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A DOLD-KAN THEOREM FOR SIMPLICIAL LIE ALGEBRAS

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ABSTRACT. We introduce and study *hypercrossed complexes of Lie algebras*, that is, non-negatively graded chain complexes of Lie algebras $L = (L_n, \partial_n)$ endowed with an additional structure by means of a suitable set of bilinear maps $L_r \times L_s \to L_n$. The Moore complex of any simplicial Lie algebra acquires such a structure and, in this way, we prove a Dold-Kan type equivalence between the category of simplicial Lie algebras and the category of hypercrossed complexes of Lie algebras. Several consequences of examining particular classes of hypercrossed complexes of Lie algebras are presented.

1. Introduction and summary

It is well known that the *Moore complex functor* N : Simpl(Ab) $\to$ Ch(Ab) sets up an equivalence between the categories of simplicial abelian groups and (non-negatively graded) chain complexes. This seminal fact was first proven independently by A. Dold and D. Kan [Dold (1958), Kan (1958)], and was soon extended to the general ground context of abelian categories in [Dold-Puppe (1961)]. Since then, the existence of any generalization for simplicial objects in non-abelian categories has been extensively studied, and a long list of relevant Dold-Kan type theorems can be found in the literature. We refer the reader to the recent paper [Bourn (2007)], where D. Bourn proves that the Moore complex functor is monadic when the ground category is semi-abelian (like groups or Lie algebras), which means that the category of simplicial objects in the category is equivalent to the category of algebras for the induced monad on the category of internal chain complexes. Also of interest is the subsequent paper by S. Lack and R. Street [Lack, Street (2015)] generalizing Bourn's theorem both to more general source categories and settings others than simplicial ones.

In [Carrasco, Cegarra (1991)], the authors gave an extension of the Dold-Kan theorem for arbitrary simplicial groups (not just abelian ones). Our result there states that the Moore complex functor $N : \mathrm{Simpl(Gp)} \to \mathrm{Ch(Gp)}$ underlies an enriched one

$$N : \mathrm{Simpl(Gp)} \to \mathrm{HXCh(Gp)}, \tag{1}$$

which establishes an equivalence between the category of simplicial groups and the category of *hypercrossed complexes* of groups, that is, chain complexes of groups (G_n, ∂_n) endowed with an additional structure in the form of certain binary operations $G_r \times G_s \to G_n$,

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satisfying suitable axioms. The stratagem was to codify all the operations acquired by the Moore complex of a simplicial group that are needed to rebuild it up to isomorphism.

Both the notion of hypercrossed complexes of groups and the above equivalence (1) transpose without difficulty to the context of Lie groups, so that there is an equivalence

$$N : \mathrm{Simpl}(\mathrm{Lie\,Gp}) \to \mathrm{HXCh}(\mathrm{Lie\,Gp}), \tag{2}$$

between the category of simplicial Lie groups and the category of hypercrossed complexes of Lie groups. Lie's second and Cartan-Lie's theorems assure us that the Lie functor establishes an equivalence between the category of simply connected Lie groups and the category of finite-dimensional real Lie algebras. Then, by transporting through this Lie's equivalence, we can derive an adequate infinitesimal counterpart of the notion of hypercrossed complexes of Lie groups, *differential hypercrossed complexes*, or *hypercrossed complexes of finite-dimensional real Lie algebras*, as well as obtain from (2) a Dold-Kan type equivalence

$$N : \mathrm{Simpl}({\scriptstyle \mathrm{f.\,d.}}\,\mathrm{Lie}\,\mathbb{R}\text{-}\mathrm{Alg}) \to \mathrm{HXCh}({\scriptstyle \mathrm{f.\,d.}}\,\mathrm{Lie}\,\mathbb{R}\text{-}\mathrm{Alg}), \tag{3}$$

between the category of simplicial finite-dimensional real Lie algebras and the category of differential hypercrossed complexes.

A similar route can be followed starting with complex Lie groups and ending with an equivalence like (3) but replacing $\mathbb{R}$ with the field $\mathbb{C}$ of complex numbers. Furthermore, by (3), for example, simplicial finite-dimensional real Lie algebras with Moore complexes trivial at dimensions higher than 1 correspond to *differential crossed modules* [Baez (2002)], and those with trivial Moore complexes at dimensions higher than 2 correspond to *differential* 2*-crossed modules* [Martins, Picken (2011)], [Jurco (2012)]. However, these are, respectively, particular instances of *crossed modules of Lie algebras* and 2*-crossed modules of Lie algebras*, defined in the context of abstract Lie algebras over any commutative ring R in [Kassel, Loday (1982)] and [Ellis (1993)], where the authors capture the structure of a Moore complex of Lie R-algebras of length 1 or 2 in their definitions of crossed module and 2-crossed module, respectively.

Consequently, we were persuaded of the potential interest of developing a theory of hypercrossed complexes for abstract Lie algebras over any ground commutative ring R, with the aim of obtaining the equivalence (3) merely as a particular case of a general Dold-Kan type correspondence,

$$N : \mathrm{Simpl}(\mathrm{Lie}\ R\text{-}\mathrm{Alg}) \to \mathrm{HXCh}(\mathrm{Lie}\ R\text{-}\mathrm{Alg}), \tag{4}$$

which is underlain by the Dold-Kan correspondence $N : \mathrm{Simpl}(R\text{-}\mathrm{Mod}) \to \mathrm{Ch}(R\text{-}\mathrm{Mod})$. This is the main goal of this paper.

Of course, Bourn's aforementioned theorem applies in our context of Lie algebras. Therefore, the monad induced by the Moore complex functor on the category of chain complexes of Lie algebras gives "hypercrossed complexes". However, their explicit (computational) description here is outside the scope of the Bourn theory (at least until a full analysis of the actual form of this monad is made).

The plan of this paper, in summary, is as follows. After this introductory section, the paper is organized into six sections, each with subsections:

Section 2 is preparatory. Its three subsections are dedicated to setting some notations and needed basic constructions concerning the simplicial category Δ, the total order on the sets $S(n)$ of all surjections $[n] \twoheadrightarrow [r]$ in Δ, and the category of simplicial Lie algebras.

In Section 3 we introduce and study *higher semidirect products* of Lie algebras. This is a key construction in our discussion since just the structure of a 2^n-semidirect product is acquired by the algebra of n-simplices $\mathcal{L}_n$ of any simplicial Lie algebra $\mathcal{L}$ when it is analyzed through its Moore complex, as we show in Section 5. In a first subsection here, we provide a needed brief account of the Schreier theory for extensions of Lie algebras.

Section 4 is entirely devoted to presenting our linear models for simplicial Lie algebras, the *hypercrossed complexes of Lie algebras*. These are pairs (L, Φ) consisting of chain complexes of Lie algebras $L = (L_n, \partial_n)$ enriched with an additional structure by means of a suitable set Φ of bilinear maps $L_r \times L_s \to L_n$, which satisfy some well behaved properties summarized in eight axioms. Some technical results are also included.

Section 5 contains our main result in the paper- the *enriched Dold-Kan equivalence* (4). To obtain it, we dedicate a first subsection to analyzing the structure of the algebras of n-simplices in any simplicial Lie algebra. As mentioned above, we conclude that they are higher semidirect products. In the second subsection we introduce the *enriched Moore complex* $N(\mathcal{L}) = (N(\mathcal{L}), \Phi(\mathcal{L}))$ of any simplicial Lie algebra $\mathcal{L}$, and we prove that $N(\mathcal{L})$ is actually a hypercrossed complex of Lie algebras. Finally, in the third subsection we show that, from any hypercrossed complex, one can build a simplicial Lie algebra whose enriched Moore complex takes us back to our given hypercrossed complex, up to isomorphism. Thus, the functor N in (4) is an equivalence of categories.

In Section 6, we show the consequences of examining certain particular classes of hypercrossed complexes of Lie algebras, like as chain complexes of modules, crossed complexes of Lie algebras, hypercrossed crossed of Lie algebras with finitely many non trivial terms, crossed modules of Lie algebras, and braided or symmetric crossed modules of Lie algebras. By the enriched Dold-Kan equivalence (4), these are respectively related to simplicial modules, Dakin's simplicial T-complexes [Dakin (1977)] of Lie algebras, Duskin-Schanuel's m-hypergroupoids [Duskin (1975), Glenn (1982)] of Lie algebras, 1-hypergroupoids of Lie algebras (= nerves of internal groupoids in the category of Lie algebras), and simplicial deloopings of 1-hypergroupoids of Lie algebras.

Finally, Section 7 is devoted to gathering most of the more technical proofs, which we have placed there so as not to hamper the flow of the paper.

Throughout the paper, all modules and Lie algebras are to be understood as being over a (any) fixed ground commutative ring R.

2. Simplicial preliminaries

We employ the standard symbolism and nomenclature which can be found in texts on simplicial theory (as in [Goerss, Jardine (1999)] and [May (1967)], for example). For definiteness or emphasis we state the following.

2.1. On the simplicial category. Let Δ denote the simplicial category, whose objects are the ordered sets $[n] = \{0, \dots, n\}$, and whose arrows are the (weakly) monotone maps between them.

For any map $\alpha : [n] \to [r]$ in Δ, we call r *range* of α, and denote it by r_α. Also, we will denote by $R(\alpha)$ the ordered set of positive integers defined by

$$R(\alpha) = \{i \mid i < n \text{ and } \alpha(i) = \alpha(i+1)\}.$$

If $R(\alpha) = \{i_1, \dots, i_p\}$, we always assume that the elements are written in order, that is, with $i_1 < \cdots < i_p$.

The *coface* maps, that is, the injections $[n-1] \hookrightarrow [n]$ in Δ which omit the ith element, are denoted by $\delta_i : [n-1] \hookrightarrow [n]$, while the *codegeneracy* maps, that is, the surjections $[n+1] \twoheadrightarrow [n]$ with $R(\sigma_i) = \{i\}$, are denoted by $\sigma_i : [n+1] \twoheadrightarrow [n]$. Recall that these maps satisfy the well-know *cosimplicial identities*: $\delta_j \delta_i = \delta_i \delta_{j-1}$, etc.

Any monotonic $\alpha : [n] \to [m]$ has a unique factorization

$$\alpha = \delta_{k_q} \cdots \delta_{k_1} \sigma_{i_1} \cdots \sigma_{i_p} \tag{5}$$

with $0 \leq k_1 < \cdots < k_q \leq m$, $0 \leq i_1 < \cdots < i_p < n$, and $n + q = m + p$ [Mac Lane (2013), Chap. VII, §5, Lemma], where $\{i_1, \dots, i_p\} = R(\alpha)$, and $\{k_1, \dots, k_q\}$ is the ordered set of elements of $[m]$ not in the image $\alpha[n]$ of α. Since the cosimplicial identities suffice to put any composite of δ's and σ's into the canonical form (5), the simplicial category Δ is generated by the coface maps $\delta_i : [n-1] \hookrightarrow [n]$ and the codegeneracy maps $\sigma_i : [n+1] \twoheadrightarrow [n]$, $0 \leq i \leq n$, subject to the cosimplicial identities.

For any integer $n \geq 0$, we write $\Delta_{\leq n}$ for the full subcategory of Δ with objects $[0], \dots [n]$, and denote the inclusion functor by $\imath_n : \Delta_{\leq n} \hookrightarrow \Delta$.

2.2. On the totally ordered sets $S(n)$. For every integer $n \geq 0$, let $S(n)$ denote the set of all surjections $\alpha : [n] \twoheadrightarrow [r_\alpha]$ in Δ, with domain $[n]$. Let us stress that if $R(\alpha) = \{i_1, \dots, i_p\}$, then $r_\alpha = n - p$ and we have the canonical representation $\alpha = \sigma_{i_1} \cdots \sigma_{i_p}$.

The set $S(n)$ admits an *antilexicographic order*, defined as follows: Given $\alpha, \beta \in S(n)$ with $R(\alpha) = \{i_1, \dots, i_p\}$ and $R(\beta) = \{j_1, \dots, j_q\}$, we say that $\alpha < \beta$ if $i_1 = j_1$, ..., $i_k = j_k$ but $i_{k+1} > j_{k+1}$ ($k \geq 0$) or $i_1 = j_1$, ..., $i_p = j_p$ and $p < q$. This is a total order in $S(n)$. The least element of $S(n)$ is the identity map, denoted by

$$n = id_{[n]} : [n] \to [n],$$

while

$$\omega_n = \sigma_0 \cdots \sigma_{n-1} : [n] \twoheadrightarrow [0]$$

is the greatest one. For example, the order in $S(4)$ is

$$\begin{aligned} &4 < \sigma_3 < \sigma_2 < \sigma_2\sigma_3 < \sigma_1 < \sigma_1\sigma_3 < \sigma_1\sigma_2 < \sigma_1\sigma_2\sigma_3 < \sigma_0 \\ &\quad < \sigma_0\sigma_3 < \sigma_0\sigma_2 < \sigma_0\sigma_2\sigma_3 < \sigma_0\sigma_1 < \sigma_0\sigma_1\sigma_3 < \sigma_0\sigma_1\sigma_2 < \sigma_0\sigma_1\sigma_2\sigma_3 = \omega_4. \end{aligned}$$

It is easily verified that, for any $0 \leq j < n$, the correspondence

$$S(n-1) \to S(n), \quad \mu \mapsto \mu\sigma_j,$$

is an order preserving injective map, whose image consists of those $\alpha \in S(n)$ such that $j \in R(\alpha)$. Thus, we have an isomorphism of ordered sets

$$S(n-1) \cong \{\alpha \in S(n) \mid j \in R(\alpha)\}, \quad \mu \mapsto \mu\sigma_j, \tag{6}$$

whose inverse is the map

$$\{\alpha \in S(n) \mid j \in R(\alpha)\} \cong S(n-1), \quad \alpha \mapsto \alpha\delta_j.$$

Similarly, if for each surjective monotone map $\mu : [n-1] \twoheadrightarrow [r-1]$, we define

$$\mu_+ : [n] \twoheadrightarrow [r]$$

by $\mu_+(i) = \mu(i)$ if $i < n$, and $\mu_+(n) = r$, then we have an order preserving bijection

$$S(n-1) \cong \{\alpha \in S(n) \mid n-1 \notin R(\alpha)\}, \quad \mu \mapsto \mu_+, \tag{7}$$

whose inverse is the map

$$\{\alpha \in S(n) \mid n-1 \notin R(\alpha)\} \to S(n-1), \quad \alpha \mapsto \alpha_-,$$

where, for any surjection $\alpha : [n] \twoheadrightarrow [r]$ with $n-1 \notin R(\alpha)$, $\alpha_- : [n-1] \twoheadrightarrow [r-1]$ is defined by $\alpha_-(i) = \alpha(i)$ for any $i \in [n-1]$.

2.3. On simplicial Lie algebras. The category of simplicial Lie algebras, denoted by Simpl(Lie Alg), is the category of contravariant functors $\mathcal{L} : \Delta^{op} \to \text{Lie Alg}$, from the simplicial category into the category of Lie algebras. If $\mathcal{L}$ is any simplicial Lie algebra, and $\alpha : [n] \to [m]$ is any map in Δ, then we write $\alpha^* : \mathcal{L}_m \to \mathcal{L}_n$ for the induced homomorphism $\mathcal{L}(\alpha) : \mathcal{L}[m] \to \mathcal{L}[n]$. Since the category Δ is generated by the coface and codegeneracy maps, in order to define a simplicial Lie algebra $\mathcal{L}$, it suffices to write down its Lie algebras of n-simplices $\mathcal{L}_n$, $n \geq 0$, together with homomorphisms

$$\begin{aligned} d_i &= \delta_i^* : \mathcal{L}_n \to \mathcal{L}_{n-1}, \quad 0 \leq i \leq n \quad \text{(face homomorphisms)} \\ s_i &= \sigma_i^* : \mathcal{L}_n \to \mathcal{L}_{n+1}, \quad 0 \leq i \leq n \quad \text{(degeneracy homomorphisms)} \end{aligned}$$

satisfying the simplicial identities: $d_i d_j = d_{j-1} d_i$ if $i < j$, etc.

For each integer $n \geq 0$, we write $\text{Simpl}_{\leq n}(\text{Lie Alg})$ for the category of n-truncated simplicial Lie algebras, that is, the category of functors $\mathcal{L} : \Delta^{op}_{\leq n} \to \text{Lie Alg}$.

Every simplicial Lie algebra $\mathcal{L}$ gives rise to an n-truncated one $\imath_n^* \mathcal{L}$ by composition with the inclusion functor $\imath_n : \Delta_{\leq n} \hookrightarrow \Delta_{\leq n+1}$. This is the *$n$-truncation functor*

$$\imath_n^* : \text{Simpl}(\text{Lie Alg}) \to \text{Simpl}_{\leq n}(\text{Lie Alg}),$$

whose right adjoint, the *n-coskeleton functor*,

$$\mathrm{cosk}^n : \mathrm{Simpl}_{\leq n}(\mathrm{Lie\,Alg}) \to \mathrm{Simpl}(\mathrm{Lie\,Alg}),$$

can be described by iteration of successive *simplicial kernels* [Duskin (1975)]. Recall that, given any n-truncated simplicial Lie algebra, $\mathcal{L}$, its $(n+1)$-simplicial kernel, $\Delta_{n+1}(\mathcal{L})$, is the Lie subalgebra of the product $\mathcal{L}_n^{n+2}$ defined by

$$\Delta_{n+1}(\mathcal{L}) = \{(x_0, \cdots, x_{n+1}) \mid d_i x_j = d_{j-1} x_i \text{ for } 0 \leq i < j \leq n+1\}.$$

There are homomorphisms $d_i : \Delta_{n+1}(\mathcal{L}) \to \mathcal{L}_n$, $0 \leq i \leq n+1$, the projections

$$d_i(x_0, \ldots, x_{n+1}) = x_i,$$

and $s_j : \mathcal{L}_n \to \Delta_{n+1}(\mathcal{L})$, $0 \leq j \leq n$, defined by

$$s_j x = (s_{j-1} d_0 x, \ldots, s_{j-1} d_{j-1} x, x, x, s_j d_{j+1} x, \ldots, s_j d_n x),$$

making of

$$\begin{tikzcd} \Delta_{n+1}(\mathcal{L}) \arrow[r, shift left, "d_0"] \arrow[r, phantom, "\cdots"] \arrow[r, shift right, "d_{n+1}"'] & \mathcal{L}_n \arrow[l, bend right, "s_0"'] \arrow[l, bend right=60, "s_n"'] \arrow[r, shift left, "d_0"] \arrow[r, phantom, "\cdots"] \arrow[r, shift right, "d_n"'] & \mathcal{L}_{n-1} \arrow[l, bend right, "s_0"'] \arrow[l, bend right=60, "s_{n-1}"'] & \cdots & \mathcal{L}_1 \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & \mathcal{L}_0 \arrow[l, bend right, "s_0"'] \end{tikzcd}$$

an $(n+1)$-truncated simplicial Lie algebra.

3. Higher semidirect products of Lie algebras

When a simplicial Lie algebra is analyzed through its Moore complex, each Lie algebra of n-simplices has a certain structure in terms of the Lie algebras which appear in the Moore complex. This structure is that we present here under the name of *higher semidirect product of Lie algebras*, and it is perhaps of interest to Lie algebra theorists independent of its use here. For a natural precedent of this notion in group theory we refer the reader to [Carrasco, Cegarra (1991)] and [Antokoletz (2008)].

3.1. Schreier theory for extensions of Lie algebras. The long-known results of Schreier on group extensions [Schreier (1926)] can be stated in the same manner for extensions of Lie algebras. We provide below a brief account of this theory, and for more details we refer to [Alekseevsky, Michor, Ruppert (2004)], [Inassaridze, Khmaladze, Ladra (2008)], [Frégier (2014)], and the references therein.

Let $L_1 \overset{u_1}{\hookrightarrow} L \overset{\pi_2}{\twoheadrightarrow} L_2$ be an extension of Lie algebras with a linear map $u_2 : L_2 \hookrightarrow L$ such that $\pi_2 u_2 = id_{L_2}$, so that $L = u_1 L_1 \oplus u_2 L_2$, as a module, and $u_1 L_1$ is an ideal of L. There is a pair of induced bilinear maps

$$(\Phi : L_1 \times L_2 \to L_1,\ \phi : L_2 \times L_2 \to L_1) \tag{8}$$

determined by the equations

$$\Phi(x, y) = u_1[u_1x, u_2y], \quad \phi(y, y') = u_1\big([u_2y, u_2y'] - u_2[y, y']\big),$$

which satisfy

$$\Phi([x, x'], y) = [x, \Phi(x', y)] - [x', \Phi(x, y)], \tag{9}$$

$$\phi(y, y') = 0, \tag{10}$$

$$\Phi(\Phi(x, y), y') - \Phi(\Phi(x, y'), y) = \Phi(x, [y, y']) + [x, \phi(y, y')], \tag{11}$$

$$\begin{aligned}\phi(y, [y', y'']) + \phi(y', [y'', y]) + \phi(y'', [y, y'])\\ = \Phi(\phi(y, y'), y'') + \Phi(\phi(y', y''), y) + \Phi(\phi(y'', y), y').\end{aligned} \tag{12}$$

The mapping Φ can be rephrased in terms of derivations of the Lie algebra L_1, as condition (9) says that, for any $y \in L_2$, $\Phi(-, y) : L_1 \to L_1$ is a derivation. Thus Φ can be presented as a linear map $\Phi : L_2 \to \mathrm{Der}(L_1)$, $y \mapsto \Phi(-, y)$, and condition (11) can be written as

$$[\Phi(-, y), \Phi(-, y')] = \Phi(-, [y, y']) + ad_{\phi(y,y')}.$$

The pair of bilinear maps (Φ, ϕ) in (8) determines the structure of the Lie algebra extension, as $L = u_1L_1 \oplus u_2L_2$ with Lie bracket

$$[u_1x + u_2y, u_1x' + u_2y'] = u_1\big([x, x'] + \Phi(x, y) - \Phi(x', y) + \phi(x', y')\big) + u_2[y, y'],$$

and one can easily check that, if (Φ, ϕ) satisfies (9)-(12), then the above formula gives a Lie algebra structure on $L = u_1L_1 \oplus u_2L_2$. Such a pair is called a 2-*cocycle of* L_2 *with coefficients in* L_1, and the set of all these is denoted by $\mathrm{Z}^2(L_2, L_1)$.

If $(\Phi, \phi), (\Psi, \psi) \in \mathrm{Z}^2(L_2, L_1)$ are two such 2-cocycles, it is plain to see that they give isomorphic extensions of L_2 by L_1 if and only if there is a linear map $\rho : L_2 \to L_1$ such that

$$\begin{aligned}&\Phi(x, y) = \Psi(x, y) + [x, \rho y],\\ &\phi(y, y') = [\rho y, \rho y'] - \rho[y, y'] + \Psi(\rho y, y') - \Psi(\rho y', y) + \psi(y, y'),\end{aligned}$$

the corresponding isomorphism being

$$L = u_1L_1 \oplus u_2L_2 \to L' = u_1L_1 \oplus u_2L_2, \quad u_1x + u_2y \mapsto u_1(x + \rho y) + u_2y.$$

We say that ρ makes the 2-cocycles *cohomologous*, and the quotient set

$$\mathrm{H}^2(L_2, L_1) = \mathrm{Z}^2(L_2, L_1)/\text{cohomology}$$

is the *second (non-abelian) cohomology set of* L_2 *with coefficients in* L_1. In this way, there is a natural bijection

$$\mathrm{Ext}(L_2, L_1) \cong \mathrm{H}^2(L_2, L_1).$$

Under this correspondence, a 2-cocycle $(\Phi, \phi) \in \mathrm{Z}^2(L_2, L_1)$ gives rise to a split extension, that is, to a *semidirect product* Lie algebra

$$L = u_1 L_1 \rtimes u_2 L_2,$$

if and only if (Φ, ϕ) is cohomologous to a 2-cocycle of the form $(\Psi, 0)$, where the map $\Psi : L_2 \to \mathrm{Der}(L_1)$, $y \mapsto \Psi(-, y)$, is an homomorphism (so that L_1 is a right Lie L_2-algebra, with action $x \cdot y = \Psi(x, y)$).

3.2. Higher semidirect products. Throughout this section, $S = (S, \leq)$ denotes any given *totally ordered finite set* with r elements, denoted by $\alpha, \beta, \ldots$, and whose least and greatest elements are respectively written as 1_S and ω_S. Of course $S \cong \{1, \ldots, r\}$, but the content of the following sections will justify to consider here arbitrary totally ordered finite sets.

Suppose that L is a Lie algebra and $\{L_\alpha\}_{\alpha \in S}$ a family of Lie algebras, indexed by the totally ordered set S, with monomorphisms of Lie algebras $u_\alpha : L_\alpha \hookrightarrow L$, such that L, as a module, is the internal direct sum of the submodules $u_\alpha L_\alpha$,

$$L = \bigoplus_{\alpha \in S} u_\alpha L_\alpha.$$

This means that each $x \in L$ can be written as a sum $x = \sum_{\alpha \in S} u_\alpha x_\alpha$ with each $x_\alpha \in L_\alpha$, and this representation is unique. Then, we say that

3.3. Definition. *L is an r-semidirect product of the Lie algebras L_α, $\alpha \in S$, and we write*

$$L = \bigrtimes_{\alpha \in S} u_\alpha L_\alpha,$$

if, for every $\beta \in S$, the submodule $\bigoplus_{\alpha \leq \beta} u_\alpha L_\alpha$ is an ideal of L.

