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EXTENSIVE CATEGORIES AND THE SIZE OF AN ORBIT

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ABSTRACT. It is well known how to compute the number of orbits of a group action. A related problem, apparently not in the literature, is to determine the number of elements in an orbit. The theory that addresses this question leads to orbital extensive categories and to combinatorial aspects of such categories.

The vertices of a regular icosahedron will be colored black or white. How many icosahedra exist if opposite vertices have different color? This is easily found from the "Burnside-Frobenius lemma" (which is due exclusively to Frobenius [6]): there are 4. For each of these objects, we might also wish to know how many differently-appearing positions they appear in, that is, how many elements do these orbits have? The theory developed in this paper gives that 2 of the orbits have 5 elements and the other 2 have 10. If vertices can be black, white or red, there are 804 objects of which 18 have 5-element orbits, 18 have 10-element orbits and, quite surprisingly, the remaining 768 have 60-element orbits. This problem will be solved in the last section together with additional remarks that indicate how to generalize to solve similar counting problems.

The theory develops from a general look at atoms in extensive categories and their relation to conjugacy classes of subgroups of a certain group. This is the main focus of this paper. Atoms in an extensive category have been studied by others [5, 3].

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1. Combinatorial Categories

1.1. DEFINITION. *A **ranged extensive category** is a locally-small category which satisfies the following four axioms.*

(RE.1) *Finite colimits exist.*

(RE.2) *Any morphism pulls back finite coproducts to finite coproducts.*

(RE.3) *Every morphism factors as $i\,p$ with i a coproduct injection and p epic.*

(RE.4) *The pullback of an epic under a coproduct injection is again an epic.*

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For convenience, we assume that, like the category of sets, a ranged extensive category contains only one initial object.

Coproducts and (RE.2) define an **extensive category**. For basic facts about extensive categories see [2]. The astute reader will note that a good deal of the theory below does not require coequalizers, but these are assumed because the resulting categories lie closer to the applications we have in mind.

Evidently, any Boolean topos is a ranged extensive category.

$\mathrm{Summ}(X)$ denotes the subobjects of X represented by a coproduct injection; note that coproduct injections *are* monic by Lemma 1.6 below. It is well known that, in any extensive category, $\mathrm{Summ}(X)$ is a Boolean algebra. (*Boolean categories* are studied in [9]. An extensive category is the same thing as a Boolean category in which every morphism is total and deterministic [9, Corollary 12.3]). $\mathrm{Summ}(X)$ is a Boolean algebra in any Boolean category [9, Theorem 5.11]). Note that for $A_1, \ldots, A_n \in \mathrm{Summ}(X)$, $A_i \to X$ is a coproduct if and only if in $\mathrm{Summ}(X)$, $A_i \cap A_j = \emptyset$ whenever $i \neq j$ and $A_1 \cup \ldots \cup A_n = X$.

1.2. Definition. *A* ***combinatorial category*** *is a ranged extensive category $\mathcal{C}$ in which every hom-set $\mathcal{C}(X,Y)$ is finite.*

For the balance of the section, we work in a combinatorial category $\mathcal{C}$.

1.3. Lemma. *For each object X in $\mathcal{C}$,* $\mathrm{Summ}(X)$ *is a finite Boolean algebra.*

Proof. Suppose $\mathrm{Summ}(X)$ were infinite. Then as is well-known (see [8, Proposition 3.4]), there exists $A_1, A_2, \ldots$ in $\mathrm{Summ}(X)$ with $A_i \wedge A_j = 0$ whenever $i \neq j$ and each $A_i \neq 0$. For any n, $X = A_1 \vee A_2 \vee \cdots \vee A_n \vee (A_1 \vee \cdots \vee A_n)'$ (where $(\cdot)'$ denotes Boolean algebra complement). As the two coproduct injections $A_i \to A_i + A_i$ are distinct for each i, $\mathcal{C}(X, X+X)$ has at least 2^n elements. This is a contradiction, since n is arbitrary. ∎

It follows that every object is uniquely a finite coproduct of atoms, where an **atom** is an object with exactly two summands, that is, an object $A \neq 0$ such that $0 \longrightarrow A \xleftarrow{\mathrm{id}} A$ is the only coproduct decomposition.

1.4. Lemma. *If $0 \to B$ is epic, $B = 0$.*

Proof. If $B \neq 0$ there exist two distinct maps $B \to B + B$. ∎

1.5. Lemma. *If $f : X \to Y$ is epic and $0 \neq Q \in \mathrm{Summ}(Y)$ then the pullback $f^{-1}(Q) \in \mathrm{Summ}(X)$ is also not 0.*

Proof. Consider the pullback

$$\begin{tikzcd} f^{-1}(Q) \arrow[r, "g"] \arrow[d] & Q \arrow[d] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

By (RE.4), g is epic. Now use Lemma 1.4. ∎

1.6. Lemma. *Every coproduct injection is an equalizer.*

Proof. Given a coproduct $P \xrightarrow{i} X \xleftarrow{i'} P'$ define F as shown.

$$
\begin{tikzcd}
P \arrow[r, "i"] \arrow[d, "i"'] & X \arrow[d, "F"] & P' \arrow[l, "i'"'] \arrow[d, "i"] \\
X \arrow[r, "\mathrm{in}_1"'] & X + X & X \arrow[l, "\mathrm{in}_2"]
\end{tikzcd}
$$

As both rows are coproducts, both squares must be pullbacks ([2, Proposition 2.2]). Thus $i = \mathrm{eq}(F, \mathrm{in}_1)$. ∎

1.7. Lemma. *Given a commutative square*

$$
\begin{tikzcd}
A \arrow[r, "p"] \arrow[d, "f"'] & B \arrow[d, "g"] \\
C \arrow[r, "i"'] & D
\end{tikzcd}
$$

with p epic and i a coproduct injection, there exists a diagonal fill-in $t : B \to C$ with $it = g$, $tp = f$.

Proof. By Lemma 1.6, write $i = \mathrm{eq}(\alpha, \beta)$. Then $\alpha gp = \alpha if = \beta if = \beta gp$ so $\alpha g = \beta g$ since p is epic. The desired t is induced by the equalizer property. ∎

1.8. Proposition. *If $f : A \to B$ is epic with A an atom, then B is also an atom.*

Proof. Let $0 \neq Q \in \mathrm{Summ}(B)$ and show $Q = B$. As A is an atom, it follows from Lemma 1.5 that $f^{-1}(Q) = A$, so we have a pullback

$$
\begin{tikzcd}
A \arrow[r, "\mathrm{id}"] \arrow[d] & A \arrow[d, "f"] \\
Q \arrow[r] & B
\end{tikzcd}
$$

As f is epic, $Q \to B$ is epic. By Lemma 1.6, we are done. ∎

1.9. Definition. *Let $\mathcal{A}$ be a choice of one atom from each isomorphism class of atoms in $\mathcal{C}$. Define $At(\mathcal{C})$ to be the full subcategory $\mathcal{A}$.*

The structure of $\mathcal{C}$ is determined by $\mathrm{At}(\mathcal{C})$ as follows. Consider a morphism $f : A \to B_1 + \cdots + B_m$ with A an atom in $\mathrm{At}(\mathcal{C})$. As the image of f can be taken as an atom in $\mathrm{At}(\mathcal{C})$ by Proposition 1.8, there exists a unique j such that f factors through B_j. Let $\{A_1, \ldots A_m\}$ be the set of atoms of $\mathrm{Summ}(X)$, and let $\{B_1, \ldots, B_n\}$ be the atoms of $\mathrm{Summ}(Y)$, both taken in $\mathrm{At}(\mathcal{C})$. Thus $X = A_1 + \cdots + A_m$, $Y = B_1 + \cdots + B_n$. To determine a morphism $f : X \to Y$, let $\alpha : \{1, \ldots, m\} \to \{1, \ldots, n\}$ be the function such that f maps A_i into $B_{\alpha i}$. Then f is determined by the function α and the resulting maps $A_i \to B_{\alpha i}$ in $\mathrm{At}(\mathcal{C})$. By the same reasoning, composition of morphisms is determined by the composition in $\mathrm{At}(\mathcal{C})$.

