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ON REFLECTIVE-COREFLECTIVE EQUIVALENCE AND ASSOCIATED PAIRS

ERIK BÉDOS, S. KALISZEWSKI, AND JOHN QUIGG

Abstract. We show that a reflective/coreflective pair of full subcategories satisfies a "maximal-normal"-type equivalence if and only if it is an associated pair in the sense of Kelly and Lawvere.

1. Introduction

In a recent paper [1] we explored a special type of category equivalence between reflective/coreflective pairs of subcategories that we first encountered in the context of crossed-product duality for C^*-algebras. Because our main example of this phenomenon involved categories of maximal and normal C^*-coactions of locally compact groups, we called it a "maximal-normal"-type equivalence.

Since then, F.W. Lawvere has drawn our attention to [3], where G.M. Kelly and he introduced the concept of *associated pairs* of subcategories. The purpose of this short note is to show that these two notions of equivalence are the same: a reflective/coreflective pair of full subcategories satisfies the "maximal-normal"-type equivalence considered in [1] if and only if it is an associated pair in the sense of [3].

As operator algebraists, we had hoped with [1] to initiate a cross-fertilization between operator algebras and category theory, and we are grateful to Ross Street for the role he has played in helping this happen. Our understanding of the operator-algebraic examples has certainly been deepened by this connection; ideally, the techniques and examples of "maximal-normal"-type equivalence will in turn provide a way of looking at associated pairs that will also be useful to category theorists.

2. Maximal-normal equivalences and associated pairs

Our conventions regarding category theory follow [4]; see also [1]. Throughout this note, we let $\mathcal{M}$ and $\mathcal{N}$ denote full subcategories of a category $\mathcal{C}$, with $\mathcal{N}$ reflective and $\mathcal{M}$ coreflective. The inclusion functors $I : \mathcal{M} \to \mathcal{C}$ and $J : \mathcal{N} \to \mathcal{C}$ are then both full and faithful. We also use the following notation:

- $N : \mathcal{C} \to \mathcal{N}$ is a reflector and $\theta : 1_{\mathcal{C}} \to JN$ denotes the unit of the adjunction $N \dashv J$;

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- $M\colon \mathcal{C} \to \mathcal{M}$ is a coreflector and $\psi\colon IM \to 1_{\mathcal{C}}$ denotes the counit of the adjunction $I \dashv M$.

In [1, Corollary 4.4] we showed that the adjunction $NI \dashv MJ$ is an adjoint equivalence between $\mathcal{M}$ and $\mathcal{N}$ if and only if

(I) for each $y \in \operatorname{Obj}\mathcal{N}$, (y, ψ_y) is an initial object in the comma category $My \downarrow \mathcal{N}$; and

(F) for each $x \in \operatorname{Obj}\mathcal{M}$, (x, θ_x) is a final object in the comma category $\mathcal{M} \downarrow Nx$.

In all our examples in [1], the adjoint equivalence $NI \dashv MJ$ between $\mathcal{M}$ and $\mathcal{N}$ was what we called the "maximal-normal" type (recall that this terminology was motivated by the particular example of maximal and normal coactions on C^*-algebras; see [1, Corollary 6.16]): in addition to (I) and (F), such an adjunction satisfies

(A) for each $z \in \operatorname{Obj}\mathcal{C}$, $(Nz, \theta_z \circ \psi_z)$ is an initial object in $Mz \downarrow \mathcal{N}$.

Equivalently, by [1, Theorem 3.4], (I) and (F) hold, and

(B) for each $z \in \operatorname{Obj}\mathcal{C}$, $(Mz, \theta_z \circ \psi_z)$ is a final object in $\mathcal{M} \downarrow Nz$.

In fact, conditions (A) and (B) alone suffice:

2.1. PROPOSITION. *The adjunction $NI \dashv MJ$ between $\mathcal{M}$ and $\mathcal{N}$ is a "maximal-normal" adjoint equivalence if and only if (A) and (B) hold.*

PROOF. By [1, Theorem 4.3], (I) is equivalent to

(I′) for each $y \in \operatorname{Obj}\mathcal{N}$, $N\psi_y : NMy \to Ny$ is an isomorphism,

while (F) is equivalent to

(F′) for each $x \in \operatorname{Obj}\mathcal{M}$, $M\theta_x : Mx \to MNx$ is an isomorphism.

On the other hand, by [1, Theorem 3.4], (A) is equivalent to

(A′) for each $z \in \operatorname{Obj}\mathcal{C}$, $N\psi_z$ is an isomorphism,

while (B) is equivalent to

(B′) for each $z \in \operatorname{Obj}\mathcal{C}$, $M\theta_z$ is an isomorphism.

Now clearly, (A′) implies (I′) and (B′) implies (F′), so (A) implies (I) and (B) implies (F). ■

We now recall from [2, 3] that a morphism f in $\mathcal{C}(x,y)$ and an object z of $\mathcal{C}$ are said to be *orthogonal* when the map $\Phi_{f,z}$ from $\mathcal{C}(y,z)$ into $\mathcal{C}(x,z)$ given by $\Phi_{f,z}(g) = g \circ f$ is a bijection. The collection of all morphisms in $\mathcal{C}$ that are orthogonal to every object of $\mathcal{N}$ is denoted by $\mathcal{N}^{\perp}$.

