm{Id}_f \circ_{8,9} \alpha_{4,7,10,11}"'] \\ (f \circ_1 g) \circ_{12} (h \circ_{10} i) \arrow[rr, Rightarrow, "\alpha_{1,12,11,9}"'] & & f \circ_9 (g \circ_{11} (h \circ_{10} i)) \end{tikzcd}$$

Sufficient to show $((1 \cdot 2) \cdot 3) \cdot (\alpha_{1,2,4,5} \circ_{3,6} \mathrm{Id}_i \cdot (\alpha_{5,6,7,8} \cdot \mathrm{Id}_f \circ_{8,9} \alpha_{4,7,10,11})) = ((1 \cdot 2) \cdot 3)) \cdot (\alpha_{2,3,10,12} \cdot \alpha_{1,12,11,9})$.

$$
\begin{aligned}
&((1\cdot 2)\cdot 3)\cdot(\alpha_{1,2,4,5}\circ_{3,6}\mathrm{Id}_i\cdot(\alpha_{5,6,7,8}\cdot\mathrm{Id}_f\circ_{8,9}\alpha_{4,7,10,11}))\\
&\qquad=(((1\cdot 2)\cdot 3)\cdot\alpha_{1,2,4,5}\circ_{3,6}\mathrm{Id}_i)\cdot(\alpha_{5,6,7,8}\cdot\mathrm{Id}_f\circ_{8,9}\alpha_{4,7,10,11})\\
&\qquad=((1\cdot 2)\cdot(3\cdot\alpha_{1,2,4,5}\circ_{3,6}\mathrm{Id}_i))\cdot(\alpha_{5,6,7,8}\cdot\mathrm{Id}_f\circ_{8,9}\alpha_{4,7,10,11})\\
&\qquad=((1\cdot 2)\cdot(\alpha_{1,2,4,5}\cdot(\mathrm{Id}_i\cdot 6)))\cdot(\alpha_{5,6,7,8}\cdot\mathrm{Id}_f\circ_{8,9}\alpha_{4,7,10,11})\\
&\qquad=((1\cdot 2)\cdot(\alpha_{1,2,4,5}\cdot 6))\cdot(\alpha_{5,6,7,8}\cdot\mathrm{Id}_f\circ_{8,9}\alpha_{4,7,10,11})\\
&\qquad=(((1\cdot 2)\cdot\alpha_{1,2,4,5})\cdot 6)\cdot(\alpha_{5,6,7,8}\cdot\mathrm{Id}_f\circ_{8,9}\alpha_{4,7,10,11})\\
&\qquad=((4\cdot 5)\cdot 6)\cdot(\alpha_{5,6,7,8}\cdot\mathrm{Id}_f\circ_{8,9}\alpha_{4,7,10,11})\\
&\qquad=(4\cdot(5\cdot 6))\cdot(\alpha_{5,6,7,8}\cdot\mathrm{Id}_f\circ_{8,9}\alpha_{4,7,10,11})\\
&\qquad=((4\cdot(5\cdot 6))\cdot\alpha_{5,6,7,8})\cdot\mathrm{Id}_f\circ_{8,9}\alpha_{4,7,10,11}\\
&\qquad=(4\cdot((5\cdot 6)\cdot\alpha_{5,6,7,8}))\cdot\mathrm{Id}_f\circ_{8,9}\alpha_{4,7,10,11}\\
&\qquad=(4\cdot(7\cdot 8))\cdot\mathrm{Id}_f\circ_{8,9}\alpha_{4,7,10,11}\\
&\qquad=((4\cdot 7)\cdot 8)\cdot\mathrm{Id}_f\circ_{8,9}\alpha_{4,7,10,11}\\
&\qquad=(4\cdot 7)\cdot(8\cdot\mathrm{Id}_f\circ_{8,9}\alpha_{4,7,10,11})\\
&\qquad=(4\cdot 7)\cdot(\mathrm{Id}_f\cdot(\alpha_{4,7,10,11}\cdot 9))\\
&\qquad=(4\cdot 7)\cdot(\alpha_{4,7,10,11}\cdot 9)\\
&\qquad=((4\cdot 7)\cdot\alpha_{4,7,10,11})\cdot 9\\
&\qquad=(10\cdot 11)\cdot 9
\end{aligned}
$$

$$
\begin{aligned}
&((1\cdot 2)\cdot 3)\cdot(\alpha_{2,3,10,12}\cdot\alpha_{1,12,11,9})\\
&\qquad=(1\cdot(2\cdot 3))\cdot(\alpha_{2,3,10,12}\cdot\alpha_{1,12,11,9})\\
&\qquad=((1\cdot(2\cdot 3))\cdot\alpha_{2,3,10,12})\cdot\alpha_{1,12,11,9}\\
&\qquad=(1\cdot((2\cdot 3)\cdot\alpha_{2,3,10,12}))\cdot\alpha_{1,12,11,9}\\
&\qquad=(1\cdot(10\cdot 12))\cdot\alpha_{1,12,11,9}\\
&\qquad=(10\cdot(1\cdot 12))\cdot\alpha_{1,12,11,9}\\
&\qquad=10\cdot((1\cdot 12)\cdot\alpha_{1,12,11,9})\\
&\qquad=10\cdot(11\cdot 9)\\
&\qquad=(10\cdot 11)\cdot 9
\end{aligned}
$$

Identity Triangle:

$$
\begin{tikzcd}
(f \circ_s 1_{B,p}) \circ_t g \arrow[dr, Rightarrow, "\lambda_{s,p}\circ_{t,w}\mathrm{Id}_g"] \arrow[dd, Rightarrow, "\alpha_{s,t,u,v}"'] & \\
& f \circ_w g \\
f \circ_v (1_{B,p} \circ_u g) \arrow[ur, Rightarrow, "\mathrm{Id}_f\circ_{v,w}\rho_{u,p}"'] &
\end{tikzcd}
$$

It is sufficient to show $(p\cdot(s\cdot t))\cdot(\alpha_{s,t,u,v}\cdot \mathrm{Id}_f\circ_{v,w}\rho_{u,p}) = (p\cdot(s\cdot t))\cdot\lambda_{s,p}\circ_{t,w}\mathrm{Id}_g$.

$$\begin{aligned}
&(p\cdot(s\cdot t))\cdot(\alpha_{s,t,u,v}\cdot \mathrm{Id}_f\circ_{v,w}\rho_{u,p})\\
&\quad=((p\cdot(s\cdot t))\cdot\alpha_{s,t,u,v})\cdot \mathrm{Id}_f\circ_{v,w}\rho_{u,p}\\
&\quad=(p\cdot((s\cdot t)\cdot\alpha_{s,t,u,v}))\cdot \mathrm{Id}_f\circ_{v,w}\rho_{u,p}\\
&\quad=(p\cdot(u\cdot v))\cdot \mathrm{Id}_f\circ_{v,w}\rho_{u,p}\\
&\quad=((p\cdot u)\cdot v)\cdot \mathrm{Id}_f\circ_{v,w}\rho_{u,p}\\
&\quad=(p\cdot u)\cdot(v\cdot \mathrm{Id}_f\circ_{v,w}\rho_{u,p})\\
&\quad=(p\cdot u)\cdot(\mathrm{Id}_f\cdot(\rho_{u,p}\cdot w))\\
&\quad=(p\cdot u)\cdot(\rho_{u,p}\cdot w)\\
&\quad=((p\cdot u)\cdot\rho_{u,p})\cdot w\\
&\quad=\mathrm{Id}_g\cdot w\\
&\quad=w
\end{aligned}$$

$$\begin{aligned}
&(p\cdot(s\cdot t))\cdot\lambda_{s,p}\circ_{t,w}\mathrm{Id}_g\\
&\quad=((p\cdot s)\cdot t)\cdot\lambda_{s,p}\circ_{t,w}\mathrm{Id}_g\\
&\quad=(p\cdot s)\cdot(t\cdot\lambda_{s,p}\circ_{t,w}\mathrm{Id}_g)\\
&\quad=(p\cdot s)\cdot(\lambda_{s,p}\cdot(\mathrm{Id}_g\cdot w))\\
&\quad=(p\cdot s)\cdot(\lambda_{s,p}\cdot w)\\
&\quad=((p\cdot s)\cdot\lambda_{s,p})\cdot w\\
&\quad=\mathrm{Id}_f\cdot w\\
&\quad=w
\end{aligned}$$

3.9. Saturation: The ana-bicategory $\mathcal{A}$ constructed is saturated.

Consider $p\in|1_A|(1)$, $1_{A,p}=f$, and $\phi: f\cong g$, then $p\cdot\phi$ is an universal. Hence, there is a (*unique*) $q\in|1_A|(1)$, such that $q=p\cdot\phi$.

Similarly, consider $s\in|\circ_{A,B,C}|(f,g)$, $f\circ_s g=h$, and $\phi: h\cong i$, then $s\cdot\phi$ is an universal 2-cell. Hence, there is a (*unique*) $t\in|\circ_{A,B,C}|(f,g)$, such that $t=s\cdot\phi$.

3.10. Theorem. *Construction $(-)^*$ transforms 2D-multitopic category to a saturated ana-bicategory.*

4. Ana-bicategory to 2D-multitopic category

In this section, the construction of a 2D-multitopic category from the ana-bicategory is given. This construction will be denoted as $\mathcal{A}\xmapsto{(-)^\#}\mathcal{A}^\#$, where $\mathcal{A}$ is the given ana-bicategory. For simplicity, in this chapter $\mathcal{A}^\#$ will be denoted by $\mathcal{M}$. This construction is more complicated than $(-)^*$, the reason being while the construction $(-)^*$ involved a process of truncation, $(-)^\#$ involves building up of 2-cells and showing global condition of universality. The construction is similar to the term-model construction in logic.

4.1. Data Construction:

0-Cells:

$$\mathrm{Cell}_0(\mathcal{M}) = \mathcal{O}(\mathcal{A})$$

1-Cells:

$$\mathrm{Cell}_1(\mathcal{M}) = \bigcup_{A,B\in\mathcal{O}(\mathcal{A})} \mathcal{O}(\mathcal{A}(A,B))$$

2-Cells: 2-cells are defined as equivalence classes of ordered labelled typed trees. The set of such trees is denoted as Υ and the equivalence relation as $\simeq\subset \Upsilon\times\Upsilon$.

The ordering means that the children of any node has left to right ordering that can not be permuted. The labels on the nodes are either 2-cells, 0-specifications or 2-specifications. Labels on the edges are the 1-cells or 0-cells (if node below is 0-specification). The labels on edges will not be shown except for the outermost ones since they can be recovered from the node labels. The degree of each node is at most 2. We will assume that the degree of the node with 0-specification to be 0 even though it has an edge coming in.

Each tree has a type. The set of types is $\Theta\subset \mathrm{Cell}_1(\mathcal{M})^*\times\mathrm{Cell}_1(\mathcal{M})$, where $\mathrm{Cell}_1(\mathcal{M})^*$ is the set of composable strings of $\mathrm{Cell}_1(\mathcal{M})$ (*pasting diagrams*). If τ is the type of the tree T, then, define $\mathrm{dom}(T) = \pi_1(\tau)$ and $co\mathrm{dom}(T) = \pi_2(\tau)$. $\cdot$ is the concatenation operation on the strings in $\mathrm{Cell}_1(\mathcal{M})^*$.

Υ and the type of trees in Υ are recursively defined as follows:

1. If $A\in\mathcal{O}(\mathcal{A})$, $p\in|1_A|(1)$ and $f = 1_{A,p}$, then

$$\begin{tikzcd} {[A]} \arrow[d, dash] \\ p \arrow[d, dash] \\ (f) \end{tikzcd}$$

is a tree of type $(A \overset{\epsilon}{-\,-\,-} A,\ A\xrightarrow{f} A)$.

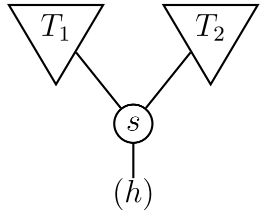
2. If $f\in\mathcal{A}(A,B)$, then

$$\begin{tikzcd} (f) \arrow[d, dash] \\ (f) \end{tikzcd}$$

is a tree of type $(A\xrightarrow{f} B,\ A\xrightarrow{f} B)$.

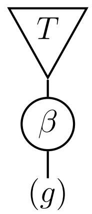
3. If T_1 is a tree of type $(A\xrightarrow{l_1} B,\ A\xrightarrow{f} B)$, T_2 a tree of type $(B\xrightarrow{l_2} C,\ B\xrightarrow{g} C)$ and $s\in|\circ_{A,B,C}|(f,g)$, where $f\in\mathcal{A}(A,B)$, $g\in\mathcal{A}(B,C)$, $A,B,C\in\mathcal{O}(\mathcal{A})$ and

$h = f \circ_s h$, then



is a tree of type $(A \xrightarrow{l_1 \cdot l_2} C, A \xrightarrow{h} C)$.

4. If T is a tree of type $(A \xrightarrow{l} B, A \xrightarrow{f} B)$, and $\beta \in \mathcal{A}(A, B)[f, g]$, where $f, g \in \mathcal{A}(A, B)$, $A, B \in \mathcal{O}(\mathcal{A})$, then



is a tree of type $(A \xrightarrow{l} B, A \xrightarrow{g} B)$.

These trees can be thought of 2 dimensional pasting diagrams in multitopic category in which 0-cells and 1-cells are as in $\mathcal{A}$ and 2-cells are 0-specifications, 2-specifications and 2-cells.

Now the equivalence relation $\simeq \subset \Upsilon \times \Upsilon$ is defined. This is done in terms of elementary tree transformations $T_1 \longrightarrow T_2$. Define $\simeq$ to be the transitive closure of $\longrightarrow$, hence

$$\simeq = \longrightarrow^*$$

Each elementary tree transformation step is invertible. Each step is labelled as $\overline{XX}$ and its inverse as $\underline{XX}$. The elementary steps are classified according to their origin in ana-bicategory.

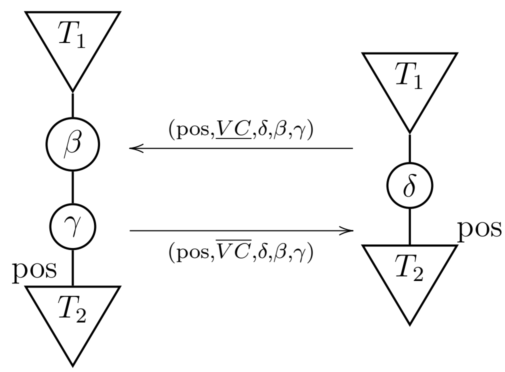
In these elementary steps the position ($pos \in \{u, l, r\}^*$) indicates where the transformation is applied. So pos is a string from alphabet $\{u, l, r\}$, which gives the position relative to the root node. The logic is simple. Start from the root node of the tree. Read pos from left to right and on seeing u move up, on l move left and on r move right. If such a move is not possible for any part of the string pos then the position is invalid. Denote $\wp(T)$ to be the set of all valid positions in T.