We continue our study of $\mathcal{C}$.

1.10. Lemma. *Every object of $\mathcal{C}$ is Dedekind-finite, that is, every monic endomorphism is an isomorphism.*

Proof. For $f : X \to X$, as $\mathcal{C}(X, X)$ is a finite semigroup, $n > 1$ exists with f^n idempotent. If f is monic then f^n is a monic idempotent, hence the identity. This shows that $f^{-1} = f^{n-1}$. ∎

1.11. Proposition. *Every endomorphism of an atom is an isomorphism.*

Proof. Every endomorphism of an atom is epic by (RE.3) and Proposition 1.8. Now use the dual of the proof of the previous lemma. ∎

1.12. Corollary. *If A, B are non-isomorphic atoms and there exists $A \to B$, then no $B \to A$ exists.* ∎

1.13. Definition. *Let $\mathcal{C}_{iso}$ denote the category of isomorphisms of $\mathcal{C}$. Let* ***Fin*** *denote the category of finite sets and functions. In the spirit of the species invented by Joyal [7, 1], define a* ***species on*** *$\mathcal{C}$ to be a functor $\mathcal{C}_{iso} \to$* ***Fin****. A species gives us something to count.*

1.14. Definition. *A species F on $\mathcal{C}^{op} \times \mathcal{C}$ is* ***binomial*** *if there are natural isomorphisms*

$$F(X,\ Y + Z) \quad \cong \quad \sum_{P \in Summ(X)} F(P, Y) \times F(P', Z)$$

1.15. Example. The hom-set species $(X, Y) \mapsto \mathcal{C}(X, Y)$ is binomial. For consider the pullbacks

$$\begin{tikzcd} P \arrow[r, "i"] \arrow[d, "g"'] & X \arrow[d, "f"] & P' \arrow[l, "i'"'] \arrow[d, "h"] \\ X \arrow[r, "\mathrm{in}_1"'] & Y + Z & Z \arrow[l, "\mathrm{in}_2"] \end{tikzcd}$$

As the category is extensive, for each f there exists unique P giving rise to g, h by pullback. Conversely, given P, g, h, unique f is induced by the coproduct and the squares must be pullbacks.

1.16. Example. The species $X, Y \mapsto$ coproduct injections $X \to Y$ is binomial. The argument is that of the previous example, noting that f is a coproduct injection if and only if g, h are. A formal proof appears in [9, Lemma 5.2, Observation 5.7], noting that extensive categories are a special case of the "Boolean categories" studied there.

1.17. Example. The species $X, Y \mapsto$ epics $X \to Y$ is binomial because a coproduct of epics is epic in any category.

1.18. Example. **Fin** is a combinatorial category.

Let $|X|$ denote the cardinality of X. Treating a natural number n as the finite set $\{1, \ldots, n\}$, a species F is binomial if and only if $p_n(y) = |F(n, y)|$ satisfies $p_n(y+z) = \sum p_k(y)\, p_{n-k}(z)$, so this will be a sequence of binomial type providing $p_n(y)$ is a degree-n polynomial. Examples 1.15, 1.16 yield that y^n and the falling factorials $y(y-1)\cdots(y-n+1)$ are polynomial sequences of binomial type. The first case proves the binomial theorem. The second case is well-known. The only atom of **Fin** is the terminal object 1. Thus At(**Fin**) is the 1-morphism category. The description of **Fin** in terms of its atom category then gives the morphisms as functions $\alpha : m \to n$ which, of course, indeed are the morphisms of this category.

The next example plays an important role in this paper.

1.19. Example. For G a group, the Boolean topos of finite right G-sets is a combinatorial category. We denote this category as $\mathbf{Fin}^G$.

Axiom (RE.3) is the usual image factorization in a topos, since monics are coproduct injections in a Boolean topos. The atoms are the 1-orbit G-sets, corresponding to the finite quotients of G (considered as a G-set where the right action gh is defined as the multiplication in the group). The resulting category of atoms will be characterized later in this paper.

1.20. Example. Any slice category $\mathcal{C}/X$ is again a combinatorial category. Any finite product of combinatorial categories is combinatorial.

We note that for G, H non-trivial groups, $\mathbf{Fin}^G \times \mathbf{Fin}^H$ is never of form $\mathbf{Fin}^K$ since no nonzero object admits a map to both atoms $(G, 0)$ and $(0, K)$.

1.21. Definition. *Denote by $\|X\|$ the number of atoms in* Summ(X).

A number of properties are immediate from the fact that every finite Boolean algebra is isomorphic to 2^n. We immediately have

- $\|X + Y\| = \|X\| + \|Y\|$.
- How many summands does X have? $|\mathrm{Summ}(X)| = 2^{\|X\|}$.
- If $\|X\| = n$, how many summands have k atoms? $\binom{n}{k}$.
- Up to isomorphism, how many objects have k atoms if there are N different atoms? $\binom{N+k-1}{N-1}$.
- The number of coproduct decompositions of X is the Bell number $B_{\|X\|}$.
- If $f : X \to Y$ is epic, Lemma 1.5 gives $\|Y\| \le \|X\|$.

In **Fin**, let $\operatorname{surj}(m, y)$ be the number of epics $X \to Y$ where X has m elements and Y has y elements. Since $\operatorname{surj}(m, y) = \emptyset$ if $m < y$ or $m > 0$ with $y = 0$, and $\operatorname{surj}(m, 1) = 1$, the binomial property gives

$$\operatorname{surj}(m, y+1) \;=\; \sum_{k=y}^{m-1} \binom{m}{k} \operatorname{surj}(k, y)$$

2. Orbit Objects

2.1. Definition. *In any category, an object G is a* ***generator*** *if whenever $f, g : X \to Y$ with $f \neq g$ there exists $x : G \to X$ with $fx \neq gx$. An* ***orbit object*** *is an object G which is both an atom and a generator and which satisfies the property that given $a, b : G \to A$ with A an atom, there exists $g : G \to G$ with $ag = b$. An* ***orbital extensive category*** *is a combinatorial category with an orbit object.*

In this section, we work in an orbital extensive category $\mathcal{C}$ with orbit object G. It will shortly be seen that such G is unique.

2.2. Example. The terminal object 1 is an orbit object for **Fin**.

2.3. Example. A finite group G (qua G-set with multiplication as action) is the orbit object for $\mathbf{Fin}^G$. For let $x, y : G \to A$ where $aG = A$ for every $a \in A$. Let $e \in G$ be the unit. There exists $g \in G$ with $x(e) = y(e)\, g$. Let $\lambda_g : G \to G$ be the equivariant left translation $\lambda_g(h) = gh$. Then $(y\, \lambda_g)(h) = y(gh) = y(eg)\, h = (y(e))\, gh = ((y(e)\, g)\, h = x(e)\, h = x(h)$ shows that $y\lambda_g = x$ as desired.

$\mathbf{Fin}^G$ constitutes a variety of universal algebras with unary operations indexed by G. For G as a G-set, the operations are the right translations whereas, as above, the endomorphisms are the left translations.

2.4. Lemma. *If $X \neq 0$ there exists $G \to X$.*

Proof. The injections $X \to X + X$ are distinct. ∎

2.5. Proposition. *All orbit objects are isomorphic.*

Proof. Immediate from Corollary 1.12 and Lemma 2.4. ∎

Since the category $\mathbf{Fin}^G$ determines the group G, there is no possibility of a "Morita theorem". That is, if $\mathbf{Fin}^G$, $\mathbf{Fin}^H$ are equivalent categories, $G \cong H$ as groups.

