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BICAT IS NOT TRIEQUIVALENT TO GRAY

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ABSTRACT. **Bicat** is the tricategory of bicategories, homomorphisms, pseudonatural transformations, and modifications. **Gray** is the subtricategory of 2-categories, 2-functors, pseudonatural transformations, and modifications. We show that these two tricategories are not triequivalent.

1. BACKGROUND. Weakening the notion of 2-category by replacing all equations between 1-cells by suitably coherent isomorphisms gives the notion of *bicategory* [1]. The analogous weakening of a 2-functor is called a *homomorphism* of bicategories, and the weakening of a 2-natural transformation is a *pseudonatural transformation.* There are also *modifications* between 2-natural or pseudonatural transformations, but this notion does not need to be weakened. The bicategories, homomorphisms, pseudonatural transformations, and modifications form a tricategory (a weak 3-category) called **Bicat**.

The subtricategory of **Bicat** containing only the 2-categories as objects, and only the 2-functors as 1-cells, but with all 2-cells and 3-cells between them, is called **Gray**. As well as being a particular tricategory, there is another important point of view on **Gray**. The category **2-Cat** of 2-categories and 2-functors is cartesian closed, but it also has a different symmetric monoidal closed structure [3], for which the internal hom $[\mathscr{A}, \mathscr{B}]$ is the 2-category of 2-functors, pseudonatural transformations, and modifications between $\mathscr{A}$ and $\mathscr{B}$. A category enriched over **2-Cat** with respect to this closed structure is called a *Gray-category.* A Gray-category has 2-categories as hom-objects, so is a 3-dimensional categorical structure, and it can be seen as a particular sort of tricategory. The closed structure of **2-Cat** gives it a canonical enrichment over itself and the resulting Gray-category is just **Gray**. **Gray** is also sometimes used as a name for **2-Cat** with this monoidal structure.

A homomorphism of bicategories $T : \mathscr{A} \to \mathscr{C}$ is called a *biequivalence* if it induces equivalences $T_{A,B} : \mathscr{A}(A,B) \to \mathscr{B}(TA,TB)$ of hom-categories for all objects $A, B \in \mathscr{C}$ (T is *locally an equivalence*), and every object $C \in \mathscr{C}$ is equivalent in $\mathscr{C}$ to one of the form TA (T is *biessentially surjective on objects*). We then write $\mathscr{A} \sim \mathscr{B}$. Every bicategory is equivalent to a 2-category [5].

A trihomomorphism of tricategories $T : \mathscr{A} \to \mathscr{C}$ is called a *triequivalence* if it induces biequivalences $T_{A,B} : \mathscr{A}(A,B) \to \mathscr{B}(TA,TB)$ of hom-bicategories for all objects $A, B \in \mathscr{A}$ (T is *locally a biequivalence*), and every object $C \in \mathscr{C}$ is biequivalent in $\mathscr{C}$ to one

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of the form TA (T is *triessentially surjective on objects*). It is not the case that every tricategory is triequivalent to a 3-category, but every tricategory is triequivalent to a Gray-category [2].

Perhaps since a **Gray**-category is a category enriched in the monoidal category **Gray**, and a tricategory can be seen as some sort of "weak **Bicat**-category", it has been suggested that **Bicat** might be triequivalent to **Gray**, and indeed Section 5.6 of [2] states that this is the case. We prove that it is not. First we prove:

2. LEMMA. *The inclusion* **Gray** $\to$ **Bicat** *is not a triequivalence.*

PROOF. If it were then each inclusion $\mathbf{Gray}(\mathscr{A},\mathscr{B}) \to \mathbf{Bicat}(\mathscr{A},\mathscr{B})$ would be a biequivalence, and so each homomorphism (pseudofunctor) between 2-categories would be pseudonaturally equivalent to a 2-functor. This is not the case. For example (see [4, Example 3.1]), let $\mathscr{A}$ be the 2-category with a single object $*$, a single non-identity morphism $f : * \to *$ satisfying $f^2 = 1$, and no non-identity 2-cells (the group of order 2 seen as a one-object 2-category); and let $\mathscr{B}$ be the 2-category with a single object $*$, a morphism $n : * \to *$ for each integer n, composed via addition, and an isomorphism $n \cong m$ if and only if $n - m$ is even (the "pseudo-quotient of $\mathbb{Z}$ by $2\mathbb{Z}$"). There is a homomorphism $\mathscr{A} \to \mathscr{B}$ sending f to 1; but the only 2-functor $\mathscr{A} \to \mathscr{B}$ sends f to 0, so this homomorphism is not pseudonaturally equivalent to a 2-functor. ∎

3. THEOREM. **Gray** *is not triequivalent to* **Bicat**.

PROOF. Suppose there were a triequivalence $\Phi : \mathbf{Gray} \to \mathbf{Bicat}$. We show that Φ would be biequivalent to the inclusion, so that the inclusion itself would be a triequivalence; but by the lemma this is impossible.

The terminal 2-category 1 is a terminal object in **Gray**, so must be sent to a "triterminal object" $\Phi 1$ in **Bicat**; in other words, $\mathbf{Bicat}(\mathscr{B}, \Phi 1)$ must be biequivalent to 1 for any bicategory $\mathscr{B}$. For any 2-category $\mathscr{A}$, we have biequivalences

$$\mathscr{A} \sim \mathbf{Gray}(1,\mathscr{A}) \sim \mathbf{Bicat}(\Phi 1, \Phi\mathscr{A}) \sim \mathbf{Bicat}(1, \Phi\mathscr{A}) \sim \Phi\mathscr{A}$$

where the first is the isomorphism coming from the monoidal structure on **Gray**, the second is the biequivalence on hom-bicategories given by Φ, the third is given by composition with the biequivalence $\Phi 1 \sim 1$, and the last is a special case of the biequivalence $\mathbf{Bicat}(1,\mathscr{B}) \sim \mathscr{B}$ for any bicategory, given by evaluation at the unique object $*$ of 1. All of these biequivalences are "natural" in a suitably weak tricategorical sense, and so Φ is indeed biequivalent to the inclusion. ∎

4. REMARK. The most suitable weak tricategorical transformation is called a tritransformation. The axioms are rather daunting, but really the coherence conditions are not needed here. We only need the obvious fact that for any 2-functor $T : \mathscr{A} \to \mathscr{B}$, the square

$$\begin{tikzcd} \mathscr{A} \arrow[r, "\sim"] \arrow[d, "T"'] & \Phi\mathscr{A} \arrow[d, "\Phi T"] \\ \mathscr{B} \arrow[r, "\sim"'] & \Phi\mathscr{B} \end{tikzcd}$$

commutes up to equivalence.

The fact that every bicategory is biequivalent to a 2-category is precisely the statement that the inclusion **Gray** $\to$ **Bicat** is triessentially surjective on objects, but as we saw in the lemma, it is not locally a biequivalence. On the other hand Gordon, Power, and Street construct in [2] a trihomomorphism **st** : **Bicat** $\to$ **Gray** which is locally a biequivalence (it induces a biequivalence on the hom-bicategories). They do this by appeal to their Section 3.6, but this does not imply that **st** is a triequivalence, as they claim, and by our theorem it cannot be one. In fact Section 5.6 is not used in the proof of the main theorem of [2], it is only used to construct the tricategory **Bicat** itself, and this does not need **st** to be a triequivalence.

By the coherence result of [2], **Bicat** is triequivalent to *some* Gray-category; and by the fact that **st** is locally a biequivalence, **Bicat** is triequivalent to a full sub-Gray-category of **Gray**, but it is not triequivalent to **Gray** itself.

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