This condition is equivalent to the requirement that, for any $\beta < \gamma$,

$$[u_\beta L_\beta, u_\gamma L_\gamma] \subseteq \bigoplus_{\alpha \leq \beta} u_\alpha L_\alpha.$$

(Observe that the order in which the subalgebras L_α appear is essential).

Suppose that $L = \bigrtimes_{\alpha \in S} L_\alpha$ is an r-semidirect product of the Lie algebras L_α, as above. If, for any $\beta, \gamma \in S$, $y \in L_\beta$, $z \in L_\gamma$, we write

$$[u_\beta y, u_\gamma z] = \sum_{\alpha \in S} u_\alpha \Phi_\alpha^{\beta,\gamma}(y, z),$$

with $\Phi_\alpha^{\beta,\gamma}(x, y) \in L_\alpha$, then we have defined bilinear maps

$$\Phi_\alpha^{\beta,\gamma} : L_\beta \times L_\gamma \to L_\alpha, \qquad \alpha, \beta, \gamma \in S, \tag{13}$$

which completely determine the Lie algebra structure of L, to which we refer as the *structure bilinear maps of the r-semidirect product Lie algebra L*. These structure maps are easily seen to satisfy the following nine conditions:

- For any $\alpha, \beta \in S$, and $y, y' \in L_\beta$,

$$\Phi_\alpha^{\beta,\beta}(y, y') = \begin{cases} [y, y'] & \text{if } \alpha = \beta, \\ 0 & \text{if } \alpha \neq \beta. \end{cases} \tag{14}$$

- For α and $\beta < \gamma$, $y \in L_\beta$, $z \in L_\gamma$,

$$\Phi_\alpha^{\beta,\gamma}(y, z) = 0 \quad \text{if} \quad \alpha > \beta. \tag{15}$$

$$\Phi_\alpha^{\gamma,\beta}(z, y) = -\Phi_\alpha^{\beta,\gamma}(y, z) \tag{16}$$

(Thus, among all these maps $\Phi_\alpha^{\beta,\gamma}$, the relevant ones are those with $\alpha \leq \beta < \gamma$.)

- For $\alpha < \beta$, $x, x' \in L_\alpha$, $y, y' \in L_\beta$,

$$\Phi_\alpha^{\alpha,\beta}([x, x'], y) = [x, \Phi_\alpha^{\alpha,\beta}(x', y)] - [x', \Phi_\alpha^{\alpha,\beta}(x, y)]. \tag{17}$$

$$\Phi_\alpha^{\alpha,\beta}(x, [y, y']) = \Phi_\alpha^{\alpha,\beta}(\Phi_\alpha^{\alpha,\beta}(x, y), y') - \Phi_\alpha^{\alpha,\beta}(\Phi_\alpha^{\alpha,\beta}(x, y'), y). \tag{18}$$

(Thus, each $\Phi_\alpha^{\alpha,\beta}$ defines a right Lie algebra action of L_β on L_α.)

- For $\alpha < \beta < \gamma$, $y, y' \in L_\beta$, $z, z' \in L_\gamma$,

$$\Phi_\alpha^{\beta,\gamma}(y, [z, z']) = \sum_{\alpha \leq \xi \leq \beta} \Phi_\alpha^{\xi,\gamma}(\Phi_\xi^{\beta,\gamma}(y, z), z') - \sum_{\alpha \leq \xi \leq \beta} \Phi_\alpha^{\xi,\gamma}(\Phi_\xi^{\beta,\gamma}(y, z'), z). \tag{19}$$

$$\Phi_\alpha^{\beta,\gamma}([y, y'], z) = \sum_{\alpha \leq \xi < \beta} \Phi_\alpha^{\xi,\beta}(\Phi_\xi^{\beta,\gamma}(y, z), y') - \sum_{\alpha \leq \xi < \beta} \Phi_\alpha^{\xi,\beta}(\Phi_\xi^{\beta,\gamma}(y', z), y). \tag{20}$$

- For $\alpha < \beta < \gamma$, $x \in L_\alpha$, $y, \in L_\beta$, $z, \in L_\gamma$,

$$[x, \Phi_\alpha^{\beta,\gamma}(y, z)] + \sum_{\alpha < \xi \leq \beta} \Phi_\alpha^{\alpha,\xi}(x, \Phi_\xi^{\beta,\gamma}(y, z)) = \Phi_\alpha^{\alpha,\gamma}(\Phi_\alpha^{\alpha,\beta}(x, y), z) - \Phi_\alpha^{\alpha,\beta}(\Phi_\alpha^{\alpha,\gamma}(x, z), y). \tag{21}$$

- For $\alpha < \beta < \gamma < \delta$, $y \in L_\beta$, $z, \in L_\gamma$, $t \in L_\delta$,

$$\begin{aligned} &\sum_{\alpha < \xi \leq \beta} \Phi_\alpha^{\xi,\gamma}(\Phi_\xi^{\beta,\delta}(y, t), z) + \sum_{\beta < \xi \leq \gamma} \Phi_\alpha^{\beta,\xi}(y, \Phi_\xi^{\gamma,\delta}(z, t)) \\ &\qquad = \sum_{\alpha \leq \xi \leq \beta} \Phi_\alpha^{\xi,\delta}(\Phi_\xi^{\beta,\gamma}(y, z), t) + \sum_{\alpha < \xi < \beta} \Phi_\alpha^{\xi,\beta}(\Phi_\xi^{\gamma,\delta}(z, t), y). \end{aligned} \tag{22}$$

And conversely, given a family of Lie algebras $\{L_\alpha\}_{\alpha \in S}$, a module $L = \bigoplus_{\alpha \in S} u_\alpha L_\alpha$ which is direct sum of the modules L_α with inclusions $u_\alpha : L_\alpha \hookrightarrow L$, and a set of bilinear maps

$$\{\Phi_\alpha^{\beta,\gamma} : L_\beta \times L_\gamma \to L_\alpha\}_{\alpha,\beta,\gamma \in S} \tag{23}$$

satisfying the conditions (14)-(22), then there is a Lie algebra structure on L, with Lie bracket given by the formula

$$\Big[\sum_{\alpha\in S}u_\alpha x_\alpha,\sum_{\alpha\in S}u_\alpha y_\alpha\Big]=\sum_{\alpha\in S}u_\alpha\Big([x_\alpha,y_\alpha]+\sum_{\substack{\beta,\gamma\,\in\,S\\ \alpha\,\leq\,\beta\,<\,\gamma}}\big(\Phi_\alpha^{\beta,\gamma}(x_\beta,y_\gamma)-\Phi_\alpha^{\beta,\gamma}(y_\beta,x_\gamma)\big)\Big),\qquad(24)$$

such that each $u_\alpha:L_\alpha\hookrightarrow L$ becomes a monomorphism of Lie algebras and $L=\bigtimes_{\alpha\in S}u_\alpha L_\alpha$ is an r-semidirect product with set of structure bilinear maps (23).

Let $L=\bigtimes_{\alpha\in S}u_\alpha L_\alpha$ be an r-semidirect product of Lie algebras L_α, as above. The submodule of L direct sum of the L_α with $\alpha\in S\setminus\{1_S\}$ is then an $(r-1)$-semidirect product of the Lie algebras L_α with $\alpha>1_S$, written as $\bigtimes_{1_S<\alpha}u_\alpha L_\alpha$, whose structure maps are those $\Phi_\alpha^{\beta,\gamma}$ in (23) with $1_S<\alpha,\beta,\gamma$. This is not a Lie subalgebra, but a quotient of L. Indeed, we have a (non-necessarily split) extension of Lie algebras

$$L_{1_S}\xhookrightarrow{u_{1_S}}L=\bigtimes_{\alpha\in S}u_\alpha L_\alpha\xrightarrow{\pi}\bigtimes_{1_S<\alpha}u_\alpha L_\alpha,\qquad \pi(\textstyle\sum_{\alpha\in S(n)}u_\alpha x_\alpha)=\sum_{1_S<\alpha}u_\alpha x_\alpha\ ,\qquad(25)$$

and the partial set of structure maps $\{\Phi_{1_S}^{\beta,\gamma}:L_\beta\times L_\gamma\to L_{1_S}\}_{1_S\leq\beta<\gamma}$ deserves to be referred as the 2-*cocycle defining the r-semidirect product $L=\bigtimes_{\alpha\in S}u_\alpha L_\alpha$ as an extension of the $(r-1)$-semidirect product $\bigtimes_{1_S<\alpha\in S}u_\alpha L_\alpha$ by L_{1_S}*. This is why: If we introduce the maps

$$\Phi_{1_S}:L_{1_S}\times\Big(\bigtimes_{1_S<\alpha}u_\alpha L_\alpha\Big)\to L_{1_S},\quad \phi_{1_S}:\Big(\bigtimes_{1_S<\alpha}u_\alpha L_\alpha\Big)\times\Big(\bigtimes_{1_S<\alpha}u_\alpha L_\alpha\Big)\to L_{1_S},$$

defined, for any $x\in L_{1_S}$, $\mathbf{x}=\sum_{1_S<\alpha}u_\alpha x_\alpha$, $\mathbf{y}=\sum_{1_S<\alpha}u_\alpha y_\alpha\in\bigtimes_{1_S<\alpha}u_\alpha L_\alpha$, by

$$\left\{\begin{array}{l}\Phi_{1_S}(x,\mathbf{x})=\sum\limits_{1_S<\alpha}\Phi_{1_S}^{1_S,\alpha}(x,x_\alpha),\\ \phi_{1_S}(\mathbf{x},\mathbf{y})=\sum\limits_{1_S<\beta<\gamma}\Phi_{1_S}^{\beta,\gamma}(x_\beta,y_\gamma)-\Phi_{1_S}^{\beta,\gamma}(y_\beta,x_\gamma).\end{array}\right.\qquad(26)$$

then the bracket operator of the Lie algebra extension $L=\bigtimes_{\alpha\in S}u_\alpha L_\alpha$ is given, in terms of the brackets $[x,y]$ of L_{1_S} and $[\mathbf{u},\mathbf{v}]$ of $\bigtimes_{1_S<\alpha}L_\alpha$ by the formula

$$\big[u_{1_S}x+\mathbf{u},u_{1_S}y+\mathbf{v}\big]=u_{1_S}\big([x,y]+\Phi_{1_S}(x,\mathbf{v})-\Phi_{1_S}(y,\mathbf{u})+\phi_{1_S}(\mathbf{u},\mathbf{v})\big)+[\mathbf{u},\mathbf{v}].$$

This means that the pair (Φ,ϕ) is just the Schreier 2-cocycle (8) associated to the extension of Lie algebras (25), and thus we are legitimated to think of the set $\{\Phi_{1_S}^{\beta,\gamma}\}_{1_S\leq\beta<\gamma}$ as a 2-cocycle of the Lie algebra $\bigtimes_{1_S<\alpha}u_\alpha L_\alpha$ with coefficients in L_{1_S} [1].

[1]However, it should be stressed that *not every* 2-*cocycle* $(\Phi,\phi)\in \mathrm{Z}^2(\bigtimes_{1_S<\alpha}u_\alpha L_\alpha,L_{1_S})$ *produces a Lie algebra extension which is an r-semidirect product of the Lie algebras L_α, $\alpha\in S$*, since, for example, the requirement $\phi_{1_S}(u_\alpha x_\alpha,u_\alpha y_\alpha)=0$ for any $\alpha>1_S$ and $x_\alpha,y_\alpha\in L_\alpha$ (check in the formula (26)) does not necessarily holds.

Similarly, we can consider $\rtimes_{\alpha<\omega_S} u_\alpha L_\alpha$, the $(r-1)$-semidirect product of the Lie algebras L_α with $\alpha < \omega_S$, the greatest index of S, whose structure maps are those $\Phi_\alpha^{\beta,\gamma}$ with $\gamma < \omega_S$. In this case, we find an extension of Lie algebras

$$\rtimes_{\alpha<\omega_S} u_\alpha L_\alpha \hookrightarrow L = \rtimes_{\alpha\in S} u_\alpha L_\alpha \underset{\leftarrow}{\overset{\pi}{\twoheadrightarrow}} L_{\omega_S}, \qquad \pi\big(\textstyle\sum_{\alpha\in S(n)} u_\alpha x_\alpha\big) = x_{\omega_S},$$

which is split, as $u_{\omega_S} : L_{\omega_S} \hookrightarrow L$ is a Lie algebra monomorphism. Hence, L is an ordinary semidirect product of L_{ω_S} by $\rtimes_{\alpha<\omega_S} u_\alpha L_\alpha$, and thus we can see it as an iterated ordinary semidirect product. However, *the converse is false*, as for example in an iterated ordinary semidirect product Lie algebra, say $L = (L_1 \rtimes L_2) \rtimes L_3$, the Lie subalgebra L_1 is not required to be an ideal.

Example: 3-semidirect products. A Lie algebra L is a 3-semidirect product of subalgebras L_1, L_2 and L_3, $L = L_1 \rtimes L_2 \rtimes L_3$, if $L = L_1 \oplus L_2 \oplus L_3$ as a module, and both submodules L_1 and $L_1 + L_2$ are ideals. Then, there are four bilinear maps

$$\begin{cases} \Phi_2^{2,3} : L_2 \times L_3 \to L_2, & (y,z) \mapsto y \cdot z, \\ \Phi_1^{1,2} : L_1 \times L_2 \to L_1, & (x,y) \mapsto x \cdot y, \\ \Phi_1^{1,3} : L_1 \times L_3 \to L_1, & (x,z) \mapsto x \cdot z, \\ \Phi_1^{2,3} : L_2 \times L_3 \to L_1, & (y,z) \mapsto \langle y,z\rangle, \end{cases}$$

the three first ones defining right Lie actions, and the last of them satisfying the equations

$$\begin{aligned} &\langle [y,y'],z\rangle = \langle y,z\rangle \cdot y' - \langle y',z\rangle \cdot y, \\ &\langle y,[z,z']\rangle = \langle y,z\rangle \cdot z' - \langle y,z'\rangle \cdot z + \langle y\cdot z, z'\rangle - \langle y \cdot z', z\rangle, \\ &[x,\langle y,z\rangle] = (x\cdot y)\cdot z - (x\cdot z)\cdot y - x\cdot(y\cdot z), \end{aligned}$$

in whose terms the Lie bracket product on L is given by $(x, x' \in L_1,\ y, y' \in L_2,\ z, z' \in L_3)$

$$\begin{aligned} [x+y+z, x'+y'+z'] =& \big([x,x'] + x\cdot y' - x'\cdot y + x\cdot z' - x'\cdot z + \langle y,z'\rangle - \langle y',z\rangle\big) \\ &+ \big([y,y'] + y\cdot z' - y'\cdot z\big) + [z,z']. \end{aligned}$$

We have a 2-cocycle $(\Phi_2^{2,3}, 0) \in \mathrm{Z}^2(L_3, L_2)$, defining the ordinary semidirect product Lie algebra $L_2 \rtimes L_3$ as an extension of L_3 by L_2, and a 2-cocycle $(\Phi, \phi) \in \mathrm{Z}^2(L_2 \rtimes L_3, L_1)$, defining L as an extension of $L_2 \rtimes L_3$ by L_1, given by

$$\begin{cases} \Phi(x, y+z) = x\cdot y + x\cdot z, \\ \phi\big(y+z, y'+z'\big) = \langle y,z'\rangle - \langle y',z\rangle. \end{cases}$$

Let us stress that, in general, the short exact sequence $L_1 \hookrightarrow L_1 \rtimes L_2 \rtimes L_3 \twoheadrightarrow L_2 \rtimes L_3$ is not split, so there is no action of $L_2 \rtimes L_3$ on L_1 such that $L_1 \rtimes (L_2 \rtimes L_3) = L_1 \rtimes L_2 \rtimes L_3$. However, we have the identity $L_1 \rtimes L_2 \rtimes L_3 = (L_1 \rtimes L_2) \rtimes L_3$, where L_3 acts on $L_1 \rtimes L_2$ by

$$(x+y)\cdot z = x\cdot z + \langle y,z\rangle + y\cdot z.$$

4. Hypercrossed complexes of Lie algebras

We present here our linear models for simplicial Lie algebras, the *hypercrossed complexes of Lie algebras*. To help motivate the reader we refer to Section 5, where we discuss some of the examples to which our results are applied. Such as simplicial Lie algebras can be view as the infinitesimal replica of simplicial 1-connected Lie groups, hypercrossed complexes of Lie algebras can be regarded as the infinitesimal counterpart of hypercrossed complexes of 1-connected Lie groups [Carrasco, Cegarra (1991), Jurco (2012)], so the name of *differential hypercrossed complexes* for them could be also appropriate. Indeed, this last terminology is used in [Martins, Picken (2011)] for the particular cases of crossed modules and 2-crossed modules of Lie algebras, which are termed differential crossed and 2-crossed modules respectively (see Subsection 6.12 for details).

Given a chain complex of Lie algebras

$$L = \cdots \to L_{n+1} \xrightarrow{\partial} L_n \xrightarrow{\partial} L_{n-1} \cdots \to L_1 \xrightarrow{\partial} L_0,$$

for any surjection $\alpha : [n] \twoheadrightarrow [r]$ in Δ, we write L_α by L_r. That is,

$$L_\alpha = L_{r_\alpha} \qquad (\alpha \in S(n)). \tag{27}$$

It will be convenient to have the following concepts.

4.1. Definition. *An* enriched complex of Lie algebras *is a pair (L, Φ), where L is a chain complex of Lie algebras, and $\Phi = \{\Phi_n^{\beta,\gamma} : L_\beta \times L_\gamma \to L_n\}$ a set of bilinear maps, one for each $n \geq 1$ and $\beta, \gamma \in S(n)$ such that $n \leq \beta < \gamma$ and $R(\beta) \cap R(\gamma) = \emptyset$.*

4.2. Definition. *Given any enriched complex of Lie algebras (L, Φ), we define its* extended set of structure bilinear maps *$\Phi^{\text{ext}} \supseteq \Phi$,*

$$\Phi^{\text{ext}} = \left\{\Phi_\alpha^{\beta,\gamma} : L_\beta \times L_\gamma \to L_\alpha,\ \alpha, \beta, \gamma \in S(n),\ n \geq 0\right\}, \tag{28}$$

as follows: For any integer $n \geq 0$, the additional maps $\Phi_n^{\beta,\gamma} : L_\beta \times L_\gamma \to L_n$ are defined by

$$\Phi_n^{n,n}(x, y) = [x, y], \tag{29}$$

$$\Phi_n^{\beta,\gamma}(x, y) = -\Phi_n^{\gamma,\beta}(y, x) \quad \textit{if} \quad \beta > \gamma. \tag{30}$$

$$\Phi_n^{\beta,\gamma}(x, y) = 0 \quad \textit{if} \quad R(\beta) \cap R(\gamma) \neq \emptyset. \tag{31}$$

and, for each integer $n \geq 1$, and each $\alpha \in S(n)$ with $\alpha > n$, the additional bilinear maps $\Phi_\alpha^{\beta,\gamma} : L_\beta \times L_\gamma \to L_\alpha$ are defined, by recursion on n, by

- *if $R(\beta) \cap R(\gamma) \neq \emptyset$, then*

$$\Phi_\alpha^{\beta,\gamma}(x, y) = \begin{cases} 0 & \textit{if} \quad R(\alpha) \cap R(\beta) \cap R(\gamma) = \emptyset, \\ \Phi_{\alpha\delta_k}^{\beta\delta_k, \gamma\delta_k}(x, y) & \textit{if} \quad k = \min R(\alpha) \cap R(\beta) \cap R(\gamma) \neq \emptyset. \end{cases} \tag{32}$$

- *if* $R(\beta) \cap R(\gamma) = \emptyset$, *then,*

$$\Phi_\alpha^{\beta,\gamma}(x,y) = \sum_{i \in R(\alpha,\beta,\gamma)} (-1)^{m-i}\ \Phi_{\alpha\delta_m}^{\beta\delta_i,\gamma\delta_i}(x,y), \tag{33}$$

where $m = \min R(\alpha)$, *and*

$$R(\alpha,\beta,\gamma) = \big\{ i \mid 1 \le i \le m,\ (i, i-1) \in R(\beta) \times R(\gamma) \cup R(\gamma) \times R(\beta) \big\}. \tag{34}$$

Note that, by definition, if $R(\beta) \cap R(\gamma) = \emptyset$ then, for any $\alpha > n$ such that $R(\alpha,\beta,\gamma) = \emptyset$, $\Phi_\alpha^{\beta,\gamma}(x,y) = 0$. In particular,

$$\Phi_\alpha^{n,\gamma}(x,y) = 0, \quad \text{for any} \ \ \alpha,\gamma \in S(n) \ \ \text{such that } n < \alpha. \tag{35}$$

By induction on n, it can be easily proven that, for any $\alpha,\beta,\gamma \in S(n)$, $x \in L_\beta$, $y \in L_\gamma$,

$$\Phi_\alpha^{\beta,\gamma}(x,y) = -\Phi_\alpha^{\gamma,\beta}(y,x). \tag{36}$$

We are now ready to state the main notion in this paper.

4.3. Definition. *A* hypercrossed complex of Lie algebras *is an enriched complex* (L,Φ) *subject to the following eight conditions, for all* $n \ge 0$*:*

- *For any* $\beta \in S(n)$ *with* $n < \beta$, $x, x' \in L_n$, $y, y' \in L_\beta$,

$$\Phi_n^{n,\beta}([x,x'],y) = [x, \Phi_n^{n,\beta}(x',y)] - [x', \Phi_n^{n,\beta}(x,y)]. \tag{37}$$

$$\Phi_n^{n,\beta}(x,[y,y']) = \Phi_n^{n,\beta}(\Phi_n^{n,\beta}(x,y),y') - \Phi_n^{n,\beta}(\Phi_n^{n,\beta}(x,y'),y). \tag{38}$$

Thus, each $\Phi_n^{n,\beta} : L_n \times L_\beta \to L_n$, $\beta > n$, defines a Lie algebra action of L_β on L_n.

- *For any* $\beta,\gamma \in S(n)$ *with* $n < \beta < \gamma$ *and* $R(\beta) \cap R(\gamma) = \emptyset$, $y, y' \in L_\beta$, *and* $z, z' \in L_\gamma$,

$$\Phi_n^{\beta,\gamma}(y,[z,z']) = \sum_{\xi \le \beta} \Phi_n^{\xi,\gamma}(\Phi_\xi^{\beta,\gamma}(y,z),z') - \sum_{\xi \le \beta} \Phi_n^{\xi,\gamma}(\Phi_\xi^{\beta,\gamma}(y,z'),z). \tag{39}$$

$$\Phi_n^{\beta,\gamma}([y,y'],z) = \sum_{\xi < \beta} \Phi_n^{\xi,\beta}(\Phi_\xi^{\beta,\gamma}(y,z),y') - \sum_{\xi < \beta} \Phi_n^{\xi,\beta}(\Phi_\xi^{\beta,\gamma}(y',z),y). \tag{40}$$

Note that the equations (39) and (40) above always hold for any $\beta,\gamma \in S(n)$ such that $n < \beta < \gamma$ and $R(\beta) \cap R(\gamma) \ne \emptyset$.

- *For any* $\beta,\gamma \in S(n)$ *with* $n < \beta < \gamma$, $x \in L_n$, $y \in L_\beta$, *and* $z \in L_\gamma$,

$$[x, \Phi_n^{\beta,\gamma}(y,z)] + \sum_{n < \xi \le \beta} \Phi_n^{n,\xi}(x, \Phi_\xi^{\beta,\gamma}(y,z)) = \Phi_n^{n,\gamma}(\Phi_n^{n,\beta}(x,y),z) - \Phi_n^{n,\beta}(\Phi_n^{n,\gamma}(x,z),y). \tag{41}$$

- *For any $\beta,\gamma,\delta \in S(n)$ with $n<\beta<\gamma<\delta$, $y \in L_\beta$, $z \in L_\gamma$, and $t \in L_\delta$,*

$$\sum_{n<\xi\leq\beta} \Phi_n^{\xi,\gamma}(\Phi_\xi^{\beta,\delta}(y,t),z) + \sum_{\beta<\xi\leq\gamma} \Phi_n^{\beta,\xi}(y,\Phi_\xi^{\gamma,\delta}(z,t)) \\ = \sum_{\xi\leq\beta} \Phi_n^{\xi,\delta}(\Phi_\xi^{\beta,\gamma}(y,z),t) + \sum_{n<\xi<\beta} \Phi_n^{\xi,\beta}(\Phi_\xi^{\gamma,\delta}(z,t),y). \tag{42}$$

- *For any $\beta,\gamma \in S(n)$ with $R(\beta)\cap R(\gamma)=\emptyset$, $y \in L_{r_\beta+1}$, and $z \in L_\gamma$,*

$$\Phi_n^{\beta,\gamma}(\partial y,z) = \Phi_{\sigma_n}^{\beta_+,\gamma\sigma_n}(y,z) + \partial\Phi_{n+1}^{\beta_+,\gamma\sigma_n}(y,z). \tag{43}$$

- *For any $\beta,\gamma \in S(n)$ with $R(\beta)\cap R(\gamma)=\emptyset$ and $n<\beta<\gamma$, $y \in L_{r_\beta+1}$, and $z \in L_{r_\gamma+1}$,*

$$\Phi_n^{\beta,\gamma}(\partial y,\partial z) = \Phi_{\sigma_n}^{\beta_+,\gamma_+}(y,z) + \partial\Phi_{n+1}^{\beta_+,\gamma_+}(y,z). \tag{44}$$

By a *morphism of hypercrossed complexes of Lie algebras* $f:(L,\Phi)\to(L',\Phi')$ we mean a morphism of chain complexes $f: L\to L'$ compatible with the bilinear operators $\Phi_n^{\beta,\gamma}$ in the natural sense as it might be expected; that is, for any $n\geq 1$, and $\alpha,\beta\in S(n)$ with $n<\alpha<\beta$ and $R(\alpha)\cap R(\beta)=\emptyset$,

$$f\Phi_n^{\beta,\gamma}(x,y) = \Phi_n'^{\beta,\gamma}(fx,fy) \qquad (x\in L_{r(\beta)},\ y\in L_{r(\gamma)}).$$

We denote by HXCh(Lie Alg) the so defined category of hypercrossed complexes.