If T is a tree and *pos* a valid position in T, then $T[pos]$ is the subtree of T at position *pos*. The notation reminds us of the fact that pos is the index for the trees, just like numbers are indexes for the sequences.

If pos is a string and pos' is a prefix for it, then $\text{pos} - \text{pos}'$ denotes the string such that $\text{pos}' \cdot (\text{pos} - \text{pos}') = \text{pos}$, where $\cdot$ is the string concatenation operation.

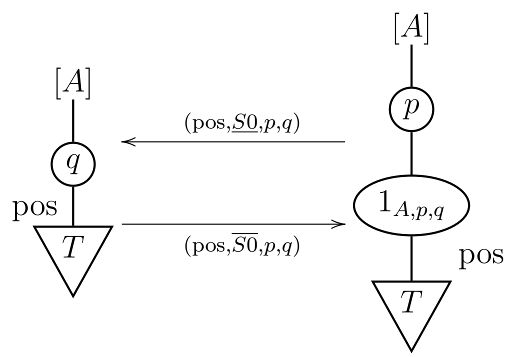
Composition Law: This comes from composition in category $\mathcal{A}(-,-)$.

1. (VC) This transformation is the replacement of two nodes representing arrows in category $\mathcal{A}(-,-)$, with their composite. Let $\beta \cdot \gamma = \delta$ in $\mathcal{A}(-,-)$.

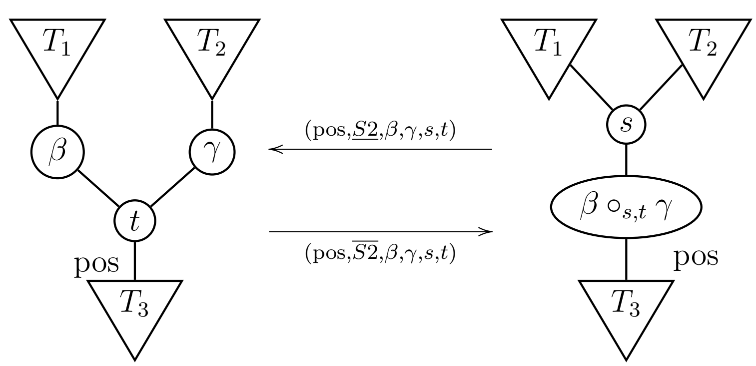


Structural Laws: These are the laws that change the specifications used and the skeleton of the tree. There are two laws for the 0-specifications and 2-specifications, and three laws for the three natural anaisomorphisms α, λ and ρ.

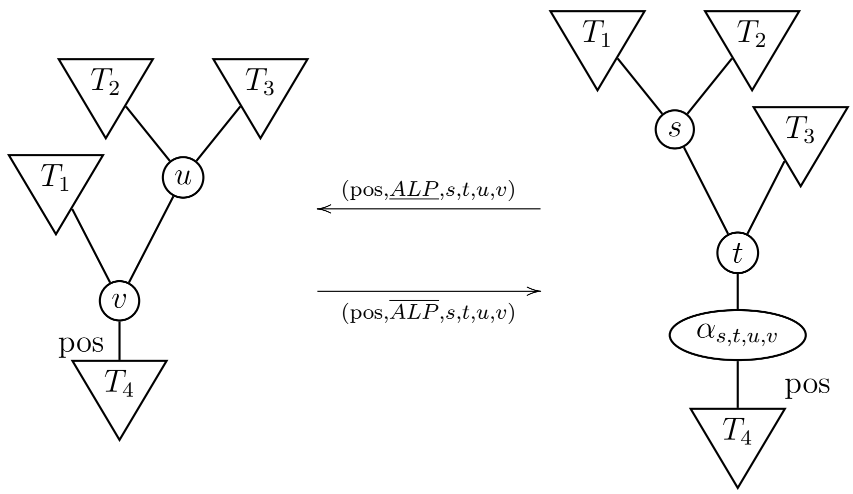
1. (S0) This transformation is for changing the 0-specifications. Let $p, q \in |1_A|(1)$.



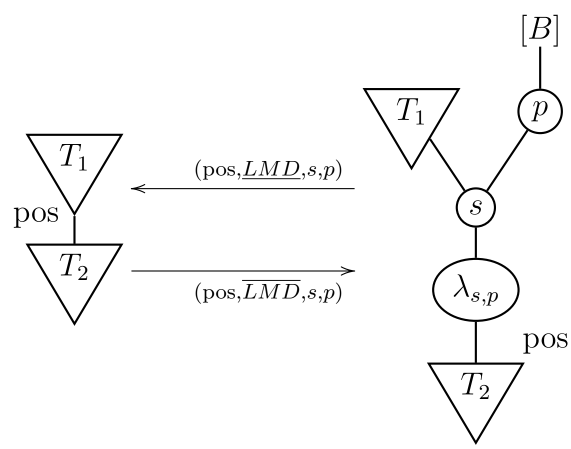
2. (S2) This transformation is for changing the 2-specifications. Let $s \in |\circ_{A,B,C}|(f_1, g_1)$, $t \in |\circ_{A,B,C}|(f_2, g_2)$, $\beta \in \mathcal{A}(A,B)[f_1, f_2]$ and $\gamma \in \mathcal{A}(B,C)[g_1, g_2]$.



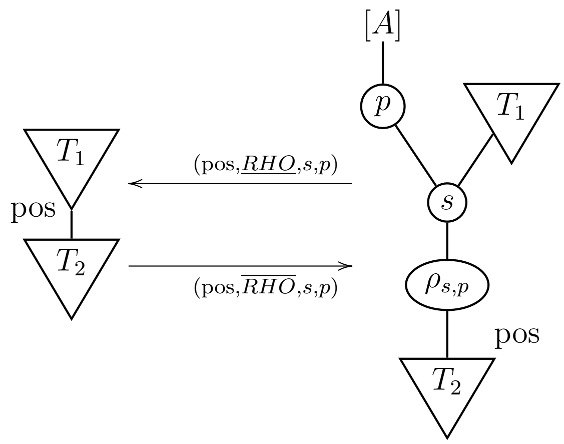
3. (ALP) This transformation changes the shape of the tree. It changes the tree from right oriented one to left oriented. Let $s \in |\circ_{A,B,C}|(f,g)$, $t \in |\circ_{A,C,D}|(f \circ_s g, h)$, $u \in |\circ_{B,C,D}|(g,h)$, $v \in |\circ_{A,B,D}|(f, g \circ_u h)$.



4. (LMD) This transformation eliminates the 0-specification on the right of a 2-specification. Let $s \in |\circ_{A,B,B}|(f, 1_{B,p})$, $p \in |1_B|(1)$.

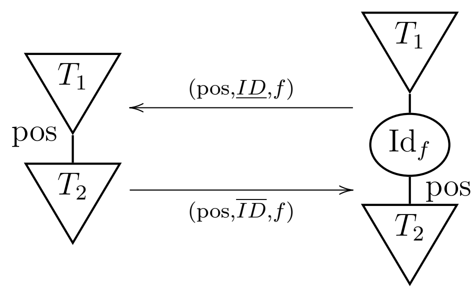


5. (RHO) This transformation eliminates the 0-specification on the left of a 2-specification. Let $s \in |\circ_{A,A,B}|(1_{A,p}, f)$, $p \in |1_A|(1)$.



Identity Law: This law is reflective of the fact that composition with identity does not affect the 2-cell.

1. (ID) This transformation introduces the identity node into the tree. Let $co\mathrm{dom}(T_1) = f$, then



Now we can define a set of two cells to be

$$\mathrm{Cell}_2(\mathcal{M}) = \Upsilon / \simeq$$

4.2. Composition: Composition of trees is defined as a typed concatenation of two trees. Let T_1 be a tree such that $co\mathrm{dom}(T_1) = f$, T_2 and pos be such that $T_2[pos] = (f)$. Then $T = T_1 \odot_{pos} T_2$ is a tree such that at position pos in T_2, T_1 is attached. (The composition can be thought of as composition of 2-PD's.)

Formally,

$$(\forall \mathrm{pos}' \in \wp(T_2)) \qquad T[\mathrm{pos}'] = \begin{cases} T_2[\mathrm{pos}'] & \text{if pos}' \text{ is not a prefix of pos} \\ T_1 \odot_{\mathrm{pos}''} T_2[\mathrm{pos}'] & \text{otherwise, where pos}'' = \mathrm{pos} - \mathrm{pos}' \end{cases}$$

4.3. Lemma. $\mathrm{Cell}_1(\mathcal{M})$ *and* $\mathrm{Cell}_2(\mathcal{M})$ *along with composition* $\odot_{-}$ *is a multicategory.*

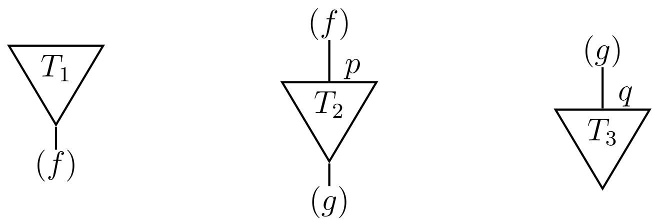
Proof.

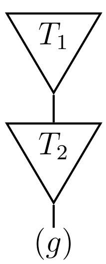
Well defined: Composition is well defined i.e. $T_1 \simeq T_1'$ and $T_2 \simeq T_2'$, then $T_1 \odot_p T_2 \simeq T_1' \odot_q T_2'$.

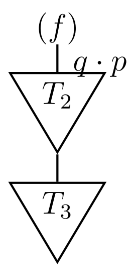
Let $\langle t1_i \rangle$ be transformations for $T_1 \longrightarrow T_1'$, $\langle t2_i \rangle$ be transformations for $T_2 \longrightarrow T_2'$. $\langle t2_i \rangle$ transforms position p to q, and $p * \langle t1_i \rangle$ be sequence of transformations with p prefixed to every transformation in $\langle t1_i \rangle$. Now, $p * \langle t1_i \rangle \cdot \langle t2_i \rangle$ is transformations for $T_1 \odot_p T_2 \longrightarrow T_1' \odot_q T_2'$.

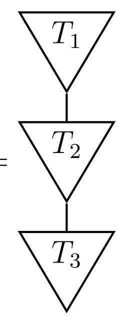
Associativity: Composition is associative.

Let T_1, T_2, T_3 be trees, $p \in \wp(T_2)$ and $q \in \wp(T_3)$ such that $codom(T_1) = f$, $codom(T_2) = g$, $T_2[p] = (f)$ and $T_3[q] = (g)$. Then $(T_1 \odot_p T_2) \odot_q T_3 = T_1 \odot_{q \cdot p} (T_2 \odot_q T_3)$. In pictures,



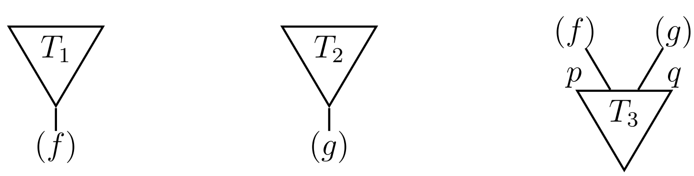
$T_1 \odot_p T_2 =$ 

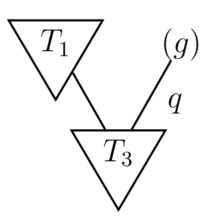
$T_2 \odot_q T_3 =$ 

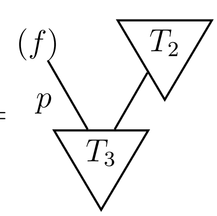
$(T_1 \odot_p T_2) \odot_q T_3 = T_1 \odot_{q \cdot p} (T_2 \odot_q T_3) =$ 

Commutativity: Composition is commutative.

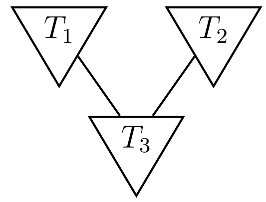
Let T_1, T_2, T_3 be trees and $p, q \in \wp(T_3)$ such that $codom(T_1) = f$, $codom(T_2) = g$, $T_3[p] = (f)$ and $T_3[q] = (g)$. Then $T_1 \odot_p (T_2 \odot_q T_3) = T_2 \odot_q (T_1 \odot_p T_3)$. In pictures,



$T_1 \odot_p T_3 =$ 

$T_2 \odot_q T_3 =$ 

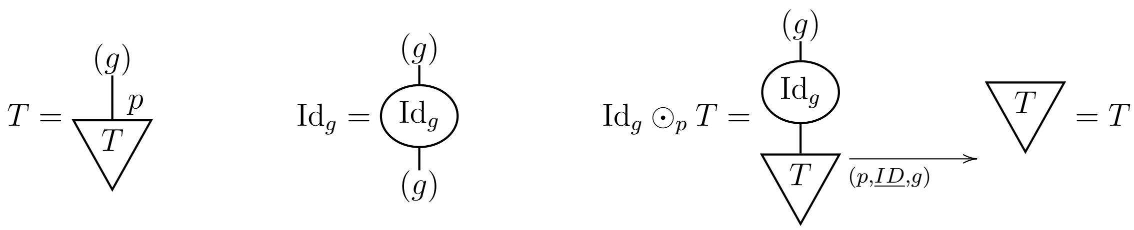
$$T_1 \odot_p (T_2 \odot_q T_3) = T_2 \odot_q (T_1 \odot_p T_3) =$$



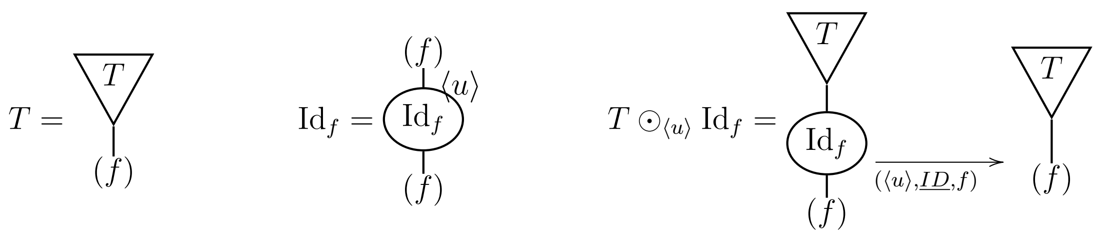
Identity: Composition respects identity laws.

