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$*$-AUTONOMOUS CATEGORIES: ONCE MORE AROUND THE TRACK

To Jim Lambek on the occasion of his 75th birthday

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ABSTRACT. This represents a new and more comprehensive approach to the $*$-autonomous categories constructed in the monograph [Barr, 1979]. The main tool in the new approach is the Chu construction. The main conclusion is that the category of separated extensional Chu objects for certain kinds of equational categories is equivalent to two usually distinct subcategories of the categories of uniform algebras of those categories.

1. Introduction

The monograph [Barr, 1979] was devoted to the investigation of $*$-autonomous categories. Most of the book was devoted to the discovery of $*$-autonomous categories as full subcategories of seven different categories of uniform or topological algebras over concrete categories that were either equational or reflective subcategories of equational categories. The base categories were:

1. vector spaces over a discrete field;
2. vector spaces over the real or complex numbers;
3. modules over a ring with a dualizing module;
4. abelian groups;
5. modules over a cocommutative Hopf algebra;
6. sup semilattices;
7. Banach balls.

For definitions of the ones that are not familiar, see the individual sections below. These categories have a number of properties in common as well as some important differences. First, there are already known partial dualities, often involving topology.

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It is these partial dualities that we wish to extend. Second, all are symmetric closed monoidal categories. All but one are categories of models of a commutative theory and get their closed monoidal structure from that (see 3.7 below). The theory of Banach balls is really different from first six and is treated in detail in [Barr, Kleisli, to appear].

What we do here is provide a uniform treatment of the first six examples. We show that in each case, there is a $*$-autonomous category of uniform space models of the theory. In most cases, this is equivalent to the topological space models. The main tool used here is the so-called Chu construction as described in an appendix to the 1979 monograph, [Chu, 1979]. He described in detail a very general construction of a large class of $*$-autonomous categories. He starts with any symmetric monoidal category $\mathcal{V}$ and any object $\perp$ therein chosen as dualizing object to produce a $*$-autonomous category denoted $\mathrm{Chu}(\mathcal{V}, \perp)$. The simplicity and generality of this construction made it appear at the time unlikely that it could have any real interest beyond its original purpose, namely showing that there was a plenitude of $*$-autonomous categories. We describe this construction in Section 2.

A preliminary attempt to carry out this approach using the Chu construction appeared in [Barr, 1996], limited to only two of the seven example categories (vector spaces over a discrete field and abelian groups). The arguments there were still very *ad hoc* and depended on detailed properties of the two categories in question. In this article, we prove a very general theorem that appeals to very few special properties of the examples.

In 1987 I discovered that models of Jean-Yves Girard's linear logic were $*$-autonomous categories. Within a few years, Vaughan Pratt and his students had found out about the Chu construction and were studying its properties intensively ([Pratt, 1993a,b, 1995, Gupta, 1994]). One thing that especially struck me was Pratt's elegant, but essentially obvious, observation that the category of topological spaces can be embedded fully into $\mathrm{Chu}(\mathbf{Set}, 2)$ (see 2.2). The real significance—at least to me—of this observation is that putting a Chu structure on a set can be viewed as a kind of generalized topology.

A reader who is not familiar with the Chu construction is advised at this point to read Section 2. Thinking of a Chu structure as a generalized topology leads to an interesting idea which I will illustrate in the case of topological abelian groups. If T is an abelian group (or, for that matter, a set), a topology is given by a collection of functions from the point set of T to the Sierpinski space—the space with two points, one open and the other not. From a categorical point of view, might it not make more sense to replace the functions to a set by group homomorphisms to a standard topological group, thus creating a definition of topological group that was truly intrinsic to the category of groups. If, for abelian groups, we take this "standard group" to be the circle group $\mathbf{R}/\mathbf{Z}$, the resultant category is (for separated groups) a certain full subcategory of $\mathrm{Chu}(\mathbf{Ab}, \mathbf{R}/\mathbf{Z})$ called $\mathrm{chu}(\mathbf{Ab}, \mathbf{R}/\mathbf{Z})$. This category is not the category of topological abelian groups. Nonetheless the category of topological abelian groups has an obvious functor into $\mathrm{chu}(\mathbf{Ab}, \mathbf{R}/\mathbf{Z})$ and this functor has both a left and a right adjoint, each of which is full and faithful. Thus the category $\mathrm{chu}(\mathbf{Ab}/, \mathbf{R}/\mathbf{Z})$ is equivalent to two distinct two full subcategories of abelian groups, each of which is thereby $*$-autonomous. In fact, any topological abelian group that can be embedded algebraically and topologically into a product of locally compact groups has both a finest and a coarsest topology that induce the same set of characters. The two subcategories consist of all those that have the finest topology and those that have the coarsest. These are the images of the left and right adjoint, respectively.

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2. The Chu construction

There are many references to the Chu construction, going back to [Chu, 1979], but see also [Barr, 1991], for example. In order to make this paper self-contained, we will give a brief description here. We stick to the symmetric version, although there are also non-symmetric variations.

2.1. The category $\mathrm{Chu}(\mathcal{V},\perp)$. Suppose that $\mathcal{V}$ is a symmetric closed monoidal category and $\perp$ is a fixed object of $\mathcal{V}$. An object of $\mathrm{Chu}(\mathcal{V},\perp)$ is a pair (V,V') of objects of $\mathcal{V}$ together with a homomorphism, called a **pairing**, $\langle -,-\rangle : V\otimes V' \longrightarrow \perp$. A morphism $(f,f'):(V,V')\longrightarrow(W,W')$ consists of a pair of arrows $f:V\longrightarrow W$ and $f':W'\longrightarrow V'$ in $\mathcal{V}$ that satisfy the symbolic identity $\langle fv,w'\rangle = \langle v,f'w'\rangle$. Diagrammatically, this can be expressed as the commutativity of the diagram

$$\begin{tikzcd}
V\otimes W' \arrow[r, "V\otimes f'"] \arrow[d, "f\otimes W'"'] & V\otimes V' \arrow[d, "\langle -,-\rangle"] \\
W\otimes W' \arrow[r, "\langle -,-\rangle"'] & \perp
\end{tikzcd}$$

Using the transposes $V\longrightarrow V' \multimap \perp$ and $V'\longrightarrow V\multimap\perp$ of the structure maps, this condition can be expressed as the commutativity of either of the squares

$$\begin{tikzcd}
V \arrow[r, "f"] \arrow[d] & W \arrow[d] \\
V'\multimap\perp \arrow[r, "f'\multimap\perp"'] & W'\multimap\perp
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
W' \arrow[r, "f'"] \arrow[d] & V' \arrow[d] \\
W\multimap\perp \arrow[r, "f\multimap\perp"'] & V\multimap\perp
\end{tikzcd}$$

A final formulation of the compatibility condition is that

$$\begin{tikzcd} \mathrm{Hom}((V,V'),(W,W')) \arrow[r] \arrow[d] & \mathrm{Hom}(V,W) \arrow[d] \\ \mathrm{Hom}(W',V') \arrow[r] & \mathrm{Hom}(V\otimes W',\perp) \end{tikzcd}$$

is a pullback.