The next result will come in quite handy.

4.4. Lemma. *Let (L,Φ) be a hypercrossed complex of Lie algebras. Then,*

- *for any $\alpha,\beta,\gamma\in S(n)$, $y\in L_{r_\beta+1}$, and $z\in L_\gamma$,*

$$\Phi_\alpha^{\beta,\gamma}(\partial y,z) = \Phi_{\alpha\sigma_n}^{\beta_+,\gamma\sigma_n}(y,z) + \partial\Phi_{\alpha_+}^{\beta_+,\gamma\sigma_n}(y,z). \tag{45}$$

- *for any $\alpha,\beta,\gamma\in S(n)$, $y\in L_{r_\beta+1}$, and $z\in L_{r_\gamma+1}$,*

$$\Phi_\alpha^{\beta,\gamma}(\partial y,\partial z) = \Phi_{\alpha\sigma_n}^{\beta_+,\gamma_+}(y,z) + \partial\Phi_{\alpha_+}^{\beta_+,\gamma_+}(y,z). \tag{46}$$

Proof. This is given in Subsection 7.1. ∎

5. A Dold-Kan type theorem for simplicial Lie algebras

We state and prove here the main result of the paper; namely, there is an *enriched Dold-Kan correspondence* establishing an equivalence between the category of simplicial Lie algebras and the category of hypercrossed complexes of Lie algebras.

5.1. Structure of the Lie algebra of n-simplices of a simplicial Lie algebra. If $\mathcal{L}$ is any simplicial Lie algebra, we denote by $N(\mathcal{L})$ its *Moore complex*; that is, $N(\mathcal{L}) = L$ is the chain complex of Lie algebras

$$L = \cdots \to L_{n+1} \xrightarrow{\partial} L_n \to \cdots \to L_1 \to L_0$$

with $L_0 = \mathcal{L}_0$,

$$L_{n+1} = \bigcap_{i=0}^{n} \ker(d_i : \mathcal{L}_{n+1} \to \mathcal{L}_n),$$

and differential $\partial : L_{n+1} \to L_n$ given by restriction of $d_{n+1} : \mathcal{L}_{n+1} \to \mathcal{L}_n$.

For any $\alpha \in S(n)$, we have induced monomorphism $\alpha^* : \mathcal{L}_{r_\alpha} \hookrightarrow \mathcal{L}_n$ which, recalling from (27) the notation $L_\alpha = L_{r_\alpha}$, restricts to $L_\alpha \subseteq \mathcal{L}_{r_\alpha}$ giving a canonical injection

$$\alpha^* : L_\alpha \hookrightarrow \mathcal{L}_n,$$

and thus, every $\alpha^* L_\alpha \subseteq \mathcal{L}_n$ is a Lie subalgebra isomorphic to L_α.

If $\mathcal{L}$ is any simplicial module, it is part of the classical Dold-Kan-Puppe Theorem (see [Goerss, Jardine (1999), Chap. III, §2, Prop. 2.2], for a recent proof) that $\mathcal{L}$ is uniquely determined by its Moore complex L, as each $\mathcal{L}_n$ is the internal direct sum of the submodules $\alpha^* L_\alpha$,

$$\mathcal{L}_n = \bigoplus_{\alpha \in S(n)} \alpha^* L_\alpha, \tag{47}$$

and the face and degeneracy operators

$$d_i : \mathcal{L}_n \to \mathcal{L}_{n-1},\ s_j : \mathcal{L}_{n-1} \to \mathcal{L}_n, \quad 0 \le i \le n,\ 0 \le j \le n-1, \tag{48}$$

satisfy the formulas

$$d_i(\alpha^* x) = \begin{cases} (\alpha\delta_i)^* x & \text{if} \quad i \text{ or } i-1 \in D(\alpha), \\ 0 & \text{if} \quad i < n \ \text{ and } i, i-1 \notin D(\alpha), \\ \alpha^*_- \partial x & \text{if} \quad i = n \ \text{ and } n-1 \notin D(\alpha). \end{cases} \tag{49}$$

$$s_j(\mu^* y) = (\mu\sigma_j)^* x. \tag{50}$$

Next fact is crucial for our deliberations.

5.2. Proposition. *Let $\mathcal{L}$ be a simplicial module with Moore complex $N(\mathcal{L}) = L$.*

Suppose that, for any $m \le n$, the module $\mathcal{L}_m$ has a given Lie algebra structure, such that the face and degeneracy operators $d_i : \mathcal{L}_m \to \mathcal{L}_{m-1}$ and $s_j : \mathcal{L}_{m-1} \to \mathcal{L}_m$ are all homomorphisms of Lie algebras. That is, we are assuming that the n-truncation $i_n^* \mathcal{L}$ of $\mathcal{L}$, is actually an n-truncated simplicial Lie algebra.

Then, the Lie algebra $\mathcal{L}_n$ is a 2^n-semidirect product of the subalgebras $\alpha^* L_\alpha$, with α following the antilexicographic order of $S(n)$. That is,

$$\mathcal{L}_n = \rtimes_{\alpha \in S(n)} \alpha^* L_\alpha$$

with structure determined by the bilinear maps

$$\Phi_\alpha^{\beta,\gamma} : L_\beta \times L_\gamma \to L_\alpha, \qquad \alpha, \beta, \gamma \in S(n), \tag{51}$$

such that

$$[\beta^* x, \gamma^* y] = \sum_{\alpha \in S(n)} \alpha^* \Phi_\alpha^{\beta,\gamma}(x, y), \qquad x \in L_\beta,\, y \in L_\gamma.$$

Proof. We must prove that, for any $\xi \in S(n)$, the submodule $\sum_{\alpha<\xi} \alpha^* L_\alpha \subseteq \mathcal{L}_n$ is an ideal. To do that, we show below that

$$\sum_{\alpha<\xi} \alpha^* L_\alpha = \bigcap_{\beta \geq \xi} \ker(d_\beta : \mathcal{L}_n \to \mathcal{L}_{r_\beta}), \tag{52}$$

where, for any $\beta \in S(n)$ with $R(\beta) = \{j_1, \ldots, j_q\}$, $d_\beta = d_{j_1} \cdots d_{j_q} : \mathcal{L}_n \to \mathcal{L}_{r_\beta}$.

Suppose first $\alpha, \beta \in S(n)$, with $\alpha < \beta$. If $R(\alpha) = \{i_1, \ldots, i_p\}$ and $R(\beta) = \{j_1, \ldots, j_q\}$, then there is a k such that $i_1 = j_1$, ...,$i_k = j_k$, but $i_{k+1} > j_{k+1}$, and the simplicial identities imply that, for any $x_\alpha \in L_\alpha$,

$$\begin{aligned} d_\beta \alpha^* x_\alpha &= d_{j_1} \cdots d_{j_k} d_{j_{k+1}} \cdots d_{j_q} s_{i_p} \cdots s_{i_{k+1}} s_{j_k} \cdots s_{j_1}(x_\alpha) \\ &= d_{j_{k+1}-k} \cdots d_{j_q-k} s_{i_p-k} \cdots s_{i_{k+1}-k}(x_\alpha) \\ &= d_{j_{k+2}-k-1} \cdots d_{j_q-k-1} s_{i_p-k-1} \cdots s_{i_{k+1}-k-1}(d_{j_{k+1}-k} x_\alpha) = 0. \end{aligned}$$

Thus, we have that $\sum_{\alpha<\xi} \alpha^* L_\alpha \subseteq \bigcap_{\beta\geq\xi} \ker(d_\beta)$.

We now prove (52) by induction (in the reverse order) on $\xi \in S(n)$. For $\xi = \omega_n$, the greatest element in $S(n)$, and any $x = \sum_{\alpha \in S(n)} \alpha^* x_\alpha$, we have

$$d_{\omega_n} x = \sum_{\alpha \in S(n)} d_{\omega_n} \alpha^* x_\alpha = d_{\omega_n} \omega_n^* x_{\omega_n} = d_0 \cdots d_{n-1} s_{n-1} \cdots s_0(x_{\omega_n}) = x_{\omega_n}.$$

Hence, $x \in \ker(d_{\omega_n})$ if and only if $x_{\omega_n} = 0$, that is, if and only if $x \in \sum_{\alpha<\omega_n} \alpha^* L_\alpha$. Proceeding inductively, let us assume that the equality in (52) holds for some $\xi \in S(n)$. Then, for ξ' the precedent of ξ in $S(n)$, we have

$$\bigcap_{\beta\geq\xi'} \ker(d_\beta) = \ker(d_{\xi'}) \cap \bigcap_{\beta\geq\xi} \ker(d_\beta) = \ker(d_{\xi'}) \cap \sum_{\alpha<\xi} \alpha^* L_\alpha,$$

so that any element $x \in \bigcap_{\beta\geq\xi'} \ker(d_\beta)$ is of the form $x = \sum_{\alpha<\xi} \alpha^* x_\alpha$, with $x_\alpha \in L_\alpha$, and satisfies that

$$0 = d_{\xi'} x = \sum_{\alpha<\xi'} d_{\xi'} \alpha^* x_\alpha + d_{\xi'} \xi'^*(x_{\xi'}) = 0 + x_{\xi'} = x_{\xi'}.$$

Therefore, $x \in \sum_{\alpha<\xi'} \alpha^* L_\alpha$ as required. ∎

The following technical lemma is useful later.

5.3. Lemma. *Let $\mathcal{L}$ be a simplicial Lie algebra and let $N(\mathcal{L}) = L$ be its Moore complex. The structure bilinear maps $\Phi_\alpha^{\beta,\gamma} : L_\beta \times L_\gamma \to L_\alpha$ in (51) satisfy the following equations*

$$\Phi_\alpha^{\beta,\gamma}(x,y) = \begin{cases} 0 & \text{if} \quad m = 0, \\ \Phi_{\alpha\delta_m}^{\beta\delta_m,\gamma\delta_m}(x,y) - \Phi_{\alpha\delta_m\sigma_{m-1}}^{\beta,\gamma}(x,y) & \text{if} \quad m \in R(\alpha,\beta,\gamma) \\ -\Phi_{\alpha\delta_m\sigma_{m-1}}^{\beta,\gamma}(x,y) & \quad \text{otherwise}, \end{cases} \tag{53}$$

for any $\alpha, \beta, \gamma \in S(n)$, with $n < \alpha$ and $R(\beta) \cap R(\gamma) = \emptyset$, where $m = \min R(\alpha)$, and $R(\alpha,\beta,\gamma)$ is the set defined in (34).

Proof. This is given in Subsection 7.2. ∎

5.4. The enriched Moore complex of a simplicial Lie algebra. Recalling from Definition 4.1 the notion of enriched complex of Lie algebras, we state the following

5.5. Definition. *The* enriched Moore complex *of the simplicial Lie algebra $\mathcal{L}$ is the pair*

$$N^{\mathrm{e}}(\mathcal{L}) = (L, \Phi), \tag{54}$$

where $L = N(\mathcal{L})$ is the Moore complex of $\mathcal{L}$, and $\Phi = \Phi(\mathcal{L})$ is the set of those structure bilinear maps of the Lie algebras $\mathcal{L}_n$ in (51), $\Phi_n^{\beta,\gamma} : L_\beta \times L_\gamma \to L_n$, with $n \geq 1$ and such that $n \leq \beta < \gamma$ and $R(\beta) \cap R(\gamma) = \emptyset$.

With the following technical lemma we will be ready to prove the main result of this subsection, stated in Proposition 5.7 below.

5.6. Lemma. *Let $\mathcal{L}$ be a simplicial Lie algebra and let $N^{\mathrm{e}}(\mathcal{L}) = (L, \Phi)$ be its enriched Moore complex. For every integer $n \geq 0$, the structure bilinear maps $\Phi_\alpha^{\beta,\gamma} : L_\beta \times L_\gamma \to L_\alpha$ of the 2^n-semidirect product Lie algebra $\mathcal{L}_n$ in (51) are precisely those belonging to the extended set of maps Φ^{ext} in (28) of (L, Φ).*

Proof. This is given in Subsection 7.3. ∎

5.7. Proposition. *The enriched Moore complex of any simplicial Lie algebra is a hypercrossed complex of Lie algebras.*

Proof. After Lemma 5.6, equalities (37)-(42) hold thanks to equalities (17)-(22). Hence, it remains to verify only the equations (43) and (44).

For, observe first that for any $\beta,\gamma\in S(n)$, $y\in L_{r(\beta)+1}$ and $z\in L_\gamma$, we have

$$\begin{aligned}
[\beta^*\partial y,\gamma^*z] &\overset{(7)}{=} [d_{n+1}\beta_+^*y, d_{n+1}(\gamma\sigma_n)^*z] = d_{n+1}[\beta_+^*x,(\gamma\sigma_n)^*y]\\
&= d_{n+1}\sum_{\xi\in S(n+1)}\xi^*\Phi_\xi^{\beta_+,\gamma\sigma_n}(y,z) = \sum_{\xi\in S(n+1)} d_{n+1}\xi^*\Phi_\xi^{\beta_+,\gamma\sigma_n}(y,z)\\
&= \sum_{\substack{\xi\in S(n+1)\\ n\notin R(\xi)}}\xi_-^*\partial\Phi_\xi^{\beta_+,\gamma\sigma_n}(y,z) + \sum_{\substack{\xi\in S(n+1)\\ n\in R(\xi)}}(\xi\delta_n)^*\Phi_\xi^{\beta_+,\gamma\sigma_n}(y,z)\\
&\overset{(7),(6)}{=} \sum_{\alpha\in S(n)}\alpha^*\big(\partial\Phi_{\alpha_+}^{\beta_+,\gamma\sigma_n}(y,z)+\Phi_{\alpha\sigma_n}^{\beta_+,\gamma\sigma_n}(y,z)\big).
\end{aligned}$$

As, on the other hand, $[\beta^*\partial y,\gamma^*z]=\sum_{\alpha\in S(n)}\alpha^*\Phi_\alpha^{\beta,\gamma}(\partial y,z)$, by comparing the component at $\alpha=n$, we get the required equality in (43), $\Phi_n^{\beta,\gamma}(\partial y,z)=\Phi_{\sigma_n}^{\beta_+,\gamma\sigma_n}(y,z)+\partial\Phi_{n+1}^{\beta_+,\gamma\sigma_n}(y,z)$.

Similarly, for any $\beta,\gamma\in S(n)$, $y\in L_{r_\beta+1}$ and $z\in L_{r_\gamma+1}$, we have

$$\begin{aligned}
[\beta^*\partial y,\gamma^*\partial z] &\overset{(7)}{=} [d_{n+1}\beta_+^*y, d_{n+1}\gamma_+^*z] = d_{n+1}[\beta_+^*x,\gamma_+^*y]\\
&= d_{n+1}\sum_{\xi\in S(n+1)}\xi^*\Phi_\xi^{\beta_+,\gamma_+}(y,z) = \sum_{\xi\in S(n+1)} d_{n+1}\xi^*\Phi_\xi^{\beta_+,\gamma_+}(y,z)\\
&= \sum_{\substack{\xi\in S(n+1)\\ n\notin R(\xi)}}\xi_-^*\partial\Phi_\xi^{\beta_+,\gamma_+}(y,z) + \sum_{\substack{\xi\in S(n+1)\\ n\in R(\xi)}}(\xi\delta_n)^*\Phi_\xi^{\beta_+,\gamma_+}(y,z)\\
&\overset{(7),(6)}{=} \sum_{\alpha\in S(n)}\alpha^*\big(\partial\Phi_{\alpha_+}^{\beta_+,\gamma_+}(y,z)+\Phi_{\alpha\sigma_n}^{\beta_+,\gamma_+}(y,z)\big).
\end{aligned}$$

As, on the other hand, $[\beta^*\partial y,\gamma^*\partial z]=\sum_{\alpha\in S(n)}\alpha^*\Phi_\alpha^{\beta,\gamma}(\partial y,\partial z)$, by comparing the component at $\alpha=n$, we get the required equality in (44). ∎

Lemma 5.6 above is also key to prove the following fact, later needed.

5.8. Lemma. *Let $\mathcal{L}$ be a simplicial module with Moore complex $N(\mathcal{L})=L$. Suppose that, for any $m\leq n$, the module $\mathcal{L}_m$ has a given Lie algebra structure such that the face and degeneracy operators $d_i:\mathcal{L}_m\to\mathcal{L}_{m-1}$ and $s_j:\mathcal{L}_{m-1}\to\mathcal{L}_m$ are all homomorphisms of Lie algebras. Let $\{\Phi_\alpha^{\beta,\gamma}:L_\beta\times L_\gamma\to L_\alpha\}_{\alpha,\beta,\gamma\in S(n)}$ be the set of structure bilinear maps (51) of $\mathcal{L}_n=\bigotimes_{\alpha\in S(n)}\alpha^*L_\alpha$, as in Proposition 5.2. Then,*

(i) *The bilinear maps $\Phi_\tau^{\eta,\mu}:L_\eta\times L_\mu\to L_\tau$, defined for $\tau,\eta,\mu,\in S(n+1)\setminus\{n+1\}$ by the formulas (32) and (33), satisfy the equations (14)-(22). Hence, the submodule of $\mathcal{L}_{n+1}=\bigoplus_{\tau\in S(n+1)}\tau^*L_\tau$ generated by the degenerated elements, $D_{n+1}(\mathcal{L})=\bigoplus_{n+1<\tau}\tau^*L_\tau$, has a canonical structure of Lie algebra, which is a $(2^{n+1}-1)$-semidirect product of the Lie algebras L_τ, with $\tau\in S(n+1)$, $\tau>n+1$,*

$$D_{n+1}(\mathcal{L})=\bigotimes_{n+1<\tau}\tau^*L_\tau$$

with bracket $[-,-]_{D_{n+1}(\mathcal{L})}$ *defined, for* $\eta,\mu\in S(n+1)\setminus\{n+1\}$, $x\in L_\eta$, $y\in L_\mu$, *by*

$$[\eta^*x,\mu^*y]_{D_{n+1}(\mathcal{L})}=\sum_{\tau>n+1}\tau^*\Phi^{\eta,\mu}_\tau(x,y), \tag{55}$$

(ii) *The restricted linear maps below are homomorphisms of Lie algebras.*

$$d_i: D_{n+1}(\mathcal{L})\to\mathcal{L}_n,\quad s_i:\mathcal{L}_n\to D_{n+1}(\mathcal{L}),\quad 0\le i\le n.$$

PROOF. (i) The n-truncated simplicial Lie algebra $\imath_n^*\mathcal{L}$ is itself the n-truncation of the simplicial Lie algebra $\mathcal{L}'=\operatorname{cosk}^n\imath_n^*\mathcal{L}$, whose Moore complex is the chain complex of Lie algebras

$$L'=\cdots\to 0\to 0\to L'_{n+1}\to L_n\to L_{n-1}\to\cdots\to L_0,$$

whose n-truncation $L_n\to\cdots\to L_0$ is the same as the n-truncation of L,

$$L'_{n+1}=\{(0,\dots,0,x)\in\Delta_{n+1}(\imath_n^*\mathcal{L})\}=\bigcap_{i=0}^{n}\ker(d_i:\mathcal{L}_n\to\mathcal{L}_{n-1})=\operatorname{Ker}(\partial:L_n\to L_{n-1}),$$

and the boundary $\partial: L'_{n+1}\to L_n$ the inclusion map.

By Proposition 5.2 (ii), the Lie algebra $\mathcal{L}'_{n+1}=\Delta_{n+1}(\imath_n^*\mathcal{L})$ is a 2^{n+1}-semidirect product of the subalgebras $\tau^*L'_\tau$, with $\tau\in S(n+1)$. Indeed,

$$\Delta_{n+1}(\imath_n^*\mathcal{L})=\rtimes\!\!\!\!\ltimes_{\tau\in S(n+1)}\tau^*L'_\tau,$$

where, by Lemma 5.6, the corresponding structure bilinear maps $\Phi^{\eta,\mu}_\tau: L'_\eta\times L'_\mu\to L'_\tau$, $\tau,\eta,\mu\in S(n+1)$, satisfy the equations (29)-(33). Among them, those $\Phi^{\eta,\mu}_\tau: L_\eta\times L_\mu\to L_\tau$ with $\tau,\eta,\mu>n+1$ verify the equations (32) and (33), and therefore they coincide with those considered in the hypothesis of the lemma. It follows that $\{\Phi^{\eta,\mu}_\tau,\ \tau,\eta,\mu\in S(n+1)\setminus\{n+1\}\}$ is just the set of structure bilinear maps of a $(2^{n+1}-1)$-semidirect product Lie algebra structure on $D_{n+1}(\mathcal{L})=\bigoplus_{n+1<\tau}\tau^*L_\tau$. Thus, $D_{n+1}(\mathcal{L})=\rtimes\!\!\!\!\ltimes_{\tau>n+1}L_\tau$ is a (2^n-1)-semidirect product Lie algebra with bracket (55).

Note that, as in (25), $\Delta_{n+1}(\imath_n^*\mathcal{L})$ is a 2^n-semidirect product extension of the Lie algebra $D_{n+1}(\mathcal{L})$ by L'_{n+1},

$$L'_{n+1}\hookrightarrow\Delta_{n+1}(\imath_n^*\mathcal{L})=\rtimes\!\!\!\!\ltimes_{\tau\in S(n+1)}L_\tau\xrightarrow{\pi}\!\!\!\!\to D_{n+1}(\mathcal{L})=\rtimes\!\!\!\!\ltimes_{n+1<\tau}L_\tau\ .$$

Hence, for any $0\le i\le n$, the linear maps in (ii) are homomorphisms of Lie algebras, as the triangles below commute and the others linear maps therein are. ■

$$\begin{tikzcd} \Delta_{n+1}(\imath_n^*\mathcal{L}) \arrow[r, two heads, "\pi"] \arrow[dr, "d_i"'] & D_{n+1}(\mathcal{L}) \arrow[d, "d_i"] \\ & \mathcal{L}_n \end{tikzcd} \qquad \begin{tikzcd} & \mathcal{L}_n \arrow[dl, "s_i"'] \arrow[dr, "s_i"] & \\ \Delta_{n+1}(\imath_n^*\mathcal{L}) \arrow[rr, two heads, "\pi"] & & D_{n+1}(\mathcal{L}) \end{tikzcd}$$

5.9. The enriched Dold-Kan correspondence. The enriched Moore complex (L,Φ) of a simplicial Lie algebra $\mathcal{L}$, (54), depends naturally on $\mathcal{L}$, and the correspondence which assigns to each simplicial Lie algebra its enriched Moore complex is the function on objects of a functor

$$N^{\mathrm{e}} : \mathrm{Simpl}(\mathrm{Lie\,Alg}) \to \mathrm{HXCh}(\mathrm{Lie\,Alg}), \tag{56}$$

to which we refers as the *enriched Dold-Kan correspondence* for Lie algebras.

5.10. Theorem. *The enriched Dold-Kan correspondence* (56) *is an equivalence of categories.*

Proof. Since the functor N^{e} is clearly full and faithful, we only have to prove that, for any hypercrossed complex of Lie algebras (L',Φ'), there is simplicial Lie algebra $\mathcal{L}$ with an isomorphism of hypercrossed complexes of Lie algebras $N^{\mathrm{e}}(\mathcal{L}) \cong (L',\Phi')$.

For any given such a (L',Φ'), by the classical Dold-Kan-Puppe theorem, we can choose a simplicial module $\mathcal{L}$, with Moore complex $N(\mathcal{L}) = L$ say, together with an isomorphism of chain complexes of modules $L \cong L'$ (indeed, such $\mathcal{L}$ is, up to isomorphism, the simplicial module described in (47), (49), and (50)). Now, we transport along the isomorphism $L \cong L'$ both the Lie algebra structures on the modules L'_n as the set of structure bilinear maps Φ' on the chain complex L' to Lie algebra structures on the modules L_n and a set of structure bilinear maps Φ on L, such that the isomorphism becomes an isomorphism of hypercrossed complexes of Lie algebras, $(L,\Phi) \cong (L',\Phi')$. Then, the proof of the theorem will be complete if we show that there is a structure of simplicial Lie algebra on the simplicial module $\mathcal{L}$ such that $N^{\mathrm{e}}(\mathcal{L}) = (L,\Phi)$.

To do that, it suffices to prove that, for any integer $n \geq 0$, the module $\mathcal{L}_n = \bigoplus_{\alpha\in S(n)} \alpha^* L_\alpha$ has a structure of Lie algebra with bracket defined by the formula

$$[\beta^* x, \gamma^* y] = \sum_{\alpha\in S(n)} \alpha^* \Phi_\alpha^{\beta,\gamma}(x,y), \qquad \beta,\gamma \in S(n), x \in L_\beta,\, y \in L_\gamma,$$

where the maps $\Phi_\alpha^{\beta,\gamma}$ are those belonging to the extended set of maps Φ^{ext} (28) of (L,Φ), as well as the face and degeneracy linear maps, $d_i : \mathcal{L}_n \to \mathcal{L}_{n-1}$ and $s_j : \mathcal{L}_{n-1} \to \mathcal{L}_n$, are actually homomorphisms of Lie algebras.