Let T be a tree and $p \in \wp(T)$ such that $co\mathrm{dom}(T) = f$ and $T[p] = (g)$. Also Id_f and Id_g be identity trees for f and g respectively.

1. $\mathrm{Id}_g \odot_p T \simeq T$.



2. $T \odot_{\langle u \rangle} \mathrm{Id}_f \simeq T$.



■

For any well ordered set I, define

$$L_I = \{(i,j) | (i,j) \in I \times I \wedge j = S_I(i)\}$$

$$\nabla_{2,I} = \{(i,j) | (i,j) \in I \times I \wedge i \leq j\}$$

and

$$\nabla_{3,I} = \{(i,j,k) | (i,j,k) \in I \times I \times I \wedge i \leq j \leq k\}$$

where S_I is the successor function inside I. Note $L_I \subset \nabla_{2,I}$. Given $m, m' \in I$ such that $m < m'$, then define $[m, m']_I = \{i \in I | m \leq i \wedge i \leq m'\}$ and $]m, m'[_I = \{i \in I | i \leq m \vee m' \leq i\}$. Given a well ordered set I and $j \notin I$, define $j \blacktriangleleft I$ to be the extension of I with j such that $j < I$. Analogously, $I \blacktriangleleft j$. Given two well ordered sets I and J, define $I \blacktriangleleft J$ to be well ordered set such that $I < J$ and ordering within elements of I and J are preserved.

4.4. Definition. *Pasting diagram (PD) is a triple* $\bar{f} = (I, f_O, f_A)$, *where*

$$\begin{array}{ll} f_O & : I \longrightarrow \mathrm{Cell}_0(\mathcal{M}) \\ f_A & : L_I \longrightarrow \mathrm{Cell}_1(\mathcal{M}) \end{array}$$

such that

$$(\forall (i,j) \in L_I)(f_A(i,j) : f_O(i) \longrightarrow f_O(j))$$

Define $|\bar{f}| = |I|$ *and* $\bar{f}|_J = (J, f_O|_J, f_A|_J)$.

Also, we say $(I, g_O, g_A) = \bar{g} \leq \bar{f} = (J, f_O, f_A)$,

$$I \subset J \wedge g_O = f_O|_I \wedge g_A = f_A|_I$$

This defines a partial order on pasting diagrams. Given pasting diagram $\bar{f} = (I, f_O, f_A)$ and $m, m' \in I$, define $\bar{g} = \bar{f} \uparrow_{(m,m')} = ([m, m']_I, g_O|_{[m,m']_I}, g_A|_{[m,m']_I})$.

We now define a complete set of specifications for a given pasting diagram. Intuitively this is a collection of 0-specifications and 2-specifications that fit together to define the composition of 1-cells such that between any pair of 0-cells there is an unique 1-cell. The basic idea in this definition is that all α, ρ, λ's associated with these specifications are identities.

4.5. Definition. *I indexed set of specifications is a quadruple* $S = (\Theta, \mathcal{F}, S0, S2)$,

$$\begin{array}{rl} \Theta & : I \longrightarrow Ob(\bar{f}) \\ \mathcal{F} & : \nabla_{2,I} \longrightarrow Ar(\bar{f}) \\ S0 & : I \longrightarrow |1_-| \\ S2 & : \nabla_{3,I} \longrightarrow |\circ_{-,-,-}| \end{array}$$

such that

1. $\mathcal{F}(i,j) : \Theta(i) \longrightarrow \Theta(j)$,

2. $S0(i) \in |1_{\Theta(i)}|(1)$,

3. $\mathcal{F}(i,i) = 1_{\Theta(i),S0(i)}$,

4. $S2(i,j,k) \in |\circ_{\Theta(i),\Theta(j),\Theta(k)}|(\mathcal{F}(i,j), \mathcal{F}(j,k))$,

5. $\mathcal{F}(i,k) = \mathcal{F}(i,j) \circ_{S2(i,j,k)} \mathcal{F}(j,k)$.

Given I indexed set of specifications S, for any non empty $J \subseteq I$, the subsystem $S|_J$ is the restriction of S to J.

$S \uparrow_{(m,m')}$ is the upper half of S from m to m', i.e. $S \uparrow_{(m,m')} = S|_{[m,m']_I}$. $S \downarrow_{(m,m')}$ is the lower half of S from m to m' i.e. $S \downarrow_{(m,m')} = S|_{]m,m'[_I}$.

4.6. Definition. *I indexed set of specifications $S = (\Theta, \mathcal{F}, S0, S2)$ is said to be a set of specifications for pasting diagram $\bar{f} = (I, f_O, f_A)$ if*

1. $\Theta = f_O$
2. $\mathcal{F}|_{L_I} = f_A$

4.7. Definition. *α-coherent systems: I indexed set of specifications S is said to be an α-coherent system if*

$$\alpha_{S2'(i,j,k),S2'(i,k,l),S2'(j,k,l),S2'(i,j,l)} = \mathrm{Id}_{\mathcal{F}'(i,l)}$$

for all $i, j, k, l \in I$ such that $i \leq j \leq k \leq l$.

4.8. Definition. *ρ-coherent systems: I indexed set of specifications S is said to be an ρ-coherent system if*

$$\rho_{S2'(i,i,j),S0'(i)} = \mathrm{Id}_{\mathcal{F}'(i,j)}$$

for all $i, j \in J$ such that $i \leq j$.

4.9. Definition. *λ-coherent systems: I indexed set of specifications S is said to be an λ-coherent system if*

$$\lambda_{S2'(i,j,j),S0'(j)} = \mathrm{Id}_{\mathcal{F}'(i,j)}$$

for all $i, j \in J$ such that $i \leq j$.

4.10. Definition. *I indexed set of specifications $S = (\Theta, \mathcal{F}, S0, S2)$ is a complete set of specifications whenever it is α, ρ, λ-coherent system.*

4.11. Lemma. *Given an object A and $p \in |1_A|(1)$, there exists $s \in |\circ_{A,A,A}|(1_{A,p}, 1_{A,p})$ such that $1_{A,p} \circ_s 1_{A,p} = 1_{A,p}$ and $\rho_{s,p} = \lambda_{s,p} = \alpha_{s,s,s,s} = \mathrm{Id}_{1_{A,p}}$*

Proof. From 2.2 $\exists s \in |\circ_{A,A,A}|(1_{A,p}, 1_{A,p})$ such that $\rho_{s,p} = \mathrm{Id}_{1_{A,p}}$. Using coherence we have

$$\begin{tikzcd} 1_{A,p} \circ_s 1_{A,p} \arrow[r, Rightarrow, shift left, "\rho_{s,p}"] \arrow[r, Rightarrow, shift right, "\lambda_{s,p}"'] & 1_{A,p} \end{tikzcd}$$

So, $\lambda_{s,p} = \rho_{s,p} = \mathrm{Id}_{1_{A,p}}$. Using coherence again,

$$\begin{tikzcd} (1_{A,p} \circ_s 1_{A,p}) \circ_s 1_{A,p} \arrow[d, Rightarrow, "\alpha_{s,s,s,s}"'] \arrow[dr, Rightarrow, "\lambda_{s,p} \circ_{s,s} \mathrm{Id}_{1_{A,p}} = \mathrm{Id}_{1_{A,p}}"] & \\ & 1_{A,p} \circ_s 1_{A,p} \\ 1_{A,p} \circ_s (1_{A,p} \circ_s 1_{A,p}) \arrow[ur, Rightarrow, "\mathrm{Id}_{1_{A,p}} \circ_{s,s} \rho_{s,p} = \mathrm{Id}_{1_{A,p}}"'] & \end{tikzcd}$$

So, $\alpha_{s,s,s,s} = \mathrm{Id}_{1_{A,p}}$. ∎

4.12. Lemma. *Given $\bar{f} = (I, f_O, f_A)$, a complete set of specifications $S = (\Theta, \mathcal{F}, S0, S2)$ for $\bar{f}$, and $\bar{g} = (J, g_O, g_A)$ such that*

1. $J = I \blacktriangleleft j$,
2. $\bar{f} = \bar{g}_I$

Then there exists a complete set of specifications $S' = (\Theta', \mathcal{F}', S0', S2')$ for $\bar{g}$ such that

$$S = S'|_I$$

Proof. In this proof for simplicity we use natural numbers. Since I and J are well ordered sets, we have $I \cong \{0, ..., n\}$ and $J \cong \{0, ..., n+1\}$. Hence, we use natural numbers as indexes in this proof locally.

Let

1. $\Theta'(i) = \Theta(i), 0 \le i \le n$, and $\Theta'(n+1) = g_O(n+1)$
2. $\mathcal{F}'(i,j) = \mathcal{F}(i,j), 0 \le i \le j \le n$, and $\mathcal{F}(n, n+1) = g_A(n)$
3. $S0'(i) = S0(i), 0 \le i \le n$
4. $S2'(i,j,k) = S2(i,j,k), 0 \le i \le j \le k \le n$

In this proof we locally use the following abbreviations:

$$\begin{aligned}
\mathcal{F}'_{i,j} &:= \mathcal{F}'(i,j) \\
\circ_{i,j,k} &:= \circ_{S2'(i,j,k)} \\
\circ_{i,j,k} &:= \circ_{S2'(i,j,k),S2'(i,j,k)} \\
\rho_{i,j} &:= \rho_{S2'(i,i,j),S0'(i)} \\
\lambda_{i,j} &:= \lambda_{S2'(i,j,j),S0'(j)} \\
\alpha_{i,j,k,l} &:= \alpha_{S2'(i,j,k),S2'(i,k,l),S2'(j,k,l),S2'(i,j,l)} \\
1_{\Theta'(i)} &:= 1_{\Theta'(i),S0'(i)} \\
\mathrm{Id}_{i,j} &:= \mathrm{Id}_{\mathcal{F}'(i,j)}
\end{aligned}$$

Base Step: Choose any $S0'(n+1)$. From 4.11 we have $S2'(n+1, n+1, n+1)$ such that $\rho_{n+1,n+1} = \lambda_{n,n+1} = \alpha_{n+1,n+1,n+1,n+1} = \mathrm{Id}_{n+1,n+1}$

Induction Step: For any $0 \le i \le n$, suppose we completed the following process for all $i < j \le n+1$, we choose $S2'(i, n, n+1) \in |\circ_{\Theta'(i),\Theta'(n),\Theta'(n+1)}|(\mathcal{F}'(i,n), \mathcal{F}'(n, n+1))$, and let $\mathcal{F}'(i, n+1) = \mathcal{F}'(i,n) \circ_{i,n,n+1} \mathcal{F}'(n, n+1)$.

Now we need to define $S2'(i, j, n+1)$ for $i \le j \le n+1 \wedge j \ne n$.

1. (I1) For $j = n+1$, from saturation we have an unique $S2'(i, n+1, n+1)$ such that

$$\lambda_{i,n+1} = \mathrm{Id}_{i,n+1}$$

2. (I2) For $j = i$, from saturation we have an unique $S2'(i,i,n+1)$ such that

$$\rho_{i,n+1} = \mathrm{Id}_{i,n+1}$$

3. (I3) For any $i < j < n$, from saturation we have an unique $S2'(i,j,n+1)$ such that

$$\alpha_{i,j,n,n+1} = \mathrm{Id}_{i,n+1}$$

The selection satisfies ρ and λ coherence. We need to show the α coherence for $i \leq j \leq k \leq l = n+1$. This has 8 cases depending on where equalities and inequalities lie. All of the following diagrams commute due to coherence.

Case 1. $i < j < k < l = n+1$, $k \neq n$

$$\begin{tikzcd}
& ((\mathcal{F}'_{i,j} \circ_{i,j,k} \mathcal{F}'_{j,k}) \circ_{i,k,n} \mathcal{F}'_{k,n}) \circ_{i,n,n+1} \mathcal{F}'_{n,n+1} \arrow[dl, Rightarrow, "\alpha_{i,j,k,n} \circ_{i,n,n+1} \mathrm{Id}_{n,n+1}"'] \arrow[ddr, Rightarrow, "\alpha_{i,k,n,n+1}"] & \\
(\mathcal{F}'_{i,j} \circ_{i,j,n} (\mathcal{F}'_{j,k} \circ_{j,k,n} \mathcal{F}'_{k,n})) \circ_{i,n,n+1} \mathcal{F}'_{n,n+1} \arrow[dd, Rightarrow, "\alpha_{i,j,n,n+1}"'] & & \\
& & (\mathcal{F}'_{i,j} \circ_{i,j,k} \mathcal{F}'_{j,k}) \circ_{i,k,n+1} (\mathcal{F}'_{k,n} \circ_{k,n,n+1} \mathcal{F}'_{n,n+1}) \arrow[ddl, Rightarrow, "\alpha_{i,j,k,n+1}"] \\
\mathcal{F}'_{i,j} \circ_{i,j,n+1} ((\mathcal{F}'_{j,k} \circ_{j,k,n} \mathcal{F}'_{k,n}) \circ_{j,n,n+1} \mathcal{F}'_{n,n+1}) \arrow[dr, Rightarrow, "\mathrm{Id}_{i,j} \circ_{i,j,n+1} \alpha_{j,k,n,n+1}"'] & & \\
& \mathcal{F}'_{i,j} \circ_{i,j,n+1} (\mathcal{F}'_{j,k} \circ_{j,k,n+1} (\mathcal{F}'_{k,n} \circ_{k,n,n+1} \mathcal{F}'_{n,n+1})) &
\end{tikzcd}$$

Now,

1. $$\alpha_{i,j,k,n} \circ_{i,n,n+1} \mathrm{Id}_{n,n+1} = \mathrm{Id}_{i,n+1}$$
   because $\alpha_{i,j,k,n} = \mathrm{Id}_{i,n}$ from specifications S.

2. $$\alpha_{i,j,n,n+1} = \mathrm{Id}_{i,n+1}$$
   from choice in item **(I3)** above.

3. $$\mathrm{Id}_{i,j} \circ_{i,j,n+1} \alpha_{j,k,n,n+1} = \mathrm{Id}_{i,n+1}$$
   because $\alpha_{j,k,n,n+1} = \mathrm{Id}_{j,n+1}$, by induction as $j > i$.

4. $$\alpha_{i,k,n,n+1} = \mathrm{Id}_{i,n+1}$$
   from choice in item **(I3)** above.