The internal hom is gotten by using an internalization of the last formulation. Define the object $[(V,V'),(W,W')]$ of $\mathcal{V}$ as a pullback

$$\begin{tikzcd} {[(V,V'),(W,W')]} \arrow[r] \arrow[d] & V \multimap W \arrow[d] \\ W' \multimap V' \arrow[r] & V\otimes W' \multimap \perp \end{tikzcd}$$

and then define

$$(V,V') \multimap (W,W') = ([(V,V'),(W,W')], V\otimes W')$$

Since the dual of (V,V') is (V',V), it follows from the interdefinability of tensor and internal hom in a $*$-autonomous category that the tensor product is

$$(V,V')\otimes(W,W') = (V\otimes W, [(V,V'),(W',W)])$$

The result is a $*$-autonomous category. See [Barr, 1991] for details.

2.2. The category Chu(**Set**, 2). An object of Chu(**Set**, 2) is a pair (S,S') together with a function $S\times S' \longrightarrow 2$. This is equivalent to a function $S' \longrightarrow 2^S$. When this function is injective we say that (S,S') is extensional and then S' is, up to isomorphism, a set of subsets of S. Moreover, one easily sees that if (S,S') and (T,T') are both extensional, then a function $f\colon S\longrightarrow T$ is the first component of some $(f,f')\colon(S,S')\longrightarrow(T,T')$ if and only if $U\in T'$ implies $f^{-1}(U)\in S'$ and then $f'=f^{-1}$ is uniquely determined. This explains Pratt's full embedding of topological spaces into Chu(**Set**, 2).

2.3. The category chu$(\mathcal{V},\perp)$. Suppose $\mathcal{V}$ is a closed symmetric monoidal category as above and suppose there is a factorization system $\mathcal{E}/\mathcal{M}$ on $\mathcal{V}$. (See [Barr, 1998] for a primer on factorization systems.) In general we suppose that the arrows in $\mathcal{E}$ are epimorphisms and that those in $\mathcal{M}$ are compatible with the internal hom in the sense that if $V\longrightarrow V'$ belongs to $\mathcal{M}$, then for any object W, the induced $W\multimap V\longrightarrow W\multimap V'$ also belongs to $\mathcal{M}$. In all the examples here, $\mathcal{E}$ consists of the surjections (regular epimorphisms) and $\mathcal{M}$ of the injections (monomorphisms), for which these conditions are automatic. An object (V,V') of the Chu category is said to be $\mathcal{M}$-**separated**, or simply separated, if the transpose $V\longrightarrow V'\multimap\perp$ is in $\mathcal{M}$ and $\mathcal{M}$-**extensional**, or simply extensional, if the other transpose $V'\longrightarrow V\multimap\perp$ is in $\mathcal{M}$. We denote by $\mathrm{Chu}_s(\mathcal{V},\perp)$, $\mathrm{Chu}_e(\mathcal{V},\perp)$, and $\mathrm{Chu}_{se}(\mathcal{V},\perp)$ the full subcategories of separated, extensional, and separated and extensional, respectively, objects of $\mathrm{Chu}(\mathcal{V}, \perp)$. Following Pratt, we usually denote the last of these by $\mathrm{chu}(\mathcal{V}, \perp)$.

The relevant facts are

1. The full subcategory $\mathrm{Chu}_s(\mathcal{V}, \perp)$ of separated objects is a reflective subcategory of $\mathrm{Chu}(\mathcal{V}, \perp)$ with reflector s.

2. The full subcategory $\mathrm{Chu}_e(\mathcal{V}, \perp)$ of extensional objects is a coreflective subcategory of $\mathrm{Chu}(\mathcal{V}, \perp)$ with coreflector e.

3. If (V, V') is separated, so is $e(V, V')$; if (V, V') is extensional, so is $s(V, V')$.

4. Therefore $\mathrm{chu}(\mathcal{V}, \perp)$ is both a reflective subcategory of the extensional category and a coreflective subcategory of the separated subcategory. It is, in particular, complete and cocomplete.

5. The tensor product of extensional objects is extensional and the internal hom of an extensional object into a separated object is separated.

6. Therefore by using $s(- \otimes -)$ as tensor product and $r(- \multimap -)$ as internal hom, the category $\mathrm{chu}(\mathcal{V}, \perp)$ is $*$-autonomous.

For details, see [Barr, 1998].

3. Topological and uniform space objects

3.1. Topology and duality. In a $*$-autonomous category we have, for any object A, that $A^* \cong \top \multimap A^* \cong A \multimap \top^*$. If we denote $\top^*$ by $\perp$, we see that the duality has the form $A \mapsto A \multimap \perp$. The object $\perp$ is called the **dualizing object** and, as we will see, the way (or at least a way) of creating a duality is by finding a dualizing object in some closed monoidal category.

In order that a category have a duality realized by an internal hom, there has to be a way of constraining the maps so that the dual of a product is a sum. In an additive category, for example, this happens without constraint for finite products, but not normally for infinite ones. A natural constraint is topological. If, for example, the dualizing object is finite discrete, then any continuous map from a product can depend on only finite many of the coordinates. For example, even for ordinary topological spaces, for a continuous function $f: \prod X_i \longrightarrow 2$, $f^{-1}(0)$ has the form $Y \times \prod_{i \notin J} X_i$ where J is a finite subset of I and Y is a subset of the finite product $\prod_{i \in J} X_i$. But then $f^{-1}(1) = Z \times \prod_{i \notin J} X_i$, where $Z = \left(\prod_{i \in J} X_i\right) - Y$. If two elements of the product $x = (x_i)_{i \in I}$ and $x' = (x'_i)_{i \in I}$ are elements such that $x_i = x'_i$ for $i \in J$ then either $x \in Y$ and $x' \in Y$ or $x \in Z$ and $x' \in Z$, but in either case $fx = fx'$. Thus f depends only the coordinates belonging to J, which means f factors through the finite product $\prod_{i \in J} X_i$. A similar argument works if 2 is replaced by any finite discrete set.

3.2. Uniformity. Despite the examples above there are reasons for thinking that the technically "correct" approach to duality is through the use of uniform structures. A very readable and informative account of uniform spaces is in [Isbell, 1964]. However, Isbell uses uniform covers as his main definition. Normally I prefer uniform covers to the approach using entourages, but for our purposes here entourages are more appropriate.

