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A TWO-DIMENSIONAL BIRKHOFF'S THEOREM

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Abstract. Birkhoff's variety theorem from universal algebra characterises equational subcategories of varieties. We give an analogue of Birkhoff's theorem in the setting of enrichment in categories. For a suitable notion of an equational subcategory we characterise these subcategories by their closure properties in the ambient algebraic category.

1. Introduction

In this paper we will state and give a proof of a 2-dimensional analogue of the Birkhoff theorem from universal algebra. Recall that in the ordinary setting, Birkhoff's theorem characterises equational subcategories of algebraic categories. An algebraic category can be viewed as a category $\mathsf{Alg}(T)$ of algebras for a strongly finitary monad T on Set. (Note that a monad is strongly finitary if its underlying functor is strongly finitary, i.e., if it preserves sifted colimits [1].) A full subcategory $\mathscr{A}$ of $\mathsf{Alg}(T)$ is said to be an *equational subcategory* of $\mathsf{Alg}(T)$ if it is (equivalent to) the category $\mathsf{Alg}(T')$ of algebras for a strongly finitary monad T', where T' is constructed by "adding new equations" to the monad T. More precisely, we ask T' to be a quotient of T, meaning that there is a monad morphism $e : T \longrightarrow T'$ that is moreover a regular epimorphism. The resulting algebraic functor

$$\mathsf{Alg}(e) : \mathsf{Alg}(T') \longrightarrow \mathsf{Alg}(T)$$

then exhibits $\mathsf{Alg}(T')$ as an equational subcategory of $\mathsf{Alg}(T)$. Every such subcategory $\mathsf{Alg}(T') \longrightarrow \mathsf{Alg}(T)$ has nice closure properties with respect to the inclusion into $\mathsf{Alg}(T)$. The content of Birkhoff's theorem is that equational subcategories can be characterised by these closure properties. In essence, this theorem holds since algebraic categories are well-behaved with respect to quotients (regular epis) – they are *exact categories* [1].

Taking inspiration from the ordinary case, we want to give a characterisation of equational subcategories of algebraic categories in the enriched setting. Namely, we shall mainly work with categories enriched in the symmetric monoidal closed category $\mathscr{V} = \mathsf{Cat}$ and we will accordingly use the enriched notions of a functor, natural transformation, etc.

Analogously to the ordinary case, in defining the notion of an equational subcategory of $\mathsf{Alg}(T)$ the idea is again to consider "quotients" $e : T \longrightarrow T'$ of strongly finitary 2-monads.

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Any subcategory $\mathsf{Alg}(e) : \mathsf{Alg}(T') \longrightarrow \mathsf{Alg}(T)$ exhibited by a quotient $e : T \longrightarrow T'$ is an equational subcategory of $\mathsf{Alg}(T)$.

Unlike to the $\mathscr{V} = \mathsf{Set}$ case, it is not immediately clear that some well-behaved notion of a quotient of strongly finitary 2-monads should exist. In $\mathscr{V} = \mathsf{Set}$, the quotients come as the epi part of the (regular epi, mono) factorisation system, and they are computed as certain colimits, the coequalisers. The solution in $\mathscr{V} = \mathsf{Cat}$ is to mimic this approach. Thus we should study factorisation systems on Cat (and the respective notions of a quotient), and find out which factorisation systems "lift up" from the category Cat to Cat-enriched algebraic categories. That is, we want to find factorisation systems that render the algebraic categories over Cat *exact* in some suitably generalised sense. This would allow us to talk about quotients of strongly finitary 2-monads while preserving the good behaviour of quotients as in Cat.

Recent advances in the theory of 2-dimensional exactness (see [6]) show that there are at least three notions of a quotient coming from three factorisation systems $(\mathcal{E}, \mathcal{M})$ on Cat, for which algebraic categories over Cat are exact:

1. (surjective on objects, injective on objects and fully faithful),
2. (bijective on objects, fully faithful),
3. (bijective on objects and full, faithful).

(For the $\mathcal{E}$ parts of the above systems, we will use the standard abbreviations, namely s.o. for surjective on objects, b.o. for bijective on objects, and b.o. full for bijective on objects and full.) We show that the 2-category $\mathsf{Mnd}_{\mathsf{sf}}(\mathsf{Cat})$ of strongly finitary 2-monads over Cat is exact in the sense of [6] with respect to all the three factorisation systems above as well.

We focus on the factorisation system (b.o. full, faithful). Unlike the other two systems, it corresponds to a meaningful notion of an equational subcategory, and it allows us to prove the 2-dimensional Birkhoff theorem by arguments very similar to those used in the proof of the ordinary Birkhoff theorem. For this factorisation system, the exactness of $\mathsf{Mnd}_{\mathsf{sf}}(\mathsf{Cat})$ implies that a monad morphism $e : T \longrightarrow T'$ is a quotient if and only if $e_C : TC \longrightarrow T'C$ is a b.o. full functor in Cat for every category C. We shall often use this "pointwise" nature of quotient monad morphisms.

The main result of the paper characterises equational subcategories of algebraic categories as those that are closed under products, quotients, subalgebras and sifted colimits. This is a characterisation in the spirit of the ordinary Birkhoff theorem. In the universal algebraic formulations, only the first three closure properties are demanded, and are dubbed "HSP" conditions. However, it was found out in [2] that even in the ordinary case, the property of being closed under *filtered* colimits is necessary when dealing with infinitely-sorted algebras. It is thus not surprising that the additional requirement of closedness under *sifted* colimits might be needed in the 2-dimensional case: the *finitary* and *strongly finitary* 2-monads no longer coincide in Cat (see Remark 3.4 for a distinguishing example), and we are dealing with the strongly finitary ones. The choice of working with strongly finitary 2-monads is fairly natural, since the 2-category $\mathsf{Mnd}_{\mathsf{sf}}(\mathsf{Cat})$ is equivalent to the 2-category Law of Cat-enriched one-sorted algebraic theories (also dubbed *Lawvere 2-theories*) [14].

In the final section we conclude with a few remarks on possible future work and on the other two factorisation systems on Cat. These systems are more poorly behaved, and thus the corresponding Birkhoff-type theorem would be of a weaker nature.

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2. Kernels and quotients in 2-categories

We shall make heavy use of factorisation systems in discussing and proving the Birkhoff theorem. The study of factorisation systems in general 2-categories is more involved than in the ordinary case. Following the exposition in [6], we first recall the definitions of enriched orthogonality and enriched factorisation systems in a general $\mathscr{V}$-category for a symmetric monoidal closed base category $\mathscr{V}$. Then we introduce *kernel-quotient systems* that generalise the correspondence between regular epimorphisms and kernels in exact categories, and we use this notion to introduce the (b.o. full, faithful) factorisation system on Cat. This factorisation system lifts up to a large class of algebraic categories, as is shown in Theorem 2.8. As an important corollary we show in Proposition 2.10 that the 2-category of strongly finitary monads on Cat inherits the (b.o. full, faithful) factorisation system, allowing us to study quotients of monads.

2.1. DEFINITION. *A morphism $f : A \longrightarrow B$ in a $\mathscr{V}$-category $\mathscr{C}$ is $\mathscr{V}$-orthogonal to $g : C \longrightarrow D$ (denoted by $f \perp g$) if the diagram*

$$\begin{array}{ccc} \mathscr{C}(B,C) & \xrightarrow{\mathscr{C}(B,g)} & \mathscr{C}(B,D) \\ {\scriptstyle \mathscr{C}(f,C)}\downarrow & & \downarrow{\scriptstyle \mathscr{C}(f,D)} \\ \mathscr{C}(A,C) & \xrightarrow[\mathscr{C}(A,g)]{} & \mathscr{C}(A,D) \end{array}$$

is a pullback in $\mathscr{V}$. Given a class $\mathcal{G}$ of morphisms of $\mathscr{C}$, we define two classes of morphisms $\mathscr{V}$-orthogonal to those in $\mathcal{G}$:

- $\mathcal{G}^{\downarrow} := \{m \mid \forall g \in \mathcal{G} : g \perp m\}$,
- $\mathcal{G}^{\uparrow} := \{e \mid \forall g \in \mathcal{G} : e \perp g\}$.

Given an object C of $\mathscr{C}$, the morphism $f : A \longrightarrow B$ is orthogonal to C if f is orthogonal to id_C, i.e., if the precomposition map

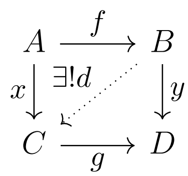
$$\mathscr{C}(f,C) : \mathscr{C}(B,C) \longrightarrow \mathscr{C}(A,C)$$

is invertible (i.e., an isomorphism). We denote this fact by $f \perp C$.

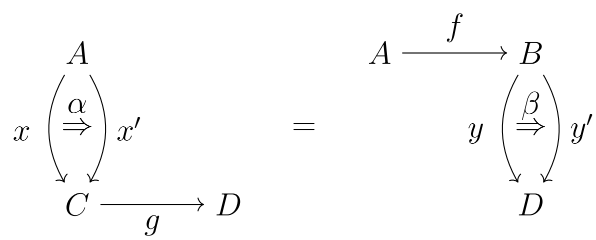
Let $\mathcal{E}$ *and* $\mathcal{M}$ *be two classes of morphisms of* $\mathscr{C}$. *We say that* $(\mathcal{E}, \mathcal{M})$ *is a* $\mathscr{V}$-factorisation system *if*

1. $\mathcal{M} = \mathcal{E}^{\downarrow}$,
2. $\mathcal{E} = \mathcal{M}^{\uparrow}$, *and*
3. *every morphism* f *in* $\mathscr{C}$ *can be factorised as the composition* $m \cdot e$ *of a morphism* m *in* $\mathcal{M}$ *and a morphism* e *in* $\mathcal{E}$.