5. Since the diagram above commutes, we have

$$\alpha_{i,j,k,n+1} = \mathrm{Id}_{i,n+1}$$

Case 2. $i < j < k = l = n+1$

$$\begin{tikzcd}
(\mathcal{F'}_{i,j} \circ_{i,j,n+1} \mathcal{F'}_{j,n+1}) \circ_{i,n+1,n+1} 1_{\Theta'(n+1)} \arrow[dd, Rightarrow, "\alpha_{i,j,n+1,n+1}"'] \arrow[dr, Rightarrow, "\lambda_{i,n+1}"] & \\
& \mathcal{F'}_{i,j} \circ_{i,j,n+1} \mathcal{F'}_{j,n+1} \\
\mathcal{F'}_{i,j} \circ_{i,j,n+1} (\mathcal{F'}_{j,n+1} \circ_{i,n+1,n+1} 1_{\Theta'(n+1)}) \arrow[ur, Rightarrow, "\mathrm{Id}_{i,j} \circ_{i,j,n+1} \lambda_{j,n+1}"] &
\end{tikzcd}$$

Now,

1. $$\lambda_{i,n+1} = \mathrm{Id}_{i,n+1}$$

   from choice in item **(I1)** above.

2. $$\mathrm{Id}_{i,j} \circ_{i,j,n+1} \lambda_{j,n+1} = \mathrm{Id}_{i,n+1}$$

   because $\lambda_{j,n+1} = \mathrm{Id}_{j,n+1}$, by induction as $j > i$.

3. Since the diagram above commutes, we have

$$\alpha_{i,j,n+1,n+1} = \mathrm{Id}_{i,n+1}$$

Case 3. $i < j = k < l = n+1$

$$\begin{tikzcd}
(\mathcal{F'}_{i,j} \circ_{i,j,j} 1_{\Theta'(j)}) \circ_{i,j,n+1} \mathcal{F'}_{j,n+1} \arrow[dd, Rightarrow, "\alpha_{i,j,j,n+1}"'] \arrow[dr, Rightarrow, "\lambda_{i,j} \circ_{i,j,n+1} \mathrm{Id}_{j,n+1}"] & \\
& \mathcal{F'}_{i,j} \circ_{i,j,n+1} \mathcal{F'}_{j,n+1} \\
\mathcal{F'}_{i,j} \circ_{i,j,n+1} (1_{\Theta'(j)} \circ_{i,j,n+1} \mathcal{F'}_{j,n+1}) \arrow[ur, Rightarrow, "\mathrm{Id}_{i,j} \circ_{i,j,n+1} \rho_{j,n+1}"] &
\end{tikzcd}$$

Now,

1. $$\lambda_{i,j} \circ_{i,j,n+1} \mathrm{Id}_{j,n+1} = \mathrm{Id}_{i,n+1}$$

   because $\lambda_{i,j} = \mathrm{Id}_{i,j}$ from choice in item **(I1)** above.

2. $$\mathrm{Id}_{i,j} \circ_{i,j,n+1} \rho_{j,n+1} = \mathrm{Id}_{i,n+1}$$

   because $\rho_{j,n+1} = \mathrm{Id}_{j,n+1}$, by induction as $j > i$.

3. Since the diagram above commutes, we have

   $$\alpha_{i,j,j,n+1} = \mathrm{Id}_{i,n+1}$$

Case 4. $i < j = k = l = n + 1$

$$\begin{tikzcd}
(\mathcal{F}'_{i,n+1} \circ_{i,n+1,n+1} 1_{\Theta'(n+1)}) \circ_{i,n+1,n+1} 1_{\Theta'(n+1)} \arrow[d, Rightarrow, "\alpha_{i,n+1,n+1,n+1}"] \arrow[dr, Rightarrow, "\lambda_{i,n+1}"] & \\
& \mathcal{F}'_{i,n+1} \circ_{i,n+1,n+1} 1_{\Theta'(n+1)} \\
\mathcal{F}'_{i,n+1} \circ_{i,n+1,n+1} (1_{\Theta'(n+1)} \circ_{n+1,n+1,n+1} 1_{\Theta'(n+1)}) \arrow[ur, Rightarrow, "\mathrm{Id}_{i,n+1} \circ_{i,n+1,n+1} \rho_{n+1,n+1}"'] &
\end{tikzcd}$$

Now,

1. $$\lambda_{i,n+1} = \mathrm{Id}_{i,n+1}$$

   from choice in item **(I1)** above.

2. $$\mathrm{Id}_{i,j} \circ_{i,n+1,n+1} \rho_{n+1,n+1} = \mathrm{Id}_{i,n+1}$$

   because $\rho_{n+1,n+1} = \mathrm{Id}_{j,n+1}$ from 4.11.

3. Since the diagram above commutes, we have

   $$\alpha_{i,n+1,n+1,n+1} = \mathrm{Id}_{i,n+1}$$

Case 5. $i = j < k < l = n+1$

$$\begin{tikzcd}
(1_{\Theta'(i)} \circ_{i,i,k} \mathcal{F'}_{i,k}) \circ_{i,k,n+1} \mathcal{F'}_{k,n+1} \arrow[d, Rightarrow, "\alpha_{i,i,k,n+1}"'] \arrow[dr, Rightarrow, "\rho_{i,k} \circ_{i,k,n+1} \mathrm{Id}_{k,n+1}"] & \\
1_{\Theta'(i)} \circ_{i,i,n+1} (\mathcal{F'}_{i,k} \circ_{i,k,n+1} \mathcal{F'}_{k,n+1}) \arrow[r, Rightarrow, "\rho_{i,n+1}"'] & \mathcal{F'}_{i,k} \circ_{i,k,n+1} \mathcal{F'}_{k,n+1}
\end{tikzcd}$$

Now,

1.
$$\rho_{i,k} \circ_{i,k,n+1} \mathrm{Id}_{k,n+1} = \mathrm{Id}_{i,n+1}$$
because $\rho_{i,k} = \mathrm{Id}_{i,k}$ from specifications S.

2.
$$\rho_{i,n+1} = \mathrm{Id}_{i,n+1}$$
from choice in item **(I2)** above.

3. Since the diagram above commutes, we have
$$\alpha_{i,i,k,n+1} = \mathrm{Id}_{i,n+1}$$

Case 6. $i = j < k = l = n+1$

$$\begin{tikzcd}
(1_{\Theta'(i)} \circ_{i,i,n+1} \mathcal{F'}_{i,n+1}) \circ_{i,n+1,n+1} 1_{\Theta'(n+1)} \arrow[d, Rightarrow, "\alpha_{i,i,n+1,n+1}"'] \arrow[dr, Rightarrow, "\rho_{i,n+1} \circ_{i,n+1,n+1} \mathrm{Id}_{1_{\Theta'(n+1)}}"] & \\
1_{\Theta'(i)} \circ_{i,i,n+1} (\mathcal{F'}_{i,n+1} \circ_{i,n+1,n+1} 1_{\Theta'(n+1)}) \arrow[r, Rightarrow, "\rho_{i,n+1}"'] & \mathcal{F'}_{i,n+1} \circ_{i,n+1,n+1} 1_{\Theta'(n+1)}
\end{tikzcd}$$

Now,

1.
$$\rho_{i,n+1} \circ_{i,n+1,n+1} \mathrm{Id}_{1_{\Theta'(n+1)}} = \mathrm{Id}_{i,n+1}$$
because $\rho_{i,n+1} = \mathrm{Id}_{i,n+1}$ from choice in item **(I2)** above.

2.
$$\rho_{i,n+1} = \mathrm{Id}_{i,n+1}$$
from choice in item **(I2)** above.

3. Since the diagram above commutes, we have
$$\alpha_{i,i,n+1,n+1} = \mathrm{Id}_{i,n+1}$$

Case 7. $i = j = k < l = n+1$

$$\begin{tikzcd}
(1_{\Theta'(i)} \circ_{i,i,i} 1_{\Theta'(i)}) \circ_{i,i,n+1} \mathcal{F}'_{i,n+1} \arrow[d, Rightarrow, "\alpha_{i,i,i,n+1}"'] \arrow[dr, Rightarrow, "\rho_{i,i} \circ_{i,i,n+1} \mathrm{Id}_{i,n+1}"] & \\
& 1_{\Theta'(i)} \circ_{i,i,n+1} \mathcal{F}'_{i,n+1} \\
1_{\Theta'(i)} \circ_{i,i,n+1} (1_{\Theta'(i)} \circ_{i,i,n+1} \mathcal{F}'_{i,n+1}) \arrow[ur, Rightarrow, "\rho_{i,n+1}"'] &
\end{tikzcd}$$

Now,

1.
$$\rho_{i,i} \circ_{i,i,n+1} \mathrm{Id}_{i,n+1} = \mathrm{Id}_{i,n+1}$$
because $\rho_{i,i} = \mathrm{Id}_{i,i}$ from 4.11 above.

2.
$$\rho_{i,n+1} = \mathrm{Id}_{i,n+1}$$
from choice in item **(I2)** above.

3. Since the diagram above commutes, we have
$$\alpha_{i,i,i,n+1} = \mathrm{Id}_{i,n+1}$$

Case 8. $i = j = k = l = n+1$
$$\alpha_{n+1,n+1,n=1,n+1} = \mathrm{Id}_{n+1,n+1}$$
from 4.11.

■

4.13. Lemma. *Given $\bar{f} = (I, f_O, f_A)$, a complete set of specifications $S = (\Theta, \mathcal{F}, S0, S2)$ for $\bar{f}$ and $\bar{g} = (J, g_O, g_A)$ such that*

1. $J = j \blacktriangleleft I$,

2. $\bar{f} = \bar{g}|_I$

Then there exists a complete set of specifications $S' = (\Theta', \mathcal{F}', S0', S2')$ for $\bar{g}$ such that

$$S = S'|_I$$

Proof. Symmetric to 4.12. ∎

4.14. Lemma. *Given $\bar{f} = (I, f_O, f_A)$, a complete set of specifications $S = (\Theta, \mathcal{F}, S0, S2)$ for $\bar{f}$, and $\bar{g} = (J, g_O, g_A)$ such that*

1. $J = L \blacktriangleleft I \blacktriangleleft R$,

2. $\bar{f} = \bar{g}|_I$

Then there exists a complete set of specifications $S' = (\Theta', \mathcal{F}', S0', S2')$ for $\bar{g}$ such that

$$S = S'|_I$$

Proof. Use 4.12 to extend to the right and 4.13 to extend to the left. ∎

4.15. Lemma. *Given a pasting diagram $\bar{f} = (I, f_O, f_A)$, there is a complete set of specifications $S = (\Theta, \mathcal{F}, S0, S2)$ for $\bar{f}$.*

Proof. We use induction on size of $|I|$.

1. Base case: Use 4.11.

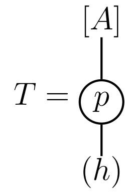
2. Induction step: Use 4.12.

∎

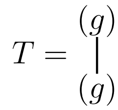
4.16. Normal form for complete specification trees: Given a PD $\bar{f}$, and a complete set of specifications S for $\bar{f}$, we have trees with only specification nodes that are constructed from S; they use only specifications appearing in S, placed in the same way as in S and contains no 2-cell nodes. Call such a tree a *specification tree* and denote a set of such trees as Υ_S ($\mathrm{dom}(T) = \bar{f}$ whenever $T \in \Upsilon_S$). All such $T \in \Upsilon_S$ are equivalent to a unique tree in normal form in Υ_S as shown below. We denote the normal form tree for S by $\coprod_S$.

4.17. Definition. *A tree $T \in \Upsilon_S$ is said to be in normal form if and only if*

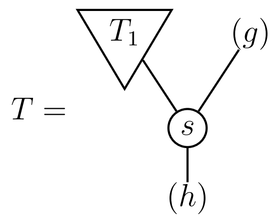
1. $|\bar{f}| = 0$, *and*



2. *or,* $|\bar{f}| = 1$, *and*



3. *or,* $|\bar{f}| > 1$, *and*



and T_1 is in normal form.

4.18. Lemma. *Given a complete specification set S and $T \in \Upsilon_S$, then $T \simeq \coprod_S$, i.e. $T \longrightarrow^* \coprod_S$.*

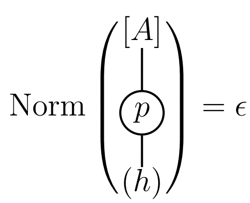
Proof. An algorithm for converting a specification tree into its normal form is given. It actually produces a witness for converting T into its normal form. This is called Norm and has type

$$\mathrm{Norm} : \Upsilon_S \longrightarrow (\to^*)$$

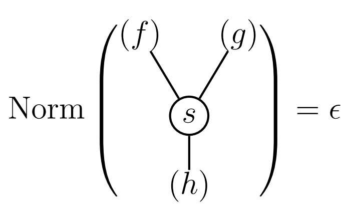
where $(\to^*)$ is set of sequences of elementary transformations.

Norm is defined recursively as:

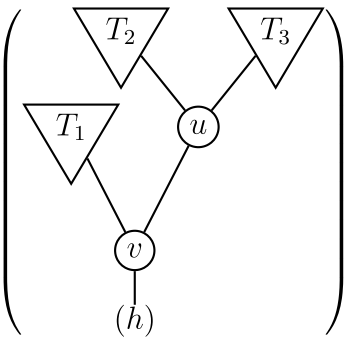
1.

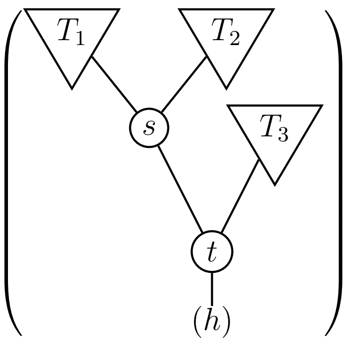


2.

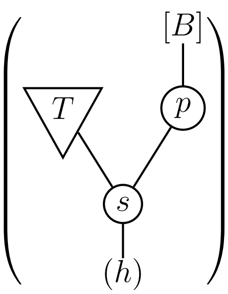
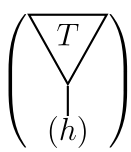


3.