In $\mathbf{Fin}^G$, G is free (on one generator). In general, we have

2.6. Proposition. *G is projective.*

Proof. Let $f : X \to Y$ be epic and let $y : G \to Y$. Factor $y = G \xrightarrow{p} A \xrightarrow{i} Y$ as in (RE.3) so that A is an atom by Proposition 1.8 and i is a summand. By Lemma 1.5, for the pullback P shown in the following diagram,

$$\begin{tikzcd} G \arrow[r, "x"] & P \arrow[r, "j"] \arrow[d, "h"] & X \arrow[d, "f"] \\ G \arrow[u, "g"] \arrow[r, "p"'] & A \arrow[r, "i"'] & Y \end{tikzcd}$$

$P \neq 0$. By Lemma 2.4 there exists $x : G \to P$. As G is an orbit object, there exists g with $p = hxg$. Thus $y = f(jxg)$ as desired. ∎

2.7. Proposition. *The atoms are precisely the (epic) quotients of G.*

Proof. Each quotient of G is an atom by Proposition 1.8. Conversely, if A is an atom then $A \neq 0$ so there exists $G \to A$ and every such morphism is epic. ∎

For each atom A we choose a morphism $\theta_A : G \to A$. All such choices are isomorphic by the definition of orbit object, noting that all endomorphisms of G are isomorphisms.

2.8. Proposition. *For any atom A, the slice category $\mathcal{C}/A$ is a combinatorial category with orbit object (G, θ_A).*

Proof. Let $f, g : (X, a) \to (Y, b)$ with $f \neq g$. Let $t : G \to X$ with $ft \neq gt$ and let $h : G \to G$ with $ath = \theta_A$. Then $th : (G, \theta_A) \to (X, a)$ with $fth \neq gth$, and this shows that (G, θ_A) is a generator. Now let $x, y : (G, \theta_A) \to (B, a)$ with (B, a) an atom of $\mathcal{C}/A$. Then B is an atom of $\mathcal{C}$ so there exists $g : G \to G$ with $G \xrightarrow{g} G \xrightarrow{x} G = G \xrightarrow{y} G$. We have $\theta_A\, g = axg = ay = \theta_A$, so g is a morphism $(G, \theta_A) \to (G, \theta_A)$. ∎

2.9. Definition. *An* ***element*** *of an object X is a morphism $x : G \to X$. We write $x \in X$.*

We denote by $|X|$ the number of elements of X.

Each atom A in $\mathrm{Summ}(X)$ induces the element $G \xrightarrow{\theta_A} A \longrightarrow X$ of X. As distinct atoms are disjoint, different atoms induce different elements. This shows that $\|X\| \leq |X|$. If A is an atom, $\|A\| = 1$ but since A may have many automorphisms $a, b, \ldots$ (as will be seen later) and since then $\theta_A, a\,\theta_A, b\,\theta_A, \ldots$ are all distinct, A may have many elements.

2.10. Proposition. *If A, B are atoms, $|\mathcal{C}(A, B)| \leq |G|$.*

Proof. Let $f : A \to B$. As G is an orbit object, there exists

$$\begin{tikzcd} G \arrow[r, "g_f"] \arrow[d, "\theta_A"'] & G \arrow[d, "\theta_B"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

g_f as shown making the square commute. If $f_1 : A \to B$ with $f \neq f_1$, $f\theta_A \neq f_1\theta_A$ (as θ_A is epic) so $\theta_B\, g_f \neq \theta_B\, g_{f_1}$ and $g_f \neq g_{f_1}$ in particular. ∎

We do not know if $|\mathcal{C}(A,B)|$ is always a divisor of $|G|$, but that will be established below in a number of special cases.

It will be seen below that every orbital extensive category has finite limits. Anticipating that result, we consider some results concerning products and the terminal object.

We clearly have the "rules of sum and product":

- $|X + Y| = |X| + |Y|$
- $|X \times Y| = |X|\,|Y|$

2.11. Lemma. *The terminal object* 1 *is an atom. Moreover,* $\|1\| = 1 = |1|$.

Proof. Each atom of 1 induces an image factorization of $G \to 1$ and there is only one such factorization. The second statement is then obvious. ∎

3. The Category of Atoms

In this section, we shall characterize orbital extensive categories. Fix an orbital extensive category $\mathcal{C}$.

3.1. Definition. *Let* G *be a finite group with unit* e. *The* ***spread*** *of* G *is the finite category* G^σ *whose objects are the subgroups of* G *and in which a morphism* $H \to K$ *is a right coset* Ka *such that* $H \subset a^{-1}Ka$. *Composition is group multiplication,* $H \xrightarrow{Ka} K \xrightarrow{Lb} L = Lba : H \to L$ *and* $K \xrightarrow{K} K$ *serves as the identity.*

Of course, it must be checked that these constructions are well defined. To begin, if $H \subset a^{-1}Ka$ and $Ka = Kb$, there exists $k \in K$ with $b = ka$. Then $b^{-1}Kb = (ka)^{-1}Kka = a^{-1}k^{-1}Kka = a^{-1}Ka \supset H$, so the definition of the morphisms makes sense. Next consider $H \xrightarrow{Ka} K \xrightarrow{Lb} L$. We have $H \subset a^{-1}Ka \subset a^{-1}b^{-1}Lba = (ba)^{-1}Lba$, so indeed $H \xrightarrow{Lba} L$ is a morphism. It must still be shown that $Lb_1a_1 = Lba$ if $Ka = Ka_1$, $Lb = Lb_1$; to that end, $(b_1a_1)(ba)^{-1} = b_1a_1a^{-1}b^{-1} \subset b_1Kb^{-1} \subset L$. Since $Ke = K$ and $K \subset e^{-1}Ke$, it is clear that the $K \xrightarrow{K} K$ is the identity for composition.

3.2. Lemma. G *is the unique terminal object of* G^σ. $\{e\}$ *is the unique object of* G^σ *with* $|G|$ *endomorphisms.* ∎

It follows from this lemma that G can be reconstructed from G^σ as the endomorphism monoid of the object with the largest number of endomorphisms.

The proof of the next result is routine.

3.3. PROPOSITION. *For H, K subgroups of G, the following statements are equivalent.*

1. *There exist morphisms $H \to K$, $K \to H$ in G^σ.*
2. *H, K are conjugate subgroups.*
3. *$H \cong K$ in G^σ.* ■

We follow the usual mathematical practice of using the same notation for a group and for its set of elements. In this sense, an orbit object G has elements all $G \to G$ which comprise a group which we also call G.

We can now state the main result of this section. It entails that the number of atoms of an orbital extensive category, up to isomorphism, is finite and at most the number of conjugacy classes of subgroups of G. It also gives that the number of different orbital extensive categories with orbit object isomorphic to a given finite group is finite.