For any equational category, a **uniform space object** is an object of the category equipped with a uniformity for which the operations of the theory are uniform. A morphism of uniform space objects is a uniform function that is also a morphism of models of the theory. Topological space objects are defined similarly. Any uniformity leads to a topological space object in a canonical way and uniform functions become continuous functions for that canonical topology. But not every topology comes from a uniformity and, if it does, it is not necessarily from a unique one. For example, the metric on the space of integers and on the set of reciprocals of integers both give the discrete topology, but the associated uniformities are quite different. (Metric spaces have a canonical uniformity. See Isbell's book for details.)

If, however, there is an abelian group structure among the operations of an equational theory, there is a canonical uniformity associated to every topology. Namely, for each open neighborhood U of the group identity take $\{(x,y) \mid xy^{-1} \in U\}$ as an entourage. Moreover, a homomorphism of the algebraic structure between uniform space objects is continuous if and only if it is uniform. Thus there is no difference, in such cases, between categories of uniform and topological space objects. Obviously, the category of topological space objects is more familiar. However, one of our categories, semilattices, does not have an abelian group structure and for that reason, we have cast our main theorem in terms of uniform structure. There is another, less important, reason. At one point, in dealing with topological abelian groups, it becomes important that the circle group is complete and completeness is a uniform, not topological, notion.

If $\mathcal{V}$ is an equational category, we denote by $\mathrm{Un}(\mathcal{V})$, the category of **uniform objects** of $\mathcal{V}$. We let $|-|\colon \mathrm{Un}(\mathcal{V}) \longrightarrow \mathcal{V}$ to be functor that forgets the uniform structure.

3.3. Small entourage. Let A be a uniform $\mathcal{V}$ object. An entourage $E \subseteq A \times A$ is called a **small entourage** if it contains no subobject of $A \times A$ that properly contains the diagonal and if any homomorphism $f\colon B \longrightarrow A$ of uniform $\mathcal{V}$ objects for which $(f \times f)^{-1}(E)$ is an entourage of B is uniform.

3.4. In all the examples we will be considering, there will be a given class of uniform objects $\mathcal{D}$ and we will be dealing with the full subcategory $\mathcal{A}$ of $\mathrm{Un}(\mathcal{V})$ consisting of those objects **strongly cogenerated** by $\mathcal{D}$, which is to say that those that can algebraically and uniformly embedded into a product of objects of $\mathcal{D}$.

3.5. Half-additive categories. A category is called **half-additive** if its hom functor factors through the category of commutative monoids. It is well known that in any such category finite sums are also products (see, for example, [Freyd, Scedrov, 1990], 1.59). Of course, additive categories are half-additive. Of the six categories considered here, five are additive and one, semilattices, is not additive, but is still half-additive. In fact, a semilattice is a commutative monoid in which every element is idempotent. This monoid structure can be equally well viewed as sup or inf.

3.6. The closed monoidal structure. The categories we are dealing with are all symmetric closed monoidal. With one exception, the closed structure derives from their being models of a commutative theory.

3.7. Commutative theories. A **commutative theory** is an equational theory whose operations are homomorphisms ([Linton, 1966]). Thus in any abelian group G, as contrasted with a non-abelian group, the multiplication $G \times G \longrightarrow G$ is an abelian group homomorphism, as are all the other operations.

3.8. THEOREM. [Linton] *The category of models of a commutative theory has a canonical structure of a symmetric closed monoidal category.*

PROOF. Suppose $\mathcal{V}$ is the category of models and $U: \mathcal{V} \longrightarrow \mathbf{Set}$ is the underlying set functor with left adjoint F. If V and W are objects of $\mathcal{V}$, then $W \multimap V$ is a subset of V^{UW} defined as the simultaneous equalizer, taken over all operations ω of the theory, of

$$\begin{tikzcd} V^{UW} \arrow[r] \arrow[dr, "V^{\omega}"'] & (V^n)^{(UW)^n} \arrow[d, "\omega^{(UW)^n}"] \\ & V^{(UW)^n} \end{tikzcd}$$

Here n is the arity of ω and the top arrow is raising to the nth power. Since the theory is commutative, ω is a homomorphism and so the equalizer is a limit of a diagram in $\mathcal{V}$ and hence lies in $\mathcal{V}$. In particular, the internal hom of two objects of $\mathcal{V}$ certainly lies in $\mathcal{V}$. The free object on one generator is the unit for this internal hom. As for the tensor product, $V \otimes W$ is constructed as a quotient of $F(UV \times UW)$, similar to the usual construction of the tensor product of two abelian groups. ■

3.9. PROPOSITION. *If A and B are objects of an equational category $\mathcal{V}$ equipped with uniformities for which their operations are uniform, then the set of uniform morphisms from $A \longrightarrow B$ is a subobject of $|A| \multimap |B|$ and thus the category of uniform $\mathcal{V}$ objects is enriched over $\mathcal{V}$. It also has tensors and cotensors from $\mathcal{V}$.*

PROOF. Let $[A, B]$ denote the set of uniform homomorphisms from A to B. For each n-ary operation ω, the arrow $\omega B: B^n \longrightarrow B$ is a uniform homomorphism and hence an arrow $[A, B]^n \cong [A, B^n] \longrightarrow [A, B]$ is induced by ωB and we define this as $\omega[A, B]$. This presents $[A, B]$ as a subobject of $|A| \multimap |B|$ so that it also satisfies all the equations of the theory and is thus an object of $\mathcal{V}$. The cotensor A^V is given by the object $V \multimap |A|$ equipped with the uniformity induced by A^{UV}. The tensor is constructed using the adjoint functor theorem with all uniformities on $V \otimes |A|$ as solution set. ■

4. The main theorem

We are now ready to state our main theorem.

4.1. THEOREM. *Suppose $\mathcal{V}$ is an equational category equipped with a closed monoidal structure, $\mathcal{D}$ is a class of uniform space objects of $\mathcal{V}$ and $\mathcal{A}$ is the full subcategory of the category of uniform space objects of $\mathcal{V}$ that is strongly cogenerated by $\mathcal{D}$. Suppose that $\perp$ is an object of $\mathcal{A}$ with the following properties*

1. *$\mathcal{V}$ is half-additive.*
2. *$\mathcal{D}$ is closed under finite products.*
3. *$\perp$ has a small entourage.*
4. *The natural map $\top \longrightarrow [\perp, \perp]$ is an isomorphism.*
5. *If D is an object of $\mathcal{D}$, $A \subseteq D$ is a subobject, then the induced arrow $[D, \perp] \longrightarrow [A, \perp]$ is surjective.*
6. *For every object D of $\mathcal{D}$, the natural evaluation map $D \longrightarrow \perp^{[D,\perp]}$ is injective.*
7. *$\mathcal{A}$ is enriched over $\mathcal{V}$ and has cotensors from $\mathcal{V}$.*

Then, using the regular-epic/monic factorization system, the canonical functor $P \colon \mathcal{A} \longrightarrow \mathrm{chu}(\mathcal{V}, |\perp|)$ defined by $P(A) = (|A|, [A, \perp])$ has a right adjoint R and a left adjoint L, each of which is full and faithful.