We proceed recursively on n. For $n = 0$ there is nothing to prove, as $\mathcal{L}_0 = L_0$ is a Lie algebra with bracket $[x,y] = \Phi_0^{0,0}(x,y)$, see (29). Making hypothesis of induction, let us assume that the n-truncation $\imath_n^*\mathcal{L}$ of $\mathcal{L}$ is an n-truncated simplicial Lie algebra, where the structure of each $\mathcal{L}_m$, $m \leq n$, as a 2^m-semidirect product $\mathcal{L}_m = \rtimes_{\alpha\in S(m)} \alpha^* L_\alpha$, according to Proposition 5.2, is given by the bilinear maps $\Phi_\alpha^{\beta,\gamma} \in \Phi^*$ for $\alpha,\beta,\gamma \in S(m)$. Hence, Lemma 5.8(i) asserts us that the bilinear maps of Φ^{ext}

$$\Phi_\tau^{\eta,\mu} : L_\eta \times L_\mu \to L_\tau, \quad \tau,\eta,\mu \in S(n+1) \setminus \{n+1\},$$

verify the equations (14)-(22), and that they form the set of structure maps of a $(2^{n+1}-1)$-semidirect product Lie algebra $D_{n+1}(\mathcal{L}) = \rtimes_{n+1<\tau}\tau^*L_\tau$, whose underlaying module is the submodule $D_{n+1}(\mathcal{L}) = \bigoplus_{n+1<\tau}\tau^*L_\tau$ of $\mathcal{L}_{n+1} = \bigoplus_{\tau\in S(n+1)}\tau^*L_\tau = L_{n+1}\oplus D_{n+1}(\mathcal{L})$.

But then, from equations (29)-(33) and (37)-(42), it is easily seen that the whole set of all maps $\Phi_\tau^{\eta,\mu}\in\Phi^{\mathrm{ext}}$ with $\eta,\mu\in S(n+1)$ satisfy the equations (14)-(22). Therefore, there is a Lie algebra structure on $\mathcal{L}_{n+1}$, with Lie bracket given by the formula

$$[\eta^*x,\mu^*y] = \sum_{\tau\in S(n+1)}\tau^*\Phi_\tau^{\eta,\mu}(x,y), \qquad \eta,\mu\in S(n+1), x\in L_\eta,\ y\in L_\mu,$$

such that $\mathcal{L}_{n+1} = \rtimes_{\tau\in S(n+1)}\tau^*L_\tau$ is a 2^{n+1}-semidirect product with structure bilinear maps the $\Phi_\tau^{\eta,\mu}\in\Phi^{\mathrm{ext}}$ above. Note that, as in (25), $\mathcal{L}_{n+1}$ is a 2^{n+1}-semidirect product extension of $D_{n+1}(\mathcal{L})$ by L_{n+1},

$$L_{n+1}\hookrightarrow \mathcal{L}_{n+1} = \rtimes_{\tau\in S(n+1)} L_\tau \overset{\pi}{\twoheadrightarrow} D_{n+1}(\mathcal{L}) = \rtimes_{n+1<\tau} L_\tau\ ,$$

defined by the 2-cocycle $\{\Phi_{n+1}^{\eta,\mu} : L_\eta\times L_\mu\to L_{n+1}\}_{n+1\le\eta<\mu}$.

We now show that the linear maps $s_i:\mathcal{L}_n\to\mathcal{L}_{n+1}$ in (48), $0\le i\le n$, are homomorphisms of Lie algebras: By Lemma 5.8(ii), we know that, when we restrict their ranges to $D_{n+1}(\mathcal{L})$, then they are homomorphism of Lie algebras $s_i:\mathcal{L}_n\to D_{n+1}(\mathcal{L})$. Therefore, for any $\beta,\gamma\in S(n)$, $x\in L_\beta$, $y\in L_\gamma$, we have

$$s_i[\beta^*x,\gamma^*y] = [s_i\beta^*x, s_i\gamma^*y]_{D_{n+1}(\mathcal{L})} = [(\beta\sigma_i)^*x,(\gamma\sigma_i)^*y]_{D_{n+1}(\mathcal{L})} \overset{(55)}{=} \sum_{n+1>\tau}\tau^*\Phi_\tau^{\beta\sigma_i,\alpha\sigma_i}(x,y).$$

As the Lie bracket of $s_i\beta^*x$ and $s_i\gamma^*x$ in $\mathcal{L}_{n+1}$ is

$$\begin{aligned}[s_i\beta^*x, s_i\gamma^*y] &= [(\beta\sigma_i)^*x,(\gamma\sigma_i)^*y] = \sum_{\tau\in S(n+1)}\tau^*\Phi_\tau^{\beta\sigma_i,\alpha\sigma_i}(x,y)\\ &= \Phi_{n+1}^{\beta\sigma_i,\alpha\sigma_i}(x,y) + \sum_{n+1>\tau}\tau^*\Phi_\tau^{\beta\sigma_i,\alpha\sigma_i}(x,y) = \Phi_{n+1}^{\beta\sigma_i,\alpha\sigma_i}(x,y) + s_i[\beta^*x,\gamma^*y],\end{aligned}$$

we get the required equality $s_i[\beta^*x,\gamma^*y] = [s_i\beta^*x, s_i\gamma^*y]$ since $\Phi_{n+1}^{\beta\sigma_i,\alpha\sigma_i}(x,y) = 0$ by (31).

Finally, we prove that the linear maps $d_i:\mathcal{L}_{n+1}\to\mathcal{L}_n$ in (48) are homomorphisms of Lie algebras: For $0\le i\le n$, the result follows from Lemma 5.8(ii), since the triangles below commute.

$$\begin{tikzcd} & \mathcal{L}_{n+1} \arrow[dl, "\pi"'] \arrow[dr, "d_i"] & \\ D_{n+1}(\mathcal{L}) \arrow[rr, "d_i"] & & \mathcal{L}_n \end{tikzcd}$$

Hence, it only remains to prove that $d_{n+1}:\mathcal{L}_{n+1}\to\mathcal{L}_n$ is a homomorphism of Lie algebras. That is, for any $\eta,\mu\in S(n+1)$, $x\in L_\eta$, $y\in L_\mu$,

$$d_{n+1}[\eta^*x,\mu^*x] = [d_{n+1}\eta^*x, d_{n+1}\mu^*y]. \tag{57}$$

To do that, we distinguish four cases.

(a) Case: $n \in R(\eta)$ and $n \in R(\mu)$. According to (6), we can write $\eta = \beta\sigma_n$ and $\mu = \gamma\sigma_n$, where $\beta = \eta\delta_n$ and $\gamma = \eta\sigma_n$. Then, as we have already proven that $s_n = \sigma_n^* : \mathcal{L}_n \to \mathcal{L}_{n+1}$ is a homomorphism of Lie algebras, we have

$$\begin{aligned} d_{n+1}[\eta^*x, \mu^*y] &= d_{n+1}[\sigma_n^*\beta^*x, \sigma_n^*\gamma^*y] = d_{n+1}[s_n\beta^*x, s_n\gamma^*y] = d_{n+1}s_n[\beta^*x, \alpha^*y] \\ &= [\beta^*x, \alpha^*y] = [d_{n+1}s_n\beta^*x, d_{n+1}s_n\gamma^*y] = [d_{n+1}\eta^*x, d_{n+1}\mu^*y]. \end{aligned}$$

To treat with the other cases, let us first compute $d_{n+1}[\eta^*x, \mu^*x]$ as follows,

$$\begin{aligned} d_{n+1}[\eta^*x, \mu^*x] &= d_{n+1} \sum_{\tau \in S(n+1)} \tau^*\Phi_\tau^{\eta,\mu}(x,y) = \sum_{\tau \in S(n+1)} d_{n+1}\tau^*\Phi_\tau^{\eta,\mu}(x,y) \\ &= \sum_{\substack{\tau \in S(n+1) \\ n \in R(\tau)}} (\tau\delta_{n+1})^*\Phi_\tau^{\eta,\mu}(x,y) + \sum_{\substack{\tau \in S(n+1) \\ n \notin R(\tau)}} \tau_-^*\partial\Phi_\tau^{\eta,\mu}(x,y) \\ &\overset{(6),(7)}{=} \sum_{\alpha \in S(n)} \alpha^*\Phi_{\alpha\sigma_n}^{\eta,\mu}(x,y) + \sum_{\alpha \in S(n)} \alpha^*\partial\Phi_{\alpha_+}^{\eta,\mu}(x,y) \\ &= \sum_{\alpha \in S(n)} \alpha^*\big(\Phi_{\alpha\sigma_n}^{\eta,\mu}(x,y) + \partial\Phi_{\alpha_+}^{\eta,\mu}(x,y)\big). \end{aligned} \tag{58}$$

(b) Case: $n \notin R(\eta)$ and $n \in R(\mu)$. In this case, by the bijections (6) and (7), we can write $\eta = \beta_+$ and $\mu = \gamma\sigma_n$, where $\beta = \eta_-$ and $\gamma = \mu\delta_{n+1}$. Hence,

$$[d_{n+1}\eta^*x, d_{n+1}\mu^*y] = [\eta_-^*\partial x, (\mu\delta_{n+1})^*y] = [\beta^*\partial x, \gamma^*y] = \sum_{\alpha \in S(n)} \alpha^*\Phi_\alpha^{\beta,\gamma}(\partial x, y),$$

$$d_{n+1}[\eta^*x, \mu^*y] \overset{(58)}{=} \sum_{\alpha \in S(n)} \alpha^*\big(\Phi_{\alpha\sigma_n}^{\beta_+,\gamma\sigma_n}(x,y) + \partial\Phi_{\alpha_+}^{\beta_+,\gamma\sigma_n}(x,y)\big),$$

and, by comparison, we see that equation (57) holds by equations (45) in Lemma 4.4.

(c) Case: $n \in R(\eta)$ and $n \notin R(\mu)$. This follows from the case (b) above:

$$d_{n+1}[\eta^*x, \mu^*y] = -d_{n+1}[\mu^*y, \eta^*x] \overset{(b)}{=} -[d_{n+1}\mu^*y, d_{n+1}\eta^*x] = [d_{n+1}\eta^*x, d_{n+1}\mu^*y].$$

(d) Case: $n \notin R(\eta)$ and $n \notin R(\mu)$. By the bijection (7), we can write $\eta = \beta_+$ and $\mu = \gamma_+$, where $\beta = \eta_-$ and $\gamma = \eta_-$. Hence,

$$[d_{n+1}\eta^*x, d_{n+1}\mu^*y] = [\eta_-^*\partial x, \mu_-^*\partial y] = [\beta^*\partial x, \gamma^*\partial y] = \sum_{\alpha \in S(n)} \alpha^*\Phi_\alpha^{\beta,\gamma}(\partial x, \partial y),$$

$$d_{n+1}[\eta^*x, \mu^*y] \overset{(58)}{=} \sum_{\alpha \in S(n)} \alpha^*\big(\Phi_{\alpha\sigma_n}^{\beta_+,\gamma_+}(x,y) + \partial\Phi_{\alpha_+}^{\beta_+,\gamma_+}(x,y)\big).$$

and, by comparison, equation (57) follows from equations (46) in Lemma 4.4. ∎

6. Some particular cases

In this section, hypercrossed complexes of Lie algebras are specialized, and it is shown the meaning of the enriched Dold-Kan equivalence in these particular examples.

6.1. Chain complexes of modules and simplicial modules. As usually, we identify any module with the abelian Lie algebra that itself defines when it is endowed with the identically zero Lie bracket. Thus, we regard the category of simplicial modules, denoted by Simpl(Mod), as the full subcategory of the category of simplicial Lie algebras defined by the simplicial abelian Lie algebras.

Similarly, we regard the category of chain complexes of modules, Ch(Mod), as the full subcategory of the category of hypercrossed complexes defined by those of the form $(L, 0)$, where L is any chain complex of modules and all the maps in the set of structure bilinear maps $\Phi = 0$ are the constantly zero maps.

By construction, the square

$$\begin{tikzcd} \mathrm{Simpl(Mod)} \arrow[r, "N"] \arrow[d, hook] & \mathrm{Ch(Mod)} \arrow[d, hook] \\ \mathrm{Simpl(Lie\,Alg)} \arrow[r, "N^{\mathrm{e}}"] & \mathrm{HXCh(Lie\,Alg)} \end{tikzcd}$$

becomes commutative, and thus *the equivalence of Theorem 5.10 restricts to that given by the classical Dold-Kan-Puppe correspondence.*

6.2. Crossed complexes and simplicial T-complexes of Lie algebras. Crossed modules of Lie algebras were first defined by C. Kassel and J.L. Loday in [Kassel, Loday (1982)]. In [Porter (1987)], T. Porter shows how these can be seen as internal categories in the category of Lie algebras. We recall below their definition, as well as the notion of crossed complexes of Lie algebras, which are the analogous to crossed complexes of groups by J.H.C. Whitehead [Whitehead (1949)].

6.3. Definition. *A crossed module of Lie algebras consists of a homomorphism of Lie algebras $\partial : L_1 \to L_0$ together with a right Lie action of L_0 on L_1, denoted by $(y_1, y_0) \mapsto y_1 \cdot y_0$, such that*

$$\partial(y_1 \cdot y_0) = [\partial y_1, y_0], \quad y_1 \cdot \partial y_1' = [y_1, y_1'],$$

for any $x \in L_0$ and $y, y' \in L_1$.

A crossed complex of Lie algebras consists of a chain complex of Lie algebras

$$L = \cdots \to L_2 \to L_1 \to L_0,$$

where L_n is abelian for $n \geq 2$, together with a right Lie action of L_0 on each L_n, for $n \geq 1$, denoted by $(y_n, y_0) \mapsto y_n \cdot y_0$, such that, for any $y_0 \in L_0$, $y_1, y_1' \in L_1$, and $y_n \in L_n$ with $n \geq 2$,

$$\partial(y_1 \cdot y_0) = [\partial y_1, y_0], \quad y_1' \cdot \partial y_1 = [y_1', y_1], \quad \partial(y_n \cdot y_0) = \partial(y_n) \cdot y_0, \quad y_n \cdot \partial y_1 = 0. \tag{59}$$

Crossed complexes of Lie algebras are related to hypercrossed complexes as follows.

6.4. PROPOSITION. *A crossed complex of Lie algebras L is the same thing as a hypercrossed complex of Lie algebras (L,Φ) where $\Phi_n^{\beta,\gamma}=0 : L_\beta\times L_\gamma\to L_n$, for any $\beta,\gamma\in S(n)$ with $R(\beta)\cap R(\gamma)=\emptyset$ and $n<\beta<\gamma$. Indeed, such a hypercrossed complex is identified with that crossed complex defined by the complex L and the right actions of L_0 on each L_n defined by*

$$y_n\cdot y_0=\Phi_n^{n,\omega_n}(y_n,y_0). \tag{60}$$

PROOF. This is given in Subsection 7.4. ∎

Thanks to the above fact, the category of crossed complex of Lie algebras, denoted by XCh(Lie Alg), can be regarded as a full subcategory of the category of hypercrossed complexes.

Next, we shall identify the category of *simplicial T-complexes of Lie algebras* as the full subcategory of the category of simplicial Lie algebras to which this subcategory correspond by the enriched Dold-Kan-Puppe equivalence (56).

Simplicial T-complexes were introduced by N.K. Dakin as a short of simplicial sets with canonical fillers [Dakin (1977)]. More precisely, recall that a simplicial set X is said to satisfy the extension condition if every k-horn has a filler; i.e., if for any n-tuple $(x_0,\dots,x_{k-1},x_{k+1},\dots,x_n)$ of $n-1$ simplices x_i of X such that $d_ix_j=d_{j-1}x_i$ if $i<j$ and $i,j\neq k$, there is an n-simplex x such that $d_ix=x_i$ for $i\neq k$. A *simplicial T-complex* is a pair (X,T) where X is a simplicial set and $T=\bigcup_{n\geq 1}T_n$ with $T_n\subseteq X_n$, a set of simplices of X called *thin*, such that

(T1) all degenerate elements of X are thin,

(T2) any horn in X has a unique thin filler,

(T3) if all faces but one of a thin simplex are thin, then so also is the remaining face.

A simplicial T-complex in any algebraic category $\mathcal{C}$ is defined to be a simplicial T-complex (X,T) in which X is a simplicial object in $\mathcal{C}$ and each T_n a subobject of X_n. For instance, simplicial T-complexes of groups were first studied in [Ashley (1989)], where they are shown to be equivalent to crossed complexes of groups by mean of a generalized Dold-Kan correspondence, see also [Nan Tie I (1989), Nan Tie II (1989)]. Below is the counterpart for simplicial T-complexes of Lie algebras.

By [Ashley (1989), Chap. 3, Proposition 1.2], if $(\mathcal{L},T)$ is any simplicial T-complex of Lie algebras, then every thin simplex is a sum of degenerated elements, and therefore $T_n=D_n(\mathcal{L})$, the submodule of $\mathcal{L}_n$ generated by the degenerated n-simplices, for any $n\geq 1$. Hence, we say that a simplicial Lie algebra $\mathcal{L}$ is a *simplicial T-complex of Lie algebras* when the pair $\big(\mathcal{L},D=\bigcup_{n\geq 1}D_n(\mathcal{L})\big)$ is. In this way, the category of simplicial T-complexes of Lie algebras, denoted by T-Simpl(Lie Alg), is a full subcategory of the category of simplicial Lie algebras.

6.5. THEOREM. *The equivalence of Theorem 5.10 restricts by giving an equivalence between the category of simplicial* T-*complexes of Lie algebras and the category of crossed* *complexes of Lie algebras:*

$$\begin{tikzcd} \text{T-Simpl(Lie Alg)} \arrow[r, "N^{\mathrm{e}}", "\sim"'] \arrow[d, hook] & \text{XCh(Lie Alg)} \arrow[d, hook] \\ \text{Simpl(Lie Alg)} \arrow[r, "N^{\mathrm{e}}", "\sim"'] & \text{HXCh(Lie Alg)}. \end{tikzcd}$$

PROOF. Let $\mathcal{L}$ be any simplicial Lie algebra with enriched Moore complex $N^{\mathrm{e}}(\mathcal{L}) = (L, \Phi)$. By [Ashley (1989), Chap. 3, Theorem 1.5], the pair $(\mathcal{L}, D)$ always satisfies the axioms (T1), (T2) and (T3). Therefore, $\mathcal{L}$ is a simplicial T-complex of Lie algebras if and only if each submodule $D_n = \bigoplus_{\beta > n} \beta^* L_\beta$ of $\mathcal{L}_n = \bigoplus_{\beta \in S(n)} \beta^* L_\beta$ is actually a Lie subalgebra, that is, if and only if $[\beta^* x, \gamma^* y] \in D_n$ for any $\beta, \gamma \in S(n)$ with $\beta, \gamma > n$, $x \in L_\beta$, $y \in L_\gamma$. By the construction in (51) of the bilinear maps $\Phi_\alpha^{\beta,\gamma}$, this last requirement is equivalent to have $\Phi_n^{\beta,\gamma}(x,y) = 0$. Since, by (16), $\Phi_n^{\beta,\gamma}(x,y) = -\Phi_n^{\gamma,\beta}(y,x)$ if $\beta > \gamma$, and, by Lemma 5.6 and (31), $\Phi_n^{\beta,\gamma}(x,y) = 0$ whenever $R(\beta) \cap R(\gamma) \neq \emptyset$, we can conclude that $\mathcal{L}$ is a simplicial T-complex of Lie algebras if and only if $\Phi_n^{\beta,\gamma} = 0$, for any $\beta, \gamma \in S(n)$ with $R(\beta) \cap R(\gamma) = \emptyset$ and $n < \beta < \gamma$. That is, after Proposition 6.4, $\mathcal{L}$ is a simplicial T-complex of Lie algebras if and only if $N^{\mathrm{e}}(\mathcal{L})$ is a crossed complex of Lie algebras. ∎

6.6. HYPERGROUPOIDS OF LIE ALGEBRAS. For m any positive integer, m-dimensional hypergroupoids were first defined by J. Duskin and S. Schanuel [Duskin (1975)], and systematically studied by P.G. Glenn in [Glenn (1982)]. These are simplicial sets in which, for any $n > m$ and $0 \leq k \leq n$, any k-horn has a unique filler. Let

$$m\text{-Hypgd(Lie Alg)}$$

denote the category of *m-hypergroupoids of Lie algebras*, that is, the full subcategory of Simp(Lie Alg) defined by those simplicial Lie algebras whose underlaying simplicial set is an m-hypergroupoid.

6.7. LEMMA. *A simplicial Lie algebra is an m-hypergroupoid if and only if it has trivial Moore complex at dimension higher than m.*

PROOF. It is a direct consequence of [Bullejos, Cegarra, Duskin (1993), Lemma 1.1]. ∎

Hence, if we state

6.8. DEFINITION. *An m-hypercrossed complex of Lie algebras is an hypercrossed complex (L, Φ) such that $L_k = 0$ for all $k > m$.*

and we denote by m-HXCh(Lie Alg) the full subcategory of HXCh(Lie Alg) with objects the m-hypercrossed complexes of Lie algebras, we get from Theorem 5.10 the following

6.9. THEOREM. *The enriched Dold-Kan correspondence of Theorem 5.10 restricts to an equivalence between the categories of m-hypergroupoids and of m-hypercrossed complexes* *of Lie algebras:*

$$\begin{array}{ccc} m\text{-Hypgd(Lie Alg)} & \xrightarrow[\sim]{N^{\mathrm{e}}} & m\text{-HXCh(Lie Alg)} \\ \downarrow & & \downarrow \\ \text{Simpl(Lie Alg)} & \xrightarrow[\sim]{N^{\mathrm{e}}} & \text{HXCh(Lie Alg)}. \end{array}$$

The m-hypergroupoids and the m-hypercrossed complexes of Lie algebras naturally arise in the classification of simplicial Lie algebras by their homotopy m-type. To explain this, briefly, recall that the homotopy groups of a simplicial Lie algebra $\mathcal{L}$ are defined as the homology groups of its Moore complex:

$$\pi_n(\mathcal{L}) = H_n(N(\mathcal{L})), \qquad n \geq 0,$$

and that a simplicial map $f : \mathcal{L} \to \mathcal{L}'$ between simplicial Lie algebras is a weak m-equivalence if it induces isomorphisms $\pi_n(f) : \pi_n(\mathcal{L}) \cong \pi_n(\mathcal{L}')$ for all $0 \leq n \leq m$. Let

$$\mathrm{Ho}_m\mathrm{Simpl(Lie\,Alg)},$$

denote the category of fractions of Simpl(Lie Alg) where the weak m-equivalences have been inverted; that is, the homotopy category of m-types of simplicial Lie algebras. By the enriched Dold-Kan correspondence, these homotopy categories easily find corresponding equivalent categories in the context of hypercrossed complexes of Lie algebras: A morphism $f : (L, \Phi) \to (L', \Phi)$ in HXCh(Lie Alg) is said to be a weak m-equivalence if the underlying chain morphism $f : L \to L'$ induces isomorphisms $H_n(f) : H_n(L) \cong H_n(L')$ for all $0 \leq n \leq m$. The *homotopy category of m-types of hypercrossed complexes of Lie algebras*, denoted by

$$\mathrm{Ho}_m\mathrm{HXCh(Lie\,Alg)},$$

is then defined to be the category of fractions of the category HXCh(Lie Alg) where all weak m-equivalences have been inverted. Let also

$$\mathrm{Ho\,m\text{-}HXCh(Lie\,Alg)}, \quad \mathrm{Ho}\,m\text{-}\mathrm{Hypgd(Lie\,Alg)},$$

be the respective localized categories of the categories m-HXCh(Lie Alg) and m-Hypgd(Lie Alg) over all their respective weak equivalences.

6.10. Lemma. *The category* m-HXCh(Lie Alg) *is a reflective subcategory of* HXCh(Lie Alg). *For any hypercrossed complex of Lie algebras* (L, Φ)*, the reflection arrow pr* : $(L, \Phi) \to (\overline{L}, \overline{\Phi})$ *is a weak m-equivalence.*

Proof. If (L, Φ) is any given hypercrossed complex of Lie algebras, the image $\partial L_{m+1} \subseteq L_m$ is an ideal: For any $y \in L_{m+1}$ and $z \in L_m$,

$$[\partial y, z] \overset{(29)}{=} \Phi_m^{m,m}(\partial y, z) \overset{(43)}{=} \Phi_{\sigma_m}^{m+1,\sigma_m}(y, z) + \partial\, \Phi_{m+1}^{m+1,\sigma_m}(y, z) \overset{(33)}{=} \partial\, \Phi_{m+1}^{m+1,\sigma_m}(y, z).$$

Let $\overline{L} = \cdots 0 \to L_m/\partial L_{m+1} \to L_{m-1} \to \cdots \to L_0$ be the complex of Lie algebras with the same $(m-1)$-truncation as L, $\overline{L}_m = L_m/\partial L_{m+1}$, the differential $\partial : L_m/\partial L_{m+1} \to$

L_{m-1} is the induced by the differential $\partial : L_m \to L_{m-1}$ of L, and $\overline{L}_k = 0$ for all $k > m$. Let $(\overline{L}, \overline{\Phi})$ the m-hypercrossed complex in which, for any $n < m$, $\overline{\Phi}_n^{\beta,\gamma} = \Phi_n^{\beta,\gamma}$, and the bilinear maps $\overline{\Phi}_m^{\beta,\gamma}$ are the given by the composition of the corresponding $\Phi_m^{\beta,\gamma} : L_\beta \times L_\gamma \to L_m$ with the projection map $pr : L_m \to L_m/\partial L_{m+1}$.