3.4. THEOREM. *Let $\mathcal{C}$ be an orbital extensive category with orbit object G. Then the full subcategory $At(\mathcal{C})$ is isomorphic to a full reflective subcategory of G^σ.*

PROOF. The elements of G act (on the right) on the elements of an object X by

$$xg \;=\; G \xrightarrow{g} G \xrightarrow{x} X$$

In particular, each element $x \in X$ induces its stability subgroup $\{g \in G : xg = x\}$, an object of G^σ. Now recall that for each atom A we have chosen a particular element $\theta_A \in A$. Define a functor $\Psi : \mathrm{At}(\mathcal{C}) \to G^\sigma$ by $\Psi(A) = H_A$, where H_A is the stability subgroup

$$H_A \;=\; \{g \in G : G \xrightarrow{g} G \xrightarrow{\theta_A} A \;=\; G \xrightarrow{\theta_A} A\}$$

Define Ψ on maps as follows. Given $f : A \to B$ there exists $a \in G$ such that the following square commutes:

$$\begin{tikzcd} G \arrow[r, "a"] \arrow[d, "\theta_A"'] & G \arrow[d, "\theta_B"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

This is because G is an orbit object. Define $\Psi(f) = H_A \xrightarrow{H_B a} H_B$. To check that this is well-defined, let $g \in H_A$. Then

$$\theta_B\, aga^{-1} = f\theta_A\, ga^{-1} = f\theta_A\, a^{-1} = \theta_B\, aa^{-1} = \theta_B$$

which shows that $H_A \subset a^{-1}H_B a$. To see that the map depends only on f and not on the choice of a, suppose also $\theta_B\, b = f\theta_A$. Then $\theta_B\, ba^{-1} = f\theta_A\, a^{-1} = \theta_B\, aa^{-1} = \theta_B$, and $ba^{-1} \in H_B$ so that $H_B\, a = H_B\, b$. That Ψ is functorial is immediate. Further, the fact that each θ_A is epic guarantees that Ψ is faithful.

We next construct a left adjoint to Ψ. Let $H \in G^\sigma$ be any subgroup. As H is a finite set of endomorphisms of G, it has a coequalizer $p : G \to A$ in $\mathcal{C}$ and A is an atom by Proposition 1.8. By orbit object, there exists $a \in G$ with $\theta_A a = p$. For $h \in H$, $\theta_A\, ah = \theta_A p = p = \theta_A a$, so $\theta_A\, aha^{-1} = \theta_A\, aa^{-1} = \theta_A$ and $aha^{-1} \in H_A$. To construct the left adjoint on H it suffices to construct it on the isomorphic object aHa^{-1} so we assume without loss of generality, that $H \subset H_A$. Now suppose that B is an atom and that $H_B\, b : H \to H_B$ is a morphism in G^σ. Consider the diagram in $\mathcal{C}$ in which H means a set of maps.

$$\begin{tikzcd} G \arrow[r, "H"] & G \arrow[r, "b"] \arrow[d, "\theta_A"'] & B \arrow[d, "\theta_B"] \\ & A \arrow[r, "f"'] & B \end{tikzcd}$$

Here, f is to be constructed. For $h \in H$ there exists $x \in H_B$ with $h = b^{-1}xb$. Then $\theta_B\, bh = \theta_B\, bb^{-1}xb = \theta_B\, xb = \theta_B\, b$. As $\theta_A = a^{-1}p$, θ_A is also a coequalizer of H so f is induced as above. It follows that

$$\begin{tikzcd} H \arrow[r, "H_A"] \arrow[dr, "H_B\, b"'] & H_A \arrow[d, "H_B\, b"] \\ & H_B \end{tikzcd}$$

establishes the adjointness. Applying the same argument to $H = H_C$ for any atom C shows that $\theta_C = \mathrm{coeq}(H_C)$ and that Ψ is full. In particular, if $H_A = H_B$ then there exist $A \to B \to A$ so that $A \cong B$. The proof is complete. ∎

For H any subgroup of the finite group G, the right coset space $G\backslash H$ is a G-set with action $(Ha)g = H(ag)$ and $He = H$ has stability subgroup $\{g \in G : Hg = H\} = H$. Moreover, $G\backslash H$ has one orbit, so is an atom of $\mathbf{Fin}^G$. The previous theorem then gives that $\mathrm{At}(\mathbf{Fin}^G) \cong G^\sigma$.

We continue to work in an orbital extensive category $\mathcal{C}$ with orbit object G.

Observe that $\mathcal{C}$ embeds $\mathbf{Fin}^G$ in a natural way via $\Phi(X) = |X|$ with right action xg the composition $G \xrightarrow{g} G \xrightarrow{x} X$. We have Φ is a functor over $\mathbf{Fin}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\Phi"] \arrow[dr, "|\cdot|"'] & \mathbf{Fin}^G \arrow[d] \\ & \mathbf{Fin} \end{tikzcd}$$

since if $f : X \to Y$ in $\mathcal{C}$, $(fx)g = f(xg)$ by the associativity of composition in $\mathcal{C}$, that is, f is equivariant.

By the definition of a category, $X \neq Y \Rightarrow |X| \cap |Y| = \emptyset$. We assumed at the beginning that a ranged extensive category has only one initial object, and this is the only object X with $|X| = 0$. It follows that Φ is injective on objects.

By the rule of sum above, $|\cdot|$ preserves finite coproducts, so Φ also does.

$\Phi(G)$ is G acting on itself by multiplication.

$\Phi(f)$ is surjective if f is epic since G is projective by Proposition 2.6. Thus Φ maps quotients of G to quotients of G and so maps atoms to atoms.

We are ready for

3.5. Theorem. $\Phi : \mathcal{C} \to \mathbf{Fin}^G$ *is a full surjection-reflective subcategory closed under finite coproducts and* G*-invariant subsets.*

Proof. For A an atom of $\mathcal{C}$ let $\theta_A : G \to A$ in $\mathcal{C}$ so that $\theta_A : \Phi(G) \to \Phi(A)$ in $\mathbf{Fin}^G$. $\{g : g\theta_A = \theta_A\}$ is the same stability subgroup in both categories. This shows that Φ maps $\mathrm{At}(\mathcal{C})$ to a full subcategory of $\mathrm{At}(\mathbf{Fin}^G) \cong G^\sigma$, by Theorem 3.4. From the process by which a combinatorial category is determined by its atom category, it follows that $\mathcal{C}$ is a full subcategory of $\mathbf{Fin}^G$.

In a finite Boolean algebra, there is only one way to write the greatest element as a supremum of atoms, namely as the supremum of all the atoms. It follows at once that $\mathrm{Summ}(X)$, $\mathrm{Summ}(\Phi(X))$ have the same atoms, and so the same summands since a summand is but an arbitrary supremum of atoms. In particular, $\mathcal{C}$ is closed under summands.

Since any coproduct of G-sets each one of which has a surjective reflection in $\mathcal{C}$ itself has a surjective reflection in $\mathcal{C}$, the proof will be complete if we show that each atom B of $\mathbf{Fin}^G$ has a surjective reflection in $\mathcal{C}$. By Theorem 3.4, B has a reflection $p : B \to B^\diamond$ in $\mathrm{At}(\mathcal{C})$. Such p is surjective, being a morphism between atoms. We will show that this is the desired reflection. Let $f : B \to \Phi(X)$ in $\mathbf{Fin}^G$. By (RE.3) for $\mathbf{Fin}^G$, factor $f = B \xrightarrow{q} A \xrightarrow{i} \Phi(X)$ with A an atom and i a summand. As we have already shown, i is a summand of $\mathcal{C}$ which is a quotient of G so A is an atom of $\mathcal{C}$. Thus there exists $\psi : B^\diamond \to A$ with $\psi i = q$ so $(i\psi)p = f$. As p is surjective, such $i\psi$ is unique. ∎

3.6. Corollary. $\mathcal{C}$ *has finite limits, and every monic in* $\mathcal{C}$ *is a summand. It follows that* $\mathcal{C}$ *is lextensive, and so distributive by [2, Proposition 4.5].*

Proof. $\mathbf{Fin}^G$ is a Boolean topos, so has finite limits and all its monics are summands. Any full reflective subcategory is closed under limits. Any monic in a full reflective category is monic in the ambient category. Since $\mathcal{C}$ is closed under summands, we are done. ∎

4. Some Counting Results for Orbital Extensive Categories

We continue to work in an orbital extensive category $\mathcal{C}$ with orbit object G.