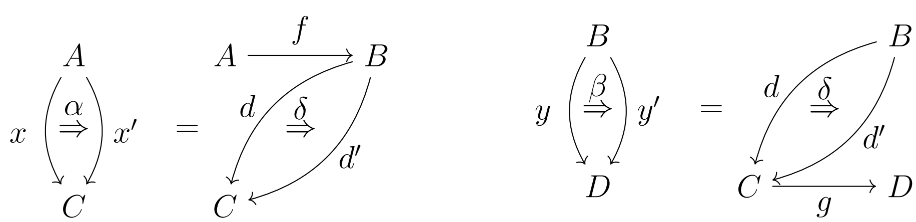
2.2. Example. We examine when two morphisms $f : A \longrightarrow B$ and $g : C \longrightarrow D$ are orthogonal in $\mathscr{C}$ for the case of $\mathscr{V} = \mathsf{Cat}$. Firstly, the morphisms have to satisfy the usual diagonal fill-in property



for every pair $x : A \longrightarrow C$ and $y : B \longrightarrow D$ of morphisms in $\mathscr{C}$. Let us denote by $d : A \longrightarrow D$ the diagonal fill-in for x and y, and denote by $d' : A \longrightarrow D$ the diagonal fill-in for x' and y'. The second requirement on f and g to be orthogonal is that they satisfy the diagonal *2-cell* property: for every pair $\alpha : x \Rightarrow x'$ and $\beta : y \Rightarrow y'$ of 2-cells such that



there has to exist a *unique* 2-cell $\delta : d \Rightarrow d'$ such that the equalities



hold. Similarly, a morphism $f : A \longrightarrow B$ in $\mathscr{C}$ is orthogonal to an object C of $\mathscr{C}$ if

1. for every $g : A \longrightarrow C$ there exists a unique morphism $h : B \longrightarrow C$ such that $h \cdot f = g$, and

2. for every 2-cell $\alpha : g \Rightarrow g'$ there exists a unique 2-cell $\beta : h \Rightarrow h'$ such that $\beta * f = \alpha$ holds.

We now recall from [6] the notion of a *kernel-quotient system*. This notion generalises the notions of a kernel and its induced quotient, and allows treating factorisation systems in enriched categories parametric in the choice of the shape of "kernel data". Importantly, this approach covers the motivating ordinary (regular epi, mono) factorisation system on Set as well as the three factorisation systems on Cat that are mentioned in the introduction.

In the following, we will restrict ourselves to $\mathscr{V}$ being a locally finitely presentable category as a monoidal closed category in the sense of [9], as we will need to impose a finiteness condition on the kernel-quotient system.

Let us denote by $\mathbb{2}$ the free $\mathscr{V}$-category on a morphism $1 \longrightarrow 0$. We let $\mathscr{F}$ be a finitely presentable $\mathscr{V}$-category containing $\mathbb{2}$ as a full subcategory. Then there is the obvious inclusion $J : \mathbb{2} \longrightarrow \mathscr{F}$ and the inclusion $I : \mathscr{K} \longrightarrow \mathscr{F}$ of the full subcategory $\mathscr{K}$ of $\mathscr{F}$ spanned by all objects of $\mathscr{F}$ except 0. We call the data (J, I) a *kernel-quotient system*, and the role of $\mathscr{K}$ is, informally, to give the "shape" of the kernels. Given a complete and cocomplete $\mathscr{V}$-category $\mathscr{C}$, there is a chain of adjunctions as in the following diagram:

$$[\mathbb{2}, \mathscr{C}] \underset{\mathrm{Ran}_J}{\overset{[J,\mathscr{C}]}{\underset{\perp}{\leftrightarrows}}} [\mathscr{F}, \mathscr{C}] \underset{[I,\mathscr{C}]}{\overset{\mathrm{Lan}_I}{\underset{\perp}{\leftrightarrows}}} [\mathscr{K}, \mathscr{C}]$$

We denote the composite adjunction by

$$[\mathbb{2}, \mathscr{C}] \underset{K}{\overset{Q}{\underset{\perp}{\leftrightarrows}}} [\mathscr{K}, \mathscr{C}]$$

and call it the *kernel-quotient* adjunction for $\mathscr{F}$.

In [6] the authors give a weaker definition of kernel-quotient adjunction to capture the cases where $\mathscr{C}$ is not complete and cocomplete. We do not need to introduce this weaker notion, as the 2-categories $\mathscr{C}$ in our examples always satisfy the completeness conditions.

2.3. Definition. *Given a complete and cocomplete $\mathscr{V}$-category $\mathscr{C}$ together with the kernel-quotient adjunction for $\mathscr{F}$, we say that an object X in $[\mathscr{K}, \mathscr{C}]$ is an* $\mathscr{F}$-kernel *if it is in the essential image of K. Any arrow $f : A \longrightarrow B$ in $\mathscr{C}$ is called an* $\mathscr{F}$-quotient map *if it is in the essential image of Q, and it is called* $\mathscr{F}$-monic *if the morphism $K(id_A, f) : K(id_A) \longrightarrow K(f)$ is an isomorphism.*

2.4. Example. The motivating example of a kernel-quotient adjunction in the ordinary setting ($\mathscr{V} = \mathsf{Set}$) is given by taking the category $\mathscr{F}$ to be of the shape

$$2 \rightrightarrows 1 \longrightarrow 0$$

with J and I being the obvious embeddings. Here the adjunction $Q \dashv K$ acts as follows. The functor Q sends a parallel pair $X = (f, g)$ to a coequaliser QX of the parallel pair (f, g). A morphism $f : A \longrightarrow B$ in $\mathscr{C}$, thus an object in $[2, \mathscr{C}]$, is sent by K to the kernel $Kf = (k_1, k_2)$ of f. The $\mathscr{F}$-monic morphisms are precisely the monomorphisms in this example.

The kernel-quotient system in the previous example allows factoring every morphism in $\mathscr{C}$ as a regular epimorphism followed by a (not necessarily monomorphic) morphism. Since both the functors $I : \mathscr{K} \longrightarrow \mathscr{F}$ and $J : 2 \longrightarrow \mathscr{F}$ are injective on objects and fully faithful, the functors Ran_J and Lan_I can *always* be taken as strict sections of the functors $[J, \mathscr{C}]$ and $[I, \mathscr{C}]$, respectively. Then the kernel-quotient adjunction $Q \dashv K$ may be taken to commute with the evaluation functors $[2, \mathscr{C}] \longrightarrow \mathscr{C}$ and $[\mathscr{K}, \mathscr{C}] \longrightarrow \mathscr{C}$ that evaluate at the object 1. This results in the counit ε of $Q \dashv K$ having the following form for all objects f in $[2, \mathscr{C}]$:

$$\begin{array}{ccc} A & \xrightarrow{id_A} & A \\ {\scriptstyle QKf}\downarrow & & \downarrow{\scriptstyle f} \\ C & \xrightarrow[\varepsilon_f]{} & B \end{array}$$

Thus we have a factorisation

$$f = \varepsilon_f \cdot QKf,$$

and QKf is a regular epi, being a coequaliser of the parallel pair Kf. If the morphism ε_f is a mono for every f, we obtain a Set-factorisation system (regular epi, mono) on $\mathscr{C}$.

The above construction of the morphism ε_f is analogous in the case of enrichment in a general $\mathscr{V}$. We say that *$\mathscr{F}$-kernel-quotient factorisations in $\mathscr{C}$ converge immediately* whenever ε_f is $\mathscr{F}$-monic for each morphism f in $\mathscr{C}$. Whenever $\mathscr{F}$-kernel-quotient factorisations converge immediately in $\mathscr{C}$, we obtain a $\mathscr{V}$-factorisation system ($\mathscr{F}$-quotient, $\mathscr{F}$-monic) on $\mathscr{C}$ (by Proposition 4 of [6]).

2.5. Example. Given a 2-category $\mathscr{F}_{\mathsf{bof}}$ generated by

$$2 \underset{t}{\overset{s}{\rightrightarrows}} 1 \xrightarrow{\;w\;} 0 \qquad (\alpha \Downarrow \Downarrow \beta : s \Rightarrow t)$$

subject to the identity $w * \alpha = w * \beta$, we obtain the following kernel-quotient system. The $\mathscr{F}_{\mathsf{bof}}$-kernel (or *equikernel*) of a morphism $f : A \longrightarrow B$ is given by the following data:

$$E \underset{t}{\overset{s}{\rightrightarrows}} A \xrightarrow{\;f\;} B \qquad (\alpha \Downarrow \Downarrow \beta : s \Rightarrow t)$$

In Cat, the category E has as objects the parallel morphisms $p, q : a \longrightarrow b$ from A for which the equality $f(p) = f(q)$ holds in B. The morphisms between objects $p_1, q_1 : a_1 \longrightarrow b_1$ and

$p_2, q_2 : a_2 \longrightarrow b_2$ in E are the pairs (m, n) of morphisms $m : a_1 \longrightarrow a_2$ and $n : b_1 \longrightarrow b_2$, satisfying the equalities $n \cdot p_1 = p_2 \cdot m$ and $n \cdot q_1 = q_2 \cdot m$. The functors s and t then act as "source" and "target" functors. That is, given $p, q : a \longrightarrow b$ as an object in E, we have that $s(p, q) = a$ and $t(p, q) = b$. The action of s and t on morphisms is as expected: using the above notation, $s(m, n) = m : a_1 \longrightarrow a_2$ and $t(m, n) = n : b_1 \longrightarrow b_2$. The natural transformations α and β then act as "morphism projections", i.e., $\alpha(p, q) = p : a \longrightarrow b$ and $\beta(p, q) = q : a \longrightarrow b$.

Given kernel-data X in $[\mathscr{K}, \mathscr{C}]$, its $\mathscr{F}_{\mathsf{bof}}$-quotient QX is its *coequifier*, i.e., a universal morphism $c : X1 \longrightarrow C$ satisfying $c * X\alpha = c * X\beta$ (see [8] or Section 5.3 in [6]). A morphism in $\mathscr{C}$ is $\mathscr{F}_{\mathsf{bof}}$-monic precisely when it is representably faithful (i.e., faithful when $\mathscr{C} = \mathsf{Cat}$). As the coequifier morphisms are always bijective on objects and full in Cat, this hints that the $\mathscr{F}_{\mathsf{bof}}$ kernel-quotient system gives rise to the (b.o. full, faithful) factorisation system on Cat. This is indeed the case. In detail, given a functor $f : A \longrightarrow B$, we can form its equikernel E and factorise f into two functors $e : A \longrightarrow A/E$ and $m : A/E \longrightarrow B$. The category A/E is the congruence category of A having the same objects as A, with the congruence on morphisms of A generated by the pairs $p, q : a \longrightarrow b$ that are objects of E. Defining e as the canonical functor that assigns to each morphism of A its equivalence class in A/E, it is obviously bijective on objects and full. The functor m assigns to each object a of A its image $f(a)$, and to the equivalence class morphism $[p : a \longrightarrow b]$ the image $f(p) : f(a) \longrightarrow f(b)$. It follows immediately from the definition of the equikernel that m is well-defined and faithful.