It is plain to see that $(\overline{L}, \overline{\Phi})$ remains a hypercrossed complex, and that the canonical chain morphism

$$\begin{tikzcd}
\cdots \arrow[r] & L_{m+1} \arrow[r] \arrow[d] & L_m \arrow[r] \arrow[d, "pr"] & L_{m-1} \arrow[r] \arrow[d, equal] & \cdots \arrow[r] & L_0 \arrow[d, equal] \\
\cdots \arrow[r] & 0 \arrow[r] & L_m/\partial L_{m+1} \arrow[r] & L_{m-1} \arrow[r] & \cdots \arrow[r] & L_0
\end{tikzcd}$$

gives a reflection morphism of hypercrossed complexes $pr : (L, \Phi) \to (\overline{L}, \overline{\Phi})$, which is a weak m-equivalence. ∎

6.11. Proposition. *All the categories below are equivalent.*

$$\mathrm{Ho}_m\mathrm{Simpl}(\mathrm{Lie\,Alg}),\ \mathrm{Ho}\,m\text{-}\mathrm{Hypgd}(\mathrm{Lie\,Alg}),\ \mathrm{Ho}_m\mathrm{HXCh}(\mathrm{Lie\,Alg}),\ \mathrm{Ho}\,m\text{-}\mathrm{HXCh}(\mathrm{Lie\,Alg}).$$

Proof. The equivalence $\mathrm{Ho}_m\mathrm{Simpl}(\mathrm{Lie\,Alg}) \simeq \mathrm{Ho}_m\mathrm{HXCh}(\mathrm{Lie\,Alg})$ is induced by the enriched Dold-Kan-Puppe equivalence in Theorem 5.10, whose restricted equivalence between the categories of m-hypergroupoids and of m-hypercrossed complexes of Lie algebras induces the equivalence $\mathrm{Ho}\,m\text{-}\mathrm{Hypgd}(\mathrm{Lie\,Alg}) \simeq \mathrm{Ho}\,m\text{-}\mathrm{HXCh}(\mathrm{Lie})$. Finally, the equivalence $\mathrm{Ho}_m\mathrm{HXCh}(\mathrm{Lie\,Alg}) \simeq \mathrm{Ho}\,m\text{-}\mathrm{HXCh}(\mathrm{Lie\,Alg})$ follows from Lemma 6.10 above. ∎

6.12. Crossed and 2-crossed modules. For lower values of m, m-hypercrossed complexes of Lie algebras can be already found in the literature, although the identification is not trivial and requires of some work. This is the goal of this subsection.

The easier case is when $m = 1$. Recalling from Definition 6.3 the notion of crossed module of Lie algebras, we have

6.13. Proposition. *A crossed module of Lie algebras is the same thing as an 1-hypercrossed complex of Lie algebras. Indeed, such a 1-hypercrossed complex is identified with the crossed module defined by the differential $\partial : L_1 \to L_0$ and the action of L_0 on L_1 defined by*

$$y_1 \cdot y_0 \overset{(60)}{=} \Phi_1^{1,\sigma_0}(y_1, y_0).$$

Proof. This, in fact, has already been done: A crossed module of Lie algebras is the same thing as a crossed complex of Lie algebras L with $L_k = 0$ for all $k \geq 2$ which, by Proposition 6.4, is the same as a 1-hypercrossed complex of Lie algebras. ∎

Hence, if XM(Lie Alg) denotes the category of crossed modules of Lie algebras, Theorem 6.9 particularizes by giving the following well-known result [Ellis (1993), Theorem 2]:

6.14. Corollary. *The enriched Dold-Kan restricts to an equivalence between the categories of* 1*-hypergroupoids and of crossed modules of Lie algebras:*

$$\begin{tikzcd} \text{1-Hypgd(Lie Alg)} \arrow[r, "N^{\mathrm{e}}", "\sim"'] \arrow[d, hook] & \text{XM(Lie Alg)} \arrow[d, hook] \\ \text{Simpl(Lie Alg)} \arrow[r, "N^{\mathrm{e}}", "\sim"'] & \text{HXCh(Lie Alg)}. \end{tikzcd}$$

We stress the relevance of 1-hypergroupoids by recalling that they are the same as nerves of internal categories (= internal groupoids) in the category of Lie algebras [Porter (1987), Glenn (1982)].

Next, we prove that the notion of 2-hypercrossed complex of Lie algebras is equivalent to that of 2-*crossed modules of Lie algebras*, first studied by G.J. Ellis in [Ellis (1993)]. Note that the definition of a 2-crossed module of Lie algebras, that we recall below, is a differential replica of the definition of 2-crossed module of Lie groups [Martins, Picken (2011), Jurco (2012)] and, by this reason, they also appear under the name of *differential* 2*-crossed modules.*

6.15. Proposition. *A* 2*-hypercrossed complex of Lie algebras is the same thing as a* 2*-crossed module of Lie algebras, that is, a* 2*-truncated complex of Lie algebras* $L_2 \xrightarrow{\partial} L_1 \xrightarrow{\partial} L_0$*, together with right Lie actions of* L_0 *on* L_1 *and* L_2*, denoted by* $(y_n, y_0) \mapsto y_n \cdot y_0$*,* $n = 1, 2$*, and a bilinear map* $\{-,-\} : L_1 \times L_1 \to L_2$*, such that the following conditions hold* [2]*:*

$$\partial(y_1 \cdot y_0) = [\partial y_1, y_0], \tag{61}$$
$$\partial(y_2 \cdot y_0) = \partial y_2 \cdot y_0, \tag{62}$$
$$\partial\{y_1, y_1'\} = y_1 \cdot \partial y_1' - [y_1, y_1'], \tag{63}$$
$$\{\partial y_2, \partial y_2'\} = [y_2', y_2], \tag{64}$$
$$\{\partial y_2, y_1\} + \{y_1, \partial y_2\} = y_2 \cdot \partial y_1, \tag{65}$$
$$\{y_1, y_1'\} \cdot y_0 = \{y_1 \cdot y_0, y_1'\} + \{y_1, y_1' \cdot y_0\}, \tag{66}$$
$$\{y_1, [y_1', y_1'']\} = \{y_1, y_1'\} \cdot \partial y_1'' + \{[y_1, y_1'], y_1''\} - \{y_1, y_1''\} \cdot \partial y_1' - \{[y_1, y_1''], y_1'\}, \tag{67}$$
$$\{[y_1, y_1'], y_1''\} = \{y_1', \partial\{y_1, y_1''\}\} - \{y_1, \partial\{y_1', y_1''\}\}. \tag{68}$$

Indeed, any 2*-hypercrossed complex* (L, Φ) *is identified with the* 2*-crossed module defined by the differentials* $L_2 \xrightarrow{\partial} L_1 \xrightarrow{\partial} L_0$*, and the right Lie actions of* L_0 *on* L_1 *and* L_2 *and the bilinear map* $\{-,-\}$ *respectively defined by*

$$y_1 \cdot y_0 = \Phi_1^{1,\sigma_0}(y_1, y_0), \quad y_2 \cdot y_0 = \Phi_2^{2,\sigma_0\sigma_1}(y_2, y_0), \quad \{y_1, y_1'\} = \Phi_2^{\sigma_1,\sigma_0}(y_1, y_1'). \tag{69}$$

Proof. This is fully given in Subsection 7.5. However, to be used before, we shall prove here how, for any given 2-hypercrossed complex (L, Φ), the maps $\Phi_2^{\sigma_1,\sigma_0}$ and $\Phi_2^{2,\sigma_0\sigma_1}$

[2]There is a mistake in the formulation of the fourth axiom in [Ellis (1993)].

determine the two Lie actions $\Phi_2^{2,\sigma_1}, \Phi_2^{2,\sigma_0} : L_2 \times L_1 \to L_2$. In effect, letting $\Phi^{\mathrm{ext}} = \{\Phi_\alpha^{\beta,\gamma}\}$ its extended set of structure bilinear maps, we have

$$\begin{aligned}
\Phi_2^{2,\sigma_1}(y_2, y_1) &\overset{(33)}{=} -\Phi_{\sigma_2}^{\sigma_0,\sigma_1\sigma_2}(y_2, y_1) \overset{(43)}{=} -\Phi_2^{\sigma_0,\sigma_1}(\partial y_2, y_1) \overset{(30)}{=} \Phi_2^{\sigma_1,\sigma_0}(y_1, \partial y_2),\\
\Phi_2^{2,\sigma_0}(y_2, y_1) &\overset{(30)}{=} -\Phi_2^{\sigma_0,2}(y_1, y_2) \overset{(33)}{=} -\Phi_{\sigma_2}^{\sigma_0\sigma_1,\sigma_2}(y_1, y_2) \overset{(43)}{=} -\Phi_2^{\sigma_0\sigma_1,2}(\partial y_1, y_2)\\
&\overset{(30)}{=} \Phi_2^{2,\sigma_0\sigma_1}(y_2, \partial y_1).
\end{aligned}$$

Therefore, in terms of (69), we have

$$\Phi_2^{2,\sigma_1}(y_2, y_1) = \{y_1, \partial y_2\}, \quad \Phi_2^{2,\sigma_0}(y_2, y_1) = y_2 \cdot \partial y_1. \tag{70}$$

■

Therefore, if 2-XM(Lie Alg) denotes the category of 2-crossed modules of Lie algebras, Theorem 6.9 particularizes by giving the following result.

6.16. Corollary. [Ellis (1993)] *The enriched Dold-Kan-Puppe equivalence restricts to an equivalence between the categories of* 2*-hypergroupoids and of* 2*-crossed modules of Lie algebras:*

$$\begin{tikzcd}
\text{2-Hypgd(Lie Alg)} \arrow[r, "N^{\mathrm{e}}", "\sim"'] \arrow[d, hook] & \text{2-XM(Lie Alg)} \arrow[d, hook] \\
\text{Simpl(Lie Alg)} \arrow[r, "N^{\mathrm{e}}", "\sim"'] & \text{HXCh(Lie Alg)}.
\end{tikzcd}$$

6.17. Delooping: Braided and symmetric crossed modules of Lie algebras. Recall that a simplicial Lie algebra is reduced if $\mathcal{L}_0 = 0$. Let

$$\Omega : {}_{\mathrm{red}}\mathrm{Simpl}(\mathrm{Lie\,Alg}) \to \mathrm{Simpl}(\mathrm{Lie\,Alg})$$

be the simplicial *loop construction*, which on any reduced simplicial Lie algebra $\mathcal{L}$ is the simplicial Lie algebra $\Omega\mathcal{L}$ with

$$(\Omega\mathcal{L})_n = \ker(d_0 : \mathcal{L}_{n+1} \to \mathcal{L}_n),$$

and for each map $\alpha : [n] \to [r]$ in Δ the induced $(\Omega\mathcal{L})_r \to (\Omega\mathcal{L})_n$ is given by restriction of the homomorphism $(\alpha+1)^* : \mathcal{L}_{r+1} \to \mathcal{L}_{n+1}$ attached by $\mathcal{L}$ to the map $\alpha+1 : [n+1] \to [r+1]$ defined by $(\alpha+1)(0) = 0$ and $(\alpha+1)(i+1) = \alpha(i)+1$, for $0 \le i \le n$, that is,

$$(\alpha+1)(j) = \begin{cases} 0 & \text{if } \ j = 0, \\ \alpha(j-1)+1 & \text{if } \ 0 < j \le n+1. \end{cases}$$

Thus, for each $0 \le i \le n$ the face operator d_i on $(\Omega\mathcal{L})_n$ is given by restriction of d_{i+1} on $\mathcal{L}_{n+1}$, and similarly for s_i.

In this section, we pay attention to those $\mathcal{L}$ such that $\Omega\mathcal{L}$ is a 1-hypergroupoid of Lie algebras, that is, the nerve of an internal category (= internal groupoid) in the category of Lie algebras. Even more, we deal here with the categories

$$\Omega^{-p}(\text{1-Hypgd}(\text{Lie Alg}))$$

of *simplicial p-deloopings of 1-hypergroupoids of Lie algebras*, recursively defined for integers $p \geq 1$ by the pullback squares

$$\begin{tikzcd}
\Omega^{-1}(\text{1-Hypgd}(\text{Lie Alg})) \arrow[r] \arrow[d, hook] & \text{1-Hypgd}(\text{Lie Alg}) \arrow[d, hook] \\
{}_{\text{red}}\text{Simpl}(\text{Lie Alg}) \arrow[r, "\Omega"] & \text{Simpl}(\text{Lie Alg}),
\end{tikzcd}$$

$$\begin{tikzcd}
\Omega^{-p-1}(\text{1-Hypgd}(\text{Lie Alg})) \arrow[r] \arrow[d, hook] & \Omega^{-p}(\text{1-Hypgd}(\text{Lie Alg})) \arrow[d, hook] \\
{}_{\text{red}}\text{Simpl}(\text{Lie Alg}) \arrow[r, "\Omega"] & \text{Simpl}(\text{Lie Alg}).
\end{tikzcd}$$

These categories $\Omega^{-p}(\text{1-Hypgd}(\text{Lie Alg}))$ are closely related to the category of crossed modules of Lie algebras. To precise this relationship between them, first note that the loop construction above has a natural counterpart on hypercrossed complexes of Lie algebras: There is a commutative square

$$\begin{tikzcd}
{}_{\text{red}}\text{Simpl}(\text{Lie Alg}) \arrow[r, "N^{\text{e}}", "\sim"'] \arrow[d, "\Omega"'] & {}_{\text{red}}\text{HXCh}(\text{Lie Alg}) \arrow[d, "\Omega"] \\
\text{Simpl}(\text{Lie Alg}) \arrow[r, "N^{\text{e}}", "\sim"'] & \text{HXCh}(\text{Lie Alg}),
\end{tikzcd} \tag{71}$$

where N^{e} is the enriched Dold-Kan-Puppe correspondence, ${}_{\text{red}}\text{HXCh}(\text{Lie Alg})$ is the category of those hypercrossed complexes of Lie algebras (L, Φ) which are reduced in the sense that $L_0 = 0$, and

$$\Omega : {}_{\text{red}}\text{HXCh}(\text{Lie Alg}) \to \text{HXCh}(\text{Lie Alg})$$

is the *loop construction*: $(\Omega L, \Omega\Phi)$, on any reduced hypercrossed complex (L, Φ), is the hypercrossed complex with

$$(\Omega L)_n = L_{n+1}, \quad (\Omega\Phi)_n^{\alpha,\beta} = \Phi_{n+1}^{\alpha+1,\beta+1}. \tag{72}$$

Since both functors N^{e} in (71) are equivalences, it is not necessary to check the axioms for $(\Omega L, \Omega\Phi)$. One simply observes that, so defined, the square commutes.

Now, recall from Proposition 6.13 that the category 1-HXCh(Lie Alg), of 1-hypercrossed complex of Lie algebras, is identified with the category XM(Lie Alg), of crossed modules of Lie algebras. Hence, the categories $\Omega^{-p}(\text{XM}(\text{Lie Alg}))$, of *p-delooping crossed modules of Lie algebras*, can be recursively defined by the pullback squares

$$\begin{tikzcd}
\Omega^{-1}(\text{XM}(\text{Lie Alg})) \arrow[r] \arrow[d, hook] & \text{XM}(\text{Lie Alg}) \arrow[d, hook] \\
{}_{\text{red}}\text{HXCh}(\text{Lie Alg}) \arrow[r, "\Omega"] & \text{HXCh}(\text{Lie Alg}),
\end{tikzcd} \qquad \begin{tikzcd}
\Omega^{-p-1}(\text{XM}(\text{Lie Alg})) \arrow[r] \arrow[d, hook] & \Omega^{-p}(\text{XM}(\text{Lie Alg})) \arrow[d, hook] \\
{}_{\text{red}}\text{HXCh}(\text{Lie Alg}) \arrow[r, "\Omega"] & \text{HXCh}(\text{Lie Alg}).
\end{tikzcd}$$

Since, by Corollary 6.14, $N^{\mathrm{e}}: \text{1-Hypgd}(\mathrm{Lie\,Alg}) \simeq \mathrm{XM}(\mathrm{Lie\,Alg})$ is an equivalence of categories, the commutativity of the square (71) gives the following

6.18. Proposition. *The enriched Dold-Kan correspondence restricts to an equivalence*

$$N^{\mathrm{e}}: \Omega^{-p}(\text{1-Hypgd}(\mathrm{Lie\,Alg})) \simeq \Omega^{-p}(\mathrm{XM}(\mathrm{Lie\,Alg})), \quad p \geq 1.$$

It is clear that $\Omega^{-p}(\,\mathrm{XM}(\mathrm{Lie\,Alg}))$, the category of p-delooping crossed modules of Lie algebras, consists of those hypercrossed complexes (L, Φ) such that $L_k = 0$ for $k \notin \{p+1, p\}$, and therefore we have

6.19. Corollary. *The category $\Omega^{-p}(\text{1-Hypgd}(\mathrm{Lie\,Alg}))$ consists of those simplicial Lie algebras whose Moore complex is trivial at dimensions outside the range $[p+1, p]$ or, in other words, the $(p-1)$-reduced $(p+1)$-hypergroupoids of Lie algebras.*

Next, we analyze the categories $\Omega^{-p}(\mathrm{XM}(\mathrm{Lie\,Alg}))$, first for $p = 1$ and then for upper p.

6.19.1. Braided crossed modules of Lie algebras. The category of delooping crossed modules of Lie algebras, $\Omega^{-1}(\mathrm{XM}(\mathrm{Lie\,Alg}))$, consists of those hypercrossed complexes (L, Φ) such that $L_k = 0$ for all $k \notin \{2, 1\}$; that is, the category of reduced 2-hypercrossed complexes which, by Proposition 6.15, are identified with those 2-crossed modules that are reduced in the sense that they have the trivial Lie algebra at dimension zero. According the description in Proposition 6.15 of the 2-crossed modules, we have the following one for delooping crossed modules of Lie algebras

6.20. Proposition. *A delooping crossed module of Lie algebras is the same thing as a "braided crossed module of Lie algebras"* [3]*; that is, a pair*

$$(\partial : L \to M,\ \{-,-\} : M \times M \to L),$$

where ∂ is a homomorphism of Lie algebras and $\{-,-\}$ a bilinear map, such that the following four conditions hold

$$\partial\{x, x'\} = [x', x], \tag{73}$$

$$\{\partial y, \partial y'\} = [y', y], \tag{74}$$

$$\{\partial y, x\} + \{x, \partial y\} = 0, \tag{75}$$

$$\{x, [x', x'']\} + \{x'', [x, x']\} + \{x', [x, x'']\} = 0. \tag{76}$$

Indeed, each delooping crossed module of Lie algebras (L, Φ) is identified with the braided crossed module defined by the differential $\partial : L_2 \to L_1$ and the bilinear map $\{-,-\} : L_1 \times L_1 \to L_2$, defined by

$$\{y_1, y_1'\} \overset{(69)}{=} \Phi_2^{\sigma_1,\sigma_0}(y_1, y_1'). \tag{77}$$

[3] It is well known that braided strict categorical groups [Joyal, Street (1991)] are the same as braided crossed modules of groups [Brown, Gilbert (1989), Arvasi, Ulualan (2005)]. Braided crossed modules of Lie algebras can be regarded as the differential replica of the braided crossed modules of Lie groups, and this is why we have adopted the given name for them.

Note that, if (L, Φ) is any delooping crossed module, its loop 1-hypercrossed complex $\Omega(L, \Phi) = (\Omega L, \Omega\Phi)$ becomes then identified with the crossed module defined by the homomorphism $\partial : L_2 \to L_1$ and the right action of L_1 on L_2 defined by

$$y_2 \cdot y_1 \overset{(72)}{=} \Phi^{1+1,\sigma_0+1}_{1+1}(y_2, y_1) = \Phi^{2,\sigma_1}_{2}(y_2, y_1) \overset{(70)}{=} \{y_1, \partial y_2\}.$$

Thus, we have the commutative square

$$\begin{tikzcd} \Omega^{-1}(\text{1-Hygpd(Lie Alg)}) \arrow[r, "\Omega"] \arrow[d, "\underset{\sim}{N^e}"'] & \text{1-Hygpd(Lie Alg))} \arrow[d, "\underset{\sim}{N^e}"] \\ \text{BXM(Lie Alg)} \arrow[r, "\Omega"] & \text{XM(Lie Alg)}, \end{tikzcd}$$

where the *loop functor*

$$\Omega : \text{BXM(Lie Alg)} \longrightarrow \text{XM(Lie Alg)} \tag{78}$$

carries any braided crossed module $(\partial : L \to M, \{-,-\})$ to the crossed module given by the homomorphism $\partial : L \to M$ and the right action of M on L defined by $y \cdot x = \{x, \partial y\}$.

Note that the loop functor Ω in (78) is faithful but, in general, neither full nor surjective in objects. Also that a braided crossed module of Lie algebras is not properly a crossed module, but it have an "underlying" crossed module by the faithful Ω construction.

6.20.1. Symmetric crossed modules of Lie algebras. Recall that the category of p-delooping crossed modules of Lie algebras, $\Omega^{-p}(\text{XM(Lie Alg)})$, consists of those hypercrossed complexes (L, Φ) such that $L_k = 0$ for all $k \notin \{p+1, p\}$.

6.21. Proposition. *For any integer $p \geq 2$, a p-delooping crossed module of Lie algebras is the same thing as a "symmetric crossed module of Lie algebras"* [4]*, that is, a pair*

$$(\partial : L \to M, \{-,-\} : M \times M \to L),$$

where ∂ is a homomorphism of Lie algebras and $\{-,-\}$ a bilinear map satisfying

$$\partial\{x, x'\} = [x', x], \tag{79}$$
$$\{\partial y, \partial y'\} = [y', y], \tag{80}$$
$$\{x, x'\} + \{x', x\} = 0, \tag{81}$$
$$\{x, [x', x'']\} + \{x', [x'', x]\} + \{x'', [x, x']\} = 0. \tag{82}$$

Indeed, each p-delooping crossed module of Lie algebras (L, Φ) is identified with the symmetric delooping crossed module defined by the homomorphism $\partial : L_{p+1} \to L_p$ and the bilinear map

$$\{-,-\} = \Phi^{\sigma_p, \sigma_{p-1}}_{p+1} : L_p \times L_p \to L_{p+1}. \tag{83}$$

Proof. This is given in Subsection 7.6. ∎

[4] It is well known that symmetric strict categorical groups are the same as symmetric crossed modules of groups [Bullejos, Carrasco, Cegarra (1993)]. Symmetric crossed modules of Lie algebras are the differential counterpart of symmetric crossed modules of Lie groups, and this is why the name we adopt for them. But symmetric crossed modules of groups are also the same as stable crossed modules of groups [Conduché (1984)], so the name of *stable crossed modules of Lie algebras* would be also appropriate.

If (L,Φ) is any 2-delooping crossed module, identified with the symmetric crossed module $(\partial : L_3 \to L_2, \{-,-\} = \Phi_3^{\sigma_2,\sigma_1} : L_2 \times L_2 \to L_3)$, then its loop delooping crossed module $\Omega(L,\Phi) = (\Omega L, \Omega\Phi)$ is identified with the braided crossed module defined by the homomorphism $\partial : (\Omega L)_2 = L_3 \longrightarrow (\Omega L)_1 = L_2$ and the bilinear map

$$(\Omega\Phi)_2^{\sigma_1,\sigma_0} \overset{(72)}{=} \Phi_{2+1}^{\sigma_1+1,\sigma_0+1} \overset{(83)}{=} \Phi_3^{\sigma_2,\sigma_1} = \{-,-\}.$$

Thus, we have the commutative square

$$\begin{tikzcd}
\Omega^{-2}(\text{1-Hygpd(Lie Alg)}) \arrow[r, "\Omega"] \arrow[d, "N^{\mathrm{e}}"', "\sim"] & \Omega^{-1}(\text{1-Hygpd(Lie Alg)}) \arrow[d, "N^{\mathrm{e}}", "\sim"'] \\
\text{SXM(Lie Alg)} \arrow[r, hook] & \text{BXM(Lie Alg)},
\end{tikzcd}$$

where $\text{SXM(Lie Alg)} \hookrightarrow \text{BXM(Lie Alg)}$ denotes the natural embedding of the category of symmetric crossed modules into the category of braided crossed modules.