4.1. Proposition. *If A is an atom, $|A|$ divides $|G|$. If there exists $A \to B$ with A, B atoms then $|A|$ divides $|B|$.*

Proof. By Theorem 3.4 we may work in $G^\sigma = \mathrm{At}(\mathbf{Fin}^G)$. $|A| = [G : H_A]$ and the index of a subgroup is always a divisor of G. If $H_A \to H_B$ exists in G^σ, a conjugate L of H_A is a subgroup of H_B so that

$$|H_A|\,|A| = |H|\,[G : H_A] = |G| = |H_B|\,[G : H_B] = |L|\,[H_B : L]\,|B| = |H_A|\,[H_B : L]\,|B|$$

Now divide by $|H_A|$. ∎

Recall that the **normalizer** N_H of a subgroup H of G is the subset $\{g \in G : g^{-1}Hg = H\}$. This is a subgroup, being the stabilizer subgroup of H when G acts on its set of subgroups by conjugation. Evidently, H is a normal subgroup of N_H so that N_H/H is a group.

4.2. Proposition. *The endomorphism monoid (= automorphism group) Aut(A) of an atom A is isomorphic to the group N_H/H if $H = H_A$.*

Proof. If $Ha : H \to H$ then $H \subset a^{-1}Ha$ so that $H = a^{-1}Ha$ by finiteness. Thus the endomorphisms of H are precisely all Ha with $a \in N_H$. Since two a are equivalent if and only if they are equal modulo H, we are done. $|\mathrm{Aut}(A)| = [N_H : H]$ divides $|N_H|$, so divides $|G|$. ∎

4.3. Definition. *An atom A is* ***transitive*** *if whenever $x, y \in A$ there exists $\psi : A \to A$ with $\psi x = y$.*

Evidently, A is transitive if and only if for all $y \in A$ there exists $\psi : A \to A$ with

$$G \xrightarrow{\theta_A} A \xrightarrow{\psi} A \;=\; y$$

4.4. Proposition. *Let B be a transitive atom and let A be an atom such that a morphism $f : A \to B$ exists. Then $|\mathcal{C}(A, B)| = |Aut(B)|$, so $|\mathcal{C}(A, B)|$ divides $|G|$ in particular.*

Proof. The function $\mathrm{Aut}(B) \to \mathcal{C}(A, B)$ sending an automorphism $\psi : B \to B$ to the function $\psi f : A \to B$ is injective because f is epic and is surjective because B is transitive. ∎

Even if B is not transitive, the argument above shows that the orbits of $\mathcal{C}(A, B)$ under the action of $\mathrm{Aut}(B)$ all have cardinality $|\mathrm{Aut}(B)|$. Thus in general, $|\mathcal{C}(A, B)|$ is a multiple of $|\mathrm{Aut}(B)|$.

4.5. Proposition. *An atom A is transitive if and only if its stability subgroup H is a normal subgroup of G.*

Proof. The definition of transitive in the triangle on the left corresponds to the commutative triangle in G^σ on the right.

$$\begin{tikzcd} & G \arrow[dl, "x"'] \arrow[dr, "y"] & \\ A \arrow[rr, "\psi"'] & & A \end{tikzcd} \qquad \begin{tikzcd} & \{e\} \arrow[dl, "Ht"'] \arrow[dr, "Hu"] & \\ H \arrow[rr, "Hv"'] & & H \end{tikzcd}$$

Here, t, u are arbitrary elements of G since the elements of $A = G\backslash H$ are arbitrary cosets, whereas we must have $H \subset v^{-1}Hv$, that is, $v \in N_H$. The transitivity of A then amounts to

$$\forall\, t, u \in G\; \exists\, v \in N_H \text{ with } vtu^{-1} \in H$$

If this condition holds then $tu^{-1} = v^{-1}(vtu^{-1}) \in N_H H = N_H$ with t, u arbitrary, so $N_H = G$ and G is normal. Conversely, if $N_H = G$ then, given t, u, set $v = t^{-1}u$ since then $v \in G = N_H$ and $vtu^{-1} = e \in H$. ∎

4.6. THEOREM. *Let $X \neq 0$ be an object. Then every atom of $G \times X$ is isomorphic to G. Indeed, $G \xrightarrow{[\mathrm{id}_G, x]} G \times X$ $(x \in X)$ is the copower $|X| \cdot G$, so that $|G \times X| = \|X\|$.*

PROOF. Write in_x for $[\mathrm{id}_G, x]$. Each in_x is monic, hence a coproduct injection. For $x, y : G \to X$, suppose we have a pullback

$$\begin{tikzcd} G \arrow[r, "\mathrm{id}"] \arrow[d, "k"'] & G \arrow[d, "\mathrm{in}_y"] \\ G \arrow[r, "\mathrm{in}_x"'] & G \times X \end{tikzcd}$$

Then $k = \mathrm{id}$ and so $y = xk = x$. As G is an atom, if $x \neq y$, the pullback must be 0, showing that the in_x are pairwise disjoint. Define P to be the supremum of all in_x in $\mathrm{Summ}(X)$. We must show $P' = 0$. If not, there exists $t : G \to P'$. Write $[p, q] = G \xrightarrow{t} P' \to A \times X$. We have the commutative square

$$\begin{tikzcd} G \arrow[r, "p"] \arrow[d, "t"'] & G \arrow[d, "\mathrm{in}_{qp^{-1}}"] \\ P' \arrow[r] & G \times X \end{tikzcd}$$

By Lemma 1.7, the atom $\mathrm{in}_{qp^{-1}}$ is in $P \wedge P'$, the desired contradiction. ∎

This theorem has surprising consequences. Consider $\mathbf{Fin}^G$, a variety of universal algebras with G the free algebra on one generator. Let X, Y be arbitrary finite G-sets with the same cardinality. The operations on a product algebra are coordinatewise, so it is quite amazing that $X \times G \cong Y \times G$. We invite the reader to explore how the "lost information" $X \times G \to X$ is encoded in the isomorphism $X \times G \cong |X| \cdot G$.

4.7. THEOREM. *If the exponential Y^X exists in $\mathcal{C}$, $|Y^X| = |Y|^{|X|}$.*

PROOF. It is clear that $|0| = 0$ and $|1| = 1$ so the identity holds if $X = 0$. In general, we apply Lemma 4.6. $|Y^X| = |\mathcal{C}(G, Y^X)| = |\mathcal{C}(G \times X, Y)| = |\mathcal{C}(G, Y)|^{|X|} = |Y|^{|X|}$. ∎

5. Examples

In this section we construct nontrivial examples of orbital extensive categories and show that such categories need not be cartesian closed.

Throughout this section, $\mathcal{C}$ is an orbital extensive category with orbit object G. We are at liberty to think of $\mathcal{C}$ as a full reflective subcategory of $\mathbf{Fin}^G$ closed under coproducts and summands because of Theorem 3.5. We begin with a converse to that theorem.

5.1. Lemma. *Let $\mathcal{D}$ be an arbitrary full reflective subcategory of $\mathcal{C}$ which contains G and is closed under coproducts and summands. Then $\mathcal{D}$ is an orbital extensive category.*

Proof. $\mathcal{D}$ is closed under limits and has finite colimits by reflecting those of $\mathcal{C}$, so (RE.1), (RE.2) hold. Since $\mathcal{D}$ is closed under coproducts, it follows from the proof of Lemma 1.6 that all summands in $\mathcal{D}$ are equalizers in $\mathcal{D}$. Hence if $p : X \to Y$ is epic in $\mathcal{D}$, factoring p in $\mathcal{C}$ as in (RE.3) renders i an epic equalizer in $\mathcal{D}$, hence an isomorphism. Thus epics in $\mathcal{D}$ are epic in $\mathcal{C}$, so (RE.3), (RE.4) are clear. Hom-sets in $\mathcal{D}$ are finite since that holds for $\mathcal{C}$. Since atoms are quotients of G, atoms of $\mathcal{D}$ must be atoms of $\mathcal{C}$, so G is the orbit object. ∎

We can now provide a converse to Theorem 3.4.