4.2. Before beginning the proof, we make some observations. We will call a morphism $A \longrightarrow \perp$ a **functional** on A. In light of condition 5, condition 6 need be verified only for objects that are algebraically 2-generated (and in the additive case, only for those that are 1-generated) since any separating functional can be extended to all of D.

In all our examples, $\perp$ is complete and a closed subobject of an object of $\mathcal{D}$ belongs to $\mathcal{D}$ so that it is sufficient to verify condition 5 when A belongs to $\mathcal{D}$.

The conclusion of the theorem implies that the full images of both R and L are equivalent to $\mathrm{chu}(\mathcal{V}, \perp)$ and hence both image categories are $*$-autonomous.

The diagonal of A in $A \times A$ will be denoted Δ_A. We begin the proof with a lemma.

4.3. LEMMA. *Suppose that $A \subseteq \prod_{i\in I} A_i$ and $\varphi \colon A \longrightarrow \perp$ is a uniform functional. Then there is a finite subset $J \subseteq I$ such that if $\widetilde{A}$ is the image of $A \longrightarrow \prod_{i\in I} A_i \longrightarrow \prod_{i\in J} A_i$ with the subspace uniformity, then φ factors as $A \longrightarrow \widetilde{A} \xrightarrow{\widetilde{\varphi}} \perp$.*

PROOF. Let $E \subseteq \perp \times \perp$ be a small entourage. The definition of the product uniformity implies that there is a finite subset $J \subseteq I$ such that if we let $B = \prod_{i\in J} A_i$ and $C = \prod_{i\notin J} A_i$, then there is an entourage $F \subseteq B \times B$ for which $(A\times A)\cap(F\times(C\times C)) \subseteq (\varphi\times\varphi)^{-1}(E)$. But then $(A\times A)\cap(\Delta_B\times(C\times C))$ is a subobject of $A\times A$ that is included in $(\varphi\times\varphi)^{-1}(E)$. This implies that $(\varphi\times\varphi)((A\times A)\cap(\Delta_B\times(C\times C)))$ is a subobject of $\perp\times\perp$ lying between $\Delta_\perp$ and E, which is then $\Delta_\perp$. In particular, if $a = (a_i)$ and $a' = (a'_i)$ are two elements of A such that $a_i = a'_i$ for all $i \in J$, then $\varphi(a) = \varphi(a')$. Thus, ignoring the uniform structure, we can factor φ via an algebraic homomorphism $\widetilde{\varphi}\colon \widetilde{A} \longrightarrow \perp$. But $(\widetilde{A}\times\widetilde{A})\cap F \subseteq (\widetilde{\varphi}\times\widetilde{\varphi})^{-1}(E)$ which means that $\widetilde{\varphi}$ is uniform in the induced uniformity on $\widetilde{A}$. ∎

4.4. COROLLARY. *For any $A \subseteq B$ in $\mathcal{A}$, the induced $[B,\perp]\longrightarrow[A,\perp]$ is surjective.*

PROOF. Since there is an embedding $B \subseteq \prod_{i\in I} D_i$ with D_i objects of $\mathcal{D}$, it is sufficient to do this in the case that $B = \prod_{i\in I} D_i$. The lemma says that any functional in $[A,\perp]$ factors as $A \longrightarrow \widetilde{A} \longrightarrow \perp$ where, for some finite $J \subseteq I$, $\widetilde{A} \subseteq \prod_{i\in J} D_i$. The latter object is in $\mathcal{D}$ by condition 2 and the map extends to it by condition 5. ∎

4.5. COROLLARY. *For any set $\{A_i \mid i \in I\}$ of objects of $\mathcal{A}$, the canonical map $\sum_{i\in I}[A_i,\perp] \longrightarrow[\prod_{i\in I} A_i,\perp]$ is an isomorphism.*

PROOF. By taking $A = \prod_{i\in I} A_i$ in the lemma, we see that every functional on the product factors through a finite product. That is, the canonical map $\operatorname{colim}_{J\subseteq I}[\prod_{i\in J} A_i,\perp] \longrightarrow[\prod_{i\in I} A_i,\perp]$ is an isomorphism, where the colimit is taken over the finite subsets $J \subseteq I$. On the other hand, half-additivity implies that the canonical map from a finite sum to finite product is an isomorphism. Putting these together, we conclude that

$$\sum_{i\in I}[A_i,\perp] \cong \operatorname*{colim}_{J\subseteq I}\sum_{i\in J}[A_i,\perp] \cong \operatorname*{colim}_{J\subseteq I}\prod_{i\in J}[A_i,\perp] \cong \operatorname*{colim}_{J\subseteq I}\left[\sum_{i\in J} A_i,\perp\right]$$
$$\cong \operatorname*{colim}_{J\subseteq I}\left[\prod_{i\in J} A_i,\perp\right] \cong \left[\prod_{i\in I} A_i,\perp\right]$$

∎

4.6. Proof of the theorem. The right adjoint to P is defined as follows. If (V,V') is an object of $\mathrm{chu}(\mathcal{V},\perp)$, then by definition of separated $V \longrightarrow V \multimap V'$ is monic. The underlying functor from the category of uniform objects to $\mathcal{V}$ has a left adjoint and hence preserves monics so that $|V| \longrightarrow[V',\perp]$ is also monic. Since the latter is a subset of $|\perp|^{|V'|}$, we have that $|V| \subseteq |\perp^{|V'|}|$. Then we let $R(V,V')$ denote $|V|$, equipped with the uniformity induced as a uniform subspace of $\perp^{|V'|}$. Also denote by $\sigma(V,V')$ the uniformity of $R(V,V')$. This is the coarsest uniformity on V for which all the functionals in V' are uniform. A morphism $PA \longrightarrow(V,V')$ consists of an arrow $f\colon |A| \longrightarrow V$ in $\mathcal{V}$ such that for any $\varphi \in V'$ the composite $\varphi\circ f$ is uniform. This means that the composite $A \longrightarrow V \longrightarrow \perp^{UV'}$ is uniform and hence that $A \longrightarrow R(V,V')$ is. Conversely, if $f\colon A \longrightarrow R(V,V')$ is given, then we have $f\colon |A| \longrightarrow V$ such that the composite $A \longrightarrow V \longrightarrow \perp^{UV'}$ is uniform and if we follow it by the coordinate projection corresponding to $\varphi \in V'$, we get that $\varphi \oslash f$ is uniform for all $\varphi \in V'$, so that there is induced a unique arrow $V' \longrightarrow [A, \perp]$ as required. This shows that R is right adjoint to P.