Similarly, if $p \geq 3$ and (L,Φ) is any p-delooping crossed module, which is identified with the symmetric crossed module $(\partial : L_{p+1} \to L_p, \{-,-\} = \Phi_{p+1}^{\sigma_p,\sigma_{p-1}} : L_p \times L_p \to L_{p+1})$, then its loop $(p-1)$-delooping crossed module $\Omega(L,\Phi) = (\Omega L, \Omega\Phi)$ is identified with the same symmetric crossed module as (L,Φ) does, since it is defined by the homomorphism

$$\partial : (\Omega L)_p = L_{p+1} \longrightarrow (\Omega L)_{p-1} = L_p$$

together with the bilinear map

$$(\Omega\Phi)_p^{\sigma_{p-1},\sigma_{p-2}} \overset{(72)}{=} \Phi_{p+1}^{\sigma_{p-1}+1,\sigma_{p-2}+1} = \Phi_{p+1}^{\sigma_p,\sigma_{p-1}} \overset{(83)}{=} \{-,-\}.$$

Hence, we have the commutative square

$$\begin{tikzcd}
\Omega^{-p}(\text{1-Hygpd(Lie Alg)}) \arrow[r, "\Omega"] \arrow[d, "N^{\mathrm{e}}"', "\sim"] & \Omega^{-p+1}(\text{1-Hygpd(Lie Alg)}) \arrow[d, "N^{\mathrm{e}}", "\sim"'] \\
\text{SXM(Lie Alg)} \arrow[r, equal] & \text{SXM(Lie Alg)}.
\end{tikzcd}$$

To finish, we shall stress the following immediate consequences (cf. [Conduché (1984), Corollaire 3.5]).

6.22. Corollary. *The loop functor*

$$\Omega : \Omega^{-2}(\text{1-Hygpd(Lie Alg)}) \to \Omega^{-1}(\text{1-Hygpd(Lie Alg)})$$

is an embedding; that is, the loop functor establishes an embedding of the category of simplicial Lie algebras whose Moore complex is trivial at other dimensions than 2 *and* 3 *into the category of simplicial Lie algebras whose Moore complex is trivial at other dimensions than* 1 *and* 2.

For any $p \geq 3$, the functor $\Omega^{p-2} : \Omega^{-p}(\text{1-Hygpd(Lie Alg)}) \to \Omega^{-2}(\text{1-Hygpd(Lie Alg)})$ is an equivalence of categories; that is, the iterated loop functor Ω^{p-2} establishes an equivalence between the category of simplicial Lie algebras whose Moore complex is trivial at other dimensions than p and $p+1$ and the category of simplicial Lie algebras whose Moore complex is trivial at other dimensions than 2 and 3.

If $p \geq 2$, every simplicial Lie algebra whose Moore complex is trivial at other dimensions than p and $p+1$ is an ∞-loop simplicial Lie algebra.

7. Some proofs

As we said in the introduction, to avoid hampering the flow of the paper most of the more technical proofs have been brought to this section.

7.1. Proof of Lemma 4.4. First note that, due to formulas (31) and (32), both equations (43) and (44) trivially hold for any $\beta, \gamma \in S(n)$ such that $R(\beta) \cap R(\gamma) \neq \emptyset$.

Proof of (45): The equation holds if $\alpha = n$, owing to condition (43). Therefore we have to prove only the case when $\alpha > n$, that we do by induction on n. The case $n = 0$ is trivial. Then, let $\alpha, \beta, \gamma \in S(n)$, $y \in L_{r_\beta+1}$, and $z \in L_\gamma$, where $\alpha > n > 0$. We distinguish three different cases, noticing previously that the following equalities hold

$$R(\beta) \cap R(\gamma) = R(\beta_+) \cap R(\gamma\sigma_n),$$
$$R(\alpha) \cap R(\beta) \cap R(\gamma) = R(\alpha\sigma_n) \cap R(\beta_+) \cap R(\gamma\sigma_n) = R(\alpha_+) \cap R(\beta_+) \cap R(\gamma\sigma_n).$$

Case $R(\beta) \cap R(\gamma) = \emptyset$. Let $m = \min R(\alpha)$. Taking into account that, for any $i \leq m$, $\beta_+\delta_i = (\beta\delta_i)_+$, $\sigma_n\delta_i = \delta_i\sigma_{n-1}$, and $\alpha_+\delta_i = (\alpha\delta_i)_+$, we have

$$\begin{aligned}
\Phi_\alpha^{\beta,\gamma}(\partial y, z) &\overset{(33)}{=} \sum_{i \in R(\alpha,\beta,\gamma)} (-1)^{m-i}\; \Phi_{\alpha\delta_m}^{\beta\delta_i,\gamma\delta_i}(\partial y, z) \\
\text{(ind. hypoth.)} &= \sum_{i \in R(\alpha,\beta,\gamma)} (-1)^{m-i} \Big(\Phi_{\alpha\delta_m\sigma_{n-1}}^{(\beta\delta_i)_+,\gamma\delta_i\sigma_{n-1}}(y,z) + \partial\, \Phi_{(\alpha\delta_m)_+}^{(\beta\delta_i)_+,\gamma\delta_i\sigma_{n-1}}(y,z) \Big) \\
&= \sum_{i \in R(\alpha,\beta,\gamma)} (-1)^{m-i} \Phi_{\alpha\sigma_n\delta_m}^{\beta_+\delta_i,\gamma\sigma_n\delta_i}(y,z) + \partial \sum_{i \in R(\alpha,\beta,\gamma)} (-1)^{m-i} \Phi_{\alpha_+\delta_m}^{\beta_+\delta_i,\gamma\sigma_n\delta_i}(y,z) \\
&\overset{(33)}{=} \Phi_{\alpha\sigma_n}^{\beta_+,\gamma\sigma_n}(y,z) + \partial\, \Phi_{\alpha_+}^{\beta_+,\gamma\sigma_n}(y,z),
\end{aligned}$$

where, for the last equality we have used that $m = \min R(\alpha) = \min R(\alpha\sigma_n) = \min R(\alpha_+)$, and that $R(\alpha,\beta,\gamma) = R(\alpha\sigma_n, \beta_+, \gamma\sigma_n) = R(\alpha_+, \beta_+, \gamma\sigma_n)$.

Case $R(\beta) \cap R(\gamma) \neq \emptyset$ but $R(\alpha) \cap R(\beta) \cap R(\gamma) = \emptyset$. In this case, by (31) and (32), we have $\Phi_\alpha^{\beta,\gamma}(\partial y, z) = 0$, $\Phi_{\alpha\sigma_n}^{\beta_+,\gamma\sigma_n}(y,z) = 0$, and $\Phi_{\alpha_+}^{\beta_+,\gamma\sigma_n}(y,z) = 0$, so the equality in (45) trivially holds.

Case $R(\beta)\cap R(\gamma)\neq\emptyset$ and $R(\alpha)\cap R(\beta)\cap R(\gamma)\neq\emptyset$. Let $k=\min R(\alpha)\cap R(\beta)\cap R(\gamma)$. Then,

$$\begin{aligned}
\Phi_{\alpha}^{\beta,\gamma}(\partial y,z) &\overset{(32)}{=} \Phi_{\alpha\delta_k}^{\beta\delta_k,\gamma\delta_k}(\partial y,z)\\
\text{(induct. hypoth.)} &= \Phi_{\alpha\delta_k\sigma_{n-1}}^{(\beta\delta_k)_+,\gamma\delta_k\sigma_{n-1}}(y,z)+\partial\Phi_{(\alpha\delta_k)_+}^{(\beta\delta_k)_+,\gamma\delta_k\sigma_{n-1}}(y,z)\\
&= \Phi_{\alpha\sigma_n\delta_k}^{\beta_+\delta_k,\gamma\sigma_n\delta_k}(y,z)+\partial\,\Phi_{\alpha_+\delta_k}^{\beta_+\delta_k,\gamma\sigma_n\delta_k}(y,z)\\
&\overset{(32)}{=} \Phi_{\alpha\sigma_n}^{\beta_+,\gamma\sigma_n}(y,z)+\partial\,\Phi_{\alpha_+}^{\beta_+,\gamma\sigma_n}(y,z).
\end{aligned}$$

Proof of (46): First, we consider the case when $\alpha=n$. Then, the equation holds if $n<\beta<\gamma$, due to condition (44). If $\beta=n=\gamma$, we have $\Phi_n^{n,n}(\partial y,\partial z)=[\partial y,\partial z]$ and $\partial\Phi_{n+1}^{n+1,n+1}(y,z)=\partial[y,z]$, by (29). Since, by (33), $\Phi_{\sigma_n}^{n+1,n+1}(y,z)=0$, the equality in (46) follows form the fact that $\partial:L_{n+1}\to L_n$ is a homomorphism of Lie algebras. If $\beta=n<\gamma$, then

$$\begin{aligned}
\Phi_n^{n,\gamma}(\partial y,\partial z) &\overset{(43)}{=} \Phi_{\sigma_n}^{n+1,\gamma\sigma_n}(y,\partial z)+\partial\,\Phi_{n+1}^{n+1,\gamma\sigma_n}(y,\partial z)\overset{(33)}{=}\partial\,\Phi_{n+1}^{n+1,\gamma\sigma_n}(y,\partial z)\\
&\overset{(36)}{=} -\partial\,\Phi_{n+1}^{\gamma\sigma_n,n+1}(\partial z,y)\overset{(43)}{=}-\partial\big(\Phi_{\sigma_{n+1}}^{(\gamma\sigma_n)_+,\sigma_{n+1}}(z,y)+\partial\,\Phi_{n+2}^{(\gamma\sigma_n)_+,\sigma_{n+1}}(z,y)\big)\\
&\overset{(\partial^2=0)}{=} -\partial\,\Phi_{\sigma_{n+1}}^{(\gamma\sigma_n)_+,\sigma_{n+1}}(z,y)\overset{(33)}{=}-\partial\,\Phi_{n+1}^{\gamma_+,n+1}(z,y)\overset{(36)}{=}\partial\,\Phi_{n+1}^{n+1,\gamma_+}(y,z).
\end{aligned}$$

As $\Phi_{\sigma_n}^{n+1,\gamma_+}(z,y)=0$ by (33), the equation (46) also holds in this case.

Therefore we have to prove only the case when $\alpha>n$, whose proof is quite similar to the one given above for (45). We proceed by induction on n. The case $n=0$ is trivial. Then, let $\alpha,\beta,\gamma\in S(n)$, $y\in L_{r_\beta+1}$, and $z\in L_{r_\gamma}+1$, where $\alpha>n>0$. We distinguish three different cases, noticing previously that the following equalities hold

$$\begin{aligned}
&R(\beta)\cap R(\gamma)=R(\beta_+)\cap R(\gamma_+),\\
&R(\alpha)\cap R(\beta)\cap R(\gamma)=R(\alpha\sigma_n)\cap R(\beta_+)\cap R(\gamma_+)=R(\alpha_+)\cap R(\beta_+)\cap R(\gamma_+).
\end{aligned}$$

Case $R(\beta)\cap R(\gamma)=\emptyset$. Let $m=\min R(\alpha)$. Then, $m=\min R(\alpha\sigma_n)=\min R(\alpha_+)$ and $R(\alpha,\beta,\gamma)=R(\alpha\sigma_n,\beta_+,\gamma_+)=R(\alpha_+,\beta_+,\gamma_+)$, so we have

$$\begin{aligned}
\Phi_{\alpha}^{\beta,\gamma}(\partial y,\partial z) &\overset{(33)}{=} \sum_{i\in R(\alpha,\beta,\gamma)}(-1)^{m-i}\;\Phi_{\alpha\delta_m}^{\beta\delta_i,\gamma\delta_i}(\partial y,\partial z)\\
\text{(ind. hypoth.)} &= \sum_{i\in R(\alpha,\beta,\gamma)}(-1)^{m-i}\Big(\Phi_{\alpha\delta_m\sigma_{n-1}}^{(\beta\delta_i)_+,(\gamma\delta_i)_+}(y,z)+\partial\,\Phi_{(\alpha\delta_m)_+}^{(\beta\delta_i)_+,(\gamma\delta_i)_+}(y,z)\Big)\\
&= \sum_{i\in R(\alpha,\beta,\gamma)}(-1)^{m-i}\Phi_{\alpha\sigma_n\delta_m}^{\beta_+\delta_i,\gamma_+\delta_i}(y,z)+\partial\sum_{i\in R(\alpha,\beta,\gamma)}(-1)^{m-i}\Phi_{\alpha_+\delta_m}^{\beta_+\delta_i,\gamma_+\delta_i}(y,z)\\
&\overset{(33)}{=} \Phi_{\alpha\sigma_n}^{\beta_+,\gamma_+}(y,z)+\partial\,\Phi_{\alpha_+}^{\beta_+,\gamma_+}(y,z).
\end{aligned}$$

Case $R(\beta)\cap R(\gamma)\neq\emptyset$ but $R(\alpha)\cap R(\beta)\cap R(\gamma)=\emptyset$. In this case, by (31) and (32), we have $\Phi_{\alpha}^{\beta,\gamma}(\partial y,\partial z)=0$, $\Phi_{\alpha\sigma_n}^{\beta_+,\gamma_+}(y,z)=0$, and $\Phi_{\alpha_+}^{\beta_+,\gamma_+}(y,z)=0$, so the equality in (46) trivially holds.

Case $R(\beta)\cap R(\gamma)\neq\emptyset$ and $R(\alpha)\cap R(\beta)\cap R(\gamma)\neq\emptyset$. Let $k=\min R(\alpha)\cap R(\beta)\cap R(\gamma)$. Then,

$$\begin{aligned}\Phi_\alpha^{\beta,\gamma}(\partial y,\partial z) &\overset{(32)}{=} \Phi_{\alpha\delta_k}^{\beta\delta_k,\gamma\delta_k}(\partial y,\partial z)\\ \text{(ind. hypoth.)} &= \Phi_{\alpha\delta_k\sigma_{n-1}}^{(\beta\delta_k)_+,(\gamma\delta_k)_+}(y,z)+\partial\,\Phi_{(\alpha\delta_k)_+}^{(\beta\delta_k)_+,(\gamma\delta_k)_+}(y,z)\\ &= \Phi_{\alpha\sigma_n\delta_k}^{\beta_+\delta_k,\gamma_+\delta_k}(y,z)+\partial\,\Phi_{\alpha_+\delta_k}^{\beta_+\delta_k,\gamma_+\delta_k}(y,z)\overset{(32)}{=}\Phi_{\alpha\sigma_n}^{\beta_+,\gamma_+}(y,z)+\partial\,\Phi_{\alpha_+}^{\beta_+,\gamma_+}(y,z).\end{aligned}$$

7.2. Proof of Lemma 5.3. Suppose first that $m=0$. For any $x\in L_\beta$ and $y\in L_\gamma$, we have

$$d_0[\beta^*x,\,\gamma^*y]=d_0\sum_{\xi\in S(n)}\xi^*\Phi_\xi^{\beta,\gamma}(x,y)=\sum_{\substack{\xi\,\in\,S(n)\\ 0\,\in\,R(\xi)}}(\xi\delta_0)^*\Phi_\xi^{\beta,\gamma}(x,y).$$

But $d_0[\beta^*x,\,\gamma^*y]=[d_0\beta^*x,\,d_0\gamma^*y]=0$, since $0\notin R(\beta)$ or $0\notin R(\gamma)$. Hence $\Phi_\xi^{\beta,\alpha}(x,y)=0$ for any $\xi\in S(n)$ with $0\in R(\xi)$ and, particularly, $\Phi_\alpha^{\beta,\gamma}(x,y)=0$.

Let us suppose now that $m\geq 1$. Then, for any $x\in L_\beta$ and $y\in L_\gamma$, we have

$$\begin{aligned}[d_m\beta^*x,\,d_m\gamma^*y] &= d_m[\beta^*x,\,\gamma^*y]=d_m\sum_{\xi\in S(n)}\xi^*\Phi_\xi^{\beta,\gamma}(x,y)=\sum_{\xi\in S(n)}d_m\xi^*\Phi_\xi^{\beta,\gamma}(x,y)\\ &= \sum_{\substack{\xi\,\in\,S(n)\\ m\,\in\,R(\xi)\\ m\text{-}1\,\notin\,R(\xi)}}(\xi\delta_m)^*\Phi_\xi^{\beta,\gamma}(x,y)+\sum_{\substack{\xi\,\in\,S(n)\\ m\,\notin\,R(\xi)\\ m\text{-}1\,\in\,R(\xi)}}(\xi\delta_m)^*\Phi_\xi^{\beta,\gamma}(x,y)+\sum_{\substack{\xi\,\in\,S(n)\\ m\,\in\,R(\xi)\\ m\text{-}1\,\in\,R(\xi)}}(\xi\delta_m)^*\Phi_\xi^{\beta,\gamma}(x,y)\\ &\overset{(6)}{=} \sum_{\substack{\mu\,\in\,S(n\text{-}1)\\ m\text{-}1\,\notin\,R(\mu)}}\mu^*\Phi_{\mu\sigma_m}^{\beta,\gamma}(x,y)+\sum_{\substack{\mu\,\in\,S(n\text{-}1)\\ m\text{-}1\,\notin\,R(\mu)}}\mu^*\Phi_{\mu\sigma_{m\text{-}1}}^{\beta,\gamma}(x,y)+\sum_{\substack{\mu\,\in\,S(n\text{-}1)\\ m\text{-}1\,\in\,R(\mu)}}\mu^*\Phi_{\mu\sigma_{m\text{-}1}}^{\beta,\gamma}(x,y)\\ &= \sum_{\substack{\mu\,\in\,S(n\text{-}1)\\ m\text{-}1\,\notin\,R(\mu)}}\mu^*\big(\Phi_{\mu\sigma_m}^{\beta,\gamma}(x,y)+\Phi_{\mu\sigma_{m\text{-}1}}^{\beta,\gamma}(x,y)\big)+\sum_{\substack{\mu\,\in\,S(n\text{-}1)\\ m\text{-}1\,\in\,R(\mu)}}\mu^*\Phi_{\mu\sigma_{m\text{-}1}}^{\beta,\gamma}(x,y).\end{aligned}$$

Now, in the case when $m\in R(\alpha,\beta,\gamma)$, we have that $m\in R(\beta)$ and $m-1\in R(\gamma)$ or $m-1\in R(\beta)$ and $m\in R(\gamma)$, and therefore

$$[d_m\beta^*x,\,d_m\gamma^*y]=[(\beta\delta_m)^*x,\,(\gamma\delta_m)^*y]=\sum_{\mu\in S(n-1)}\mu^*\Phi_\mu^{\beta\delta_m,\gamma\delta_m}(x,y).$$

If we take $\mu=\alpha\delta_m$, we have that $m-1\notin R(\mu)$ and $\alpha=\mu\sigma_m$. Then, by comparison in the above expressions of $[d_m\beta^*x,\,d_m\gamma^*y]$, we get the claimed equality

$$\Phi_{\alpha\delta_m}^{\beta\delta_m,\gamma\delta_m}(x,y)=\Phi_\alpha^{\beta,\gamma}(x,y)+\Phi_{\alpha\delta_m\sigma_{m\text{-}1}}^{\beta,\gamma}(x,y).$$

Finally, in the case $m\notin R(\alpha,\beta,\gamma)$, we have $m\notin R(\beta)$ and $m-1\notin R(\beta)$ or $m\notin R(\gamma)$ and $m-1\notin R(\gamma)$. In both cases, $[d_m\beta^*x,\,d_m\gamma^*y]=0$. Then, by comparison as above for $\mu=\alpha\delta_m$, we conclude the required equality $0=\Phi_\alpha^{\beta,\gamma}(x,y)+\Phi_{\alpha\delta_m\sigma_{m\text{-}1}}^{\beta,\gamma}(x,y)$.

7.3. Proof of Lemma 5.6. It suffices to prove that the maps $\Phi_\alpha^{\beta,\gamma}$ satisfy the equations (29)-(33). It is clear that equations (29) and (30) hold owing to equalities in (14) and (16). To prove (31) and (32), suppose $\beta, \gamma \in S(n)$ and $j \in R(\beta) \cap R(\gamma)$. Then, by (6), $\beta = \beta\delta_j\sigma_j$, $\gamma = \gamma\delta_j\sigma_j$ and, for any $x \in L_\beta$ and $y \in L_\gamma$, we have

$$[\beta^* x, \gamma^* y] = [\sigma_j^*(\beta\delta_j)^* x,\, \sigma_j^*(\gamma\delta_j)^* y] = \sigma_j^*[(\beta\delta_j)^* x,\, (\gamma\delta_j)^* y] = \sigma_j^* \sum_{\mu \in S(n-1)} \mu^* \Phi_\mu^{\beta\delta_j,\gamma\delta_j}(x,y)$$

$$= \sum_{\mu \in S(n-1)} (\mu\sigma_j)^* \Phi_\mu^{\beta\delta_j,\gamma\delta_j}(x,y) \overset{(6)}{=} \sum_{\substack{\alpha \in S(n) \\ j \in R(\alpha)}} \alpha^* \Phi_{\alpha\delta_j}^{\beta\delta_j,\gamma\delta_j}(x,y).$$

Since $[\beta^* x, \gamma^* y] = \sum_{\alpha \in S(n)} \alpha^* \Phi_\xi^{\beta,\gamma}(x,y)$, by comparison, we conclude that

$$\Phi_\alpha^{\beta,\gamma}(x,y) = \begin{cases} \Phi_{\alpha\delta_j}^{\beta\delta_j,\gamma\delta_j}(x,y) & \text{if } \ j \in R(\alpha), \\ 0 & \text{if } \ j \notin R(\alpha), \end{cases}$$

and thus (31) and (32) follows.

We next prove (33). By (53), this is clear if $m = 0$. So we can proceed inductively on $m \geq 1$. Observe that $\min R(\alpha\delta_m\sigma_{m-1}) = m-1$ and that $\alpha\delta_m\sigma_{m-1}\delta_{m-1} = \alpha\delta_m$. Then, if we are in the case when $m \in R(\alpha,\beta,\gamma)$, since $R(\alpha,\beta,\gamma) = R(\alpha\delta_m\sigma_{m-1},\beta,\gamma) \cup \{m\}$, the equality (53) and the hypothesis of induction give

$$\begin{aligned} \Phi_\alpha^{\beta,\gamma}(x,y) &= \Phi_{\alpha\delta_m}^{\beta\delta_m,\gamma\delta_m}(x,y) - \Phi_{\alpha\delta_m\sigma_{m\text{-}1}}^{\beta,\gamma}(x,y) \\ &= \Phi_{\alpha\delta_m}^{\beta\delta_m,\gamma\delta_m}(x,y) - \sum_{i \in R(\alpha\delta_m\sigma_{m-1},\beta,\gamma)} (-1)^{m-i-1}\ \Phi_{\alpha\delta_m}^{\beta\delta_i,\gamma\delta_i}(x,y) \\ &= \Phi_{\alpha\delta_m}^{\beta\delta_m,\gamma\delta_m}(x,y) + \sum_{i \in R(\alpha\delta_m\sigma_{m-1},\beta,\gamma)} (-1)^{m-i}\ \Phi_{\alpha\delta_m}^{\beta\delta_i,\gamma\delta_i}(x,y) \\ &= \sum_{i \in R(\alpha,\beta,\gamma)} (-1)^{m-i}\ \Phi_{\alpha\delta_m}^{\beta\delta_i,\gamma\delta_i}(x,y). \end{aligned}$$

And, otherwise, if $m \notin R(\alpha,\beta,\gamma)$, since $R(\alpha,\beta,\gamma) = R(\alpha\delta_m\sigma_{m-1},\beta,\gamma)$, the equality (53) and the hypothesis of induction give

$$\begin{aligned} \Phi_\alpha^{\beta,\gamma}(x,y) = \ -\Phi_{\alpha\delta_m\sigma_{m\text{-}1}}^{\beta,\gamma}(x,y) &= \ -\sum_{i \in R(\alpha\delta_m\sigma_{m-1},\beta,\gamma)} (-1)^{m-i-1}\ \Phi_{\alpha\delta_m}^{\beta\delta_i,\gamma\delta_i}(x,y) \\ &= \sum_{i \in R(\alpha\delta_m\sigma_{m-1},\beta,\gamma)} (-1)^{m-i}\ \Phi_{\alpha\delta_m}^{\beta\delta_i,\gamma\delta_i}(x,y) = \sum_{i \in R(\alpha,\beta,\gamma)} (-1)^{m-i}\ \Phi_{\alpha\delta_m}^{\beta\delta_i,\gamma\delta_i}(x,y). \end{aligned}$$

7.4. Proof of Proposition 6.4. Let us analyze a hypercrossed complex of Lie algebras (L,Φ) satisfying the conditions in the proposition. Let $\Phi^{\text{ext}} = \left\{\Phi_\alpha^{\beta,\gamma} : L_\beta \times L_\gamma \to L_\alpha\right\}$ be its extended set of structure bilinear maps, as in Definition 4.2.