To summarise, for $\mathscr{C} = \mathsf{Cat}$ the kernel-quotient factorisations for $\mathscr{F}_{\mathsf{bof}}$ converge immediately, and they give rise to the (b.o. full, faithful) factorisation system.

The main focus of [6] is to study the generalised notions of regularity and exactness, parametric in the choice of a kernel-quotient system $\mathscr{F}$. This yields a theory of $\mathscr{F}$-regularity and $\mathscr{F}$-exactness. We do not need to introduce the theory of $\mathscr{F}$-exactness in detail. In fact, we use the results of [6] only to "lift" the (b.o. full, faithful) factorisation system of Example 2.5 on Cat to any algebraic category $\mathsf{Alg}(T)$ for a strongly finitary 2-monad T on Cat.

2.6. Remark. We say that a diagram

$$E \underset{t}{\overset{s}{\rightrightarrows}} A \qquad \alpha \Downarrow \Downarrow \beta \tag{1}$$

of kernel data is *reflexive* if there exists a morphism $i_A : A \longrightarrow E$ as in the diagram

$$E \; \begin{matrix} \xrightarrow{\quad s \quad} \\ \alpha \Downarrow \Downarrow \beta \\ \xrightarrow{\quad t \quad} \\ \xleftarrow[\quad i_A \quad]{} \end{matrix} \; A$$

that satisfies the reflexivity equalities

$$s \cdot i_A = t \cdot i_A = id_A,$$
$$\alpha * i_A = \beta * i_A = 1.$$

In Cat, the equikernel (1) of any functor $f : A \longrightarrow B$ is indeed reflexive. Recalling the description of E from Example 2.5, we see that the assignment

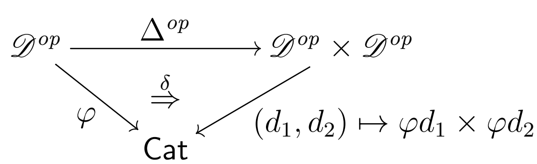
$$a \mapsto id_a, id_a : a \longrightarrow a,$$
$$m : a \longrightarrow b \mapsto (m, m)$$

defines a morphism $i_A : A \longrightarrow E$ that satisfies the reflexivity equalities.

It follows from the above remark that each b.o. full functor is the coequifier of a reflexive diagram: its equikernel. This observation is important because coequifiers of reflexive diagrams (reflexive coequifiers) are examples of *sifted* colimits. In the ordinary setting, sifted colimits are those colimits that commute with finite products in the category of sets, see [1]. In particular, Theorem 2.15 of [1] contains a useful characterisation of *diagrams* for sifted colimits. A diagram $\mathscr{D}$ is sifted if and only if it is connected and the diagonal $\Delta : \mathscr{D} \longrightarrow \mathscr{D} \times \mathscr{D}$ is final. In the case of enrichment in Cat, sifted colimits are again those colimits that commute with finite products, now in the category Cat. It is possible to characterise sifted colimits in a manner similar to the ordinary characterisation. A weight $\varphi : \mathscr{D}^{op} \longrightarrow \mathsf{Cat}$ is sifted if and only if

1. φ is *connected*, meaning that the unique 2-functor $\int^d \varphi d \longrightarrow \mathbb{1}$ is an isomorphism, and

2. the diagonal 2-functor $\Delta : \mathscr{D} \longrightarrow \mathscr{D} \times \mathscr{D}$ is *final*, meaning that the 2-cell



is a left Kan extension, where $\delta_d : \varphi d \longrightarrow \varphi d \times \varphi d$ is the diagonal functor.

This characterisation is contained, e.g., in Remark 4.2 of [7].

2.7. Remark. Recall that a 2-functor $T : \mathscr{C} \longrightarrow \mathscr{C}$ is called *strongly finitary* if it preserves sifted colimits. Using Remark 2.6 we see that every strongly finitary endo-2-functor $T : \mathsf{Cat} \longrightarrow \mathsf{Cat}$ preserves b.o. full functors, as they are coequifiers for some reflexive diagram in Cat. We will use this fact very often in the following sections.

Let us denote by $\mathsf{Cat}_{\mathsf{sf}}$ the 2-category of natural numbers $n = \{0, 1, \ldots, n-1\}$ and functions between them. There is an inclusion $\iota : \mathsf{Cat}_{\mathsf{sf}} \longrightarrow \mathsf{Cat}$ that represents n as the discrete category with the object set n, and maps a function $f : m \longrightarrow n$ to the corresponding functor with object assignment f. Theorem 8.31 of [5] states that Cat is the free cocompletion of $\mathsf{Cat}_{\mathsf{sf}}$ under sifted colimits. This observation is useful as it shows that any category C is a sifted colimit of finite discrete categories. Indeed, every discrete category is a filtered colimit of its finite discrete subcategories, and every category is a sifted colimit (a special codescent object) of discrete categories (see, e.g., Chapter 1 of [5]). Moreover, strongly finitary 2-functors $T : \mathsf{Cat} \longrightarrow \mathsf{Cat}$ correspond (up to isomorphism) to 2-functors $T \cdot \iota : \mathsf{Cat}_{\mathsf{sf}} \longrightarrow \mathsf{Cat}$, as the following diagram

$$\begin{array}{ccc} \mathsf{Cat}_{\mathsf{sf}} & \xrightarrow{\ \iota\ } & \mathsf{Cat} \\ & \overset{\lambda}{\Rightarrow} & \\ {\scriptstyle T\cdot\iota}\searrow & & \swarrow{\scriptstyle T} \\ & \mathsf{Cat} & \end{array}$$

is a left Kan extension. This correspondence is stated and proved in Corollary 8.45 of [5]. Via this correspondence we may identify the 2-category $\mathsf{StrFin}(\mathsf{Cat})$ of strongly finitary endo-2-functors of Cat with the (2-functor) 2-category $[\mathsf{Cat}_{\mathsf{sf}}, \mathsf{Cat}]$. We will use this identification in the proof of Proposition 2.10.

The factorisation system given by $\mathscr{F}_{\mathsf{bof}}$ lifts from Cat to the categories of algebras for a strongly finitary 2-monad T. We will introduce the notion of an algebraic category and then state the "lifting theorem" for $\mathscr{F}_{\mathsf{bof}}$.

For a 2-monad T on a 2-category $\mathscr{C}$, we denote the 2-category of T-algebras and their *strict* homomorphisms by $\mathsf{Alg}(T)$. Recall that a morphism $a : TA \longrightarrow A$ is a T-algebra if it satisfies the axioms

$$\begin{array}{ccc} A & \xrightarrow{\eta^T_A} & TA \\ & {\scriptstyle id_A}\searrow & \downarrow{\scriptstyle a} \\ & & A \end{array} \qquad\qquad \begin{array}{ccc} TTA & \xrightarrow{Ta} & TA \\ {\scriptstyle \mu^T_A}\downarrow & & \downarrow{\scriptstyle a} \\ TA & \xrightarrow[a]{} & A, \end{array}$$

and a morphism $h : A \longrightarrow B$ is a strict homomorphism between T-algebras (A, a) and (B, b) if it makes the usual diagram

$$\begin{array}{ccc} TA & \xrightarrow{Th} & TB \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ A & \xrightarrow[h]{} & B \end{array}$$

commute in $\mathscr{C}$. Let us recall the 2-dimensional structure of $\mathsf{Alg}(T)$. Given two T-algebras $a : TA \longrightarrow A$ and $b : TB \longrightarrow B$, and two homomorphisms $h, h' : A \longrightarrow B$ between (A, a) and (B, b), the 2-cells $\alpha : h' \Rightarrow h$ between the homomorphisms h' and h are exactly those

2-cells $\alpha : h' \Rightarrow h$ in $\mathscr{C}$ that moreover satisfy the following equality:

$$\begin{array}{ccc} TA \underset{Th}{\overset{Th'}{\underset{\Downarrow T\alpha}{\rightrightarrows}}} TB & & TA \\ \quad\quad\quad\downarrow b & = & a \downarrow \quad\quad\quad \\ \quad\quad\quad B & & A \underset{h}{\overset{h'}{\underset{\Downarrow \alpha}{\rightrightarrows}}} B \end{array}$$

For us, the algebraic 2-category $\mathsf{Alg}(T)$ is therefore what other authors commonly denote by $\mathsf{Alg}_s(T)$, see [12]. As we do not deal with the weaker kinds of morphisms, we will talk simply of homomorphisms instead of strict homomorphisms in what follows. We call the 2-categories equivalent to the 2-categories of the form $\mathsf{Alg}(T)$ *algebraic*.