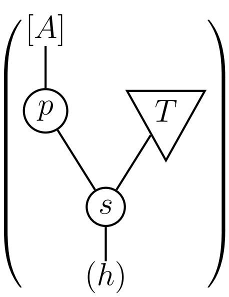
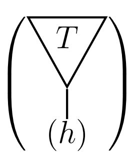
Norm 

$=$ Norm  $\cdot\, (\epsilon, \overline{ID}, h) \cdot (\epsilon, \overline{ALP}, s, t, u, v)$

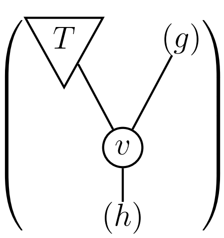
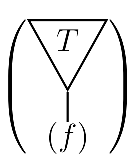
4.

Norm  $=$ Norm  $\cdot\, (\epsilon, \overline{ID}, h) \cdot (\epsilon, \underline{LMD}, s, p)$

5.

Norm  $=$ Norm  $\cdot\, (\epsilon, \overline{ID}, h) \cdot (\epsilon, \underline{RHO}, s, p)$

6.

Norm  $= \langle l \rangle *$ Norm 

Let $\langle t_i \rangle = \mathrm{Norm}(T)$. Then $\langle t_i \rangle$ is a transformation for $T \longrightarrow \coprod_S$. Hence, $T \simeq \coprod_S$. ■

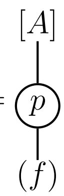
4.19. Corollary. *Given a complete specification set S and $T_1, T_2 \in \Upsilon_S$ then $T_1 \simeq T_2$.*

Proof. $T_1 \simeq \coprod_S \simeq T_2$. ■

4.20. Residue modulo complete specifications of a tree: Given a tree T, let number of 0-specifications in T be $\#_0(T)$ and number of 2-specifications in T be $\#_2(T)$. Let $\#(T) = \#_2(T) + 1 - \#_0(T)$.

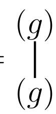
For any tree T such that $\mathrm{dom}(T) = \bar{f}$ and S being a complete specification set for $\bar{f}$, we define $[T]_S$ the residue of T modulo S recursively as below. We also produce a witness for the transformation that finds $[T]_S$, denoted as $\vee_S(T)$. Now,

1. If

$T =$ 

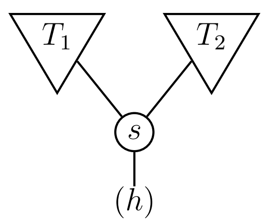
, then $[T]_S = 1_{A,S0(0),p}$ and $\vee_S(T') = (\epsilon, \overline{S0}, S0(0), p)$.

2. If

$T =$ 

then $[T]_S = \mathrm{Id}_g$ and $\vee_S(T) = (\epsilon, \overline{ID}, f)$.

3. If

$T =$ 

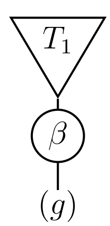
then

$$[T]_S = [T_1]_{S\uparrow\mathrm{dom}(T_1)} \circ_{S2(0,\#(T_1),\#(T_1)+\#(T_2)),s} [T_2]_{S\uparrow\mathrm{dom}(T_2)}$$

and

$$\vee_S(T) = \langle l \rangle * \vee_{S\uparrow\mathrm{dom}(T_1)}(T_1) \cdot \langle r \rangle * \vee_{S\uparrow\mathrm{dom}(T_2)}(T_2)$$
$$\cdot (\epsilon, \overline{S2}, [T_1]_{S\uparrow\mathrm{dom}(T_1)}, [T_2]_{S\uparrow\mathrm{dom}(T_2)}, S2(0, \#(T_1), \#(T_1) + \#(T_2)), s)$$

4. If

$T =$ 

then $[T]_S = [T_1]_S \cdot \beta$ and

$$\vee_S(T) = \langle u \rangle * \vee_S(T_1) \cdot (\epsilon, \overline{VC}, [T_1]_S \cdot \beta, [T_1]_S, \beta)$$

Denote by $\langle T \rangle_S$, the tree obtained using transformations $\vee_S(T)$ on T. Then note that $\langle T \rangle_S[\langle u \rangle] \in \Upsilon_S$. We denote $\langle T \rangle_S[\langle u \rangle]$ as T/S. Thus we have an equation

$$T \simeq T/S \odot_{\langle u \rangle} [T]_S$$

Let $\longrightarrow_r$ be the restriction of $\longrightarrow$ (elementary transformations) for the case where $\mathrm{pos} = \epsilon$ i.e. the transformation is applied at the root only.

4.21. Lemma. *If $T \longrightarrow_r T'$ and S a complete set of specifications for* $\mathrm{dom}(T)$*, then* $[T]_S = [T']_S$.

Proof. For simplicity in this proof, subscript S is removed from $[T]_S$.

1. $\overline{\mathbf{VC}}$ For $(\epsilon, \overline{VC}, \delta, \beta, \gamma)$, $[T] = [T_1] \cdot \beta \cdot \gamma$ and $[T'] = [T_1] \cdot \delta = [T_1] \cdot \beta \cdot \gamma$. Hence, $[T] = [T']$.
2. $\underline{\mathbf{VC}}$ For $(\epsilon, \underline{VC}, \delta, \beta, \gamma)$, proof is same as above.
3. $\overline{\mathbf{S0}}$ For $(\epsilon, \overline{S0}, p, q)$, $[T] = 1_{\Theta(i),S0(i),q}$ and $[T'] = 1_{\Theta(i),S0(i),p} \cdot 1_{\Theta(i),p,q} = 1_{\Theta(i),S0(i),q}$. Hence, $[T] = [T']$.
4. $\underline{\mathbf{S0}}$ For $(\epsilon, \underline{S0}, , p, q)$, proof is the same as above.
5. $\overline{\mathbf{S2}}$ For $(\epsilon, \overline{S2}, \beta, \gamma, s, t)$, $[T] = ([T_1] \cdot \beta) \circ_{S2(i,j,k),t} ([T_2] \cdot \gamma)$ and $[T'] = ([T_1] \circ_{S2(i,j,k),s} [T_2]) \cdot (\beta \circ_{s,t} \gamma)$. Hence, $[T] = [T']$.
6. $\underline{\mathbf{S2}}$ For $(\epsilon, \underline{S2}, \beta, \gamma, s, t)$, proof is the same as above.
7. $\overline{\mathbf{ALP}}$ For $(\epsilon, \overline{ALP}, s, t, u, v)$, $[T] = [T_1] \circ_{S2(i,j,l),v} (\gamma \circ_{S2(j,k,l),u} \delta)$ and $[T'] = ([T_1] \circ_{S2_{i,j,k},s} [T_2]) \circ_{S2(i,k,l),t} [T_3]$. From naturality we have, $[T'] = \alpha_{i,j,k,l} \cdot [T]$ and from α coherence, we have $\alpha_{i,j,k,l} = \mathrm{Id}_{i,l}$. Hence, $[T] = [T']$.
8. $\underline{\mathbf{ALP}}$ For $(\epsilon, \underline{ALP}, s, t, u, v)$, proof is the same as above.
9. $\overline{\mathbf{LMD}}$ For $(\epsilon, \overline{LMD}, s, p)$, $[T] = [T_1] \cdot \mathrm{Id}_f = [T_1]$ and $[T'] = ([T_1] \circ_{S2(i,j,j),s} 1_{\Theta(j),S0(j),p}) \cdot \lambda_{s,p}$. From naturality we have, $[T'] = \lambda_{i,j} \cdot [T]$ and λ coherence we have $\lambda_{i,j} = \mathrm{Id}_{i,j}$. Hence, $[T] = [T']$.
10. $\underline{\mathbf{LMD}}$ For $(\epsilon, \underline{LMD}, s, p)$, proof is the same as above.
11. $\overline{\mathbf{RHO}}$ For $(\epsilon, \overline{RHO}, s, p)$, $[T] = [T_1] \cdot \mathrm{Id}_f = [T_1]$ and $[T'] = (1_{\Theta(j),S0(j),p} \circ_{S2(i,i,j),s} [T_1]) \cdot \rho_{s,p}$. From naturality we have, $[T'] = \rho_{i,j} \cdot [T]$ and ρ coherence we have $\rho_{i,j} = \mathrm{Id}_{i,j}$. Hence, $[T] = [T']$.

12. $\underline{\mathbf{RHO}}$ For $(\epsilon, \underline{RHO}, s, p)$, proof is the same as above.

13. $\overline{\mathbf{ID}}$ For $(\epsilon, \overline{ID}, f)$, $[T] = [T_1]$, $[T'] = [T_1] \cdot \mathrm{Id}_f = [T_1]$.

14. $\underline{\mathbf{ID}}$ For $(\epsilon, \underline{ID}, f)$, proof is the same as above.

■

4.22. Lemma. *If* $[T_1]_S = [T_1']_S$, *then* $[T_1 \odot_p T_2]_S = [T_1' \odot_p T_2]_S$

Proof. Since T_1 is a subtree of $T_1 \odot_p T_2$ and T_1' is a subtree of $T_1' \odot_p T_2$, while evaluating $[T_1 \odot_p T_2]_S$ and $[T_1' \odot_p T_2]_S$, at a certain point we need to evaluate $[T_1]_S$ and $[T_1']_S$. But then $[T_1]_S = [T_1']_S$ and the rest of the evaluation is same for $[T_1 \odot_p T_2]_S$ and $[T_1' \odot_p T_2]_S$. Hence, $[T_1 \odot_p T_2]_S = [T_1' \odot_p T_2]_S$ ■

4.23. Corollary. *If* $T \simeq T'$ *and* S *is a complete set of specifications for* $\mathrm{dom}(T)$, *then* $[T]_S = [T']_S$.

Proof. Using 4.21 and 4.22 we have $T \longrightarrow T' \Longrightarrow [T]_S = [T']_S$. Using induction on the number of steps in $\simeq = \longrightarrow^*$, we get the required result. ■

4.24. Lemma. *Suppose* $T_1, T_2 \in \Upsilon$ *such that* $\tau(T_1) = \tau(T_2)$, *and* S *is a complete set of specifications for* $\mathrm{dom}(T_1) = \mathrm{dom}(T_2)$, *then*

$$T_1 \simeq T_2 \Longleftrightarrow [T_1]_S = [T_2]_S$$

Proof. $(\Longrightarrow)$ 4.23

$(\Longleftarrow)$

$$\begin{tikzcd}
T_1 \arrow[r, "\vee_S(T_1)"] & \langle T_1\rangle_S = T_1/S \odot_{\langle u\rangle} [T_1]_S \arrow[r, "\simeq"] & \Upsilon_S \odot_{\langle u\rangle} [T_1]_S \arrow[d, "="'] \\
T_2 \arrow[r, "\vee_S(T_2)"'] & \langle T_2\rangle_S = T_2/S \odot_{\langle u\rangle} [T_2]_S \arrow[r, "\simeq"] & \Upsilon_S \odot_{\langle u\rangle} [T_2]_S
\end{tikzcd}$$

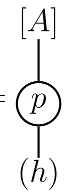
■

4.25. Definition. *A tree* T *is said to be universal if and only if* $[T]_S$ *is an isomorphism, where* S *is any complete set of specifications for* $\mathrm{dom}(T)$.

4.26. Lemma. *Given any pasting diagram* $\bar{f}$, *there is an universal 2-cell* U *such that* $\mathrm{dom}(U) = \bar{f}$.

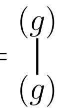
PROOF. Let S be a set complete set of specifications for $\bar{f}$, then consider $U = \Upsilon_S$. Let S' be any complete set of specifications for $\bar{f}$. We use induction on the structure of the U.

1. $|\bar{f}| = 0$,

$U =$ 

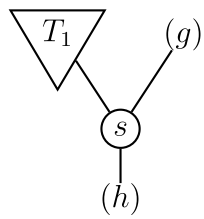
then, $[U]_{S'} = 1_{A,p',p}$ which is an isomorphism.

2. or, $|\bar{f}| = 1$, and

$T =$ 

then, $[U]_{S'} = \mathrm{Id}_f$ which is an isomorphism.

3. or, $|\bar{f}| > 1$, and

$T =$ 

then, by induction $[T_1]_{S'}$ is an isomorphism. Hence, $[T]_{S'} = [T_1]_{S'} \circ_{s',s} \mathrm{Id}_g$ is an isomorphism.

■

4.27. LEMMA. *Given any 2-cell T such that* $\mathrm{dom}(T) = \bar{f}$ *and a universal 2-cell U such that* $\mathrm{dom}(U) = \bar{g} = \bar{f} \uparrow_{(m,m')}$, *then there is a 2-cell T', such that $U \odot_{-} T' \simeq T$.*

PROOF. Let S be a set of complete set of specifications for $\bar{g} = \mathrm{dom}(U)$. Since, $\bar{g} = \bar{f} \uparrow_{(m,m')}$, we extend S to S' such that S' is complete set of specifications for $\bar{f}$. Now, let $S'' = S' \downarrow_{(m,m')}$, $\bar{h} = \bar{f}[co\mathrm{dom}(U)/(m,m')]$, $\mathrm{pos} = \langle l^{|\bar{h}|-m-1} \cdot r^{if(m=0)(0)else(1)} \rangle$, $\mathrm{pos}' = u \cdot \mathrm{pos}$ and, $\mathrm{pos}'' = \mathrm{pos}' \cdot u$. Then $U/S \odot_{\mathrm{pos}} \coprod_{S''} \in \Upsilon_{S'}$ with $\mathrm{dom}(U/S \odot_{\mathrm{pos}} \coprod_{S''}) = \bar{f}$. Now we have,

$$\begin{aligned}
T &\simeq T/S' \odot_{\langle u \rangle} [T]_{S'} \\
&\simeq (U/S \odot_{\mathrm{pos}} \textstyle\coprod_{S''}) \odot_{\langle u \rangle} [T]_{S'} \\
&\simeq (((U/S \odot_{\langle u \rangle} [U]_S) \odot_{\langle u \rangle} [U]_S^{-1}) \odot_{\mathrm{pos}} \textstyle\coprod_{S''}) \odot_{\langle u \rangle} [T]_{S'} \\
&\simeq ((U \odot_{\langle u \rangle} [U]_S^{-1}) \odot_{\mathrm{pos}} \textstyle\coprod_{S''}) \odot_{\langle u \rangle} [T]_{S'} \\
&= (U \odot_{\langle u \rangle} [U]_S^{-1}) \odot_{\mathrm{pos}'} (\textstyle\coprod_{S''} \odot_{\langle u \rangle} [T]_{S'}) \\
&= U \odot_{\mathrm{pos}''} ([U]_S^{-1} \odot_{\mathrm{pos}'} (\textstyle\coprod_{S''} \odot_{\langle u \rangle} [T]_{S'}))
\end{aligned}$$