(A) for $n \geq 2$, the Lie algebra L_n is abelian. In effect, for any $y_n, y'_n \in L_n$, we have

$$\Phi_n^{\sigma_{n-1},\sigma_{n-2}}(\partial y_n, \partial y'_n) \overset{(44)}{=} \Phi_{\sigma_n}^{\sigma_{n-1},\sigma_{n-2}}(y_n, y'_n) + \partial\Phi_{n+1}^{\sigma_{n-1},\sigma_{n-2}}(y_n, y'_n).$$

Since, by hypothesis, $\Phi_n^{\sigma_{n-1},\sigma_{n-2}}(\partial y_n, \partial y'_n) = 0 = \Phi_{n+1}^{\sigma_{n-1},\sigma_{n-2}}(y_n, y'_n)$, we get

$$0 = \Phi_{\sigma_n}^{\sigma_{n-1},\sigma_{n-2}}(y_n, y'_n) \overset{(33)}{=} \Phi_n^{n,n}(y_n, y'_n) = [y_n, y'_n].$$

(B) for any $n \geq 2$ and $\gamma \in S(n)$ with $n < \gamma < \omega_n$, $\Phi_n^{n,\gamma} = 0 : L_n \times L_\gamma \to L_n$. In effect, let us first assume that $n-1 \in R(\gamma)$ and proceed by induction on

$$k = k(\gamma) = \min\{i \mid i, i+1, \ldots, n-1 \in R(\gamma)\}.$$

Note that $k > 0$, as $\gamma \neq \omega_n = \sigma_0 \cdots \sigma_{n-2}\sigma_{n-1}$. Suppose $k = 1$ or that $k > 1$ but $k-2 \notin R(\gamma)$. Then, for any $y_n \in L_n$ and $y_\gamma \in L_\gamma$,

$$\Phi_n^{\sigma_{k-1},\gamma}(\partial y_n, y_\gamma) \overset{(43)}{=} \Phi_{\sigma_n}^{\sigma_{k-1},\gamma\sigma_n}(y_n, y_\gamma) + \partial\Phi_{n+1}^{\sigma_{k-1},\gamma\sigma_n}(y_n, y_\gamma).$$

Since, by hypothesis, $\Phi_n^{\sigma_{k-1},\gamma}(\partial y_n, y_\gamma) = 0 = \partial\Phi_{n+1}^{\sigma_{k-1},\gamma\sigma_n}(y_n, y_\gamma)$, we have

$$0 = \Phi_{\sigma_n}^{\sigma_{k-1},\gamma\sigma_n}(y_n, y_\gamma) \overset{(33)}{=} (-1)^{n-k}\,\Phi_n^{n,\gamma\sigma_n\delta_k}(y_n, y_\gamma) = (-1)^{n-k}\,\Phi_n^{n,\gamma}(y_n, y_\gamma).$$

Now, suppose $k > 1$ and $k-2 \in R(\gamma)$. As above, we have

$$\begin{aligned} 0 = \Phi_{\sigma_n}^{\sigma_{k-1},\gamma\sigma_n}(y_n, y_\gamma) &\overset{(33)}{=} (-1)^{n-k}\,\Phi_n^{n,\gamma\sigma_n\delta_k}(y_n, y_\gamma) + (-1)^{n-k+1}\,\Phi_n^{n,\gamma\sigma_n\delta_{k-1}}(y_n, y_\gamma) \\ &= (-1)^{n-k}\,\Phi_n^{n,\gamma}(y_n, y_\gamma) + (-1)^{n-k+1}\,\Phi_n^{n,\gamma\delta_{k-1}\sigma_{n-1}}(y_n, y_\gamma). \end{aligned}$$

Since $R(\gamma\delta_{k-1}\sigma_{n-1}) = \{k-1\} \cup R(\gamma) \setminus \{k-2\}$, it follows that $k(\gamma\delta_{k-1}\sigma_{n-1}) < k = k(\gamma)$. Hence, by hypothesis of induction, $\Phi_n^{n,\gamma\delta_{k-1}\sigma_{n-1}}(y_n, y_\gamma) = 0$, and therefore $\Phi_n^{n,\gamma}(y_n, y_\gamma) = 0$.

Suppose now that $n-1 \notin R(\gamma)$. Then,

$$\Phi_n^{\sigma_{n-1},\gamma}(\partial y_n, y_\gamma) \overset{(43)}{=} \Phi_{\sigma_n}^{\sigma_{n-1},\gamma\sigma_n}(y_n, y_\gamma) + \partial\Phi_{n+1}^{\sigma_{n-1},\gamma\sigma_n}(y_n, y_\gamma).$$

Since, by hypothesis, $\Phi_n^{\sigma_{n-1},\gamma}(\partial y_n, y_\gamma) = 0 = \partial\Phi_{n+1}^{\sigma_{n-1},\gamma\sigma_n}(y_n, y_\gamma)$, in the case that $n-2 \notin R(\gamma)$ we have

$$0 = \Phi_{\sigma_n}^{\sigma_{n-1},\gamma\sigma_n}(y_n, y_\gamma) \overset{(33)}{=} \Phi_n^{n,\gamma}(y_n, y_\gamma),$$

whereas, in the case when $n-2 \in R(\gamma)$,

$$0 = \Phi_{\sigma_n}^{\sigma_{n-1},\gamma\sigma_n}(y_n, y_\gamma) \overset{(33)}{=} \Phi_n^{n,\gamma}(y_n, y_\gamma) - \Phi_n^{n,\gamma\delta_{n-1}\sigma_{n-1}}(y_n, y_\gamma),$$

and we also conclude that $\Phi_n^{n,\gamma}(y_n, y_\gamma) = 0$, since $n-1 \in R(\gamma\delta_{n-1}\sigma_{n-1})$ and therefore $\Phi_n^{n,\gamma\delta_{n-1}\sigma_{n-1}}(y_n, y_\gamma) = 0$ by the already discussed case above.

(C) $\underline{\text{for any } \alpha,\beta,\gamma\in S(n) \text{ with } r_\beta\geq 2 \text{ and } \gamma<\omega_n,\ \Phi_\alpha^{\beta,\gamma}=0: L_\beta\times L_\gamma\to L_\alpha}$. To prove this, by (36), we can assume that $\beta\leq\gamma$. Note that the hypothesis $r_\beta\geq 2$ imply $n\geq 2$. For any such n, $\Phi_n^{n,n}=0$ due to (29) and (A); $\Phi_n^{n,\gamma}=0$, for $\gamma>n$, owing to (B); and also $\Phi_n^{\beta,\gamma}=0$, for $\beta>n$, by (31). Thus, the claim holds for $\alpha=n$. Now, for arbitrary $\alpha>n$ we proceed by induction on n. If $n=2$, there is only one element $\beta\in S(2)=\{2,\sigma_1,\sigma_0,\sigma_0\sigma_1\}$ with range ≥ 2, namely $\beta=2$. Then, the assert holds since, by (33), $\Phi_\alpha^{n,\gamma}=0$ for any n and γ. Now, the claim for any $n\geq 2$ easily follows by induction on n and taking into account the formulas (32) and (33).

It follows from (B) that the Lie actions $\Phi_n^{n,\omega_n}: L_n\times L_0\to L_n$, $n\geq 1$, are the unique non constantly zero structure bilinear maps in Φ. Let us denote them as in (60).

We observe now that, for this hypercrossed complex (L,Φ), the axioms (37)-(42) are trivially verified, so that they do not impose any requirement on the family of actions (60). This is evident as regards the conditions (37) and (38). To check the remaining, note that, by (31), (30) and our hypotheses, we have $\Phi_n^{\beta,\gamma}=0$ for any $\beta,\gamma\in S(n)\setminus\{n\}$. Then, looking at the equalities(39) and (40), we see that the terms on the left are both 0, since $\Phi_n^{\beta,\gamma}=0$, while the summations on the right reduce to the two summands for $\xi=n$, which are both 0, again because $\Phi_n^{\beta,\gamma}=0$. Looking now at equality (41), the first summand on the left is 0, since L_n is abelian (the existence of $\beta,\gamma\in S(n)$ with $n<\beta<\gamma$ imply that $n\geq 2$) and all the other summands on the left are also 0, since the involved $\Phi_n^{n,\xi}$ are constantly zero by (B), and, similarly, all the summands at the right are 0, since $\Phi_n^{n,\beta}=0$. Regarding finally with the equality in (42), all the summands in the first summation on the left are 0, since all the involved $\Phi_n^{\xi,\gamma}$ are constantly 0, and all in the second one are also 0, since the involved $\Phi_n^{\beta,\xi}$ are 0. Similarly, all the summands in the second summation on the right are 0, since the involved $\Phi_n^{\xi,\beta}$ are constantly zero, while the first summation on the right reduces to the summand for $\xi=n$ (since for $\xi>n$, $\Phi_n^{\xi,\delta}=0$), which is also 0 since $\Phi_n^{\beta,\gamma}=0$.

Dealing now the condition (43), we first observe that in the case when $n\leq\beta<\gamma<\omega_n$ this is trivially verified, and therefore it neither imposes any requirements on the actions (60). In effect, by (B) if $\beta=n$ or by hypothesis if $\beta>n$, the term on the left of the equality is 0, while, by (C), both summands on the right are 0 (note that $\beta\neq\omega_n$, so $r_\beta\geq 1$ and $r_{\beta+1}\geq 2$). The same thing happens in the case when $n\leq\gamma<\beta<\omega_n$. Therefore the verification of condition (43) reduces to verify the equations below, where we have taken into account (33).

$$\Phi_n^{n,\omega_n}(\partial y_{n+1},y_0)=\partial\Phi_{n+1}^{n+1,\omega_{n+1}}(y_{n+1},y_0), \tag{84}$$

$$\Phi_n^{\omega_n,n}(\partial y_1,y_n')=\Phi_{\sigma_n}^{\omega_{n+},\sigma_n}(y_1,y_n'), \tag{85}$$

$$\Phi_n^{n,n}(\partial y_{n+1},y_n)=\partial\Phi_{n+1}^{n+1,\sigma_n}(y_{n+1},y_n). \tag{86}$$

For $n=0$, by (29) and (60), the first equation (84) is equivalent to $[\partial y_1,y_0]=\partial(y_1\cdot y_0)$, while for $n\geq 1$, by (60), it is equivalent to $\partial y_{n+1}\cdot y_0=\partial(y_{n+1}\cdot y_0)$. Similarly, for $n=0$, the second equation (85) is equivalent to $[\partial y_1,y_0']=\partial(y_1\cdot y_0')$, while for $n=1$ it is equivalent to $y'_1 \cdot \partial y_1 = [y'_1, y_1]$, since

$$\Phi_1^{\sigma_0,1}(\partial y_1, y'_1) \overset{(30)}{=} -\Phi_1^{1,\sigma_0}(y'_1, \partial y_1) \overset{(60)}{=} -(y'_1 \cdot \partial y_1),$$
$$\Phi_{\sigma_1}^{\sigma_0,\sigma_1}(y_1, y'_1) \overset{(33)}{=} \Phi_1^{1,1}(y_1, y'_1) \overset{(29)}{=} [y_1, y'_1],$$

and, for $n \geq 2$ to $y'_n \cdot \partial y_1 = 0$, since

$$\Phi_n^{\omega_n,n}(\partial y_1, y'_n) \overset{(30)}{=} -\Phi_n^{n,\omega_n}(y'_n, \partial y_1) \overset{(60)}{=} -(y'_n \cdot \partial y_1),$$
$$\Phi_{\sigma_n}^{\omega_{n+},\sigma_n}(y_1, y'_n) \overset{(30)}{=} -\Phi_{\sigma_n}^{\sigma_n,\omega_{n+}}(y'_n, y_1) \overset{(33)}{=} -\Phi_n^{n,\omega_{n-1+}}(y'_n, y_1) \overset{(B)}{=} 0.$$

As regards equation (86), for $n = 0$ this becomes equivalent to $[\partial y_1, y_0] = \partial y_1 \cdot y_0$. Since, by (B), $\Phi_{n+1}^{n+1,\sigma_n}(y_{n+1}, y_0) = 0$, for $n \geq 1$ this equation is equivalent to $[\partial y_{n+1}, y_n] = 0$, which always holds if $n \geq 2$, because of the abelianity of L_n, and also for $n = 1$ as consequence of (84) and (85), since

$$[\partial y_2, y_1] = \partial y_2 \cdot \partial y_1 = \partial(y_2 \cdot \partial y_1) = \partial(0) = 0.$$

Finally, we observe that equation (44) does not impose any additional requirement: The term on the left and the second summand on the right of the equality are both 0 by hypothesis, and the first summand on the right is also 0 by (C).

All in all, an hypercrossed complex of Lie algebra (L, Φ) such that $\Phi_n^{\beta,\gamma} = 0$ for any $\beta, \gamma \in S(n)$ with $R(\beta) \cap R(\gamma) = \emptyset$ and $n < \beta < \gamma$ is the same as a complex of Lie algebras L where every L_n is abelian for $n \geq 2$, by (A), together a Lie action (60) of L_0 on L_n, for each $n \geq 1$, such that the equations in (59) hold; that is, a crossed complex.

7.5. Proof of Proposition 6.15. Let (L, Φ) be any given 2-hypercrossed complex of Lie algebras, $\Phi^{\text{ext}} = \{\Phi_\alpha^{\beta,\gamma}\}$ be its extended set of structure maps, and let us adopt the notation in (69) for the bilinear maps Φ_1^{1,σ_0}, $\Phi_2^{2,\sigma_0\sigma_1}$ and $\Phi_2^{\sigma_1,\sigma_0}$, respectively.

Let us now analyze the meaning of the axioms (37)-(44), to be a 2-hypercrossed complex, for an enriched complex (L, Φ) with $L_k = 0$ if $k \geq 3$, as above, and whose non-trivial structure bilinear maps are given by (69) and (70).

Starting with (44), the only requirement here is the equation

- $\underline{\Phi_2^{\sigma_1,\sigma_0}(\partial y_2, \partial y'_2) = \Phi_{\sigma_2}^{\sigma_1,\sigma_0}(y_2, y'_2)}$. As, by (69), $\Phi_2^{\sigma_1,\sigma_0}(\partial y_2, \partial y'_2) = \{\partial y_2, \partial y'_2\}$, while

$$\Phi_{\sigma_2}^{\sigma_1,\sigma_0}(y_2, y'_2) \overset{(33)}{=} -\Phi_2^{2,2}(y_2, y'_2) \overset{(29)}{=} -[y_2, y'_2] = [y'_2, y_2],$$

we see that the verification of condition (44) is equivalent to condition (64) in the definition of 2-crossed module: $\{\partial y_2, \partial y'_2\} = [y'_2, y_2]$.

With regard to condition (43), this is expanded in the following eight equations:

- $\underline{\Phi_0^{0,0}(\partial y_1, y_0) = \Phi_{\sigma_0}^{1,\sigma_0}(y_1, y_0) + \partial\Phi_1^{1,\sigma_0}(y_1, y_0)}$. As, by (29), $\Phi_0^{0,0}(\partial y_1, y_0) = [\partial y_1, y_0]$; by (33), $\Phi_{\sigma_0}^{1,\sigma_0}(y_1, y_0) = 0$; and, by (69), $\Phi_1^{1,\sigma_0}(y_1, y_0) = y_1 \cdot y_0$, this equation is equivalent to condition (61), that is, $\partial(y_1 \cdot y_0) = [\partial y_1, y_0]$.

- $\underline{\Phi_1^{1,\sigma_0}(\partial y_2, y_0) = \Phi_{\sigma_1}^{2,\sigma_0\sigma_1}(y_2, y_0) + \partial\Phi_2^{2,\sigma_0\sigma_1}(y_2, y_0)}$. By (33), we have $\Phi_{\sigma_1}^{2,\sigma_0\sigma_1}(y_2, y_0) = 0$. Hence, by (69), this equation can be written as (62), that is, $\partial(y_2 \cdot y_0) = \partial y_2 \cdot y_0$.
- $\underline{\Phi_1^{\sigma_0,1}(\partial y_1', y_1) = \Phi_{\sigma_1}^{\sigma_0,\sigma_1}(y_1', y_1) + \partial\Phi_2^{\sigma_0,\sigma_1}(y_1', y_1)}$. By (30) and (69), we have $\Phi_1^{\sigma_0,1}(\partial y_1', y_1) = -y_1 \cdot \partial y_1'$. By (33) and (29), $\Phi_{\sigma_1}^{\sigma_0,\sigma_1}(y_1', y_1) = [y_1', y_1]$. By (30) and (69), $\Phi_2^{\sigma_0,\sigma_1}(y_1', y_1) = -\{y_1, y_1'\}$. Therefore, the equation can be written as (63), that is, $\partial\{y_1, y_1'\} = y_1 \cdot \partial y_1' - [y_1, y_1']$.
- $\underline{\Phi_2^{\sigma_1,\sigma_0}(\partial y_2, y_1) = \Phi_{\sigma_2}^{\sigma_1,\sigma_0\sigma_2}(y_2, y_1)}$. As, by (69), $\Phi_2^{\sigma_1,\sigma_0}(\partial y_2, y_1) = \{\partial y_2, y_1\}$, while

$$\Phi_{\sigma_2}^{\sigma_1,\sigma_0\sigma_2}(y_2, y_1) \overset{(33)}{=} \Phi_2^{2,\sigma_0}(y_2, y_1) - \Phi_2^{2,\sigma_1}(y_2, y_1) \overset{(70)}{=} y_2 \cdot \partial y_1 - \{y_1, \partial y_2\},$$

the equation is equivalent to (65), that is, $\{\partial y_2, y_1\} + \{y_1, \partial y_2\} = y_2 \cdot \partial y_1$.
- $\underline{\Phi_2^{\sigma_0,\sigma_1}(\partial y_2, y_1) = \Phi_{\sigma_2}^{\sigma_0,\sigma_1\sigma_2}(y_2, y_1)}$. This equation does not impose any new requirement since, by (30), (69) and (70), $\Phi_2^{\sigma_0,\sigma_1}(\partial y_2, y_1) = -\{y_1, \partial y_2\} = \Phi_{\sigma_2}^{\sigma_0,\sigma_1\sigma_2}(y_2, y_1)$.
- $\underline{\Phi_2^{\sigma_1,2}(\partial y_2, y_2') = \Phi_{\sigma_2}^{\sigma_1,\sigma_2}(y_2, y_2')}$. This becomes equivalent to (64) since, owing to (30) and (70), $\Phi_2^{\sigma_1,2}(\partial y_2, y_2') = -\{\partial y_2, \partial y_2'\}$ while, due to (33) and (29), $\Phi_{\sigma_2}^{\sigma_1,\sigma_2}(y_2, y_2') = [y_2, y_2']$.
- $\underline{\Phi_2^{\sigma_0,2}(\partial y_2, y_2') = \Phi_{\sigma_2}^{\sigma_0,\sigma_2}(y_2, y_2')}$. This equation does not impose any new requirement since, by (33), $\Phi_{\sigma_2}^{\sigma_0,\sigma_2}(y_2, y_2') = 0$, and, by (30) and (70), $\Phi_2^{\sigma_0,2}(\partial y_2, y_2') = -y_2' \cdot \partial^2 y_2 = 0$.
- $\underline{\Phi_2^{\sigma_0\sigma_1,2}(\partial y_1, y_2) = \Phi_{\sigma_2}^{\sigma_0\sigma_1,\sigma_2}(y_1, y_2)}$. As above, this equation always holds, since, by (30), (69), (33), and (70), $\Phi_2^{\sigma_0\sigma_1,2}(\partial y_1, y_2) = -y_2 \cdot \partial y_1 = \Phi_{\sigma_2}^{\sigma_0\sigma_1,\sigma_2}(y_1, y_2)$.

Working now the condition (42), its only requirement here is the equation

- $$\underline{\Phi_2^{\sigma_1,\sigma_0}\big(\Phi_{\sigma_1}^{\sigma_1,\sigma_0\sigma_1}(y_1, y_0), y_1'\big) + \Phi_2^{\sigma_1,\sigma_0}\big(y_1, \Phi_{\sigma_0}^{\sigma_0,\sigma_0\sigma_1}(y_1', y_0)\big)}$$
$$\underline{= \Phi_2^{2,\sigma_0\sigma_1}\big(\Phi_2^{\sigma_1,\sigma_0}(y_1, y_1'), y_0\big) + \Phi_2^{\sigma_1,\sigma_0\sigma_1}\big(\Phi_{\sigma_1}^{\sigma_1,\sigma_0}(y_1, y_1'), y_0\big)}.$$

As, by (69), (32) and (31), we have

$$\Phi_2^{\sigma_1,\sigma_0}\big(\Phi_{\sigma_1}^{\sigma_1,\sigma_0\sigma_1}(y_1, y_0), y_1'\big) = \{\Phi_1^{1,\sigma_0}(y_1, y_0), y_1'\} = \{y_1 \cdot y_0, y_1'\},$$

$$\Phi_2^{\sigma_1,\sigma_0}\big(y_1, \Phi_{\sigma_0}^{\sigma_0,\sigma_0\sigma_1}(y_1', y_0)\big) = \{y_1, \Phi_1^{1,\sigma_0}(y_1', y_0)\} = \{y_1, y_1' \cdot y_0\},$$

$$\Phi_2^{2,\sigma_0\sigma_1}\big(\Phi_2^{\sigma_1,\sigma_0}(y_1, y_1'), y_0\big) = \{y_1, y_1'\} \cdot y_0, \quad \Phi_2^{\sigma_1,\sigma_0\sigma_1}\big(\Phi_{\sigma_1}^{\sigma_1,\sigma_0}(y_1, y_1'), y_0\big) = 0,$$

we see that (42) is equivalent to (66), that is: $\{y_1 \cdot y_0, y_1'\} + \{y_1, y_1' \cdot y_0\} = \{y_1, y_1'\} \cdot y_0$.

We now pay attention to conditions (39) and (40). These, in our case, consist of the following two equations

- $$\underline{\Phi_2^{\sigma_1,\sigma_0}(y_1, [y_1', y_1'']) = \Phi_2^{2,\sigma_0}(\Phi_2^{\sigma_1,\sigma_0}(y_1, y_1'), y'') - \Phi_2^{2,\sigma_0}(\Phi_2^{\sigma_1,\sigma_0}(y_1, y_1''), y')}$$
$$\underline{+\Phi_2^{\sigma_1,\sigma_0}(\Phi_{\sigma_1}^{\sigma_1,\sigma_0}(y_1, y_1'), y_1'') - \Phi_2^{\sigma_1,\sigma_0}(\Phi_{\sigma_1}^{\sigma_1,\sigma_0}(y_1, y_1''), y_1')},$$
- $$\underline{\Phi_2^{\sigma_1,\sigma_0}([y_1, y_1'], y_1'') = \Phi_2^{2,\sigma_1}(\Phi_2^{\sigma_1,\sigma_0}(y_1, y_1''), y_1') - \Phi_2^{2,\sigma_1}(\Phi_2^{\sigma_1,\sigma_0}(y_1', y_1''), y_1)},$$

which, by (69) and (70), agree with conditions (67) and (68) for a 2-crossed module.