As shown in [3, Proposition 2.1], a morphism $f : x \to y$ in $\mathcal{C}$ belongs to $\mathcal{N}^{\perp}$ if and only if f is inverted by N, that is, Nf is an isomorphism. (The standing assumption in [3] that $\mathcal{N}$ is replete is not necessary for this fact to be true. To see this, note that Nf is an isomorphism if and only if the map $\Psi_{f,z}$ from $\mathcal{N}(Ny,z)$ into $\mathcal{N}(Nx,z)$ given by $\Psi_{f,z}(h) = h \circ Nf$ is a bijection for each object z of $\mathcal{N}$. For each such z, the universal properties of θ imply that the map $\tau_{w,z}$ from $\mathcal{N}(Nw,z)$ into $\mathcal{C}(w,z)$ given by $\tau_{w,z}(g) = g \circ \theta_w$ is a bijection for each object w of $\mathcal{C}$. Now, as $\theta_y \circ f = Nf \circ \theta_x$, the diagram

$$\begin{tikzcd}
\mathcal{N}(Ny,z) \arrow[r, "\Psi_{f,z}"] \arrow[d, "\tau_{y,z}"'] & \mathcal{N}(Nx,z) \arrow[d, "\tau_{x,z}"] \\
\mathcal{C}(y,z) \arrow[r, "\Phi_{f,z}"'] & \mathcal{C}(x,z)
\end{tikzcd}$$

is readily seen to commute. It follows that $\Psi_{f,z}$ is a bijection if and only if $\Phi_{f,z}$ is a bijection. This shows that Nf is an isomorphism if and only if f is orthogonal to z for each object z of $\mathcal{N}$, *i.e.*, if and only if f belongs to $\mathcal{N}^{\perp}$.)

Similarly, a morphism f in $\mathcal{C}(x,y)$ and an object z in $\mathcal{C}$ are *co-orthogonal* when the map $g \to f \circ g$ from $\mathcal{C}(z,x)$ into $\mathcal{C}(z,y)$ is a bijection. The collection of all morphisms in $\mathcal{C}$ that are co-orthogonal to every object in $\mathcal{M}$ is denoted by $\mathcal{M}^{\top}$. Equivalently, a morphism $f : x \to y$ in $\mathcal{C}$ belongs to $\mathcal{M}^{\top}$ if and only if f is inverted by M, that is, if and only if Mf is an isomorphism.

The pair $(\mathcal{N},\mathcal{M})$ is called an *associated pair* if $\mathcal{N}^{\perp} = \mathcal{M}^{\top}$; equivalently, if for every morphism f in $\mathcal{C}$, N inverts f if and only if M does. We refer to [3, Section 2] for more information concerning this concept (in the case where both $\mathcal{M}$ and $\mathcal{N}$ are also assumed to be replete).

2.2. THEOREM. *The adjunction $NI \dashv MJ$ is a "maximal-normal" adjoint equivalence if and only if $(\mathcal{N},\mathcal{M})$ is an associated pair.*

PROOF. First assume that $(\mathcal{N},\mathcal{M})$ is an associated pair, and let x be an object in $\mathcal{C}$. As pointed out above, the map $\tau_{x,z}$ is a bijection from $\mathcal{N}(Nx,z)$ into $\mathcal{C}(x,z)$ for each object z of $\mathcal{N}$. But $\Phi_{\theta_x,z} = \tau_{x,z}$, so this means that θ_x lies in $\mathcal{N}^{\perp}$, and therefore in $\mathcal{M}^{\top}$. As $\mathcal{M}^{\top}$ consists of the morphisms in $\mathcal{C}$ that are inverted by M, we deduce that $M\theta_x$ is an isomorphism. This shows that (B′) holds, and therefore that (B) holds. The argument that (A) holds is similar, so $NI \dashv MJ$ is a "maximal-normal" adjoint equivalence by Proposition 2.1.

Now assume that the adjunction $NI \dashv MJ$ is a "maximal-normal" adjoint equivalence. Then $N \cong NIM$ by [1, Proposition 5.3], and NI is an equivalence. So for any morphism f

of $\mathcal{C}$, we have

$$\begin{aligned} Nf \text{ is an isomorphism} &\Leftrightarrow NIMf \text{ is an isomorphism} \\ &\Leftrightarrow Mf \text{ is an isomorphism.} \end{aligned}$$

Thus $(\mathcal{N}, \mathcal{M})$ is an associated pair. ∎

2.3. Remark. In the examples presented in [1, Section 6], the adjunctions $NI \dashv MJ$ are "maximal-normal" adjoint equivalences, so all the pairs $(\mathcal{N}, \mathcal{M})$ there are associated pairs. Moreover, all these pairs consist of subcategories that are easily seen to be replete. It follows from [3, Theorem 2.4] that $\mathcal{M}$ and $\mathcal{N}$ are uniquely determined as subcategories by each other, a fact that is not *a priori* obvious in any of the examples.

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Institute of Mathematics
University of Oslo
P.B. 1053 Blindern, 0316 Oslo, Norway

School of Mathematical and Statistical Sciences
Arizona State University
Tempe, AZ 85287

School of Mathematical and Statistical Sciences
Arizona State University
Tempe, AZ 85287
Email: `bedos@math.uio.no`
`kaliszewski@asu.edu`
`quigg@asu.edu`

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