Thus $T' = ([U]_S^{-1} \odot_{\mathrm{pos}'} (\coprod_{S''} \odot_{\langle u \rangle} [T]_{S'}))$ satisfies the lemma. ■

4.28. Lemma. *Given an universal 2-cell* U*, and two 2-cells* T_1 *and* T_2 *such that,* $\mathrm{dom}(U) = \bar{g}$*,* $\mathrm{dom}(T_1) = \mathrm{dom}(T_2) = \bar{f}$*,* $co\mathrm{dom}(T_1) = co\mathrm{dom}(T_2)$ *and* $co\mathrm{dom}(U) = f_A(m)$*. Then* $U \odot_{p1} T_1 \simeq U \odot_{p2} T_2$*, implies* $T_1 \simeq T_2$*.*

Proof. In this proof we use isomorphism of indexes with natural numbers. Let$S$ be complete set of specifications for $\bar{g}$. Let $\bar{h} = \bar{f}[\bar{g}/(m, m+1)]$. Let S' be an extension of S to $\bar{h}$ and $S'' = S' \downarrow_{m,m+|\bar{g}|}$. Let $p1' = \vee_{S''}(T_1)(p1)$, $p2' = \vee_{S''}(T_2)(p2)$, and $p = \langle l^{|\bar{f}|-m-1} \cdot r^{if(m=0)(0)else(1)} \rangle$. Then

$$
\begin{array}{lrl}
 & U \odot_{p1} T_1 & \simeq U \odot_{p2} T_2 \\
\Rightarrow & U \odot_{p1'} (T_1/S'' \odot_{\langle u \rangle} [T_1]_{S''}) & \simeq U \odot_{p2'} (T_2/S'' \odot_{\langle u \rangle} [T_2]_{S''}) \\
\Rightarrow & U \odot_{p'} (\coprod_{S''} \odot_{\langle u \rangle} [T_1]_{S''}) & \simeq U \odot_{p'} (\coprod_{S''} \odot_{\langle u \rangle} [T_2]_{S''}) \\
\Rightarrow & (U \odot_p \coprod_{S''}) \odot_{\langle u \rangle} [T_1]_{S''} & \simeq (U \odot_p \coprod_{S''}) \odot_{\langle u \rangle} [T_2]_{S''} \\
\Rightarrow & ((U/S \odot_{\langle u \rangle} [U]_S) \odot_p \coprod_{S''}) \odot_{\langle u \rangle} [T_1]_{S''} & \simeq ((U/S \odot_{\langle u \rangle} [U]_S) \odot_p \coprod_{S''}) \odot_{\langle u \rangle} [T_2]_{S''} \\
\Rightarrow & ((U/S \odot_p \coprod_{S''}) \odot_{\langle u \rangle} \delta) \odot_{\langle u \rangle} [T_1]_{S''} & \simeq ((U/S \odot_p \coprod_{S''}) \odot_{\langle u \rangle} \delta) \odot_{\langle u \rangle} [T_2]_{S''} \\
\Rightarrow & \delta \cdot [T_1]_{S''} & = \delta \cdot [T_2]_{S''} \\
\Rightarrow & [T_1]_{S''} & = [T_2]_{S''} \\
\Rightarrow & T_1 & \simeq T_2
\end{array}
$$

where $\delta = (\mathrm{Id}_{\mathcal{F}''(0,m)} \circ_{S2''(0,m,m+1),S2''(0,m,m+1)} [U]_S^{-1}) \circ_{S2''(0,m+1,|\bar{h}|),S2''(0,m+1,|\bar{h}|)} \mathrm{Id}_{\mathcal{F}''(m+1,|\bar{h}|)}$. Since $[U]_S$ is an isomorphism, so is δ. ∎

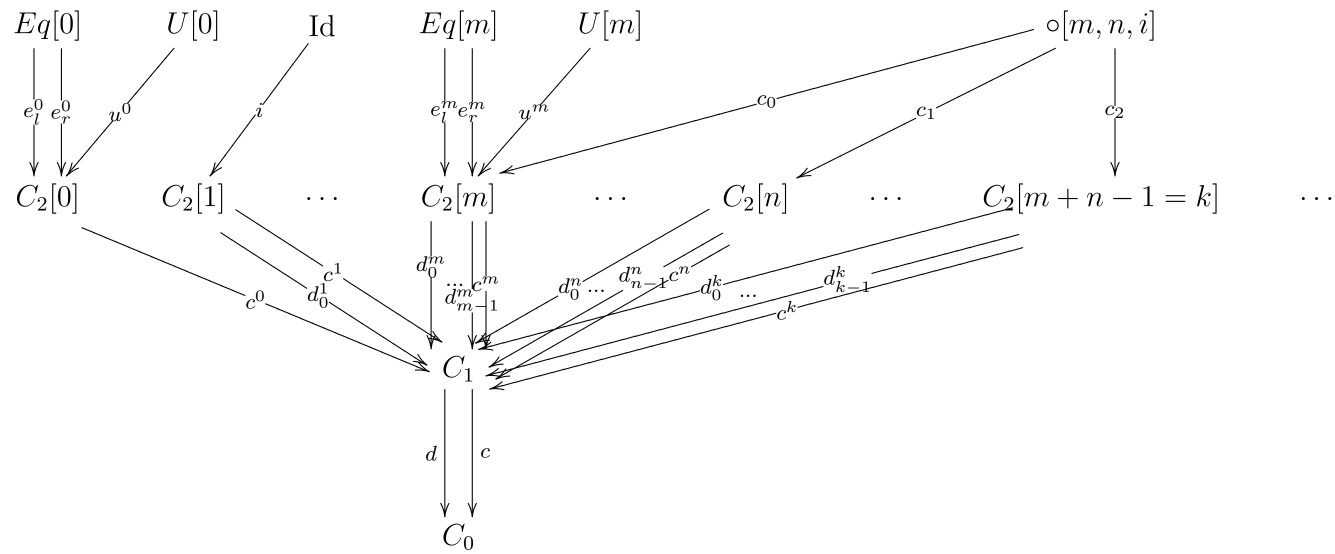
4.29. Theorem. *Construction* $(-)^{\#}$ *transforms ana-bicategory to 2D-multitopic category.*

5. Equivalence of 2D-multitopic category and ana-bicategory

In section 3 and section 4 two constructions $(-)^*$ and $(-)^{\#}$ were described. In this section we show that these constructions form adjoint pairs in the sense of FOLDS. First we take two composites $\mathcal{M} \xmapsto{(-)^*} \mathcal{M}^* \xmapsto{(-)^{\#}} \mathcal{M}^{*\#}$ and $\mathcal{A} \xmapsto{(-)^{\#}} \mathcal{A}^{\#} \xmapsto{(-)^*} \mathcal{A}^{\#*}$. $\mathcal{A} \simeq \mathcal{A}^{\#*}$ is obvious because all the data is preserved. In fact this is equality.

Non obvious equivalence is that of $\mathcal{M} \simeq \mathcal{M}^{*\#}$, on which we start to work now.

5.1. FOLDS signature. The FOLDS signature for 2D-multitopic category (L_{2D-Mlt}) is



The following equations hold for the arrows in the above one way category.

$$
\begin{aligned}
(\forall n \in \{1,2,\ldots\})(\forall 1 \leq i < n)(d_i^n \cdot d &= d_{i-1}^n \cdot c) \\
c^0 \cdot d &= c^0 \cdot c \\
(\forall n \in \{1,2,\ldots\})(d_0^n \cdot d &= c^n \cdot d) \\
(\forall n \in \{1,2,\ldots\})(d_{n-1}^n \cdot c &= c^n \cdot c) \\
i \cdot d_0^1 &= i \cdot c^1 \\
(\forall n \in \{0,1,\ldots\})(\forall p \in C_2[n] \downarrow L_{2D-Mlt})(eq_l \cdot p &= eq_r \cdot p) \\
(\forall m \in \{0,1,\ldots\})(\forall n \in \{0,1,\ldots\})(\forall 0 \leq i < n)(c_1 \cdot d_i^n &= c_0 \cdot c^m) \\
(\forall m \in \{0,1,\ldots\})(\forall n \in \{0,1,\ldots\})(\forall 0 \leq i < n)(\forall 0 \leq j < i)(c_1 \cdot d_j^n &= c_2 \cdot d_j^{m+n-1}) \\
(\forall m \in \{0,1,\ldots\})(\forall n \in \{0,1,\ldots\})(\forall 0 \leq i < n)(\forall i < j < n)(c_1 \cdot d_j^n &= c_2 \cdot d_{m+j-1}^{m+n-1}) \\
(\forall m \in \{0,1,\ldots\})(\forall n \in \{0,1,\ldots\})(\forall 0 \leq i < n)(\forall 0 \leq j < m)(c_0 \cdot d_j^m &= c_2 \cdot d_{i+j}^{m+n-1}) \\
(\forall m \in \{0,1,\ldots\})(\forall n \in \{0,1,\ldots\})(\forall 0 \leq i < n)(c_1 \cdot c^m &= c_2 \cdot c^{m+n-1})
\end{aligned}
$$

The 2D-multitopic category is L_{2D-Mlt} structure that satisfies the following axioms (Σ_{2D-Mlt}).

1. Equality:

    1. Reflexivity
    $$(\forall c \in C_2[n])(\exists e(c,c) \in Eq[n])$$

    2. Symmetry
    $$(\forall e_1(c_1,c_2) \in Eq[n])(\exists e_2(c_2,c_1) \in Eq[n])$$

    3. Transitivity
    $$(\forall e_1(c_1,c_2) \in Eq[n])(\forall e2(c_2,c_3) \in Eq[n])(\exists e_3(c_1,c_3) \in Eq[n])$$

2. Composition:

   1. Existence of composite

$$
\begin{aligned}
&(\forall a \in C_2[m])(\forall b \in C_2[n]) \\
&\quad (c^m(a) = d^m(p_i^m(b)) \\
&\quad \Longrightarrow \\
&\quad (\exists d \in C_2[m+n-1])(\exists c(a,b,d) \in \circ[m,n,i]))
\end{aligned}
$$

   2. Composition is well defined

$$
\begin{aligned}
&(\forall a \in C_2[m]) \\
&\quad (((\exists e(a,a') \in Eq[m]) \\
&\quad \wedge((\forall b \in C_2[n])(\forall d \in C_2[m+n-1])(\exists c(a,b,d) \in \circ[m,n,i]))) \\
&\quad \Longrightarrow \\
&\quad (\exists c(a',b,d) \in \circ[m,n,i]))
\end{aligned}
$$

   3. Composition is well defined

$$
\begin{aligned}
&(\forall b \in C_2[n]) \\
&\quad (((\exists e(b,b') \in Eq[n]) \\
&\quad \wedge((\forall a \in C_2[m])(\forall d \in C_2[m+n-1])(\exists c(a,b,d) \in \circ[m,n,i]))) \\
&\quad \Longrightarrow \\
&\quad (\exists c(a,b',d) \in \circ[m,n,i]))
\end{aligned}
$$

   4. Uniqueness of composition

$$
\begin{aligned}
&(\forall a \in C_2[m])(\forall b \in C_2[n]) \\
&\quad (c^m(a) = d^m(p_i^m(b)) \\
&\quad \wedge(\forall d \in C_2[m+n-1])(\forall d' \in C_2[m+n-1]) \\
&\quad\quad (\exists c(a,b,d) \in \circ[m,n,i])(\exists c'(a,b,d') \in \circ[m,n,i]) \\
&\quad \Longrightarrow \\
&\quad (\exists e(d,d') \in Eq[m+n-1]))
\end{aligned}
$$

3. Commutativity:

$$
\begin{aligned}
&(\forall a \in C_2[m])(\forall a' \in C_2[m'])(\forall b \in C_2[n]) \\
&\quad (c^m(a) = d^m(p_i^m(b)) \wedge c^m(a') = d^m(p_j^m(b)) \wedge i < j \\
&\quad \wedge(\forall ab \in C_2[m+n-1])(\forall a'b \in C_2[m'+n-1]) \\
&\quad\quad (\forall a'ab \in C_2[m'+m+n-2])(\forall aa'b \in C_2[m+m'+n-2]) \\
&\quad\quad (\exists c(a,b,ab) \in \circ[m,n,i])(\exists c'(a',b,a'b) \in \circ[m',n,i]) \\
&\quad\quad (\exists c''(a',ab,a'ab) \in \circ[m',m+n-1,j+m-1]) \\
&\quad\quad (\exists c'''(a,a'b,aa'b) \in \circ[m,m'+n-1,i]) \\
&\quad \Longrightarrow \\
&\quad (\exists e(a'ab,aa'b) \in Eq[m+m'+n-2]))
\end{aligned}
$$

4. Associativity:

$$
\begin{array}{l}
(\forall a \in C_2[l])(\forall b \in C_2[m])(\forall d \in C_2[n]) \\
\quad (c^l(a) = d^m(p_j^m(b)) \wedge c^m(b) = d^n(p_i^n(d)) \\
\quad \wedge(\forall ab \in C_2[l+m-1])(\forall bd \in C_2[m+n-1]) \\
\quad\quad (\forall (ab)d \in C_2[l+m+n-2])(\forall a(bd) \in C_2[l+m+n-1]) \\
\quad\quad (\exists c(a,b,ab) \in \circ[l,m,j])(\exists c'(b,d,bd) \in \circ[m,n,i]) \\
\quad\quad (\exists c''(ab,d,(ab)d) \in \circ[l+m-1,n,l+m+n-2]) \\
\quad\quad (\exists c'''(a,bd,a(bd)) \in \circ[l,m+n-1,i+j]) \\
\quad \Longrightarrow \\
\quad (\exists e((ab)d, a(bd)) \in Eq[l+m+n-2]))
\end{array}
$$

5. Identity:

$$
\begin{array}{l}
(\forall f \in C_1)(\exists \mathrm{Id}_f \in I)(\forall m) \\
\quad ((\forall a \in C_2[m])(\forall 0 \le i < m) \\
\quad\quad (d^n(p_i^n(a)) = f \Longrightarrow (\exists c(i(\mathrm{Id}_f), a, a) \in \circ[1,n,i])) \\
\quad\quad \wedge(\forall a \in C_2[m]) \\
\quad\quad\quad (c^n(a) = f \Longrightarrow (\exists c(a, i(\mathrm{Id}_f), a) \in \circ[n,1,0])))
\end{array}
$$

6. Universality:

   1. Definition of universal 2-cell

$$
\begin{array}{l}
Univ(u \in U[n]) \\
\textbf{iff} \\
(\forall m \ge n)(\forall a \in C_2[m])(\forall 0 \le i < m)(\forall 0 \le j < n) \\
\quad ((d_j^n(u^n(u)) = d_{i+j}^m(a)) \\
\quad \Longrightarrow \\
\quad (\exists b \in C_2[m-n+1]) \\
\quad\quad ((\exists c(u^n(u), b, a) \in \circ[n, m-n+1, i]) \\
\quad\quad \wedge(\forall b' \in C_2[m-n+1]) \\
\quad\quad\quad (\exists c'(u^n(u), b', a) \in \circ[n, m-n+1, i]) \\
\quad\quad\quad \Longrightarrow \\
\quad\quad\quad (\exists e(b, b') \in Eq[m-n+1])))
\end{array}
$$

   2. Existence of universal 2-cell for length 0 PD

$$(\forall A \in C_0)(\exists u \in U[0])(c(d^0(u^0(u))) = A \wedge Univ(u))$$

   3. Existence of universal 2-cell for length > 0 PD

$$
\begin{array}{l}
(\forall n \in \{1,2,\ldots\}) \\
\quad (\forall f_0 \in C_1)\prod_{i=1}^{i<n}((\forall f_i \in C_1)(d(f_i) = c(f_{i-1}))) \\
\quad\quad (\exists u \in U[n]) \\
\quad\quad\quad ((\forall 0 \le i < n)((d^i(u^n(u)) = f_i) \wedge Univ(u)))
\end{array}
$$

5.2. Structure. We define $\mathcal{M}$ and $\mathcal{M}^{*\#}$ as two L_{2D-Mlt} structures. The meaning of the arrows will be common for both and will be described after filling in the object descriptions.