Next we claim that PR is naturally equivalent to the identity. This is equivalent to showing any functional uniform on $R(V, V')$ already belongs to V'. But any functional $\varphi: R(V, V') \longrightarrow \perp$ extends by Corollary 4.4 to a functional on $\perp^{V'}$. From Corollary 4.5, there is a finite set of functionals $\varphi_1, \ldots, \varphi_n \in V'$ and a functional $\alpha: \perp^n \longrightarrow \perp$ such that

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\varphi"'] & \perp^{V'} \arrow[d] \\ \perp & \perp^n \arrow[l, "\alpha"] \end{tikzcd}$$

where the right hand arrow is the projection on the coordinates corresponding to $\varphi_1, \ldots, \varphi_n$. If the components of α are $\alpha_1, \ldots, \alpha_n$, then this says that $\varphi = \alpha_1 \oslash \varphi_1 + \cdots + \alpha_n \oslash \varphi_n$, which is in V'.

For an object A of $\mathcal{A}$ it will be convenient to denote RPA by σA. This is the underlying $\mathcal{V}$ object of A equipped with the weak uniformity for the functionals on A.

Define a homomorphism $f: A \longrightarrow B$ to be **weakly uniform** if the composite $A \longrightarrow B \longrightarrow \sigma B$ is uniform. This is equivalent to the assumption that for every functional $\varphi: B \longrightarrow \perp$, the composite $A \xrightarrow{f} B \xrightarrow{\varphi} \perp$ is a functional on A. It is also equivalent to the assumption that $f: \sigma A \longrightarrow \sigma B$ is uniform. Given A, let $\{A \longrightarrow A_i \mid i \in I\}$ range over the isomorphism classes of weakly uniform surjective homomorphisms out of A. Define τA as the pullback in the diagram

$$\begin{tikzcd} \tau A \arrow[r] \arrow[d] & \prod A_i \arrow[d] \\ \sigma A \arrow[r] & \prod \sigma A_i \end{tikzcd}$$

If $f: A \longrightarrow B$ is weakly uniform, it factors $A \twoheadrightarrow A' \subseteq B$ and the first arrow is weakly uniform since every uniform functional on A' extends to a uniform functional on B. Since, up to isomorphism, $A \twoheadrightarrow A'$ is among the $A \twoheadrightarrow A_i$, it follows that $f: \tau A \longrightarrow B$ is uniform. Since the identity is a weakly uniform surjection, the lower arrow in the square above is a subspace inclusion and hence so is the upper arrow. That implies that the lower arrow in the diagram of functionals

$$\begin{tikzcd} {[\prod \sigma A_i, \perp]} \arrow[r] \arrow[d] & {[\sigma A, \perp]} \arrow[d] \\ {[\prod A_i, \perp]} \arrow[r] & {[\tau A, \perp]} \end{tikzcd}$$

is a surjection. The left hand arrow is equivalent to $\sum[\sigma A_i, \perp] \longrightarrow \sum[A_i, \perp]$ (Corollary 4.5), which we have just seen is an isomorphism. Thus the right hand arrow is a surjection, while it is evidently an injection. This shows that τA has the same functionals as A. If $\widehat{A}$ were a strictly finer uniformity than that of τA on the same underlying $\mathcal{V}$ structure that had the same set of functionals as A, then the identity $A \longrightarrow \widehat{A}$ would be weakly uniform and then $\tau A \longrightarrow \widehat{A}$ would be uniform, a contradiction. Thus if we define $L(V, V') = \tau R(V, V')$ we know at least that $PL \cong \mathrm{Id}$ so that L is full and faithful. If $(f, f'): (V, V') \longrightarrow PA$ is a Chu morphism, then $f: V \longrightarrow |A|$ is a homomorphism such that for each uniform functional $\varphi: A \longrightarrow \perp$ the composite $\varphi \circ f \in V'$. Thus $R(V, V') \longrightarrow A$ is weakly uniform and hence $L(V, V') = \tau R(V, V') \longrightarrow A$ is uniform. Conversely, if $f: L(V, V') \longrightarrow A$ is uniform, then for any functional $\varphi: A \longrightarrow \perp$, the composite $\varphi \circ f$ is uniform on $L(V, V')$ and hence belongs to V' so that we have $(V, V') \longrightarrow PA$. ∎

We will say that an object A with $A = \sigma A$ has the **weak uniformity** (or **weak topology**) and that one for which $A = \tau A$ has the **Mackey uniformity** (or **Mackey topology**). The latter name is taken from the theory of locally convex topological vector spaces where a Mackey topology is characterized by the property of having the finest topology with a given set of functionals.

4.7. Exceptions. In verifying the hypotheses of Theorem 4.1, one notes that each example satisfies simpler hypotheses. And each simpler hypothesis is satisfied by most of the examples. Most of the categories are additive (exception: semilattices) and then we can use topologies instead of uniformities. In most cases, the dualizing object is discrete (exceptions: abelian groups and real or complex vector spaces) and the existence of a small entourage (or neighborhood of 0 in the additive case) is automatic. In most cases, the theory is commutative and the closed monoidal structure comes from that (exception: modules over a Hopf algebra) so that the enrichment of the uniform category over the base is automatic. Thus most of the examples are exceptional in some way (exceptions: vector spaces over a field and modules with a dualizing module), so that we may conclude that they are *all* exceptional.

5. Vector spaces: the case of a discrete field

The simplest example of the theory is that of vector spaces over a discrete field. Let K be a fixed discrete field. We let $\mathcal{V}$ be the category of K-vector spaces with the usual closed monoidal structure and let $\mathcal{D}$ be the discrete spaces. Since the category is additive, we can work with topologies rather than uniformities. We take the dualizing object $\perp$ as the field K with the discrete topology.

The conditions of Theorem 4.1 are all evident and so we conclude that the full subcategories of the category of topological K-vector spaces consisting of weakly topologized space and of Mackey spaces are $*$-autonomous.

We note that infinite dimensional discrete spaces do not have the weak uniformity. In fact the weak uniformity associated to the chu space $(V, V \multimap K)$ is V with the uniform topology in which the open subspaces are the cofinite dimensional ones. Since the 0 subspace is not cofinite dimensional, the space is not discrete. On the other hand, the map to the discrete V is weakly uniform and so the Mackey space associated is discrete.