5.2. Theorem. *Let G be a finite group and let $\mathcal{E}$ be a full reflective subcategory of G^σ containing $\{e\}$ and all isomorphisms. then there exists an orbital extensive category $\mathcal{C}$ with $At(\mathcal{C}) \cong \mathcal{E}$.*

Proof. $\mathcal{E}$ is isomorphic to the full subcategory $\mathcal{A}$ of $\mathbf{Fin}^G$ of all atoms $G\backslash H$ with H in $\mathcal{E}$. Let $\mathcal{C}$ be the full subcategory of $\mathbf{Fin}^G$ of all finite coproducts of atoms in $\mathcal{A}$. Evidentally, $\mathcal{C}$ is closed under finite coproducts and summands. By exactly the same proof as in Theorem 3.5, $\mathcal{C}$ is a full surjection-reflective subcategory of $\mathbf{Fin}^G$ containing $G\backslash\{e\} = G$. By Lemma 5.1, $\mathcal{C}$ is an orbital extensive category. That $\mathrm{At}(\mathcal{C}) \cong \mathcal{E}$ is clear from the construction. ∎

Let $(\cdot)^\diamond : \mathbf{Fin}^G \to \mathcal{C}$ denote the reflector.

5.3. Lemma. *Let C in $\mathcal{C}$ be such that $C \times (\cdot) : \mathcal{C} \to \mathcal{C}$ preserves coequalizers. Then for all X in $\mathbf{Fin}^G$, $C \times X^\diamond \;\cong\; (C \times X)^\diamond$.*

Proof. Let X, Y satisfy the condition, i.e. $C \times X^\diamond \cong (C \times X)^\diamond$, $C \times Y^\diamond \cong (C \times Y)^\diamond$. Then

$$\begin{aligned}
C \times (X+Y)^\diamond &\cong C \times (X^\diamond + Y^\diamond) \quad \text{(reflectors preserve coproducts)}\\
&\cong (C \times X^\diamond) + (C \times Y^\diamond) \quad (\mathbf{Fin}^G \text{ is a topos})\\
&\cong (C \times X)^\diamond + (C \times Y)^\diamond \;\cong\; ((C \times X) + (C \times Y))^\diamond\\
&\cong (C \times (X+Y))^\diamond
\end{aligned}$$

Thus it suffices to show that $C \times A^\diamond \cong (C \times A)^\diamond$ for A an atom of $\mathbf{Fin}^G$. There exists epic $p : G \to A$ so that $p = \mathrm{coeq}(t, u)$ with $t, u : K \to G$ the kernel pair of p in $\mathbf{Fin}^G$. As

$K \to G \times G$ is a summand, K, G are in $\mathcal{C}$. Let $\eta : A \to A^\diamond$ be the reflection, so that $\eta\, p = \operatorname{coeq}(t, u)$ in $\mathcal{C}$. As $\mathbf{Fin}^C$ is cartesian closed, $\mathrm{id} \times p = \operatorname{coeq}(\mathrm{id} \times t, \mathrm{id} \times u)$ in $\mathbf{Fin}^G$ whence $(C \times A)^\diamond = \operatorname{coeq}(id \times t, \mathrm{id} \times u)$ in $\mathcal{C}$. But by hypothesis, also $C \times A^\diamond = \operatorname{coeq}(\mathrm{id} \times t, \mathrm{id} \times u)$ in $\mathcal{C}$, so we are done. ∎

It is well known that the exponential object Y^X in $\mathbf{Fin}^G$ can be constructed as the set of all functions from the set X to the set Y with G-action $f * g \ = \ X \xrightarrow{g} X \xrightarrow{f} Y \xrightarrow{g^{-1}} Y$.

5.4. Proposition. *Suppose that $(\cdot) \times G : \mathcal{C} \to \mathcal{C}$ has a right adjoint $(\cdot)^G$. Then Y^G is also the exponential in $\mathbf{Fin}^G$.*

Proof. $\mathcal{C} \xrightarrow{(\cdot)^G} \mathcal{C} \xrightarrow{(\cdot)^\diamond} \mathbf{Fin}^G$ has left adjoint

$$\begin{array}{rcll} Y \ \mapsto \ Y^\diamond \times G & \cong & (Y \times G)^\diamond & \text{(Lemma 5.3)} \\ & \cong & (|Y| \cdot G)^\diamond & \text{(alchemy theorem)} \\ & \cong & |Y| \cdot G^\diamond & ((\cdot)^\diamond \text{ preserves coproducts}) \\ & \cong & |Y| \cdot G \ \cong \ Y \times G & \end{array}$$

which shows that the surjective reflection $Y^X \to (Y^X)^\diamond$ is the exponential in $\mathbf{Fin}^G$. But both exponentials have cardinality $|Y|^{|X|}$ by Theorem 4.7, so the reflection is an equivariant isomorphism. ∎

The proposition just proved leads to a way to create examples of orbital extensive categories that are not cartesian closed.

5.5. Theorem. *Suppose $(\cdot) \times G : \mathcal{C} \to \mathcal{C}$ has a right adjoint. Let $C \subset G$ be any conjugacy glass of G, so that C is an atom of $\mathbf{Fin}^G$. Then C is, in fact, in $\mathcal{C}$.*

Proof. G^G exists in $\mathcal{C}$ and coincides with G^G in $\mathbf{Fin}^G$ by Proposition 5.4. Let G act on itself by conjugation, $x * g = g^{-1}xg$. Let $\lambda_a\, g = ag$ be the left translation map and embed G in G^G by the injective map $\lambda : G \to G^G$, $a \mapsto \lambda_{a^{-1}}$. The action of g on G is the map $\delta_g x = x * g = g^{-1}xg$. In G^G we have $\lambda_{a^{-1}} * g \ = \ G \xrightarrow{\delta_g} G \xrightarrow{\lambda_{a^{-1}}} G \xrightarrow{\delta_{g^{-1}}} G \ = \ \lambda_{(g^{-1}ag)^{-1}} \ = \ \lambda_{(a*g)^{-1}}$ which shows that G is an invariant subset of G^G, and hence is in $\mathcal{C}$. Since an orbit of G under this action is a conjugacy class of G, C is an atom in $\mathcal{C}$. ∎

5.6. Example. $\{e\}, G$ generate a full reflective subcategory $\mathcal{C}$ of G^σ with reflection $\{e\}^\diamond = \{e\}$, $H^\diamond = G$ if $H \neq \{e\}$.

This works because G is terminal in G^σ and because there are no maps $H \to \{e\}$ if $H \neq \{e\}$. Theorem 5.2 constructs an orbital extensive category $\mathcal{C}$ with orbit object G and atoms 1 and G. If G is non-abelian, it has a conjugacy class C with $1 < |C| < |G|$. By Theorem 5.5, G^G does not exist in $\mathcal{C}$, so $\mathcal{C}$ is not cartesian closed.

5.7. Example. Every orbital extensive category $\mathcal{C}$ has a **transitive hull** $\mathcal{D}$ whose objects are all finite coproducts of the transitive atoms of $\mathcal{C}$, and such $\mathcal{D}$ is an orbital extensive category.

In view of Proposition 4.5, it suffices to see that the normal subgroups of G form a full reflective subcategory of G^σ. This is easy: set $H^\diamond$ to be the normal subgroup generated by H.