2.8. Theorem. *Let T be a strongly finitary 2-monad on $[X, \mathsf{Cat}]$ (with X an arbitrary set). Then the $\mathscr{F}_{\mathsf{bof}}$ kernel-quotient factorisations converge immediately in the 2-category* $\mathsf{Alg}(T)$ *of T-algebras. These factorisations give rise to a factorisation system: the quotient morphisms are precisely those morphisms whose underlying morphisms are pointwise bijective on objects and full, and the monic morphisms are precisely those whose underlying morphisms are pointwise faithful.*

Proof. Observe that the forgetful 2-functor $U : \mathsf{Alg}(T) \longrightarrow [X, \mathsf{Cat}]$ creates limits and sifted colimits. In particular, U creates equikernels and coequifiers of equikernels, since the equikernel is a reflexive pair in the sense of Remark 2.6. The factorisation of any morphism $h : (A, a) \longrightarrow (B, b)$ in $\mathsf{Alg}(T)$ is thus computed as in $[X, \mathsf{Cat}]$, and there the $\mathscr{F}_{\mathsf{bof}}$ factorisations converge immediately. The $\mathscr{F}_{\mathsf{bof}}$ factorisations thus converge immediately in $\mathsf{Alg}(T)$. Moreover, any $\mathscr{F}_{\mathsf{bof}}$-quotient morphism in $[X, \mathsf{Cat}]$ is pointwise bijective on objects and full, as it is a colimit and these are computed pointwise in $[X, \mathsf{Cat}]$. ∎

In the context of categories of algebras, the lifted factorisation system gives rise to the notions of a quotient algebra and a subalgebra. Let T be a strongly finitary 2-monad T on Cat, and take an algebra (A, a) from $\mathsf{Alg}(T)$. We say that (B, b) is a *subalgebra* of (A, a) if there is a homomorphism $m : (B, b) \rightarrowtail (A, a)$ with m faithful, as in the left-hand side of the diagram (2). By a *quotient algebra* of (A, a) we mean a T-algebra (B, b) together with a b.o. full morphism $h : A \twoheadrightarrow B$ in Cat that is a homomorphism, as in the right-hand side of the diagram (2).

$$\begin{array}{ccc} TB & \xrightarrow{Tm} & TA \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle a} \\ B & \underset{m}{\rightarrowtail} & A \end{array} \qquad\qquad \begin{array}{ccc} TA & \overset{Th}{\twoheadrightarrow} & TB \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ A & \underset{h}{\twoheadrightarrow} & B \end{array} \tag{2}$$

Let us remark that in the above diagram concerning quotient algebras, the morphism $Th : TA \twoheadrightarrow TB$ is indeed b.o. full by Remark 2.7 since h is b.o. full and T is strongly finitary.

2.9. Remark. Denote by N the discrete 2-category with natural numbers as objects. We have an obvious inclusion $J : N \longrightarrow \mathsf{Cat}_{\mathsf{sf}}$ that is an identity on objects (recall the description of $\mathsf{Cat}_{\mathsf{sf}}$ from Remark 2.7), and it induces a 2-functor

$$V = - \cdot J : [\mathsf{Cat}_{\mathsf{sf}}, \mathsf{Cat}] \longrightarrow [N, \mathsf{Cat}]$$

given by precomposition with J. Then let us denote by W the underlying 2-functor

$$W : \mathsf{Mnd}_{\mathsf{sf}}(\mathsf{Cat}) \longrightarrow [\mathsf{Cat}, \mathsf{Cat}] \xrightarrow{- \cdot \iota} [\mathsf{Cat}_{\mathsf{sf}}, \mathsf{Cat}]$$

mapping a strongly finitary 2-monad (T, μ, η) on Cat to its underlying endo-2-functor T and restricting it to the 2-functor $T \cdot \iota : \mathsf{Cat}_{\mathsf{sf}} \longrightarrow \mathsf{Cat}$. An argument from [13] shows that there is a chain

$$\begin{array}{c} \mathsf{Mnd}_{\mathsf{sf}}(\mathsf{Cat}) \\ H \left\uparrow \dashv \right\downarrow W \\ [\mathsf{Cat}_{\mathsf{sf}}, \mathsf{Cat}] \\ G \left\uparrow \dashv \right\downarrow V \\ [N, \mathsf{Cat}] \end{array}$$

of adjunctions with the composite adjunction

$$\begin{array}{c} \mathsf{Mnd}_{\mathsf{sf}}(\mathsf{Cat}) \\ F \left\uparrow \dashv \right\downarrow U \\ [N, \mathsf{Cat}] \end{array}$$

being monadic. Thus $\mathsf{Mnd}_{\mathsf{sf}}(\mathsf{Cat})$ is equivalent to the 2-category $[N, \mathsf{Cat}]^M$ of algebras for the 2-monad $M = UF$. The 2-category $[N, \mathsf{Cat}]^M$ is a locally finitely presentable category (in the 2-dimensional sense of [9]), and so it is complete and cocomplete. We will show that the right adjoint U preserves sifted colimits, and therefore M is strongly finitary. Then, using Theorem 2.8, we will be able to conclude that $\mathsf{Mnd}_{\mathsf{sf}}(\mathsf{Cat})$ admits the (b.o. full, faithful) factorisation system of Example 2.5.

2.10. Proposition. *The 2-monad $M = UF$ given by the adjunction*

$$\begin{array}{c} \mathsf{Mnd}_{\mathsf{sf}}(\mathsf{Cat}) \\ F \left\uparrow \dashv \right\downarrow U \\ [N, \mathsf{Cat}] \end{array}$$

is strongly finitary.

Proof. In the notation of the previous remark, U is the composite of right adjoints W and V. The 2-functor V, being defined as a precomposition with J (recall Remark 2.9), has itself a right adjoint and is therefore strongly finitary. To deduce that U preserves sifted colimits, and that M is thus strongly finitary, it is enough to show that W preserves sifted colimits. The argument can be taken almost verbatim from Section 4 of [10], where the authors show a similar result for *finitary* monads. In the following we shall identify the 2-category $[\mathsf{Cat}_{\mathsf{sf}}, \mathsf{Cat}]$ with the (2-equivalent) 2-category $\mathsf{StrFin}(\mathsf{Cat})$ of strongly finitary endo-2-functors of Cat as in Remark 2.7. Take a weight $\varphi : \mathscr{D}^{op} \longrightarrow \mathsf{Cat}$ for a sifted colimit (i.e., a sifted weight), and a diagram $D : \mathscr{D} \longrightarrow \mathsf{Mnd}_{\mathsf{sf}}(\mathsf{Cat})$ sending d to a strongly finitary 2-monad $(T_d, \mu^{T_d}, \eta^{T_d})$. Denote the weighted colimit object $\varphi * WD$ in $[\mathsf{Cat}_{\mathsf{sf}}, \mathsf{Cat}]$ by T. For every strongly finitary $S : \mathsf{Cat} \longrightarrow \mathsf{Cat}$, both $- \cdot S$ and $S \cdot -$ are again strongly finitary, the first by having a right adjoint, and the second one since colimits in $[\mathsf{Cat}_{\mathsf{sf}}, \mathsf{Cat}]$ are computed pointwise. Therefore the weighted colimit $(\varphi \times \varphi) * D'$ of the diagram $D' : \mathscr{D} \times \mathscr{D} \longrightarrow [\mathsf{Cat}_{\mathsf{sf}}, \mathsf{Cat}]$ sending (d, d') to $T_d \cdot T_{d'}$ weighted by $\varphi \times \varphi : \mathscr{D}^{op} \times \mathscr{D}^{op} \longrightarrow \mathsf{Cat}$ is the 2-functor TT. Since the diagonal 2-functor $\Delta : \mathscr{D} \longrightarrow \mathscr{D} \times \mathscr{D}$ is final with respect to the weight φ, it follows that the weighted colimit $\varphi * D'\Delta$ is also the 2-functor TT. This in turn induces a multiplication $\mu : TT \longrightarrow T$, and similarly we get the unit $\eta : Id \longrightarrow T$. Thus T carries a monad structure, and it follows that W preserves sifted colimits. ■

Consider now a quotient $e : T \twoheadrightarrow T'$ of monads T and T' in $\mathsf{Mnd}_{\mathsf{sf}}(\mathsf{Cat})$. From Theorem 2.8 it follows that e is pointwise b.o. full. That is, the functor $e_n : Tn \twoheadrightarrow T'n$ is b.o. full for every finite discrete category n. Of course, for strongly finitary monads we may state an even stronger pointwise property of quotient monad maps: given a quotient $e : T \twoheadrightarrow T'$, its component $e_C : TC \longrightarrow T'C$ is b.o. full for each *category* C. This is true since each category is a sifted colimit of finite discrete categories, and since T and T' preserve sifted colimits, see Remark 2.7.

Using the above observations, we shall see that quotients of monads correspond to equational subcategories of algebraic categories.

2.11. Remark. Let us give an algebraic meaning to the fact that a quotient $e : T \twoheadrightarrow T'$ of strongly finitary 2-monads on Cat implies that every $e_n : Tn \twoheadrightarrow T'n$ is b.o. full in Cat. Viewing the objects of Tn as n-ary terms, bijectivity on objects of e_n means that the quotient e does not postulate any new equations between terms. On the other hand, fullness of e_n means that $T'n$ is obtained from Tn by identifying morphisms in Tn. On the level of algebras, this imposes equations between *morphisms* of the underlying category of an algebra. We will make the notion of an equation precise in Section 3.

2.12. Example. Let $\mathscr{A}$ be the 2-category of categories C equipped with the following algebraic structure, subject to no axioms:

1. one nullary operation I and one binary operation $\otimes$,
2. natural transformations: the associator α with components $\alpha_{a,b,c} : (a \otimes b) \otimes c \longrightarrow a \otimes (b \otimes c)$ and the "back-associator" α' with components $\alpha'_{a,b,c} : a \otimes (b \otimes c) \longrightarrow (a \otimes b) \otimes c$,

3. natural transformations: the left and right unitors λ and ρ with the components $\lambda_a : I \otimes a \longrightarrow a$ and $\rho_a : a \otimes I \longrightarrow a$, together with "back-unitors" λ' and ρ' with the components $\lambda'_a : a \longrightarrow I \otimes a$ and $\rho'_a : a \longrightarrow a \otimes I$.

In particular, the "back-associators" and "back-unitors" are not forced to be the inverses of associators and unitors. The morphisms in $\mathscr{A}$ are those functors that preserve the algebraic structure "on the nose", and the 2-cells are monoidal natural transformations between those functors. We can obtain from $\mathscr{A}$ a full subcategory $\mathscr{B}$ spanned by "monoidal categories without coherence": that is, consider only those categories C from $\mathscr{A}$ whose associator and unitors are in fact natural *isomorphisms*, with their corresponding inverse transformations being the "back-transformations". In an informal sense, $\mathscr{B}$ is an equational subcategory of $\mathscr{A}$ defined by the equations

$$\alpha \cdot \alpha' = \alpha' \cdot \alpha = 1, \quad \lambda \cdot \lambda' = 1, \quad \lambda' \cdot \lambda = 1, \quad \rho \cdot \rho' = 1, \quad \rho' \cdot \rho = 1.$$

Let MonCat be the 2-category of monoidal categories, strict monoidal functors and monoidal natural transformations between those functors. Informally again, MonCat can be obtained as an equational subcategory of $\mathscr{B}$ by considering those categories from $\mathscr{B}$ that satisfy the usual triangle and pentagon identities.