5.3. Definition. *$\mathcal{M}$: C_0 and C_1 are $\mathrm{Cell}_0(\mathcal{M})$ and $\mathrm{Cell}_1(\mathcal{M})$. $C_2[i]$ is 2-cells with length of domain i. $U[i]$ are universals of domain length i. I is identity 2-cells. $Eq[i] = \{(c, c)|c \in C_2[i]\}$. $\circ[m, n, i] = \{(\alpha, \beta, \gamma) \in C_2[m] \times C_2[n] \times C_2[m + n - 1]|\alpha \cdot \beta = \gamma\}$.*

5.4. Definition. *$\mathcal{M}^{*\#}$: C_0 and C_1 are same as above. $C_2[i] = \{T \in \Upsilon||\mathrm{dom}(T)| = i\}$. $Eq[i] = \{(T_1, T_2)|T_1, T_2 \in C_2[i] \wedge T_1 \simeq T_2\}$. $I = \{T|T \in C_2[1] \wedge T \simeq \mathrm{Id}_f$ for some $f \in C_1\}$. $U[i] \subset C_2[i]$ are the universal arrows as defined in 4.25. $\circ[m, n, i] = \{(T_1, T_2, T_3) \in C_2[m] \times C_2[n] \times C_2[m + n - 1]|T_1 \odot T_2 \simeq T_3\}$.*

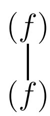
c and d map 1-cells to their domain and codomain 0-cells. d_i^m and c^m maps 2-cell to its i^{th} place in domain and to its codomain 1-cell. e_l^m and e_r^m are left and right sides of equality on 2-cells. u^m is an injection of universals into 2-cells and i is an injection of identities into $C_2[1]$.

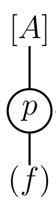
All the axioms in Σ_{2D-Mlt} are true for the structure $\mathcal{M}^{*\#}$ as has been verified in the previous section. For $\mathcal{M}$ they are automatic from the axioms of 2D-multitopic category.

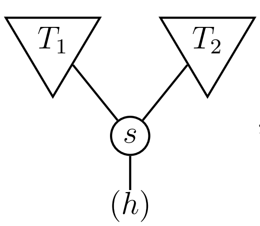
5.5. Evaluation: 0 and 1 cells of these two structures coincide as was given by the constructions in the previous chapters. For 2-cells, we define a map from $\mathcal{M}^{*\#}$ to $\mathcal{M}$ called ev, an abbreviation for evaluation, remembering the fact that trees in Υ come from 2 cells in $\mathcal{M}$ which has composition defined in it.

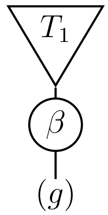
$$ev : \Upsilon \longrightarrow \mathrm{Cell}_2(\mathcal{M})$$

This is defined inductively on the structure of trees (in Υ) and we show it is invariant under the equivalence relation $\simeq$ defined in previous chapter.

1. If $T =$  , then $\mathrm{ev}(T) = \mathrm{Id}_f$.

2. If $T =$  , then $\mathrm{ev}(T) = p$.

3. If $T =$  , $\mathrm{ev}(T_1) = \alpha$ and $\mathrm{ev}(T_2) = \beta$, then $\mathrm{ev}(T) = \alpha \cdot \beta \cdot s$

4. If $T =$  and $\mathrm{ev}(T_1) = \alpha$, then $\mathrm{ev}(T) = \alpha \cdot \beta$.

5.6. Lemma.

$$T \longrightarrow_r T' \Longrightarrow \mathrm{ev}(T_1) = \mathrm{ev}(T_2)$$

Proof. To show that *ev* is invariant under $\simeq$, we show it is invariant under each elementary step. Let the resulting tree after elementary transformation of T be T'.

1. $\overline{\mathbf{VC}}$ For $(\epsilon, \overline{VC}, \delta, \beta, \gamma)$, $\mathrm{ev}(T) = \mathrm{ev}(T_1) \cdot \beta \cdot \gamma$ and $\mathrm{ev}(T') = \mathrm{ev}(T_1) \cdot \delta = \mathrm{ev}(T_1) \cdot \beta \cdot \gamma$. Hence, $\mathrm{ev}(T) = \mathrm{ev}(T')$.
2. $\underline{\mathbf{VC}}$ For $(\epsilon, \underline{VC}, \delta, \beta, \gamma)$, proof is the same as above.
3. $\overline{\mathbf{S0}}$ For $(\epsilon, \overline{S0}, p, q)$, $\mathrm{ev}(T) = q$ and $\mathrm{ev}(T') = p \cdot 1_{A,p,q} = q$. Hence, $\mathrm{ev}(T) = \mathrm{ev}(T')$.
4. $\underline{\mathbf{S0}}$ For $(\epsilon, \underline{S0}, p, q)$, proof is the same as above.
5. $\overline{\mathbf{S2}}$ For $(\epsilon, \overline{S2}, \beta, \gamma, s, t)$, $\mathrm{ev}(T) = (\mathrm{ev}(T_1) \cdot \beta) \cdot (\mathrm{ev}(T_2) \cdot \gamma) \cdot t$ and $\mathrm{ev}(T') = \mathrm{ev}(T_1) \cdot \mathrm{ev}(T_2) \cdot s \cdot (\beta \circ_{s,t} \gamma) = \mathrm{ev}(T_1) \cdot \mathrm{ev}(T_2) \cdot \beta \cdot \gamma \cdot t$. Hence, $\mathrm{ev}(T) = \mathrm{ev}(T')$.
6. $\underline{\mathbf{S2}}$ For $(\epsilon, \underline{S2}, \beta, \gamma, s, t)$, proof is the same as above.
7. $\overline{\mathbf{ALP}}$ For $(\epsilon, \overline{ALP}, s, t, u, v)$, $\mathrm{ev}(T) = \mathrm{ev}(T_1) \cdot (\mathrm{ev}(T_2) \cdot \mathrm{ev}(T_3) \cdot u) \cdot v$, and $\mathrm{ev}(T') = (\mathrm{ev}(T_1) \cdot \mathrm{ev}(T_2) \cdot s) \cdot \mathrm{ev}(T_3) \cdot t \cdot \alpha_{s,t,u,v}$. By using definition of $\alpha_{s,t,u,v}$, we have $\mathrm{ev}(T) = \mathrm{ev}(T')$.
8. $\underline{\mathbf{ALP}}$ For $(\epsilon, \underline{ALP}, s, t, u, v)$, proof is the same as above.
9. $\overline{\mathbf{LMD}}$ For $(\epsilon, \overline{LMD}, s, p)$, $\mathrm{ev}(T) = \mathrm{ev}(T_1)$ and $\mathrm{ev}(T') = (\mathrm{ev}(T_1) \cdot p \cdot s \cdot \lambda_{s,p}$. By using definition of $\lambda_{s,p}$, we have $\mathrm{ev}(T) = \mathrm{ev}(T')$.
10. $\underline{\mathbf{LMD}}$ For $(\epsilon, \underline{LMD}, s, p)$, proof is the same as above.
11. $\overline{\mathbf{RHO}}$ For $(\epsilon, \overline{RHO}, s, p)$, $\mathrm{ev}(T) = \mathrm{ev}(T_1)$ and $\mathrm{ev}(T') = p \cdot \mathrm{ev}(T_1) \cdot s \cdot \rho_{s,p} = \mathrm{ev}(T_1) \cdot (p \cdot s) \cdot \rho_{s,p}$. By using definition of $\rho_{s,p}$, we have $\mathrm{ev}(T) = \mathrm{ev}(T')$.
12. $\underline{\mathbf{RHO}}$ For $(\epsilon, \underline{RHO}, s, p)$, proof is the same as above.
13. $\overline{\mathbf{ID}}$ For $(\epsilon, \overline{ID}, f)$, $\mathrm{ev}(T) = \mathrm{ev}(T_1)$, $\mathrm{ev}(T') = \mathrm{ev}(T_1) \cdot \mathrm{Id}_f$. Hence, $\mathrm{ev}(T) = \mathrm{ev}(T')$.
14. $\underline{\mathbf{ID}}$ For $(\epsilon, \underline{ID}, f)$, proof is the same as above.

■

5.7. Lemma. *If* $\mathrm{ev}(T_1) = \mathrm{ev}(T_1')$, *then* $\mathrm{ev}(T_1 \odot_p T_2) = \mathrm{ev}(T_1' \odot_p T_2)$

Proof. Since T_1 is a subtree of $T_1 \odot_p T_2$ and T_1' is a subtree of $T_1' \odot_p T_2$, while evaluating $\mathrm{ev}(T_1 \odot_p T_2)$ and $\mathrm{ev}(T_1' \odot_p T_2)$, at a certain point we need to evaluate $\mathrm{ev}(T_1)$ and $\mathrm{ev}(T_1')$. But then $\mathrm{ev}(T_1) = \mathrm{ev}(T_1')$, and the rest of the evaluation is same for $\mathrm{ev}(T_1 \odot_p T_2)$ and $\mathrm{ev}(T_1' \odot_p T_2)$. Hence, $\mathrm{ev}(T_1 \odot_p T_2) = \mathrm{ev}(T_1' \odot_p T_2)$ ∎

5.8. Corollary.

$$T \simeq T' \Longrightarrow \mathrm{ev}(T) = \mathrm{ev}(T')$$

Proof. Using 5.6 and 5.7 we have $T \longrightarrow T' \Longrightarrow \mathrm{ev}(T) = \mathrm{ev}(T')$. Using induction on the number of steps in $\simeq = \longrightarrow^*$ we get the required result. ∎

5.9. Lemma.

$$T_1 \simeq T_2 \Longleftrightarrow \mathrm{ev}(T_1) = \mathrm{ev}(T_2)$$

Proof. ($\Longrightarrow$) 5.8

($\Longleftarrow$) Let S be a complete set of specifications for $\mathrm{dom}(T_1)$. Then we have $T_1 \simeq T_1/S \odot [T_1]_S$ and $T_2 \simeq T_2/S \odot [T_2]_S$. Since $T_1/S \simeq T_2/S$, we have $\mathrm{ev}(T_1/S) = \mathrm{ev}(T_2/S)$. Also since T_1/S is composed of only specifications(universals), $\mathrm{ev}(T_1/S)$ is universal, hence left cancellable. Thus,

$$\begin{array}{rrl} & eq(T_1) & = \mathrm{ev}(T_2) \\ \Rightarrow & \mathrm{ev}(T_1/S) \cdot [T_1]_S & = \mathrm{ev}(T_2/S) \cdot [T_2]_S \\ \Rightarrow & [T_1]_S & = [T_2]_S \\ \Rightarrow & T_1 & \simeq T_2 \end{array}$$

∎

5.10. The Span: To show FOLDS equivalence for $\mathcal{M}$ and $\mathcal{M}^{*\#}$ we need to find tuple (S, p, q) as $\mathcal{M} \xleftarrow{p} S \xrightarrow{q} \mathcal{M}^{*\#}$ such that p, q are fiberwise surjective. We will show that actually $S = \mathcal{M}^{*\#}$, $q = \mathrm{Id}$ and p is constructed using ev for 2-Cells. Since p, q are natural transformations, we use p_{C_0} etc to denote its components.

Surjectivity of q is immediate. Now we list the components of p.

$$\begin{aligned} p_{C_0} &= \mathrm{Id}_{C_0} \\ p_{C_1} &= \mathrm{Id}_{C_1} \\ p_{C_2[m]} &= \mathrm{ev}|_{C_2[m]} \\ p_{\mathrm{Id}} &= \mathrm{ev}|_{\mathrm{Id}} \\ p_{U[m]} &= \mathrm{ev}|_{U[m]} \\ p_{Eq[m]} &= (\mathrm{ev}|_{C_2[m]} \circ \pi_1, \mathrm{ev}|_{C_2[m]} \circ \pi_2) \\ p_{\circ[m,n,i]} &= (\mathrm{ev}|_{C_2[m]} \circ \pi_1, \mathrm{ev}|_{C_2[m]} \circ \pi_2, \mathrm{ev}|_{C_2[m]} \circ \pi_3) \end{aligned}$$

p being a natural transformation is obvious. p_{C_0} and p_{C_1} are obviously surjective as they are identities.

5.11. Lemma. $p_{C_2[m]} = \mathrm{ev}|_{C_2[m]}$ *is fiberwise surjective on* $\mathcal{M}(C_2[m])$.

PROOF. Since p is an identity on $C_2[m]$ and $p_{C_2[m]}$ preserves the frame for $C_2[m]$, surjectivity will imply fiberwise surjectivity.

1. $m = 0$: For any $\beta \in \mathcal{M}(C_2[0])$, let $p \in \mathcal{M}(U[0])$ such that $\mathrm{dom}(p) = \mathrm{dom}(\beta)$, then there is an unique $\gamma \in \mathcal{M}(C_2[1])$ such that $\beta = p \cdot \gamma$. Now, consider $T = p \odot_{\langle u \rangle} p \in \mathcal{M}^{*\#}(C_2[0])$, then $\mathrm{ev}(T) = p \cdot \gamma = \beta$.