A group is **Hamiltonian** if it is non-abelian and all of its subgroups are normal. It is known [4] that a finite group is Hamiltonian if and only if it is a product $Q \times A$ with Q the 8-element quaternion group and A a particular type of abelian group. It follows that for most non-abelian groups, the transitive hull is a proper subcategory of $\mathbf{Fin}^G$.

6. Identifying Atoms

In this final section, we put objects under the microscope in order to see their individual atoms. Recall the colored icosahedron of the opening paragraph. With black and white, there are 4 orbits. We now wish to determine for which subgroups H of the symmetry group G it happens that $G\backslash H$ is an orbit. Among other things, this tells us that the orbit has $[G : H]$ elements.

The icosahedron group G is a 60-element group isomorphic to the alternating group A_5. We represent this group by permutations on the 12 vertices. Label one of the vertices 1 and label the 5 adjacent vertices 2,3,4,5,6 clockwise. Label the remaining vertices so that for $1 \leq k \leq 6$, k and $k+6$ are opposite. The following table records the 5 conjugacy classes of G. Given there, is the disjoint cycle factorization of a typical element of the class, and the number of elements of the class.

$$\begin{array}{rcl}
\mathrm{e} & = & (1)\ (2)\ (3)\ (4)\ (5)\ (6)\ (7)\ (8)\ (9)\ (10)\ (11)\ (12),\ \ 1 \\
\mathrm{a} & = & (1\ 2)\ (3\ 6)\ (4\ 10)\ (5\ 11)\ (7\ 8)\ (9\ 12),\ \ 15 \\
\mathrm{b} & = & (1\ 2\ 3)\ (4\ 6\ 11)\ (5\ 10\ 12)\ (7\ 8\ 9),\ \ 20 \\
\mathrm{c} & = & (1)\ (2\ 3\ 4\ 5\ 6)\ (7)\ (8\ 9\ 10\ 11\ 12),\ \ 12 \\
\mathrm{d} & = & (1\ 8\ 3\ 5\ 12)\ (2\ 9\ 11\ 6\ 7)\ (4)\ (10),\ \ 12
\end{array}$$

The well-known Burnside-Frobenius lemma computes the number of orbits:

6.1. Theorem. *[6] Let G be a finite group and let X be a finite G-set. Let the ith conjugacy class C_i have c_i elements and choose $g_i \in C_i$, $i = 1, \ldots, k$. Define $N_i = \{x \in X : xg_i = x\}$. Then the number of orbits of X is the average number of elements of X fixed by the elements of G, namely*

$$\frac{1}{|G|} \sum_{i=1}^{k} c_i N_i$$

For "coloring problems" such as that for the icosahedron, we represent G by permutations on a set V of "vertices" so that the "colored objects" are functions $V \to C$ for a given set C of "colors". Regard C as the trivial action $C \times G \to C$, $cg = c$, that is, let $C = |C| \cdot 1$ be a copower of the terminal object in $\mathbf{Fin}^G$. The exponential object C^V then has action

$$C^V \times G \to C^V, \quad (V \xrightarrow{f} C,\, g) \mapsto G \xrightarrow{g} G \xrightarrow{f} C$$

this makes precise how elements of G move a colored object.

The opening example with G the icosahedron group permuting vertices $1, \ldots, 12$ has $X = \{f \in C^V : w, v \text{ opposite vertices} \Rightarrow f(w) \neq f(v)\}$. We calculate the number of orbits of X since the more general problem to be considered must build on the ideas used here. Write $c = |C|$. Each of the cycle decompositions for elements of G partitions V into blocks. Clearly the requirement $fg = f$ holds if and only if f assigns the same color to all elements in a block. Thus $N_e = c^6(c-1)^6$ since 6 vertices can be any color with the opposite vertices any other color. $N_a = 0$ because the block $\{4, 10\}$ contains opposite vertices which cannot be colored differently. In all other cases for g, there are six vertices in 2 blocks with all opposite vertices in two other blocks, so $N_g = c^2(c-1)^2$. By Burnside-Frobenius, the number of orbits is

$$f(c) \quad = \quad \frac{c^6(c-1)^6 + 44c^2(c-1)^2}{60}$$

and, in particular, $f(2) = 4$ as claimed. These 4 orbits are atoms $G\backslash H$ and it is our object to find out which subgroups H arise.

To begin, we observe that the conjugacy classes of a group G forms a poset $(P, \leq)$ if $H \leq K$ means that there exists $H \to K$ in G^σ. This is a poset by Proposition 3.3. Let $H_1, \ldots, H_k$ be a choice of one representative from each subgroup conjugacy class and let M be the incidence matrix of $(P, \leq)$, that is, the row-i, column-j entry is 1 if $H_i \leq H_j$, and is otherwise 0. It is well known that the vertices in V can be listed so that M is lower triangular (begin with a maximal element and proceed inductively).

For a specific example, let G be the icosahedron group. There are exactly 9 conjugacy classes of subgroups and it happens that non-conjugate subgroups have different cardinality. As $60 = 2^2 \cdot 3 \cdot 5$, Sylow theory gives three conjugacy classes for the subgroups $K4, \boldsymbol{Z}_3, \boldsymbol{Z}_5$. Here the Sylow 2-group is the Klein 4-group $K4$ since it is clear from the table above that there are no elements of order 4. List the nine subgroups in the order $G, h12, h10, h6, K4, \boldsymbol{Z}_5, \boldsymbol{Z}_3, \boldsymbol{Z}_2, \{e\}$ (the numbers indicating the number of elements in the subgroup). The incidence matrix is then

$$M \quad = \quad \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

6.2. Definition. *For any X in $\mathbf{Fin}^G$, define integer-valued functions on the subgroups of G by*

$$\begin{aligned} \psi(H) &= \text{number of orbits of } X \text{ isomorphic to } G\backslash H \\ \varphi(H) &= \sum_{H \le K} \psi(K) \end{aligned}$$

Our objective is to compute ψ. It is clear that

$$M \begin{bmatrix} \psi(H_1) \\ \cdots \\ \psi(H_k) \end{bmatrix} = \begin{bmatrix} \varphi(H_1) \\ \cdots \\ \varphi(H_k) \end{bmatrix}$$

Our approach will be to use the theory to compute $\varphi(H_i)$. As $\det(M) = 1$, M is invertible so we can then determine ψ by $\psi = M^{-1}\varphi$. This technique is known as "Möbius inversion".

Notice, by the way, that while the poset of subgroups of a group under inclusion is always a lattice, the poset of conjugacy classes for the icosahedron group fails to be. This is because $h6, h12$ have two distinct maximal lower bounds, $\mathbb{Z}_2$ and $\mathbb{Z}_3$. $(P, \le)$ just misses being a sublattice of divisors of 60 by virtue of having no G^σ-morphisms $h6 \to h12$.

We turn now to computing $\varphi(H)$.

6.3. Lemma. *For any G-set X, stability subgroups along the same orbit are conjugate, specifically, $H_{xg} = g^{-1}H_xg$.*

Proof. $h \in G_{xg} \Leftrightarrow xgh = xg \Leftrightarrow xghg^{-1} = x \Leftrightarrow ghg^{-1} \in H_x \Leftrightarrow h \in g^{-1}H_xg$. ∎

6.4. Definition. *Given X in $\mathbf{Fin}^G$ and $H \in G^\sigma$, define*

$$X_H = \{x \in X : H \le H_x\}$$

6.5. Lemma. *X_H is G-invariant.*

Proof. Let $x \in X_H$, $g \in G$. By Lemma 6.3, H_x, H_{xg} are conjugate subgroups, hence isomorphic in G^σ. Thus $H \le H_{xg}$. ∎

6.6. Proposition. *$\varphi(H)$ is the number of orbits of X_H.*

Proof. For any G-set X, the orbit of $x \in X$ is isomorphic to the G-set $G\backslash H_x$. This orbit is counted in $\varphi(H)$ precisely when $H \le H_x$, that is, precisely when $x \in X_H$. ∎

This proposition reduces the computation of $\varphi(H)$ to a Burnside-Frobenius calculation for the G-invariant set X_H.