The 2-category $\mathscr{A}$ can be easily seen to be the 2-category $\mathsf{Alg}(R)$ of algebras for a strongly finitary 2-monad R on Cat. The results of Section 3 will show that there is a chain

$$R \twoheadrightarrow S \twoheadrightarrow T$$

of quotients of strongly finitary 2-monads R, S and T for which we have the correspondences

$$\mathscr{A} \simeq \mathsf{Alg}(R), \quad \mathscr{B} \simeq \mathsf{Alg}(S), \quad \mathsf{MonCat} \simeq \mathsf{Alg}(T).$$

Moreover, the monad morphism quotients induce the inclusions

$$\mathsf{Alg}(T) \longrightarrow \mathsf{Alg}(S) \longrightarrow \mathsf{Alg}(R)$$

that correspond to the inclusions of equational subcategories $\mathsf{MonCat} \subseteq \mathscr{B} \subseteq \mathscr{A}$. The theory developed in Section 3 will make these correspondences precise.

We will end the present section with a remark stating that b.o. full morphisms are epimorphisms with respect to morphisms and 2-cells. These properties will allow us to prove a Cat-enriched Birkhoff theorem in the following section, with the proof being very much in the spirit of the proof for ordinary Birkhoff theorem. Specifically, these properties will be crucial in proving that quotients of monads induce 2-dimensionally fully faithful algebraic functors (as defined in Definition 3.1).

2.13. Remark. Given a b.o. full $h : C \longrightarrow A$ in Cat, the functor

$$\mathsf{Cat}(h, B) : \mathsf{Cat}(A, B) \longrightarrow \mathsf{Cat}(C, B)$$

is injective on objects and fully faithful for every B. The injectivity on objects of $\mathsf{Cat}(h, B)$ corresponds to h being an epimorphism in Cat, faithfulness of $\mathsf{Cat}(h, B)$ states that h is an epimorphism with respect to 2-cells, and fullness of $\mathsf{Cat}(h, B)$ corresponds to a factorisation property of h w.r.t. 2-cells.

Consider the following diagram

$$\begin{array}{ccccc} & & A & & \\ & \overset{h}{\nearrow} & & \overset{f}{\searrow} & \\ C & & \Downarrow\beta & & B \\ & \underset{h}{\searrow} & & \underset{g}{\nearrow} & \\ & & A & & \end{array}$$

in Cat with h being b.o. full. Denote by

$$K \underset{t}{\overset{s}{\rightrightarrows}} C \xrightarrow{\ h\ } A \qquad (\gamma \Downarrow \Downarrow \delta)$$

the kernel-quotient pair of h. The morphism h is a coequifier of the kernel diagram since Cat is $\mathscr{F}_{\mathsf{bof}}$-exact. Both the composites $f \cdot h$ and $g \cdot h$ also coequify the kernel diagram. By the 2-dimensional universal property of coequifiers the equality

$$\begin{array}{c} C \\ fh \left(\overset{\beta}{\Rightarrow} \right) gh \\ B \end{array} \qquad = \qquad \begin{array}{c} C \xrightarrow{\ h\ } A \\ f \left(\overset{\alpha}{\Rightarrow} \right) g \\ B \end{array}$$

holds for a unique 2-cell α. This observation equivalently says that $\mathsf{Cat}(h, B)$ is fully faithful.

3. Birkhoff theorem for the kernel-quotient system $\mathscr{F}_{\mathsf{bof}}$

In this section we first recall basic definitions concerning subcategories and equivalence of categories in the Cat-enriched setting. After a short review of the properties of algebraic categories and algebraic functors we state and prove the Birkhoff theorem for the $\mathscr{F}_{\mathsf{bof}}$ kernel-quotient system.

3.1. Definition. *A 2-functor $T : \mathscr{C} \longrightarrow \mathscr{D}$ between 2-categories $\mathscr{C}$ and $\mathscr{D}$ is called* fully faithful *if for any pair A, B of objects of $\mathscr{C}$ the action $T_{A,B} : \mathscr{C}(A, B) \longrightarrow \mathscr{D}(TA, TB)$ is an* isomorphism *of categories. We say that T exhibits $\mathscr{C}$ as a* full subcategory *of $\mathscr{D}$.*

When $\mathscr{C}$ is moreover closed in $\mathscr{D}$ under isomorphisms, we call $\mathscr{C}$ a replete *full subcategory of $\mathscr{D}$. The 2-category $\mathscr{C}$ is closed in $\mathscr{D}$ under isomorphisms if for any object A in $\mathscr{C}$ and any isomorphism $i : TA \longrightarrow D$ in $\mathscr{D}$ there exists an isomorphism $j : A \longrightarrow B$ in $\mathscr{C}$ such that $Tj = i$.*

The 2-categories $\mathscr{C}$ and $\mathscr{D}$ are equivalent *if there is a fully faithful 2-functor $T : \mathscr{C} \longrightarrow \mathscr{D}$ that is* essentially surjective*, that is, for any object D of $\mathscr{D}$ there exists an object A of $\mathscr{C}$ with TA being isomorphic to D, denoted by $TA \cong D$.*

3.2. Remark. Recall that an algebraic 2-category is a 2-category that is equivalent to a 2-category $\mathsf{Alg}(T)$ for some 2-monad T on $\mathscr{C}$. We will look at some important properties of algebraic categories and algebraic functors (functors arising from a monad morphism):

1. Consider two 2-monads T and T' on $\mathscr{C}$, and a monad morphism $e : T \longrightarrow T'$. This monad morphism gives rise to an algebraic 2-functor $\mathsf{Alg}(e) : \mathsf{Alg}(T') \longrightarrow \mathsf{Alg}(T)$ between the 2-categories $\mathsf{Alg}(T')$ and $\mathsf{Alg}(T)$ of algebras for T' and T. On objects, $\mathsf{Alg}(e)$ acts as follows:

$$\begin{array}{ccc} T'A & & TA \\ & & \downarrow e_A \\ a' \Big\downarrow & \mapsto & T'A \\ & & \downarrow a' \\ A & & A \end{array}$$

   On morphisms and 2-cells $\mathsf{Alg}(e)$ acts as an identity. A homomorphism $h : (A, a') \longrightarrow (B, b')$ between two T'-algebras $a' : T'A \longrightarrow A$ and $b' : T'B \longrightarrow B$ gets mapped to a homomorphism $h : (A, a' \cdot e_A) \longrightarrow (B, b' \cdot e_B)$ of the corresponding T-algebras. Indeed, the outer rectangle in the diagram

$$\begin{array}{ccc} TA & \xrightarrow{Th} & TB \\ e_A \downarrow & & \downarrow e_B \\ T'A & \xrightarrow{T'h} & T'B \\ a' \downarrow & & \downarrow b' \\ A & \xrightarrow[h]{} & B \end{array}$$

   clearly commutes. The same reasoning applies for the 2-cells $\alpha : h \Rightarrow h'$ between two homomorphisms $h : (A, a') \longrightarrow (B, b')$ and $h' : (A, a') \longrightarrow (B, b')$.

   The action of $\mathsf{Alg}(e)$ on morphisms and 2-cells is thus faithful for any $e : T \longrightarrow T'$.

2. The algebraic 2-category $\mathsf{Alg}(T)$ for a strongly finitary monad T on Cat is cowellpowered with respect to quotient algebras. Indeed, for every small category A there is, up to isomorphism, only a set of b.o. full functors of the form $h : A \longrightarrow B$ in Cat. Thus for a T-algebra (A, a) there is, up to isomorphism, only a set of quotients $h : (A, a) \longrightarrow (B, b)$ in $\mathsf{Alg}(T)$.

3. Given an algebraic 2-category $\mathsf{Alg}(T)$ for a strongly finitary 2-monad T on Cat, it is a standard observation that the underlying 2-functor $U : \mathsf{Alg}(T) \longrightarrow \mathsf{Cat}$ creates 2-limits. See Theorem 6.8 of [3] for a proof that U preserves these limits and observe that it can be easily modified to show that U in fact creates these limits. Since T is strongly finitary, the 2-functor U also creates sifted colimits. In particular, U creates reflexive coequifiers. That is, given a reflexive diagram

$$(K,k) \underset{h}{\overset{f}{\underset{g}{\rightrightarrows}}} (A,a) \quad (\Downarrow\Downarrow)$$

in $\mathsf{Alg}(T)$, and the coequifier of the U-image of the above diagram

$$K \underset{Uh}{\overset{Uf}{\underset{Ug}{\rightrightarrows}}} A \xrightarrow{\quad c \quad} C \quad (\Downarrow\Downarrow)$$

there exists a unique algebra (C, c) such that c is a homomorphism between (A, a) and (C, c).

We now turn to the proof of the Birkhoff theorem.

3.3. Theorem. *Let T be a strongly finitary 2-monad on* Cat *and let $\mathscr{A}$ be a full subcategory* $\mathsf{Alg}(T)$ *of the category of algebras for the 2-monad T. Then the following are equivalent:*

1. *There is a strongly finitary 2-monad T' and a b.o. full monad morphism $e : T \twoheadrightarrow T'$ such that the comparison 2-functor $\mathscr{A} \longrightarrow \mathsf{Alg}(T')$ is an equivalence.*
2. *The category $\mathscr{A}$ is closed in* $\mathsf{Alg}(T)$ *under sifted colimits, 2-products, quotient algebras, and subalgebras.*

Proof. We first prove the implication (1) $\Rightarrow$ (2) in the following manner:

(a) Given the monad morphism $e : T \twoheadrightarrow T'$, we get a 2-functor $\mathsf{Alg}(e) : \mathsf{Alg}(T') \longrightarrow \mathsf{Alg}(T)$ that we show to be fully faithful.

(b) We show that $\mathsf{Alg}(e)$ preserves sifted colimits and 2-limits.

(c) Finally we show that $\mathsf{Alg}(T')$ is closed in $\mathsf{Alg}(T)$ under subalgebras and quotient algebras.