6. Dualizing modules

The case of a vector space over a discrete field has one generalization, suggested by R. Raphael (private communication). Let R be a commutative ring. Say that an R-module $\perp$ is a **dualizing module** if it is a finitely generated injective cogenerator and the canonical map $R \longrightarrow \mathrm{Hom}_R(\perp, \perp)$ is an isomorphism. Let $\mathcal{A}$ be the category of topological (= uniform) R-modules that are strongly cogenerated by the discrete ones. Then taking the small neighborhood to be $\{0\}$ and $\mathcal{D}$ the class of discrete modules, the conditions of Theorem 4.1 are satisfied and we draw the same conclusion.

6.1. Existence of dualizing modules. Not every ring has a dualizing module. For example, no finitely generated abelian group is injective as a $\mathbf{Z}$-module, so $\mathbf{Z}$ lacks a dualizing module. On the other hand, If R is a finite dimensional commutative algebra over a field K, then $\mathrm{Hom}_K(R, K)$ is a dualizing module for K. It follows that any artinian commutative ring has a dualizing module:

6.2. Proposition. *Suppose K is a commutative ring with a dualizing module D and R is a commutative K-algebra finitely generated and projective as a K-module. Then for any finitely generated R-projective P of constant rank one, the R-module* $\mathrm{Hom}_K(P, D)$ *is a dualizing module for R.*

Proof. It is standard that such a module is injective. In fact, for an injective homomorphism $f: M \longrightarrow N$, we have that

$$\mathrm{Hom}_R(f, \mathrm{Hom}_K(P, D)) \cong \mathrm{Hom}_K(P \otimes_R f, D)$$

which is surjective since P is R-flat. Since P is finitely generated projective as an R-module, it is retract of a finite sum of copies of R. Similarly, R is a retract of a finite sum of copies of K, whence P is a retract of a finite sum of copies of K. Then $\mathrm{Hom}(P, D)$ is a retract of a finite sum of copies of D and is thus finitely generated as a K-module, *a fortiori* as an R-module. Next we note that a constantly rank one projective P has endomorphism ring R. In fact, the canonical $R \longrightarrow \mathrm{Hom}_R(P, P)$ localizes to the isomorphism $R_Q \longrightarrow \mathrm{Hom}_{R_Q}(P_Q, P_Q) \cong \mathrm{Hom}_{R_Q}(R_Q, R_Q)$ which is an isomorphism, at each prime ideal Q and hence is an isomorphism. Then we have that

$$\begin{aligned}
\mathrm{Hom}_R(\mathrm{Hom}_K(P, D), \mathrm{Hom}_K(P, D)) &\cong \mathrm{Hom}_K(P \otimes_R \mathrm{Hom}_K(P, D), D) \\
&\cong \mathrm{Hom}_R(P, \mathrm{Hom}_K(\mathrm{Hom}_K(P, D), D)) \\
&\cong \mathrm{Hom}_R(P, P) \cong R
\end{aligned}$$

since D is a dualizing module for K and P is a finitely generated K module. ∎

Whether any non-artinian commutative ring has a dualizing module is an open question. For example, the product of countably many fields does not appear to have a dualizing module. The obvious choice would be the product ring itself and, although it is injective, it is not a cogenerator since the quotient of the ring mod the ideal which is the direct sum is a module that is annihilated by every minimal idempotent so that the quotient module has no non-zero homomorphism into the ring.

7. Vector spaces: case of the real or complex field

We will treat the case of the complex field. The real case is similar. We take for $\mathcal{D}$ the class of Banach spaces and the base field $\mathbf{C}$ as dualizing object. The $\mathcal{D}$-cogenerated objects are just the spaces whose topology is given by seminorms. These are just the locally convex spaces (see [Kelly, Namioka, 1963], 2.6.4). The conditions of 4.1 follow immediately from the Hahn-Banach theorem and we conclude that the category $\mathrm{chu}(\mathcal{V}, \perp)$ is equivalent to both full subcategories of weakly topologized and Mackey spaces and that both categories are $*$-autonomous. In particular, the existence of the Mackey topology follows quite easily from this point of view.

We can also give a relatively easy proof of the fact that the Mackey topology is convergence on weakly compact, convex, circled subsets of the dual. In fact, let A be a locally convex space and A^* denote the weak dual. If $f\colon A \longrightarrow D$ is a weakly continuous map, then we have an induced map, evidently continuous in the weak topology, $f^*\colon D^* \longrightarrow A^*$ and one sees immediately that the weakly continuous seminorm induced on A by the composite $A \xrightarrow{f} D \xrightarrow{||-||} \mathbf{R}$ is simply the sup on $f^*(C)$, where C is the unit ball of D^*, which is compact in the weak topology. On the other hand, if $C \subseteq A^*$ is compact, convex, and circled, let B be the linear subspace of A^* generated by C made into a Banach space with C as unit ball. With the topology induced by that of C, so that a morphism out of B is continuous if and only if its restriction to C is, B becomes an object of the category $\underline{C}$ as described in [Barr, 1979], IV.3.10. This category consists of the mixed topology spaces whose unit balls are compact. The discussion in IV.3.16 of the same reference then implies that every functional on B^* is represented by an element of B. This means that the induced $A \longrightarrow B^*$ is weakly continuous. But B^* is a Banach space whose norm is the absolute sup on C, as is the induced seminorm on A.

8. Banach balls

A Banach ball is the unit ball of a Banach space. The conclusions, but not the hypotheses of Theorem 4.1 are valid in this case too. However, the proof is different and will appear elsewhere [Barr, Kleisli, 1999]. The proof given here of the existence of the right adjoint and the Mackey topology was first found in this context.

9. Abelian groups

The category of abelian groups is an example of the theory. For $\mathcal{D}$ we take the class of locally compact groups. The dualizing object is, as usual, the circle group, $\mathbf{R}/\mathbf{Z}$. Since the category is additive, we can deal with topologies instead of uniformities. A **small neighborhood** of 0 is an open neighborhood of 0 that contains no non-zero subgroup and for which a homomorphism to the circle is continuous if and only if the inverse image of that neighborhood is continuous.

9.1. PROPOSITION. *The image* $U \subseteq \mathbf{R}/\mathbf{Z}$ *of the interval* $(-1/3, 1/3) \subseteq \mathbf{R}$ *is a small neighborhood of 0.*

PROOF. Suppose $x \neq 0$ in U. Suppose, say, that x is in the image of a point in $(0, 1/3)$. Then the first one of x, $2x$, $4x$, $\dots$, that is larger than $1/3$ will be less than $2/3 \equiv -1/3$, which shows that U contains no non-zero subgroup.

