

## BICAT IS NOT TRIEQUIVALENT TO GRAY

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ABSTRACT. **Bicat** is the tricategory of bicategories, homomorphisms, pseudonatural transformations, and modifications. **Gray** is the subtricategory of 2-categories, 2-functors, pseudonatural transformations, and modifications. We show that these two tricategories are not triequivalent.

1. BACKGROUND. Weakening the notion of 2-category by replacing all equations between 1-cells by suitably coherent isomorphisms gives the notion of *bicategory* [1]. The analogous weakening of a 2-functor is called a *homomorphism* of bicategories, and the weakening of a 2-natural transformation is a *pseudonatural transformation*. There are also *modifications* between 2-natural or pseudonatural transformations, but this notion does not need to be weakened. The bicategories, homomorphisms, pseudonatural transformations, and modifications form a tricategory (a weak 3-category) called **Bicat**.

The subtricategory of **Bicat** containing only the 2-categories as objects, and only the 2-functors as 1-cells, but with all 2-cells and 3-cells between them, is called **Gray**. As well as being a particular tricategory, there is another important point of view on **Gray**. The category **2-Cat** of 2-categories and 2-functors is cartesian closed, but it also has a different symmetric monoidal closed structure [3], for which the internal hom  $[\mathcal{A}, \mathcal{B}]$  is the 2-category of 2-functors, pseudonatural transformations, and modifications between  $\mathcal{A}$  and  $\mathcal{B}$ . A category enriched over **2-Cat** with respect to this closed structure is called a *Gray-category*. A Gray-category has 2-categories as hom-objects, so is a 3-dimensional categorical structure, and it can be seen as a particular sort of tricategory. The closed structure of **2-Cat** gives it a canonical enrichment over itself and the resulting Gray-category is just **Gray**. **Gray** is also sometimes used as a name for **2-Cat** with this monoidal structure.

A homomorphism of bicategories  $T : \mathcal{A} \rightarrow \mathcal{C}$  is called a *biequivalence* if it induces equivalences  $T_{A,B} : \mathcal{A}(A, B) \rightarrow \mathcal{B}(TA, TB)$  of hom-categories for all objects  $A, B \in \mathcal{C}$  ( $T$  is *locally an equivalence*), and every object  $C \in \mathcal{C}$  is equivalent in  $\mathcal{C}$  to one of the form  $TA$  ( $T$  is *biessentially surjective on objects*). We then write  $\mathcal{A} \sim \mathcal{B}$ . Every bicategory is equivalent to a 2-category [5].

A trihomomorphism of tricategories  $T : \mathcal{A} \rightarrow \mathcal{C}$  is called a *triequivalence* if it induces biequivalences  $T_{A,B} : \mathcal{A}(A, B) \rightarrow \mathcal{B}(TA, TB)$  of hom-bicategories for all objects  $A, B \in \mathcal{A}$  ( $T$  is *locally a biequivalence*), and every object  $C \in \mathcal{C}$  is biequivalent in  $\mathcal{C}$  to one

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of the form  $TA$  ( $T$  is *trinessentially surjective on objects*). It is not the case that every tricategory is triequivalent to a 3-category, but every tricategory is triequivalent to a Gray-category [2].

Perhaps since a **Gray**-category is a category enriched in the monoidal category **Gray**, and a tricategory can be seen as some sort of “weak **Bicat**-category”, it has been suggested that **Bicat** might be triequivalent to **Gray**, and indeed Section 5.6 of [2] states that this is the case. We prove that it is not. First we prove:

2. LEMMA. *The inclusion  $\mathbf{Gray} \rightarrow \mathbf{Bicat}$  is not a triequivalence.*

PROOF. If it were then each inclusion  $\mathbf{Gray}(\mathcal{A}, \mathcal{B}) \rightarrow \mathbf{Bicat}(\mathcal{A}, \mathcal{B})$  would be a biequivalence, and so each homomorphism (pseudofunctor) between 2-categories would be pseudonaturally equivalent to a 2-functor. This is not the case. For example (see [4, Example 3.1]), let  $\mathcal{A}$  be the 2-category with a single object  $*$ , a single non-identity morphism  $f : * \rightarrow *$  satisfying  $f^2 = 1$ , and no non-identity 2-cells (the group of order 2 seen as a one-object 2-category); and let  $\mathcal{B}$  be the 2-category with a single object  $*$ , a morphism  $n : * \rightarrow *$  for each integer  $n$ , composed via addition, and an isomorphism  $n \cong m$  if and only if  $n - m$  is even (the “pseudo-quotient of  $\mathbb{Z}$  by  $2\mathbb{Z}$ ”). There is a homomorphism  $\mathcal{A} \rightarrow \mathcal{B}$  sending  $f$  to 1; but the only 2-functor  $\mathcal{A} \rightarrow \mathcal{B}$  sends  $f$  to 0, so this homomorphism is not pseudonaturally equivalent to a 2-functor. ■

3. THEOREM. ***Gray** is not triequivalent to **Bicat**.*

PROOF. Suppose there were a triequivalence  $\Phi : \mathbf{Gray} \rightarrow \mathbf{Bicat}$ . We show that  $\Phi$  would be biequivalent to the inclusion, so that the inclusion itself would be a triequivalence; but by the lemma this is impossible.

The terminal 2-category  $1$  is a terminal object in **Gray**, so must be sent to a “triterminal object”  $\Phi 1$  in **Bicat**; in other words,  $\mathbf{Bicat}(\mathcal{B}, \Phi 1)$  must be biequivalent to  $1$  for any bicategory  $\mathcal{B}$ . For any 2-category  $\mathcal{A}$ , we have biequivalences

$$\mathcal{A} \sim \mathbf{Gray}(1, \mathcal{A}) \sim \mathbf{Bicat}(\Phi 1, \Phi \mathcal{A}) \sim \mathbf{Bicat}(1, \Phi \mathcal{A}) \sim \Phi \mathcal{A}$$

where the first is the isomorphism coming from the monoidal structure on **Gray**, the second is the biequivalence on hom-bicategories given by  $\Phi$ , the third is given by composition with the biequivalence  $\Phi 1 \sim 1$ , and the last is a special case of the biequivalence  $\mathbf{Bicat}(1, \mathcal{B}) \sim \mathcal{B}$  for any bicategory, given by evaluation at the unique object  $*$  of  $1$ . All of these biequivalences are “natural” in a suitably weak tricategorical sense, and so  $\Phi$  is indeed biequivalent to the inclusion. ■

4. REMARK. The most suitable weak tricategorical transformation is called a tritransformation. The axioms are rather daunting, but really the coherence conditions are not needed here. We only need the obvious fact that for any 2-functor  $T : \mathcal{A} \rightarrow \mathcal{B}$ , the square

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\sim} & \Phi \mathcal{A} \\ T \downarrow & & \downarrow \Phi T \\ \mathcal{B} & \xrightarrow{\sim} & \Phi \mathcal{B} \end{array}$$

commutes up to equivalence.

The fact that every bicategory is biequivalent to a 2-category is precisely the statement that the inclusion  $\mathbf{Gray} \rightarrow \mathbf{Bicat}$  is triessentially surjective on objects, but as we saw in the lemma, it is not locally a biequivalence. On the other hand Gordon, Power, and Street construct in [2] a trihomomorphism  $\mathbf{st} : \mathbf{Bicat} \rightarrow \mathbf{Gray}$  which is locally a biequivalence (it induces a biequivalence on the hom-bicategories). They do this by appeal to their Section 3.6, but this does not imply that  $\mathbf{st}$  is a triequivalence, as they claim, and by our theorem it cannot be one. In fact Section 5.6 is not used in the proof of the main theorem of [2], it is only used to construct the tricategory  $\mathbf{Bicat}$  itself, and this does not need  $\mathbf{st}$  to be a triequivalence.

By the coherence result of [2],  $\mathbf{Bicat}$  is triequivalent to *some* Gray-category; and by the fact that  $\mathbf{st}$  is locally a biequivalence,  $\mathbf{Bicat}$  is triequivalent to a full sub-Gray-category of  $\mathbf{Gray}$ , but it is not triequivalent to  $\mathbf{Gray}$  itself.

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