Ad (a): The action of $\mathsf{Alg}(e)$ on morphisms and 2-cells is faithful by point (1) of Remark 3.2. We prove that $\mathsf{Alg}(e)$ is indeed fully faithful by showing that its action on morphisms and 2-cells is full. The fullness on morphisms comes from observing that given any diagram of the form

$$\begin{array}{ccc} TA & \xrightarrow{Th} & TB \\ {\scriptstyle e_A}\downarrow & & \downarrow{\scriptstyle e_B} \\ T'A & \xrightarrow{T'h} & T'B \\ {\scriptstyle a'}\downarrow & & \downarrow{\scriptstyle b'} \\ A & \xrightarrow[h]{} & B \end{array}$$

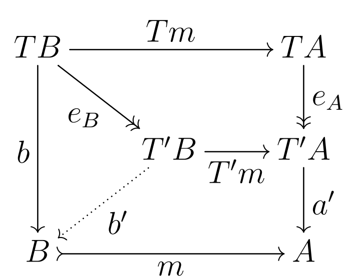
such that the outer rectangle commutes, the lower square commutes since $e_A : TA \twoheadrightarrow T'A$ is b.o. full, and thus epi. Similarly, given a 2-cell $\alpha : h \Rightarrow h'$ in $\mathsf{Alg}(T)$, it is also a 2-cell in $\mathsf{Alg}(T')$ by 2-naturality of e (saying that $e_B * T\alpha = \alpha * e_A$), and since e_A is an epimorphism on 2-cells by Remark 2.13. The algebraic 2-functor $\mathsf{Alg}(e)$ is therefore indeed fully faithful.

Ad (b): Let us denote by $U^T : \mathsf{Alg}(T) \longrightarrow \mathsf{Cat}$ and by $U^{T'} : \mathsf{Alg}(T') \longrightarrow \mathsf{Cat}$ the underlying 2-functors of $\mathsf{Alg}(T)$ and $\mathsf{Alg}(T')$. Then $U^{T'} = U^T \cdot \mathsf{Alg}(e)$. The 2-functor $U^{T'}$ preserves 2-limits and sifted colimits and U creates them. Therefore $\mathsf{Alg}(e)$ preserves 2-limits and sifted colimits.

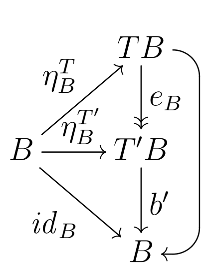
Ad (c): Now we show that the 2-category $\mathsf{Alg}(T')$ is closed in $\mathsf{Alg}(T)$ under subalgebras and quotient algebras. To this end, consider a T'-algebra (A, a') and its image $(A, a) = (A, a' \cdot e_A)$ under $\mathsf{Alg}(e)$. Given any subalgebra (B, b) of (A, a) as in the diagram

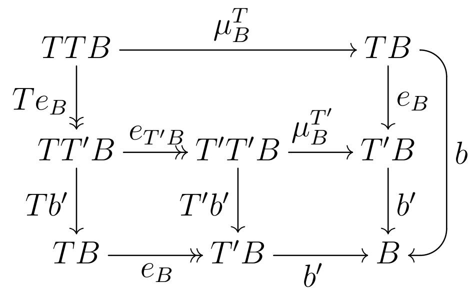
$$\begin{array}{ccc} TB & \xrightarrow{Tm} & TA \\ \Bigg\downarrow{\scriptstyle b} & & \downarrow{\scriptstyle e_A} \\ & & T'A \\ & & \downarrow{\scriptstyle a} \\ B & \underset{m}{\rightarrowtail} & A, \end{array}$$

we can use the naturality of e

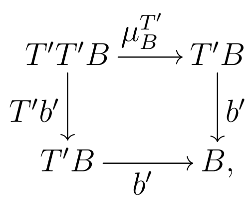


and define b' as the unique diagonal fill-in with respect to e_B and m in the above diagram. (Recall that (b.o. full, faithful) is a factorisation system on Cat.) This $b' : T'B \longrightarrow B$ is a T'-algebra. We inspect the following diagrams to see that (B, b') satisfies both algebra axioms.



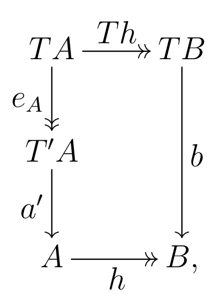


Consider the left-hand diagram. The upper triangle commutes by the unit axiom of the monad morphism e, and the outer triangle commutes since (B, b) is a T-algebra. Thus the lower triangle commutes by virtue of e_B being an epimorphism. In the right-hand diagram, the outer square commutes since (B, b) is a T-algebra. The upper rectangle is an instance of a monad morphism axiom, and the lower left square commutes by naturality of e. The morphism Te_B is b.o. full, as e_B is and T preserves b.o. full morphisms by Remark 2.7. Thus the composite morphism $e_{T'B} \cdot Te_B$ is b.o. full as well. By the cancellation property of b.o. full morphisms we obtain the commutativity of the square



and this proves that (B, b') is a T'-algebra. In conclusion, $\mathsf{Alg}(T')$ is indeed closed in $\mathsf{Alg}(T)$ under subalgebras.

The closedness of $\mathsf{Alg}(T')$ under quotient algebras in $\mathsf{Alg}(T)$ follows from closedness under limits and sifted colimits. Whenever we are given a T'-algebra (A, a') and a quotient homomorphism $h : (A, a) = (A, a' \cdot e_A) \twoheadrightarrow (B, b)$ of T-algebras as in



the kernel (K, k) of h lies in $\mathsf{Alg}(T')$. This is true since (K, k) is easily seen to be a subalgebra of the cotensor algebra $(A, a)^{\bullet \rightrightarrows \bullet}$ (where $\bullet \rightrightarrows \bullet$ denotes the obvious category), and $(A, a)^{\bullet \rightrightarrows \bullet}$ is in turn a subalgebra of the product algebra $(A, a)^2$. Since the kernel (K, k) is reflexive and as $\mathsf{Alg}(T')$ is closed in $\mathsf{Alg}(T)$ under sifted colimits, it follows that (B, b) lies in $\mathsf{Alg}(T')$.

The second part of the proof is the implication (2) $\Rightarrow$ (1). Given a strongly finitary 2-monad T and a full subcategory

$$J : \mathscr{A} \longrightarrow \mathsf{Alg}(T)$$

of $\mathsf{Alg}(T)$ that is closed under sifted colimits, 2-products, quotient algebras and subalgebras, we need to find a strongly finitary 2-monad T' such that there is a monad morphism $T \twoheadrightarrow T'$ and the comparison $\mathscr{A} \longrightarrow \mathsf{Alg}(T')$ is an equivalence. Observe that $\mathscr{A}$ is a replete subcategory of $\mathsf{Alg}(T)$ as this follows from closedness under unary products.

We will proceed as follows:

(a) We will form an *ordinary* left adjoint to J by using Freyd's Adjoint Functor Theorem [15].

(b) We will show that J preserves cotensors with $\mathbb{2}$ and that the ordinary adjunction is thus enriched in Cat, using Proposition 3.1 of [4].

(c) We will construct a monad morphism $T \twoheadrightarrow T'$ from the above adjunction and show the equivalence $\mathscr{A} \simeq \mathsf{Alg}(T')$.

Ad (a): We will show that $\mathscr{A}$ has and J preserves ordinary limits. Since J is fully faithful, it suffices to prove that $\mathscr{A}$ is closed in $\mathsf{Alg}(T)$ under ordinary limits. By assumption, $\mathscr{A}$ is closed in $\mathsf{Alg}(T)$ under 2-products. It is therefore closed under ordinary products as well, since 2-products and ordinary products coincide in Cat. We need to show that it is closed also under equalisers. To this end, consider an equaliser diagram

$$(A,a) \rightarrowtail JX \underset{Jt}{\overset{Js}{\rightrightarrows}} JY$$

in $\mathsf{Alg}(T)$. Equalisers in $\mathsf{Alg}(T)$ are computed on the level of underlying categories, which implies that $A \rightarrowtail UJX$ is faithful. Thus (A,a) is a subalgebra of JX. Since the 2-category $\mathscr{A}$ is closed under subalgebras in $\mathsf{Alg}(T)$, we proved that it is closed under equalisers as well.

To establish the existence of a left adjoint for J, we now only need to find an ordinary solution set for every object (A,a) of $\mathsf{Alg}(T)$. We claim that the solution set is the set $\{h_i : (A,a) \twoheadrightarrow JX_i \mid i \in I\}$ of all the (representatives of the) quotients of (A,a) that lie in $\mathscr{A}$. This is indeed a set due to the nature of b.o. fullness, recall point (2) of Remark 3.2. Given any morphism $f : (A,a) \longrightarrow JY$, we can factorise it to obtain a triangle

$$\begin{array}{ccc} (A,a) & \xrightarrow{\quad f \quad} & JY \\ & {\scriptstyle h}\searrow\!\!\!\!\!\searrow \quad \nearrow\!\!\!\!\rightarrowtail & \\ & (B,b) & \end{array}$$

and moreover, since (B,b) is a subalgebra of JY, we have that $(B,b) \cong JX$ holds for some X from $\mathscr{A}$, and the solution set condition is satisfied. The unit of the adjunction is constructed as follows: we take the product $\prod_{i\in I} JX_i$ of all the codomains of the quotients in the solution set, and factorise the mediating morphism $(h_i) : (A,a) \longrightarrow \prod_{i\in I} JX_i$ as in the following diagram.

$$\begin{array}{ccc} (A,a) & \xrightarrow{\quad (h_i) \quad} & \prod_{i\in I} JX_i \\ & {\scriptstyle \eta_{(A,a)}} \searrow \qquad \nearrow & \\ & JL(A,a) & \end{array}$$

Note that $\eta_{(A,a)}$ is b.o. full for every algebra (A,a).

Ad (b): Take a T-algebra (A,a) that belongs to $\mathscr{A}$ and form its cotensor $(A,a)^{\mathbb{2}}$. By means of the inclusion functor $2 \longrightarrow \mathbb{2}$, we have a canonical homomorphism $(A,a)^{\mathbb{2}} \longrightarrow (A,a)^2$ whose underlying functor is faithful, and thus renders $(A,a)^{\mathbb{2}}$ as a subalgebra of a product of algebras contained in $\mathscr{A}$. By the closure properties imposed on $\mathscr{A}$, we have that $\mathscr{A}$ is closed in $\mathsf{Alg}(T)$ under forming cotensors with $\mathbb{2}$ as well.