It is clear that the set of all $2^{-n}U$, $n = 0, 1, 2, \dots$ is a neighborhood base at 0. Suppose that $f\colon A \longrightarrow \mathbf{R}/\mathbf{Z}$ is a homomorphism such that $V = f^{-1}(U)$ is open in A. Let $V_0 = V$ and inductively choose an open neighborhood V_n of 0 so that $V_n - V_n \subseteq V_{n-1}$. Then one easily sees by induction that $V_n \subseteq f^{-1}(2^{-n}U)$. ∎

The remaining conditions of Theorem 4.1 are almost trivial, given Pontrjagin duality. The only thing of note is that if $D \in \mathcal{D}$ and $A \subseteq D$ then any continuous homomorphism $\varphi\colon A \longrightarrow \mathbf{R}/\mathbf{Z}$ can first be extended to the closure of A, since the circle is compact and hence complete in the uniformity. A closed subgroup of a locally compact group is locally compact and the duality theory of locally compact groups gives the extension to all of D.

9.2. PROPOSITION. *Locally compact groups are Mackey groups.*

PROOF. Since all the groups in $\mathcal{A}$ are embedded in a product of locally compact groups, it suffices to know that a weakly continuous map between locally compact topological groups is continuous. This is found in [Glicksberg, 1962]. ∎

9.3. Other choices for $\mathcal{D}$. One thing to note is that it is possible to choose a different category $\mathcal{A}$. The result can be a different notion of Mackey group. For instance, you could choose for $\mathcal{A}$ the subspaces of compact spaces. In that case weakly continuous coincides with continuous and weak and Mackey topologies coincide. Another possibility is to use compact and discrete spaces. It is easy to see that the real line cannot be embedded into a product of compact and discrete spaces. There are no non-zero maps to a discrete space, so it would have to embedded into a product of compact spaces. But the real line is complete, so the only way it could be embedded into a product of compact spaces would be if it were compact.

In the original monograph, countable sums of copies of $\mathbf{R}$ were permitted in $\mathcal{D}$. But the sum of countably many copies of $\mathbf{R}$ is not locally compact. Here we show that we also get a model of the theory by allowing $\mathcal{D}$ to consist of countable sums of locally compact groups. The only issue here is the injectivity of the circle. So suppose $A \subseteq D$, where $D = D_1 \oplus D_2 \oplus \cdots$ is a countable sum of locally compact groups. As above, we can suppose that A is closed in D. Let $F_n(D) = D_1 \oplus \cdots \oplus D_n$ and $F_n(A) = A \cap F_n(D)$. Every element of A is in some finite summand, so that, algebraically at least, $A = \operatorname{colim} F_n(A)$. Whether it is topologically is not important, since we will show that every continuous character on the colimit extends to a continuous character on D. What does matter is that, by definition of the topology on the countable sum, $D = \operatorname{colim} F_n(D)$ both algebraically and topologically. The square

$$\begin{tikzcd}
F_{n-1}(A) \arrow[r, tail] \arrow[d, tail] & F_n(A) \arrow[d, tail] \\
F_{n-1}(D) \arrow[r, tail] & F_n(D)
\end{tikzcd}$$

is a pullback by definition. There is no reason for it to be a pushout, but if we denote the pushout by P_n, it is trivial diagram chase to see that $P_n \rightarrowtail F_n(D)$ is injective. The group $F_n(D)$ is locally compact and so, therefore, is the closed subgroup $F_n(A)$ and so is the closure $\overline{P_n}$. Thus, taking Pontrjagin duals, all the arrows in the diagram below are surjective and the square is a pullback:

$$\begin{tikzcd}
F_n(D)^* \arrow[dr, two heads] \arrow[drr, two heads] \arrow[ddr, two heads] & & \\
& \overline{P_n}^* \arrow[r, two heads] \arrow[d, two heads] & F_{n-1}(D)^* \arrow[d, two heads] \\
& F_n(A)^* \arrow[r, two heads] & F_{n-1}(A)^*
\end{tikzcd}$$

The surjectivity of the arrow $F_n(D)^* \twoheadrightarrow \overline{P_n}^*$ implies that each square of

$$\begin{tikzcd}
\cdots \arrow[r, two heads] \arrow[d] & F_{n+1}(D)^* \arrow[r, two heads] \arrow[d] & F_n(D)^* \arrow[r, two heads] \arrow[d] & F_{n-1}(D)^* \arrow[r, two heads] \arrow[d] & \cdots \arrow[d] \\
\cdots \arrow[r, two heads] & F_{n+1}(A)^* \arrow[r, two heads] & F_n(A)^* \arrow[r, two heads] & F_{n-1}(A)^* \arrow[r, two heads] & \cdots
\end{tikzcd}$$

is a weak pullback. From this, it is a simple argument to see that the induced arrow $\lim F_n(D) \twoheadrightarrow \lim F_n(A)$ is surjective.

10. Modules over a cocommutative Hopf algebra

Modules over a Hopf algebra are not models of a commutative theory, unless the algebra should also be commutative and, even so, the closed structure does not come from there.

Thus we will have to verify directly that the category of the topological algebras is enriched over the category of discrete algebras for the theory.

There are two important special cases and we begin with brief descriptions of them.

10.1. Group representations. Let G be a group and K be a field. A K-representation of G is a homomorphism of G into the group of automorphisms of some vector space over K. Equivalently, it is the action of G on a K-vector space. A third equivalence is with a module over the group algebra $K[G]$. The category of K-representations of G is thus an equational category, that of the $K[G]$-modules, but the theory is not commutative unless G should be commutative and even in that case, we do not want that closed monoidal structure. The one we want has as tensor product of modules M and N the K-tensor product $M \otimes N = M \otimes_K N$. The G-action is the so-called diagonal action $x(m \otimes n) = xm \otimes xn$, $x \in G$, extended linearly. The internal hom takes for $M \multimap N$ the set of K-linear maps with action given by $(xf)m = x(f(x^{-1}m))$ for $x \in G$. This gives a symmetric closed monoidal structure for which the unit object is K with trivial G action, meaning every element of G acts as the identity on K.

If M and N are topological vector spaces with continuous action of G (which is assumed discrete, at least here), then it is easily seen that the set of continuous linear transformations $M \longrightarrow N$ is a G-representation with the same definition of action and we denote it by $[M, N]$ as before. Thus the category of topological (= uniform) G-modules is enriched over the category of G-modules. The cotensor is also easy. Define A^V as $|A| \multimap V$ topologized as a subspace of A^{UV}.

10.2. Lie algebras. Let K be a field and $\mathfrak{g}$ be a K-Lie algebra. A K-representation of $\mathfrak{g}$ is a Lie algebra homomorphism of $\mathfrak{g}$ into the Lie algebra of endomorphisms of a K-vector space V. In other words, for $x \in \mathfrak{g}$ and $v \in V$, there is defined a K-linear product xv in such a way that $[x, y]v = x(yv) - y(xv)$. If V and W are two such actions, there is an action on $V \otimes W = V \otimes_K W$ given by $x(v \otimes w) = xv \otimes w + v \otimes xw$. The space $V \multimap W$ of K-linear transformations has an action given by $(xf)(v) = x(fv) - f(xv)$. If $\mathfrak{g}$ acts continuously on topological vector spaces V and W, then xf is continuous when f is so that the category of topological representations is enriched over the category of representations. The cotensor works in the same way as with the groups.