Next, we go over conditions (37) and (38). These say us that the bilinear maps Φ_1^{1,σ_0} and $\Phi_2^{2,\sigma_0\sigma_1}$ in (69), and Φ_2^{2,σ_0} and Φ_2^{2,σ_1} in (70), are right Lie actions. That the first two are Lie actions of L_0 on L_1 and L_2, respectively, is part of the axiomatic for a 2-crossed complex of Lie algebras. The other two are actually Lie algebra actions as consequence of the axioms, so they do not impose new requirements. In effect, this is quite obvious with regards the map Φ_2^{2,σ_0}, and for Φ_2^{2,σ_1} we have

$$\begin{aligned}
\{y_1,\partial[y_2,y_2']\} &\overset{(65)}{=} [y_2,y_2']\cdot\partial y_1 - \{[\partial y_2,\partial y_2'],y_1\}\\
&\overset{(68)}{=} [y_2,y_2']\cdot\partial y_1 - \{\partial y_2',\partial\{\partial y_2,y_1\}\} + \{\partial y_2,\partial\{\partial y_2',y_1\}\}\\
&\overset{(64)}{=} [y_2,y_2']\cdot\partial y_1 - [\{\partial y_2,y_1\},y_2'] + [\{\partial y_2',y_1\},y_2]\\
&\overset{(65)}{=} [y_2,y_2']\cdot\partial y_1 - [y_2\partial y_1 - \{y_1,\partial y_2\},y_2'] + [y_2'\partial y_1 - \{y_1,\partial y_2'\},y_2]\\
&= [y_2,y_2']\cdot\partial y_1 - [y_2\cdot\partial y_1,y_2'] + [y_2'\cdot\partial y_1,y_2] + [\{y_1,\partial y_2\},y_2'] - [\{y_1,\partial y_2'\},y_2]\\
&= [y_2,\{y_1,\partial y_2'\}] - [y_2',\{y_1,\partial y_2\}].
\end{aligned}$$

At this point, it only remains to check condition (41), but actually this holds without any new requirement. In effect, it expands on the following three equations

- $\underline{[y_2,\Phi_2^{\sigma_1,\sigma_0}(y_1,y_1')] + \Phi_2^{2,\sigma_1}(y_2,\Phi_{\sigma_1}^{\sigma_1,\sigma_0}(y_1,y_1'))}$
  $\underline{= \Phi_2^{2,\sigma_0}(\Phi_2^{2,\sigma_1}(y_2,y_1),y_1') - \Phi_2^{2,\sigma_1}(\Phi_2^{2,\sigma_0}(y_2,y_1'),y_1)},$
- $\underline{[y_2,\Phi_2^{\sigma_1,\sigma_0\sigma_1}(y_1,y_0)] + \Phi_2^{2,\sigma_1}(y_2,\Phi_{\sigma_1}^{\sigma_1,\sigma_0\sigma_1}(y_1,y_0))}$
  $\underline{= \Phi_2^{2,\sigma_0\sigma_1}(\Phi_2^{2,\sigma_1}(y_2,y_1),y_0) - \Phi_2^{2,\sigma_1}(\Phi_2^{2,\sigma_0\sigma_1}(y_2,y_0),y_1)},$
- $\underline{[y_2,\Phi_2^{\sigma_0,\sigma_0\sigma_1}(y_1,y_0)] + \Phi_2^{2,\sigma_1}(y_2,\Phi_{\sigma_1}^{\sigma_0,\sigma_0\sigma_1}(y_1,y_0)) + \Phi_2^{2,\sigma_0}(y_2,\Phi_{\sigma_0}^{\sigma_0,\sigma_0\sigma_1}(y_1,y_0))}$
  $\underline{= \Phi_2^{2,\sigma_0\sigma_1}(\Phi_2^{2,\sigma_0}(y_2,y_1),y_0) - \Phi_2^{2,\sigma_0}(\Phi_2^{2,\sigma_0\sigma_1}(y_2,y_0),y_1)},$

and we have

$$\begin{aligned}
[y_2,\Phi_2^{\sigma_1,\sigma_0}(y_1,y_1')] + \Phi_2^{2,\sigma_1}(y_2,\Phi_{\sigma_1}^{\sigma_1,\sigma_0}(y_1,y_1')) &= [y_2,\{y_1,y_1'\}] + \{[y_1,y_1'],\partial y_2\}\\
&\overset{(64)}{=} \{\partial\{y_1,y_1'\},\partial y_2\} + \{[y_1,y_1'],\partial y_2\} \overset{(63)}{=} \{y_1\cdot\partial y_1',\partial y_2\}\\
&\overset{(66)}{=} \{y_1,\partial y_2\}\cdot\partial y_1' - \{y_1,\partial y_2\cdot\partial y_1'\} \overset{(62)}{=} \{y_1,\partial y_2\}\cdot\partial y_1' - \{y_1,\partial(y_2\cdot\partial y_1')\}\\
&= \Phi_2^{2,\sigma_0}(\Phi_2^{2,\sigma_1}(y_2,y_1),y_1') - \Phi_2^{2,\sigma_1}(\Phi_2^{2,\sigma_0}(y_2,y_1'),y_1),
\end{aligned}$$

$$\begin{aligned}
[y_2,\Phi_2^{\sigma_1,\sigma_0\sigma_1}(y_1,y_0)] + \Phi_2^{2,\sigma_1}(y_2,\Phi_{\sigma_1}^{\sigma_1,\sigma_0\sigma_1}(y_1,y_0)) &\overset{(31)}{=} \Phi_2^{2,\sigma_1}(y_2,\Phi_{\sigma_1}^{\sigma_1,\sigma_0\sigma_1}(y_1,y_0))\\
&\overset{(32)}{=} \Phi_2^{2,\sigma_1}(y_2,\Phi_1^{1,\sigma_0}(y_1,y_0)) = \{y_1\cdot y_0,\partial y_2\}\\
&\overset{(66)}{=} \{y_1,\partial y_2\}\cdot y_0 - \{y_1,\partial y_2\cdot y_0\} \overset{(62)}{=} \{y_1,\partial y_2\}\cdot y_0 - \{y_1,\partial(y_2\cdot y_0)\}\\
&= \Phi_2^{2,\sigma_0\sigma_1}(\Phi_2^{2,\sigma_1}(y_2,y_1),y_0) - \Phi_2^{2,\sigma_1}(\Phi_2^{2,\sigma_0\sigma_1}(y_2,y_0),y_1),
\end{aligned}$$

$$
\begin{aligned}
[y_2, \Phi_2^{\sigma_0,\sigma_0\sigma_1}(y_1,y_0)] + \Phi_2^{2,\sigma_1}(y_2, \Phi_{\sigma_1}^{\sigma_0,\sigma_0\sigma_1}(y_1,y_0)) + \Phi_2^{2,\sigma_0}(y_2, \Phi_{\sigma_0}^{\sigma_0,\sigma_0\sigma_1}(y_1,y_0)) \\
\overset{(32)}{=} \Phi_2^{2,\sigma_0}(y_2, \Phi_1^{1,\sigma_0}(y_1,y_0)) = y_2 \cdot \partial(y_1 \cdot y_0) \overset{(61)}{=} y_2 \cdot [\partial y_1, y_0] \\
= (y_2 \cdot \partial y_1) \cdot y_0 - (y_2 \cdot y_0) \cdot \partial y_1 \\
= \Phi_2^{2,\sigma_0\sigma_1}(\Phi_2^{2,\sigma_0}(y_2,y_1), y_0) - \Phi_2^{2,\sigma_0}(\Phi_2^{2,\sigma_0\sigma_1}(y_2,y_0), y_1).
\end{aligned}
$$

7.6. Proof of Proposition 6.21. Let (L, Φ) be any given enriched complex of Lie algebras with $L_k = 0$ for all $k \notin \{p, p+1\}$, and let $\Phi^{\text{ext}} = \left\{\Phi_\alpha^{\beta,\gamma} : L_\beta \times L_\gamma \to L_\alpha,\ \alpha,\beta,\gamma \in S(n)\right\}$ be its extended set of structure bilinear maps. We write $L = L_{p+1}$ and $M = L_p$, and adopt the notation in (83) for the bilinear map $\Phi_{p+1}^{\sigma_p,\sigma_{p-1}}$.

First, suppose that (L, Φ) is a hypercrossed complex. Then the map $\{-,-\}$ determines all the possibly non-trivial structure bilinear maps in Φ, that is, the maps $\Phi_{p+1}^{p+1,\sigma_i} : L \times M \to L$, for $0 \le i \le p$ and $\Phi_{p+1}^{\sigma_i,\sigma_j} : M \times M \to L$, for $0 \le j < i \le p$, by the equations

$$\Phi_{p+1}^{p+1,\sigma_i}(y,x) = 0, \quad \text{for } 0 \le i < p, \tag{87}$$

$$\Phi_{p+1}^{p+1,\sigma_p}(y,x) = \{x, \partial y\} \tag{88}$$

$$\Phi_{p+1}^{\sigma_{j+1},\sigma_j}(x,x') = (-1)^{p-j+1}\{x,x'\}, \tag{89}$$

$$\Phi_{p+1}^{\sigma_i,\sigma_j}(x,x') = 0, \quad \text{for } i > j+1. \tag{90}$$

In effect, for (87) and (88) we have

$$\Phi_{p+1}^{p+1,\sigma_i}(y,x) \overset{(30)}{=} -\Phi_{p+1}^{\sigma_i,p+1}(x,y) \overset{(33)}{=} -\Phi_{\sigma_{p+1}}^{\sigma_i\sigma_p,\sigma_{p+1}}(x,y) \overset{(43)}{=} -\Phi_{p+1}^{\sigma_i\sigma_p,p+1}(\partial x, y) \overset{(\partial x=0)}{=} 0,$$

$$\Phi_{p+1}^{p+1,\sigma_p}(y,x) \overset{(33)}{=} -\Phi_{\sigma_{p+1}}^{\sigma_{p-1},\sigma_p\sigma_{p+1}}(y,x) \overset{(43)}{=} -\Phi_{p+1}^{\sigma_{p-1}\sigma_p}(\partial y, x) \overset{(30)}{=} \Phi_{p+1}^{\sigma_p,\sigma_{p-1}}(x,\partial y) \overset{(83)}{=} \{x,\partial y\},$$

for (90), we distinguish the cases $i = p$ and $i < p$, and we have

$$\Phi_{p+1}^{\sigma_p,\sigma_j}(x,x') \overset{(33),(30)}{=} \Phi_{\sigma_{p+1}}^{\sigma_j\sigma_{p-1},\sigma_p\sigma_{p+1}}(x',x) \overset{(43)}{=} \Phi_{p+1}^{\sigma_j\sigma_{p-1},\sigma_p}(\partial x', x) \overset{(\partial x'=0)}{=} 0,$$

$$\Phi_{p+1}^{\sigma_i,\sigma_j}(x,x') \overset{(33)}{=} (-1)^{p-i}\Phi_{\sigma_{p+1}}^{\sigma_i\sigma_{i+1},\sigma_j\sigma_{i-1}}(x,x') \overset{(44)}{=} (-1)^{p-i}\Phi_{p+1}^{\sigma_i\sigma_{i+1},\sigma_j\sigma_{i-1}}(\partial x, \partial x') = 0,$$

while (89) for $j = p-1$ is (83) and, for $j < p-1$, we have

$$
\begin{aligned}
\Phi_{p+1}{}^{\sigma_{j+1},\sigma_j}(x,x') + (-1)^{p-j}\{x,x'\} \overset{(83)}{=} \Phi_{p+1}^{\sigma_{j+1},\sigma_j}(x,x') + (-1)^{p-j}\Phi_{p+1}^{\sigma_p,\sigma_{p-1}}(x,x') \\
\overset{(33)}{=} \Phi_{\sigma_{p+1}}^{\sigma_{j+1}\sigma_{p+1},\sigma_j\sigma_p}(x,x') \overset{(30)}{=} -\Phi_{\sigma_{p+1}}^{\sigma_j\sigma_p,\sigma_{j+1}\sigma_{p+1}}(x',x) \overset{(43)}{=} \Phi_{p+1}^{\sigma_j\sigma_p,\sigma_{j+1}}(\partial x', x) \overset{(\partial x'=0)}{=} 0.
\end{aligned}
$$

We now analyze the axioms to be a hypercrossed complex for the given enriched complex (L, Φ) as above, whose non-trivial structure bilinear maps in Φ are given by (88) and (89).

We start with (44) and (43). The only non trivially verified requirements here concern the following seven equations:

• $\underline{0 = \Phi^{\sigma_{j+1},\sigma_j}_{\sigma_p}(x,x') + \partial\,\Phi^{\sigma_{j+1},\sigma_j}_{p+1}(x,x')}$. As $\Phi^{\sigma_{j+1},\sigma_j}_{p+1}(x,x') = (-1)^{p-j+1}\{x,x'\}$, while

$$\Phi^{\sigma_{j+1},\sigma_j}_{\sigma_p}(x,x') \overset{(33)}{=} (-1)^{p-j+1}\Phi^{p,p}_{p}(x,x') \overset{(29)}{=} (-1)^{p-j+1}[x,x'],$$

we see that this equation is equivalent to (79), that is: $\partial\{x,x'\} = [x',x]$.

• $\underline{\Phi^{\sigma_{j+1},\sigma_j}_{p+1}(\partial y,\partial y') = \Phi^{\sigma_{j+1},\sigma_j}_{\sigma_{p+1}}(y,y')}$. Since $\Phi^{\sigma_{j+1},\sigma_j}_{p+1}(\partial y,\partial y') = (-1)^{p-j-1}\{\partial y,\partial y'\}$, and

$$\Phi^{\sigma_{j+1},\sigma_j}_{\sigma_{p+1}}(y,y') \overset{(33)}{=} (-1)^{p-j}\Phi^{p+1,p+1}_{p+1}(y,y') \overset{(29)}{=} (-1)^{p-j}[y,y'],$$

it follows that this equation is equivalent to (80): $\{\partial y,\partial y'\} = [y',y]$.

• $\underline{0 = \Phi^{\sigma_{k-1}\sigma_{j+1},\sigma_k\sigma_j}_{\sigma_{p+1}}(x,x')}$. Since

$$\begin{aligned}
\Phi^{\sigma_{k-1}\sigma_{j+1},\sigma_k\sigma_j}_{\sigma_{p+1}}(x,x') &\overset{(33)}{=} (-1)^{p-j}\Phi^{\sigma_{k-1},\sigma_k}_{p+1}(x,x') + (-1)^{p-k-1}\Phi^{\sigma_j,\sigma_{j-1}}_{p+1}(x,x')\\
&\overset{(30)}{=} (-1)^{p-j+1}\Phi^{\sigma_k,\sigma_{k-1}}_{p+1}(x',x) + (-1)^{p-k-1}\Phi^{\sigma_j,\sigma_{j-1}}_{p+1}(x,x')\\
&\overset{(89)}{=} (-1)^{p-j+1}(-1)^{p-k}\{x',x\} + (-1)^{p-k-1}(-1)^{p-j}\{x,x'\}\\
&= (-1)^{j+k+1}(\{x',x\} + \{x,x'\}),
\end{aligned}$$

this equation becomes equivalent to (81): $\{x,x'\} + \{x',x\} = 0$.

• $\underline{0 = \Phi^{\sigma_{k+1}\sigma_{j+1},\sigma_k\sigma_j}_{\sigma_{p+1}}(x,x')}$. Actually, this equation is always verified:

$$\begin{aligned}
\Phi^{\sigma_{k+1}\sigma_{j+1},\sigma_k\sigma_j}_{\sigma_{p+1}}(x,x') &\overset{(33)}{=} (-1)^{p-j}\Phi^{\sigma_{k+1},\sigma_k}_{p+1}(x,x') + (-1)^{p-k}\Phi^{\sigma_j,\sigma_{j-1}}_{p+1}(x,x')\\
&\overset{(89)}{=} (-1)^{p-j}(-1)^{p-k+1}\{x,x'\} + (-1)^{p-k}(-1)^{p-j}\{x,x'\}\\
&= (-1)^{j+k+1}\{x,x'\} + (-1)^{j+k}\{x,x'\} = 0.
\end{aligned}$$

• $\underline{\Phi^{p,p}_{p}(\partial y,x) = \Phi^{p+1,\sigma_p}_{\sigma_p}(y,x) + \partial\,\Phi^{p+1,\sigma_p}_{p+1}(y,x)}$. This is actually consequence of (79), since

$$\Phi^{p+1,\sigma_p}_{\sigma_p}(y,x) + \partial\,\Phi^{p+1,\sigma_p}_{p+1}(y,x) \overset{(33)}{=} \partial\,\Phi^{p+1,\sigma_p}_{p+1}(y,x) \overset{(88)}{=} \partial\{x,\partial y\} \overset{(79)}{=} [\partial y,x] \overset{(29)}{=} \Phi^{p,p}_{p}(\partial y,x).$$

• $\underline{0 = \Phi^{\sigma_{p-1},\sigma_p}_{\sigma_p}(x,x') + \partial\,\Phi^{\sigma_{p-1},\sigma_p}_{p+1}(x,x')}$. This is equivalent to (79), since

$$\begin{aligned}
\Phi^{\sigma_{p-1},\sigma_p}_{\sigma_p}(x,x') + \partial\,\Phi^{\sigma_{p-1},\sigma_p}_{p+1}(x,x') &\overset{(33),(30)}{=} \Phi^{p,p}_{p}(x,x') - \partial\,\Phi^{\sigma_p,\sigma_{p-1}}_{p+1}(x',x)\\
&\overset{(29),(89)}{=} [x,x'] - \partial\{x',x\}.
\end{aligned}$$

• $\underline{\Phi^{\sigma_p,p+1}_{p+1}(\partial y,y') = \Phi^{\sigma_p,\sigma_{p+1}}_{\sigma_{p+1}}(y,y')}$. This becomes equivalent to (80), since

$$\begin{aligned}
\Phi^{\sigma_p,p+1}_{p+1}(\partial y,y') - \Phi^{\sigma_p,\sigma_{p+1}}_{\sigma_{p+1}}(y,y') &\overset{(30),(33)}{=} -\Phi^{p+1,\sigma_p}_{p+1}(y',\partial y) - \partial\,\Phi^{p+1,p+1}_{p+1}(y,y')\\
&\overset{(88),(29)}{=} -\{\partial y,\partial y'\} - [y,y'].
\end{aligned}$$

Now we focus on axioms (40) and (39). These, in our case, consist of the following three equations, the second of them for any $j < p$ and the third one for any $j < p-1$,

- $\underline{\Phi_{p+1}^{\sigma_p,\sigma_{p-1}}([x,x'],x'') = \Phi_{p+1}^{p+1,\sigma_p}(\Phi_{p+1}^{\sigma_p,\sigma_{p-1}}(x,x''),x') - \Phi_{p+1}^{p+1,\sigma_p}(\Phi_{p+1}^{\sigma_p,\sigma_{p-1}}(x',x''),x')},$
- $\underline{\Phi_{p+1}^{\sigma_{j+1},\sigma_j}([x,x'],x'') = \Phi_{p+1}^{\sigma_{j+2},\sigma_{j+1}}(\Phi_{\sigma_{j+2}}^{\sigma_{j+1},\sigma_j}(x,x''),x') - \Phi_{p+1}^{\sigma_{j+2},\sigma_{j+1}}(\Phi_{\sigma_{j+2}}^{\sigma_{j+1},\sigma_j}(x',x''),x)},$
- $\underline{\Phi_{p+1}^{\sigma_{j+1},\sigma_j}(x,[x',x'']) = \Phi_{p+1}^{\sigma_{j+1},\sigma_j}(\Phi_{\sigma_{j+1}}^{\sigma_{j+1},\sigma_j}(x,x'),x'') - \Phi_{p+1}^{\sigma_{j+1},\sigma_j}(\Phi_{\sigma_{j+1}}^{\sigma_{j+1},\sigma_j}(x,x''),x')}.$

But all these equations become now equivalent to condition (82) for stable delooping crossed module. In effect,

$$\begin{aligned}
&\Phi_{p+1}^{p+1,\sigma_p}(\Phi_{p+1}^{\sigma_p,\sigma_{p-1}}(x,x''),x') - \Phi_{p+1}^{p+1,\sigma_p}(\Phi_{p+1}^{\sigma_p,\sigma_{p-1}}(x',x''),x) - \Phi_{p+1}^{\sigma_p,\sigma_{p-1}}([x,x'],x'')\\
&\quad\overset{(89),(88)}{=} \{x',\partial\{x,x''\}\} - \{x,\partial\{x',x''\}\} - \{[x,x'],x''\}\\
&\quad\overset{(79)}{=} \{x',[x'',x]\} - \{x,[x'',x']\} - \{[x,x'],x''\}\\
&\quad\overset{(81)}{=} \{x',[x'',x]\} + \{x,[x',x'']\} + \{x'',[x,x']\},
\end{aligned}$$

$$\begin{aligned}
&\Phi_{p+1}^{\sigma_{j+2},\sigma_{j+1}}(\Phi_{\sigma_{j+2}}^{\sigma_{j+1},\sigma_j}(x,x''),x') - \Phi_{p+1}^{\sigma_{j+2},\sigma_{j+1}}(\Phi_{\sigma_{j+2}}^{\sigma_{j+1},\sigma_j}(x',x''),x) - \Phi_{p+1}^{\sigma_{j+1},\sigma_j}([x,x'],x'')\\
&\quad\overset{(89),(33)}{=} (-1)^{p-j}\{-\Phi_p^{p,p}(x,x''),x'\} - (-1)^{p-j}\{-\Phi_p^{p,p}(x',x''),x\}\\
&\qquad\qquad - (-1)^{p-j+1}\{[x,x'],x''\}\\
&\quad\overset{(29)}{=} (-1)^{p-j}\big(\{[x'',x],x'\} + \{[x',x''],x\} + \{[x,x'],x''\}\big)\\
&\quad\overset{(81)}{=} (-1)^{p-j+1}\Big(\{x',[x'',x]\} + \{x,[x',x'']\} + \{x'',[x,x']\}\Big),
\end{aligned}$$

$$\begin{aligned}
&\Phi_{p+1}^{\sigma_{j+1},\sigma_j}(\Phi_{\sigma_{j+1}}^{\sigma_{j+1},\sigma_j}(x,x'),x'') - \Phi_{p+1}^{\sigma_{j+1},\sigma_j}(\Phi_{\sigma_{j+1}}^{\sigma_{j+1},\sigma_j}(x,x''),x') - \Phi_{p+1}^{\sigma_{j+1},\sigma_j}(x,[x',x''])\\
&\quad\overset{(89),(33)}{=} (-1)^{p-j+1}\{\Phi_p^{p,p}(x,x'),x''\} - (-1)^{p-j+1}\{\Phi_p^{p,p}(x,x''),x'\}\\
&\qquad\qquad - (-1)^{p-j+1}\{x,[x',x'']\}\\
&\quad\overset{(29)}{=} (-1)^{p-j}\big(-\{[x,x'],x''\} + \{[x,x''],x'\} + \{x,[x',x'']\}\big)\\
&\quad\overset{(81)}{=} (-1)^{p-j}\big(\{x'',[x,x']\} + \{x',[x'',x]\} + \{x,[x',x'']\}\big).
\end{aligned}$$

We now pay attention to condition (42). This reduces here to the equation

- $\underline{\Phi_{p+1}^{\sigma_i,\sigma_{i-1}}(x,\Phi_{\sigma_{i-1}}^{\sigma_{k+1},\sigma_k}(x',x'')) = \Phi_{p+1}^{\sigma_{i+1},\sigma_i}(x,\Phi_{\sigma_{i+1}}^{\sigma_{k+1},\sigma_k}(x',x''),x)}$, where $k+1 < i \le p$, which is always satisfied thanks to (81):

$$\begin{aligned}
&\Phi_{p+1}^{\sigma_i,\sigma_{i-1}}(x,\Phi_{\sigma_{i-1}}^{\sigma_{k+1},\sigma_k}(x',x'')) - \Phi_{p+1}^{\sigma_{i+1},\sigma_i}(x,\Phi_{\sigma_{i+1}}^{\sigma_{k+1},\sigma_k}(x',x''),x)\\
&\quad\overset{(89),(33)}{=} (-1)^{p-i}\{x,(-1)^{i-k}\Phi_p^{p,p}(x',x'')\} - (-1)^{p-i+1}\{(-1)^{i-k}\Phi_p^{p,p}(x',x''),x\}\\
&\quad\overset{(29)}{=} (-1)^{p-i}\{x,(-1)^{i-k}[x',x'']\} - (-1)^{p-i+1}\{(-1)^{i-k}[x',x''],x\}\\
&\quad\overset{(29)}{=} (-1)^{p-k}\big(\{x,[x',x'']\} + \{[x',x''],x\}\big) \overset{(81)}{=} 0.
\end{aligned}$$

Going now over condition (41), which neither imposes any new requirement. Indeed, this reduces to the equation

- $\underline{[y,\Phi_{p+1}^{\sigma_{j+1},\sigma_j}(x,x')]+\Phi_{p+1}^{p+1,\sigma_p}(y,\Phi_{\sigma_p}^{\sigma_{j+1},\sigma_j}(x,x'))=0}$, and we have

$$\begin{aligned}
&[y,\Phi_{p+1}^{\sigma_{j+1},\sigma_j}(x,x')]+\Phi_{p+1}^{p+1,\sigma_p}(y,\Phi_{\sigma_p}^{\sigma_{j+1},\sigma_j}(x,x'))\\
&\overset{(89),(33)}{=}[y,(-1)^{p-j+1}\{x,x'\}]+\Phi_{p+1}^{p+1,\sigma_p}(y,(-1)^{p-j+1}\Phi_p^{p,p}(x,x'))\\
&\overset{(88),(29)}{=}(-1)^{p-j+1}\big([y,\{x,x'\}]+\{[x,x'],\partial y\}\big)\\
&\overset{(79)}{=}(-1)^{p-j+1}\big([y,\{x,x'\}]+\{\partial\{x',x\},\partial y\}\big)\\
&\overset{(81),(80)}{=}(-1)^{p-j+1}\big(-[y,\{x',x\}]+[y,\{x',x\}]\big)=0.
\end{aligned}$$

Finally, it only remains to check conditions (37) and (38) which, as the last above, do not impose any additional requirement. In effect, these reduce here to the equations

- $\underline{\Phi_{p+1}^{p+1,\sigma_p}([y,y'],x)=[y,\Phi_{p+1}^{p+1,\sigma_p}(y',x)]-[y',\Phi_{p+1}^{p+1,\sigma_p}(y,x)]}$,
- $\underline{\Phi_{p+1}^{p+1,\sigma_p}(y,[x,x'])=\Phi_{p+1}^{p+1,\sigma_p}[\Phi_{p+1}^{p+1,\sigma_p}(y,x),x']-\Phi_{p+1}^{p+1,\sigma_p}[\Phi_{p+1}^{p+1,\sigma_p}(y,x'),x]}$.

But we have

$$\begin{aligned}
&\Phi_{p+1}^{p+1,\sigma_p}(y,[x,x'])-\Phi_{p+1}^{p+1,\sigma_p}[\Phi_{p+1}^{p+1,\sigma_p}(y,x),x']+\Phi_{p+1}^{p+1,\sigma_p}[\Phi_{p+1}^{p+1,\sigma_p}(y,x'),x]\\
&\overset{(88)}{=}\{[x,x'],\partial y\}-\{x',\partial\{x,\partial y\}\}+\{x,\partial\{x',\partial y\}\}\\
&\overset{(79)}{=}\{[x,x'],\partial y\}-\{x',[\partial y,x]\}+\{x,[\partial y,x']\}\\
&\overset{(81)}{=}\{\partial y,[x',x]\}+\{x',[x,\partial y]\}+\{x,[\partial y,x']\}\overset{(81)}{=}0,
\end{aligned}$$

$$\begin{aligned}
&\Phi_{p+1}^{p+1,\sigma_p}(y,[x,x'])-\Phi_{p+1}^{p+1,\sigma_p}[\Phi_{p+1}^{p+1,\sigma_p}(y,x),x']+\Phi_{p+1}^{p+1,\sigma_p}[\Phi_{p+1}^{p+1,\sigma_p}(y,x'),x]\\
&\overset{(88)}{=}\{x,[\partial y,\partial y']\}-[y,\{x,\partial y'\}]+[y',\{x,\partial y\}]\\
&\overset{(80)}{=}\{x,[\partial y,\partial y']\}-\{\partial\{x,\partial y'\},\partial y\}+\{\partial\{x,\partial y\},\partial y'\}\\
&\overset{(79)}{=}\{x,[\partial y,\partial y']\}-\{[\partial y',x],\partial y\}+\{[\partial y,x],\partial y'\}\\
&\overset{(81)}{=}\{x,[\partial y,\partial y']\}+\{\partial y,[\partial y',x]\}+\{\partial y',[x,\partial y]\}\overset{(81)}{=}0.
\end{aligned}$$

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