Ad (c): We can now define the 2-monad T' and the monad morphism $\varphi : T \longrightarrow T'$ for which we will show the equivalence $\mathscr{A} \simeq \mathsf{Alg}(T')$. Let us first settle the notation and write $(L \dashv J, \eta, \varepsilon)$ for the adjunction $L \dashv J : \mathscr{A} \longrightarrow \mathsf{Alg}(T)$, denote by $(F^T, U^T, \eta^T, \varepsilon^T)$ the adjunction $F^T \dashv U^T : \mathsf{Alg}(T) \longrightarrow \mathsf{Cat}$, and let $\mu^T : TT \longrightarrow T$ be the multiplication of the 2-monad T.

This allows us to define the 2-functor $T' := U^T JLF^T$ which is the underlying endofunctor of a 2-monad $(T', \eta^{T'}, \mu^{T'})$ with the unit $\eta^{T'}$ and the composition $\mu^{T'}$ defined by the assignments

$$\eta^{T'} := U^T \eta F^T \cdot \eta^T, \qquad \mu^{T'} := U^T J\varepsilon LF^T \cdot U^T JL\varepsilon^T JLF^T.$$

Then there is a corresponding monad morphism $\varphi = U^T \eta F^T : T \twoheadrightarrow T'$. The proof that φ is indeed a monad morphism is standard and proceeds exactly as in the non-enriched case. Moreover, φ is a quotient, since

1. $\eta_{(A,a)}$ is a quotient for each algebra (A,a), and
2. U^T preserves quotients since T does.

Let us denote by

$$\begin{array}{ccc} \mathscr{A} & \xrightarrow{\quad K \quad} & \mathsf{Alg}(T') \\ & {\scriptstyle U^T J} \searrow \qquad \swarrow {\scriptstyle U^{T'}} & \\ & \mathsf{Cat} & \end{array}$$

the ordinary comparison functor. We will apply the ordinary Beck's theorem to infer that K is an ordinary equivalence. Since $\mathscr{A}$ has and $U^T J$ preserves sifted colimits, $\mathscr{A}$ has and

$U^T J$ preserves coequalisers of reflexive pairs. Moreover, since U^T reflects isomorphisms and J is fully faithful, the composite functor $U^T J$ also reflects isomorphisms. Therefore $K : \mathscr{A} \longrightarrow \mathsf{Alg}(T')$ is indeed an equivalence in the ordinary sense.

We will now show that on objects, the inclusion $J : \mathscr{A} \longrightarrow \mathsf{Alg}(T)$ factorises, up to isomorphism, as in the following triangle:

$$\begin{array}{ccc} & & \mathsf{Alg}(T') \\ & \overset{K}{\nearrow} & \\ \mathscr{A} & & \Big\downarrow \mathsf{Alg}(\varphi) \\ & \underset{J}{\searrow} & \\ & & \mathsf{Alg}(T). \end{array}$$

Indeed, for any object A of $\mathscr{A}$ the equality

$$KA = (U^T JA, U^T J\varepsilon_A \cdot U^T JL\varepsilon^T_{JA})$$

holds. The algebra KA gets mapped by the functor $\mathsf{Alg}(\varphi)$ to an algebra with a structure map

$$\begin{aligned} U^T J\varepsilon_A \cdot U^T JL\varepsilon^T_{JA} \cdot \varphi_{U^T JA} &= U^T J\varepsilon_A \cdot U^T JL\varepsilon^T_{JA} \cdot U^T \eta_{F^T U^T JA} \\ &= U^T J\varepsilon_A \cdot U^T \eta_{JA} \cdot U^T \varepsilon^T_{JA} \\ &= U^T \varepsilon^T_{JA}, \end{aligned}$$

where the first equality holds by the definition of φ, the second one follows from naturality of η, and the third one comes from the triangle identity of $L \dashv J$. But $(U^T JA, U^T \varepsilon^T_{JA})$ is isomorphic to JA, as $(U^T JA, U^T \varepsilon^T_{JA})$ is the image of JA under the trivial comparison functor

$$I : \mathsf{Alg}(T) \longrightarrow \mathsf{Alg}(T).$$

Both J and $\mathsf{Alg}(\varphi)$ are fully faithful in Cat-enriched sense: the 2-functor J is such by assumption and $\mathsf{Alg}(\varphi)$ was proved to be fully faithful for a quotient monad morphism φ in the first part of the proof. We can conclude that the ordinary equivalence $K : \mathscr{A} \longrightarrow \mathsf{Alg}(T')$ is enriched in Cat, thus finishing the proof. ∎

3.4. Remark. A point that needs to be discussed is that we demand $\mathscr{A}$ to be closed under sifted colimits in $\mathsf{Alg}(T)$ in the characterisation of equational subcategories of $\mathsf{Alg}(T)$. It is true that in the original Birkhoff theorem there is no need to demand closedness under any class of colimits whatsoever. However, even in the ordinary case of $\mathscr{V} = \mathsf{Set}$, closedness under filtered colimits (or directed unions) is essential in the case of many-sorted universal algebra, see [2]. In the case of $\mathscr{V} = \mathsf{Cat}$, at least the requirement for closedness under filtered colimits is arguably expectable. The reason why our version of the Birkhoff theorem asks for an even stronger closure property, i.e., closedness under *sifted* colimits, is the following. While finitary and strongly finitary monads on Set coincide (every finitary monad is strongly finitary), this is not the case for 2-monads on Cat: a finitary 2-monad need not be strongly finitary. For example, the 2-monad T that gives rise to the 2-category $\mathsf{Alg}(T)$ of categories $\mathscr{C}$ equipped with one "arrow-ary" operation $\mathscr{C}^2 \longrightarrow \mathscr{C}$ is finitary, but T fails to preserve sifted colimits in general. Since we are dealing with *strongly finitary* 2-monads on Cat, being closed under *sifted* colimits is the corresponding closure property.

3.5. Remark. In our setting, the property of being closed under sifted colimits is equivalent to being closed under *conical* filtered colimits and under *codescent objects of strict reflexive coherence data* by Remark 8.44 of [5]. For our purposes, the only two important points concerning codescent objects are that

1. they are the colimit objects for a certain *sifted* diagram, and
2. in the categories $\mathsf{Alg}(T)$ for a strongly finitary 2-monad T on Cat, the *universal cocone* over such a diagram consists of a single *bijective on objects homomorphism.*

This allows us to state the conditions of our Birkhoff theorem in an alternative way. In $\mathsf{Alg}(T)$, define an algebra (B, b) to be a *(b.o.)-quotient* of (A, a) if there is a homomorphism $h : (A, a) \longrightarrow (B, b)$ that is bijective on objects. Since every b.o. full functor is b.o., we may strengthen the property of being closed under quotient algebras to the property of being closed under (b.o.)-quotient algebras, and replace the requirement for closedness under sifted colimits by closedness under filtered colimits.

It remains to argue that a full subcategory $\mathscr{A}$ of $\mathsf{Alg}(T)$ closed under 2-limits and sifted colimits is closed under (b.o.)-quotients. Given a (b.o.)-quotient $h : (A, a) \longrightarrow (B, b)$ with (A, a) contained in $\mathscr{A}$, it follows by the results of [6] (see Section 5.1 in particular) that h is the quotient of the kernel

$$h|h|h \underset{q}{\overset{p}{\rightrightarrows}} h|h \qquad h|h|h \xrightarrow{m} h|h \xleftarrow{i} (A, a), \qquad (A,a) \underset{c}{\overset{d}{\rightrightarrows}} h|h$$

where the component $h|h$ is a certain subalgebra of $(A, a)^2$, and the component $h|h|h$ is a pullback of c and d. Without loss of generality, the above kernel can be considered reflexive, and thus (B, b) belongs to $\mathscr{A}$, being a sifted colimit of algebras contained in $\mathscr{A}$.

The following alternative statement of Birkhoff theorem is a direct corollary of the above remark, and it may be more useful in practice for detecting equational subcategories of algebraic categories.

3.6. Corollary. *The full subcategory $\mathscr{A}$ of $\mathsf{Alg}(T)$ is an equational subcategory of $\mathsf{Alg}(T)$ if and only if it is closed in $\mathsf{Alg}(T)$ under 2-products, (b.o.)-quotient algebras, subalgebras and filtered colimits.*

In the ordinary setting, full algebraic subcategories induced by a quotient monad morphism can be characterised as a special kind of orthogonal subcategories. Without substantial changes to the reasoning, the same characterisation can be obtained for the case of $\mathscr{V} = \mathsf{Cat}$, as is shown below.

Given a 2-category $\mathscr{X}$ and a set $S = \{f_i : X_i \longrightarrow Y_i \mid i \in I\}$ of morphisms of $\mathscr{X}$, we will denote by $S^\perp$ the full subcategory $J : \mathscr{Y} \longrightarrow \mathscr{X}$ spanned by the objects Y that are orthogonal to all morphisms in S.