To that end, we begin by observing that $H \leq H_x \Leftrightarrow$ some subgroup conjugate to H is contained in $H_x \Leftrightarrow$ some subgroup conjugate to H fixes x.

To go further, we return to an invariant subset X of colored objects C^V where G is a group of permutations of V. Our approach in the next paragraphs is influenced by the work of [10].

If subgroup H is to fix $f \in C^V$, consider the action of H on V. For $x \in V$ and $h \in H$ we must have $fx = f(xh)$. This means that f is constant on each block of the orbit partition of V induced by the action of H.

Consider the subgroup K4 with elements e as well as the three order-2 permutations with cycle decompositions

$$\begin{gathered}(1\ 2)\ (3\ 6)\ (4\ 10)\ (5\ 11)\ (7\ 8)\ (9\ 12)\\ (1\ 7)\ (2\ 8)\ (3\ 12)\ (4\ 11)\ (5\ 10)\ (6\ 9)\\ (1\ 8)\ (2\ 7)\ (3\ 9)\ (4\ 5)\ (6\ 12)\ (10\ 11)\end{gathered}$$

The orbit partition of this $K4$ is

$$\{\{1,2,7,8\},\{3,6,9,12\},\{4,5,10,11\}\}$$

Since at least one (in fact all three) blocks have a pair of opposite vertices, no coloring we seek can be fixed by K4. By Lemma 6.3, this holds for each of the conjugates of K4. We conclude that $\psi(K4) = 0$. Similar investigations show that $\psi(\boldsymbol{Z}_2) = \psi(h6) = \psi(h12) = 0$.

One next checks that the orbits for the subgroups $h12$ and, of course, G have only one block, so can only fix a constant coloring. Since opposite vertices must be differently colored, we conclude $\psi(h12) = \psi(G) = 0$ as well.

Among the permutations $\{e, a, b, c, d\}$, a has a cycle $(4\,10)$ containing a pair of opposite vertices. For any $g \in G$, $(g^{-1}ag)g^{-1}(4) = g^{-1}a(4) = g^{-1}(10)$ and any symmetry moves opposite vertices to opposite vertices, so it follows that any of the 15 conjugates of a will also have a cycle with opposite vertices. Thus for N_g as in the Burnside-Frobenius formula, $N_g = 0$ for g conjugate to a.

We must work harder to deal with e, b, c, d and the subgroups $\{e\}$, $\boldsymbol{Z_3}$, $\boldsymbol{Z_5}$. We'll begin with $\boldsymbol{Z_3}$ which has e and two permutations of order 3 whose partition blocks are

$$\begin{gathered}\{\{1,3,4\},\{2,12,5\},\{6,11,8\},\{7,9,10\}\}\\ \{\{1,4,3\},\{2,5,12\},\{6,8,11\},\{7,10,9\}\}\end{gathered}$$

The orbit partition of $\boldsymbol{Z_3}$ is

$$\{\{1,3,4\},\{2,5,12\},\{6,8,11\},\{7,9,10\}\}$$

The cycles of the permutation b partition V into the blocks

$$\{\{1,2,3\},\{4,6,11\},\{5,10,12\},\{7,8,9\}\}$$

For both $Z3$ to fix a coloring and a to fix a coloring, the blocks of the refinement join of the two partitions (i.e. the partition of the equivalence relation generated by the union of the two equivalence relations) have all elements colored the same. Now the join of these two partitions is the one-block partition which contributes nothing. However, me must check not only against b but also for all 20 conjugates of b. It is equivalent and more efficient to check b against all 10 conjugates of $\boldsymbol{Z}_3$ (conjugacy classes of subgroups are smaller than conjugacy classes of elements). When this is done, exactly one of the 10 delivers a nontrivial partition join, and it coincides with the orbit partition of $\boldsymbol{Z}_3$ shown above. There are 2 cycles containing the opposite vertices of the other 2 cycles, leading to $c^2(c-2)^2$ colorings. When one similarly checks the ten conjugate subgroups of $\boldsymbol{Z}_3$ against c, d all partition joins have one block.

We must not forget to check the 10 conjugate subgroups against the identity permutation! Here the partition join will coincide the with orbit partition of the subgroup, which in all cases is 2 cycles with the other 2 having the opposite vertices of the second 2. Here each of the 10 subgroups contributes $c^2(c-1)^2$ colorings for a total of $10c^2(c-1)^2$. But wait! Some of these colorings may have been fixed by more than one of the 10 subgroups. This sets up a potentially disastrous inclusion-exclusion. We must subtract the effect of two subgroups at a time, then add in the effect of three subgroups as a time, and so forth. Happily, the join of the orbit partition of any two of these subgroups is found to be the one-block partition. Thus the calculation stops, and, applying the Burnside-Frobenius formula, we have found that

$$\phi(\boldsymbol{Z}_2) \;=\; \frac{10c^2(c-1)^2 + 20c^2(c-1)^2}{60} \;=\; \frac{c^2(c-1)^2}{2}$$

In fact, $\{\boldsymbol{Z}_2, K4, h10, h12, G\}$ is a complete list of all the elements $\geq \boldsymbol{Z}_2$ and $\psi(H) = 0$ for all $H \neq \boldsymbol{Z}_2$. We thus conclude

$$\psi(\boldsymbol{Z}_2) = \frac{c^2(c-1)^2}{2}$$

Which yields 30-element orbits for any number of colors ≥ 2.

An entirely similar analysis yields $\psi(\boldsymbol{Z}_5) = \frac{c^2(c-1)^2}{2}$. Since the only subgroups that can fix a valid coloring are $\boldsymbol{Z}_3, \boldsymbol{Z}_5$ and $\{e\}$, we deduce that

$$\psi(\{e\} \;=\; \frac{c^6(c-1)^6 + 44c^2(c-1)^2}{60} - c^2(c-1)^2 \;=\; \frac{c^6(c-1)^6 - 16c^2(c-1)^2}{60}$$

These formulas establish the claims made at the beginning.

References

[1] F. Bergeron, G. Labelle and P. Leroux, *Combinatorial Species and Tree-like Structures*, Cambridge University Press, 1998.

[2] A. Carboni, S. Lack and R.F.C. Walters, Introduction to extensive and distributive categories, *Journal of Pure and Applied Algebra* 84, 1993, 145–158.

[3] J. R. B. Cockett, Introduction to distributive categories, *Mathematical Structures in Computer Science* 3, 1993, 277–307.

[4] R. Dedekind, Über Gruppen, deren sämmtliche Theiler Normaltheiler sind, *Mathematische Annalen* 48, 1897, 548–561.

[5] Y. Diers, Categories of Boolean Sheaves of Simple Algebras, *Lecture Notes in Mathematics* 1187, Springer-Verlag, 1985.

[6] G. Frobenius, Über die congruenz nach einem aus zwei endlichen Gruppen gebildeten Doppelmodul, *Crelle's Journal* 101, 1887, 273-299.

[7] A. Joyal, Une théorie combinatoire des séries formelles, *Advances in Mathematics* 42, 1981, 1–82.

[8] S. Koppelberg, *Handbook of Boolean Algebras, Volume I*, North-Holland, 1989.

[9] E. Manes, *Predicate Transformer Semantics*, Cambridge University Press, 1992.

[10] G.-C. Rota and D.A. Smith, Enumeration under group action, *Annali Scuola Normale Superiore - Pisa, Classe di Scienze, Serie IV - Vol. IV, n. 4*, 1977, 637–646.

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