2. $m = 1$: For any $\beta \in \mathcal{M}(C_2[1])$, consider $T = \beta \in \mathcal{M}^{*\#}(C_2[1])$, then $\mathrm{ev}(T) = \beta$.

3. $m \geq 2$: We use induction.

    1. Base case $n = 2$: For any $\gamma \in \mathcal{M}(C_2[2])$, let $s \in \mathcal{M}(U[2])$ such that $\mathrm{dom}(s) = \mathrm{dom}(\gamma)$. Then there is $\beta \in \mathcal{M}(C_2[1])$ such that $\gamma = s \cdot \beta$. Now, tree $T = s \odot_{\langle u \rangle} \beta \in \mathcal{M}^{*\#}(C_2[2])$, is such that $\mathrm{ev}(T) = s \cdot \beta = \gamma$.

    2. Induction Step: Suppose for all $\alpha \in \mathcal{M}(C_2[n])$, there is T_α such that $\mathrm{ev}(T_\alpha) = \alpha$. Now consider $\gamma \in \mathcal{M}(C_2[n+1])$, and $s \in \mathcal{M}(U[2])$ such that $\mathrm{dom}(s) \leq_0 \mathrm{dom}(\gamma)$. Then there is $\beta \in \mathcal{M}(C_2[n])$ such that $\gamma = s \cdot \beta$. By induction hypothesis, there is a tree T_β such that $\mathrm{ev}(T_\beta) = \beta$. Let pos be such that $T_\beta[\mathrm{pos}] = co\mathrm{dom}(s)$. Then tree $T = s \odot_{\mathrm{pos}} T_\beta \in \mathcal{M}^{*\#}(C_2[n+1])$ is such that $\mathrm{ev}(T) = s \cdot \beta = \gamma$.

■

5.12. LEMMA. $p_{\mathrm{Id}} = \mathrm{ev}|_{\mathrm{Id}}$ *is fiberwise surjective.*

PROOF. Let $\mathrm{Id}_f \in \mathcal{M}(\mathrm{Id})$ and $T \in \mathcal{M}^{*\#}(C_2[1])$ such that $p_{C_2[1]}(T) = \mathrm{Id}_f$. Since, $\mathcal{M}(i)(\mathrm{Id}_f) = \mathrm{Id}_f = p_{C_2[1]}(T)$, we need to show that $T \in \mathcal{M}^{*\#}(\mathrm{Id})$, $\mathcal{M}^{*\#}(i)(T) = T$, and $p_{\mathrm{Id}}(T) = \mathrm{Id}_f$.

Since $p_{C_2[1]}(T) = \mathrm{Id}_f$, we have $T \simeq \mathrm{Id}_f$, hence $T \in \mathcal{M}^{*\#}(\mathrm{Id})$. Since $\mathcal{M}^{*\#}(i)$ is an injection, we have $\mathcal{M}^{*\#}(i)(T) = T$. Now, $p_{\mathrm{Id}}(T) = \mathrm{ev}|_{\mathrm{Id}}(T) = \mathrm{Id}_f$. ■

5.13. LEMMA. *A 2-cell $\alpha : f \Rightarrow g$ is universal in $\mathcal{M}$ if and only if α is an isomorphism in $\mathcal{M}^*$.*

PROOF. It is obvious that universals are isomorphisms (for any β, consider $\alpha^{-1} \cdot \beta$).

Suppose α is an universal in $\mathcal{M}$. Then let β be such that $\alpha \cdot \beta = \mathrm{Id}_f$. Now,

$$\begin{aligned} \alpha \cdot (\beta \cdot \alpha) &= (\alpha \cdot \beta) \cdot \alpha \\ &= \mathrm{Id}_f \cdot \alpha \\ &= \alpha \\ &= \alpha \cdot \mathrm{Id}_g \end{aligned}$$

Since, universals are left-cancellable, $\beta \cdot \alpha = \mathrm{Id}_g$. ■

5.14. LEMMA. *Given a* $T \in \mathcal{M}^{*\#}(C_2[m])$,

$$T \in \mathcal{M}^{*\#}(U[m]) \Longleftrightarrow \mathrm{ev}(T) \in \mathcal{M}(U[m])$$

PROOF. Let S be CSS for $\mathrm{dom}(T)$. Now $T \simeq T/S \odot_{\langle u \rangle} [T]_S$, hence $\mathrm{ev}(T) = \mathrm{ev}(T/S) \cdot [T]_S$. As T/S is a tree made of only specifications (universals), $\mathrm{ev}(T/S)$ is an universal.

($\Longrightarrow$) Since $T \in \mathcal{M}^{*\#}(U[m])$, $[T]_S$ is an isomorphism. So, from 5.13, $\mathrm{ev}([T]_S)$ is an universal. Hence the composite $\mathrm{ev}(T/S) \cdot \mathrm{ev}([T]_S) = \mathrm{ev}(T)$ is an universal.

($\Longleftarrow$) Now since $\mathrm{ev}(T)$ and $\mathrm{ev}(T/S)$ are universals, from 2.4, $\mathrm{ev}([T]_S)$ is an universal. Now from 5.13, $[T]_S$ is an isomorphism. Hence, $T \in \mathcal{M}^{*\#}(U[m])$. ∎

5.15. LEMMA. $p_{U[m]} = \mathrm{ev}|_{U[m]}$ *is fiberwise surjective.*

PROOF. Let $u \in \mathcal{M}(U[m])$ and $T \in \mathcal{M}^{*\#}(C_2[m])$ such that $p_{C_2[m]}(T) = u$. Since, $\mathcal{M}(u^m)(u) = u = p_{C_2[m]}(T)$, we need to show that $T \in \mathcal{M}^{*\#}(U[m])$, $\mathcal{M}^{*\#}(u^m)(T) = T$, and $p_{U[m]}(T) = u$.

Since u is an universal and $\mathrm{ev}(T) = u$, $T \in \mathcal{M}^{*\#}(U[m])$ from 5.14. Since $\mathcal{M}^{*\#}(u^m)$ is an injection, we have $\mathcal{M}^{*\#}(u^m)(T) = T$. Now, $p_{U[m]}(T) = \mathrm{ev}|_{U[m]}(T) = u$. ∎

5.16. LEMMA. $p_{Eq[m]} = (\mathrm{ev}|_{C_2[m]} \circ \pi_1, \mathrm{ev}|_{C_2[m]} \circ \pi_2)$ *is fiberwise surjective.*

PROOF. Let $(\alpha, \alpha) \in \mathcal{M}(Eq[m])$ and $T, T' \in \mathcal{M}^{*\#}(C_2[m])$ such that $p_{C_2[m]}(T) = p_{C_2[m]}(T')$ $= \alpha$. Since, $\mathcal{M}(e_l^m)((\alpha, \alpha)) = \alpha = p_{C_2[m]}(T)$, and $\mathcal{M}(e_r^m)((\alpha, \alpha)) = \alpha = p_{C_2[m]}(T')$, we need to show that $(T, T') \in \mathcal{M}^{*\#}(Eq[m])$, $\mathcal{M}^{*\#}(e_l^m)((T, T')) = T$, $\mathcal{M}^{*\#}(e_r^m)((T, T')) =$ T', and $p_{Eq[m]}((T, T')) = (\alpha, \alpha)$.

Since $\mathrm{ev}(T) = \mathrm{ev}(T')$, $T \simeq T'$ (5.9), hence $(T, T') \in \mathcal{M}^{*\#}(Eq[m])$. Since $\mathcal{M}^{*\#}(e_l^m)$ and $\mathcal{M}^{*\#}(e_r^m)$ are projections, we have $\mathcal{M}^{*\#}(e_l^m)((T, T')) = T$, $\mathcal{M}^{*\#}(e_r^m)((T, T')) = T'$. Now, $p_{Eq[m]}((T, T')) = (\mathrm{ev}|_{C_2[m]} \circ \pi_1, \mathrm{ev}|_{C_2[m]} \circ \pi_2)(T, T') = (\mathrm{ev}|_{C_2[m]}(T), \mathrm{ev}|_{C_2[m]}(T')) = (\alpha, \alpha)$. ∎

5.17. LEMMA. *If* T_1 *and* T_2 *are two composable trees at position pos, then* $\mathrm{ev}(T_1 \odot_{\mathrm{pos}} T_2) =$ $\mathrm{ev}(T_1) \cdot \mathrm{ev}(T_2)$.

PROOF. We use induction on the structure of T_2.

1. T_2 is an empty tree. Then, $\mathrm{ev}(T_1 \odot_{\mathrm{pos}} T_2) = \mathrm{ev}(T_1) = \mathrm{ev}(T_1) \cdot \mathrm{Id}_{\mathrm{codom}(T_1)} = \mathrm{ev}(T_1) \cdot$ $\mathrm{ev}(T_2)$.

2. $T_2 = T' \odot_{\langle l \rangle} (T'' \odot_{\langle r \rangle} s)$. Here we have two cases.

    1. pos begins with l. Then,

$$\begin{aligned} \mathrm{ev}(T_1 \odot_{\mathrm{pos}} T_2) &= \mathrm{ev}(T_1 \odot_{\mathrm{pos}\ \langle l \rangle} T') \cdot \mathrm{ev}(T'') \cdot s \\ &= \mathrm{ev}(T_1) \cdot \mathrm{ev}(T') \cdot \mathrm{ev}(T'') \cdot s \\ &= \mathrm{ev}(T_1) \cdot \mathrm{ev}(T_2) \end{aligned}$$

    Here, $\mathrm{ev}(T_1 \odot_{\mathrm{pos}\ \langle l \rangle} T') = \mathrm{ev}(T_1) \cdot \mathrm{ev}(T')$ as tree T' is less complex than T_2.

2. pos begins with r. Then,

$$\begin{aligned}\mathrm{ev}(T_1 \odot_{\mathrm{pos}} T_2) &= \mathrm{ev}(T') \cdot \mathrm{ev}(T_1 \odot_{\mathrm{pos}\ \langle r\rangle} T'') \cdot s \\ &= \mathrm{ev}(T') \cdot \mathrm{ev}(T_1) \cdot \mathrm{ev}(T'') \cdot s \\ &= \mathrm{ev}(T_1) \cdot \mathrm{ev}(T') \cdot \mathrm{ev}(T'') \cdot s \\ &= \mathrm{ev}(T_1) \cdot \mathrm{ev}(T_2)\end{aligned}$$

In here, $\mathrm{ev}(T_1 \odot_{\mathrm{pos}\ \langle r\rangle} T'') = \mathrm{ev}(T_1) \cdot \mathrm{ev}(T'')$ as tree T'' is less complex than T_2.

3. $T_2 = T' \odot_{\langle u\rangle} \beta$. Then,

$$\begin{aligned}\mathrm{ev}(T_1 \odot_{\mathrm{pos}} T_2) &= \mathrm{ev}(T_1 \odot_{\mathrm{pos}\ \langle u\rangle} T') \cdot \beta \\ &= \mathrm{ev}(T_1) \cdot \mathrm{ev}(T') \cdot \beta \\ &= \mathrm{ev}(T_1) \cdot \mathrm{ev}(T_2)\end{aligned}$$

■

5.18. Lemma. *$p_{\circ[m,n,i]} = (\mathrm{ev}|_{C_2[m]} \circ \pi_1, \mathrm{ev}|_{C_2[m]} \circ \pi_2, \mathrm{ev}|_{C_2[m]} \circ \pi_3)$ is fiberwise surjective.*

Proof. Let $(\alpha,\beta,\gamma) \in \mathcal{M}(\circ[m,n,i])$, $T_1 \in \mathcal{M}^{*\#}(C_2[m])$, $T_2 \in \mathcal{M}^{*\#}(C_2[n])$, and $T_3 \in \mathcal{M}^{*\#}(C_2[m+n-1])$ such that $p_{C_2[m]}(T_1) = \alpha$, $p_{C_2[n]}(T_2) = \beta$, and $p_{C_2[m+n-1]}(T_3) = \gamma$. Since, $\mathcal{M}(c_0)((\alpha,\beta,\gamma)) = \alpha = p_{C_2[m]}(T_1)$, $\mathcal{M}(c_1)((\alpha,\beta,\gamma)) = \beta = p_{C_2[n]}(T_2)$, and $\mathcal{M}(c_0)((\alpha,\beta,\gamma)) = \gamma = p_{C_2[m+n-1]}(T_3)$, we need to show that $(T_1,T_2,T_3) \in \mathcal{M}^{*\#}(\circ[m,n,i])$, $\mathcal{M}^{*\#}(c_0)((T_1,T_2,T_3)) = T_1$, $\mathcal{M}^{*\#}(c_1)((T_1,T_2,T_3)) = T_2$, $\mathcal{M}^{*\#}(c_2)((T_1,T_2,T_3)) = T_3$, and $p_{\circ[m,n,i]}((T_1,T_2,T_3)) = (\alpha,\beta,\gamma)$.

Since $\mathrm{ev}(T_3) = \gamma = \alpha \cdot \beta = \mathrm{ev}(T_1) \cdot \mathrm{ev}(T_2) = \mathrm{ev}(T_1 \odot T_2)$, we have $T_1 \odot T_2 \simeq T_3$, hence $(T_1,T_2,T_3) \in \mathcal{M}^{*\#}(\circ[m,n,i])$. Since $\mathcal{M}^{*\#}(c_0)$, $\mathcal{M}^{*\#}(c_1)$, and $\mathcal{M}^{*\#}(c_2)$ are projections, we have $\mathcal{M}^{*\#}(c_0)((T_1,T_2,T_3)) = T_1$, $\mathcal{M}^{*\#}(c_1)((T_1,T_2,T_3)) = T_2$ and $\mathcal{M}^{*\#}(c_2)((T_1,T_2,T_3)) = T_3$. Now, $p_{\circ[m,n,i]}((T_1,T_2,T_3)) = (\mathrm{ev}|_{C_2[m]} \circ \pi_1, \mathrm{ev}|_{C_2[m]} \circ \pi_2, \mathrm{ev}|_{C_2[m]} \circ \pi_3)((T_1,T_2,T_3)) = (\mathrm{ev}|_{C_2[m]}(T_1), \mathrm{ev}|_{C_2[m]}(T_2), \mathrm{ev}|_{C_2[m]}(T_3)) = (\alpha,\beta,\gamma)$. ■

5.19. Theorem. *2D-multitopic category and ana-bicategory are equivalent.*

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