3.7. Corollary. *The equational subcategories*

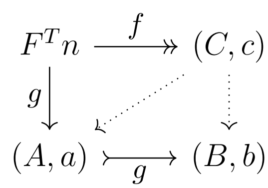
$$J : \mathscr{A} \longrightarrow \mathsf{Alg}(T)$$

of the 2-category $\mathsf{Alg}(T)$ *of algebras for a strongly finitary 2-monad* T *are precisely the orthogonal subcategories of* $\mathsf{Alg}(T)$ *of the form*

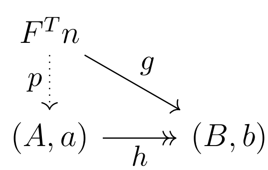
$$\mathscr{A} = \{f : F^T n \twoheadrightarrow (C, c) \mid f \in I\}^\perp = I^\perp$$

for some set I *of quotient morphisms in* $\mathsf{Alg}(T)$*. Moreover, each morphism in* I *has as its domain a free algebra on a finite discrete category.*

Proof. To see that one direction of this statement holds, observe that $\mathscr{A}$ is closed under subobjects in $\mathsf{Alg}(T)$: Given an algebra (B, b) in $\mathscr{A}$ and its subalgebra (A, a), we have for any $g : F^T n \longrightarrow (A, a)$ a situation



where the unique morphism $(C, c) \longrightarrow (B, b)$ exists since $f \perp (B, b)$, and the unique diagonal exists by the diagonal property of the factorisation system. The universal property of 2-products establishes that $\mathscr{A}$ is closed in $\mathsf{Alg}(T)$ under 2-products. To see that $\mathscr{A}$ is closed in $\mathsf{Alg}(T)$ under sifted colimits we show that $\mathsf{Alg}(T)(F^T n, -)$ preserves sifted colimits. This is the case, since $\mathsf{Alg}(T)(F^T n, -) \cong \mathsf{Cat}(n, U^T -)$ and both $\mathsf{Cat}(n, -)$ and U^T preserve sifted colimits. To show that $\mathscr{A}$ is closed in $\mathsf{Alg}(T)$ under quotients, observe first that $\mathsf{Alg}(T)(F^T n, -)$ preserves quotient maps since both U^T and $\mathsf{Cat}(n, -)$ are easily seen to preserve quotient maps. This property implies that $F^T n$ is projective with respect to quotients, and that the factorisation granted by projectivity is unique. Consider a quotient $h : (A, a) \twoheadrightarrow (B, b)$ with (A, a) in $\mathscr{A}$. To prove that (B, b) is in $\mathscr{A}$, observe that for any morphism $g : F^T n \longrightarrow (B, b)$ there is a unique morphism $p : F^T n \longrightarrow (A, a)$:



Since (A, a) is orthogonal to f, we obtain a triangle

$$\begin{array}{ccc} F^T n & \xrightarrow{f} & (C,c) \\ {\scriptstyle p}\downarrow & \swarrow {\scriptstyle o} & \\ (A,a) & & \end{array}$$

The composite $h \cdot o$ then proves that $f \perp (B, b)$. Indeed, given any other factorisation $g = i \cdot f$, the equality $i = h \cdot o$ holds since f is epi.

In the other direction, recall that reflective subcategories are *always* orthogonality classes. In our case we have that

$$\mathscr{A} = \{\eta_{(A,a)} : (A, a) \twoheadrightarrow JL(A, a) \mid (A, a) \in \mathsf{Alg}(T)\}^{\perp}.$$

We need to take a subset of the above class of morphisms such that the codomain of each morphism is a free algebra on a finite discrete category. For this, we first use that every algebra (A, a) is a sifted colimit of free algebras on finite discrete categories. Indeed, consider the full subcategory $\mathcal{G} \longrightarrow \mathsf{Alg}(T)$ spanned by algebras of the form $F^T n$ for a natural number n. By Proposition 4.2 of [11], $\mathsf{Alg}(T)$ is a free cocompletion of $\mathcal{G}$ under sifted colimits; the only interesting property to check being that the closure of $\mathcal{G}$ in $\mathsf{Alg}(T)$ under sifted colimits is the whole of $\mathsf{Alg}(T)$. Observe that a free algebra $F^T X$ on a discrete category X is a filtered colimit of free algebras on finite discrete categories, a free algebra $F^T C$ on a category C is a sifted colimit (codescent object) of free algebras on discrete categories, and any algebra (A, a) is a reflexive coequaliser of free algebras $F^T A$ on A. The result follows from this reasoning.

Secondly, if an object is orthogonal to a given set of arrows, it is orthogonal to their colimit in the category of arrows as well. Since JL preserves sifted colimits, we get that

$$\{\eta_{(A,a)} : (A, a) \twoheadrightarrow JL(A, a) \mid (A, a) \in \mathsf{Alg}(T)\}^{\perp}$$

is equal to the subcategory

$$\{\eta_{F^T n} : F^T n \twoheadrightarrow JLF^T n \mid n \in \mathsf{Cat}, n \text{ finite discrete}\}^{\perp},$$

as we needed. ∎

The above result may be reformulated to resemble the original universal algebraic formulation of Birkhoff's theorem even more. Taking again

$$\mathscr{A} = \{f : F^T n \twoheadrightarrow (C, c) \mid f \in I\}^{\perp},$$

we know that any morphism $f : F^T n \twoheadrightarrow (C, c)$ as above is the coequifier of its kernel:

$$(K,k) \underset{t}{\overset{s}{\rightrightarrows}} F^T n \xrightarrow{f} (C,c) \qquad \gamma \Downarrow \Downarrow \delta$$

Given an algebra (A, a), it is orthogonal to f precisely when each morphism $g : F^T n \longrightarrow (A, a)$ coequifies the 2-cells γ and δ. Now consider the underlying set K_0 of the category K by means of the b.o. inclusion functor $i : K_0 \longrightarrow U^T(K, k)$. Transposing this functor, we get a homomorphism $i^\sharp : F^T K_0 \longrightarrow (K, k)$ defined as the composite

$$\begin{array}{ccc} F^T K_0 & \xrightarrow{\quad i^\sharp \quad} & (K,k) \\ & {\scriptstyle F^T i} \searrow \quad \nearrow {\scriptstyle \varepsilon^T_{(K,k)}} & \\ & F^T U^T(K,k) & \end{array}$$

of two homomorphisms that are surjective on objects. The morphism $\varepsilon^T_{(K,k)}$ is surjective on objects since its underlying functor is a split epi k, and $F^T i$ is in fact b.o., because $T = U^T F^T$ as a strongly finitary monad preserves b.o. functors. A given morphism $g : F^T n \longrightarrow (A, a)$ therefore coequifies γ and δ if and only if it coequifies the whiskered 2-cells $\gamma * i^\sharp$ and $\delta * i^\sharp$:

$$F^T K_0 \xrightarrow{i^\sharp} (K,k) \underset{t}{\overset{s}{\rightrightarrows}} F^T n \qquad \gamma \Downarrow \Downarrow \delta$$

As a left adjoint, F^T preserves coproducts, and thus

$$F^T K_0 \cong F^T(\coprod_{\mathsf{obj}(K_0)} \mathbb{1}) \cong \coprod_{\mathsf{obj}(K_0)} F^T \mathbb{1}$$

holds. This allows us to reduce the pair γ and δ of 2-cells into $\mathsf{obj}(K_0)$-many pairs γ_c and δ_c of 2-cells

$$F^T \mathbb{1} \underset{t_c}{\overset{s_c}{\rightrightarrows}} F^T n \qquad \gamma_c \Downarrow \Downarrow \delta_c$$

such that a morphism $g : F^T n \longrightarrow (A, a)$ coequifies γ and δ precisely when it coequifies all the pairs γ_c and δ_c. In fact, let us call each such a pair (γ_c, δ_c) an *equation in* T and observe that it corresponds precisely to a pair of morphisms in $U^T F^T n$. Let us now say that an algebra (A, a) from $\mathsf{Alg}(T)$ *satisfies the equation* $\gamma_c = \delta_c$ if every morphism $g : F^T n \longrightarrow (A, a)$ coequifies γ_c and δ_c. We have just proved the following "universal-algebraic" version of Birkhoff's theorem.

3.8. Corollary. *A full subcategory $\mathscr{A}$ of $\mathsf{Alg}(T)$ for a strongly finitary monad T on* Cat *is closed under 2-products, subalgebras, quotient algebras and sifted colimits if and only if there is a set $E = \{\gamma_i = \delta_i \mid i \in I\}$ of equations in T such that $\mathscr{A}$ consists of algebras of $\mathsf{Alg}(T)$ that satisfy E.*

4. Concluding remarks

In this section we first discuss possible directions for future work. Then we conclude by showing that the kernel-quotient systems $\mathscr{F}_{\mathsf{bo}}$ and $\mathscr{F}_{\mathsf{so}}$ are inadequate for obtaining any kind of a well-behaved Birkhoff-type theorem.

Equational logic for the $\mathscr{F}_{\mathsf{bof}}$ factorisation system. In classical universal algebra, it is known that there is an *equational logic* that is sound and complete with respect to the notion of *equational consequence*. See Section 3.2.4 of [16] for a nice treatment. An obvious question is whether there is an "equational logic" sound and complete with respect to the notion of equational consequence that comes from our definition of what an equation is in the 2-dimensional context. Finding such a calculus is a problem for future work.

Other factorisation systems. We will discuss some problems concerning the factorisation systems (bijective on objects, fully faithful) and (surjective on objects, injective on objects and fully faithful) on Cat.

We can see immediately that the situation is very different in the case of the (b.o., f.f.) factorisation system when compared to the (b.o. full, faithful) case. Given a monad morphism $e : T \longrightarrow T'$ with $e_X : TX \longrightarrow T'X$ being b.o. for all categories X, the algebraic functor

$$\mathsf{Alg}(e) : \mathsf{Alg}(T') \longrightarrow \mathsf{Alg}(T)$$

need not be fully faithful in the 2-dimensional sense. This calls for a different approach to stating and proving a Birkhoff-style theorem. Indeed, trying to mimic the approach to the case of the (b.o. full, faithful) factorisation system breaks down at the very beginning: we will not be able to characterise equational subcategories by their closure properties, as the subcategories need not be full. Even more goes wrong: not every b.o. functor is epimorphic in Cat, and Cat is not cowellpowered with respect to b.o. quotients.

Recall from Remark 2.11 that a b.o. full quotient $e : T \twoheadrightarrow T'$ has the components $e_n : Tn \twoheadrightarrow T'n$ pointwise b.o. full in Cat. Algebraically, this specifies new equations that have to hold between morphisms in a T'-algebra. However, no such algebraic meaning can be given in the case of a b.o. quotient $e : T \twoheadrightarrow T'$. This is because the components $e_n : Tn \twoheadrightarrow T'n$ are pointwise *only b.o.* in Cat. Thus T' as a monad may contain new 2-dimensional algebraic information, and in this context it does not make sense to talk about $\mathsf{Alg}(T')$ as of an equational subcategory of $\mathsf{Alg}(T)$.

The same remarks remain true when considering the (s.o., i.o.f.f.) factorisation system. Thus both the (b.o. full, faithful) and (s.o., i.o.f.f.) factorisation systems would allow only for a very weak and rather generic Birkhoff theorem.

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