10.3. Modules over a cocommutative Hopf algebra. These two notions above come together in the notion of a module over a cocommutative Hopf algebra. Let K be a field. A **cocommutative Hopf algebra** over K is a K-algebra given by a multiplication $\mu: H \otimes H \longrightarrow H$ (all tensor products in this section are over K), a unit $\eta: K \longrightarrow H$, a comultiplication $\delta: H \longrightarrow H \otimes H$, a counit $\epsilon: H \longrightarrow K$ and an involution $\iota: H \longrightarrow H$ such that

HA–1. (H, μ, η) is an associative, unitary algebra;

HA–2. (H, δ, ϵ) is a coassociative, counitary, cocommutative coalgebra;

HA–3. δ and ϵ are algebra homomorphisms; equivalently, μ and η are coalgebra homomorphisms;

HA–4. ι makes (H, μ, η) into a group object in the category of cocommutative coalgebras.

This last condition is equivalent to the commutativity of

$$\begin{tikzcd} H \arrow[r, "\delta"] \arrow[d, "\eta \circ \epsilon"'] & H \otimes H \arrow[d, "1 \otimes \iota"] \\ H & H \otimes H \arrow[l, "\mu"] \end{tikzcd}$$

The leading examples of Hopf algebras are group algebras and the enveloping algebras of Lie algebras. If G is a group, the group algebra $K[G]$ is a Hopf algebra with $\delta(x) = x \otimes x$, $\epsilon(x) = 1$ and $\iota(x) = x^{-1}$, all for $x \in G$ and extended linearly. In the case of a Lie algebra $\mathfrak{g}$, the definitions are $\delta(x) = 1 \otimes x + x \otimes 1$, $\epsilon(x) = 0$ and $\iota(x) = -x$, all for $x \in \mathfrak{g}$.

10.4. The general case. Let H be a cocommutative Hopf algebra. By an H-**module** we simply mean a module over the algebra part of H. If M and N are modules, we define $M \otimes N$ to be the tensor product over K with H action given by the composite

$$H \otimes M \otimes N \xrightarrow{\delta \otimes 1 \otimes 1} H \otimes H \otimes M \otimes N \longrightarrow H \otimes M \otimes H \otimes N \longrightarrow M \otimes N$$

The second arrow is the symmetry isomorphism of the tensor and the third is simply the two actions. We define $M \multimap N$ to be the set of K-linear arrows with the action $H \otimes (M \multimap N) \longrightarrow M \multimap N$ the transpose of the arrow $H \otimes M \otimes (M \multimap N) \longrightarrow N$ given by

$$\begin{aligned} H \otimes M \otimes (M \multimap N) &\xrightarrow{\delta \otimes 1 \otimes 1} H \otimes H \otimes M \otimes (M \multimap N) \\ &\xrightarrow{1 \otimes \iota \otimes 1 \otimes 1} H \otimes H \otimes M \otimes (M \multimap N) \\ &\longrightarrow H \otimes M \otimes (M \multimap N) \longrightarrow H \otimes N \longrightarrow N \end{aligned}$$

The third arrow is the action of H on M, the fourth is evaluation and the fifth is the action of H on N.

That this gives an autonomous category can be shown by a long diagram chase. The tensor unit is the field with the action $xa = \epsilon(x)a$.

We have to show that the category of topological modules is enriched over the category of modules. We can describe the enriched structure as consisting of the continuous linear maps with the module structure given as before, that is by conjugation. The continuity of the module structure guarantees that the action preserves continuity. From then on the argument is the same. The dualizing object is the discrete field K which has a small neighborhood and the rest of the argument is the same. The cotensor is just as in the case of group representations.

The class $\mathcal{D}$ consists of the discrete objects. The dualizing object is the tensor unit. Since the internal hom is just that of the vector spaces, the conditions of Theorem 4.1 are easy.

11. Semilattices

By **semilattice** we will mean inf semilattice, which is a partially ordered set in which every finite set of elements has an inf. It is obviously sufficient that there be a top element and that every pair of elements have an inf. The category is obviously equivalent to that of sup semilattices, since you can turn the one upside down to get the other. The category is equational having a single constant, 1 (the top element) and a single binary operation $\wedge$ which is unitary (with respect to 1), commutative, associative and idempotent. (In fact, sup semilattices have exactly the same description—it all depends on how you interpret the operations.)

Semilattices do not form an additive category, but they are half-additive since they are commutative monoids. The dualizing object is the 2 element chain with the discrete uniformity, which evidently has a small entourage. Since it is the tensor unit, condition 4 of Theorem 4.1 is satisfied. For $\mathcal{D}$, we take the class of discrete lattices. We need show only conditions 5 and 6.

Suppose we have an inclusion $L_1 \subseteq L_2$ of discrete semilattices and $f\colon L_1 \longrightarrow \perp$ is a semilattice homomorphism. We will show that if $x \in L_2 - L_1$, then f can be extended to the semilattice generated by L_1 and x. This semilattice is $L_1 \cup (L_1 \wedge x)$. We first define $fx = 1$ unless there are elements $a, b \in L_1$ such that $fa = 1$, $fb = 0$ and $a \wedge x \le b$ in which case we define $fx = 0$. Then we define $f(a \wedge x) = fa \wedge fx$ for any $a \in L_1$. The only thing we have to worry about is if $a \wedge x \in L_1$ for some $a \in L_1$. If $fa = 0$, then $f(a \wedge x) \le fa$ so that $f(a \wedge x) = 0 = fa \wedge fx$. If $fa = 1$, then either $f(a \wedge x) = 1$ or taking $b = a \wedge x$ we satisfy the condition for defining $fx = 0$ and then $0 = f(a \wedge x) = fa \wedge fx$ as required. The rest of the argument is a routine application of Zorn's lemma. This completes the proof of 5. Now 6 follows immediately since given any two elements of a discrete lattice, they generate a sublattice of at most 4 four elements and it is easy to find a separating functional on that sublattice.

12. The category of δ-objects

We will very briefly explain why the categories of δ-objects considered in [Barr, 1979] is also $*$-autonomous. Of course, it is likely more interesting that the larger categories constructed here are $*$-autonomous, but in the interests of recovering all the results from the monograph, we include it.

An object T is called ζ-**complete** if it is injective with respect to dense subobjects of compact objects. That is, if in any diagram

$$\begin{tikzcd} C_0 \arrow[r] \arrow[d] & C \\ T & \end{tikzcd}$$

with C compact and C_0 a dense subobject, can be completed by an arrow $C \longrightarrow T$. The object T is ζ^*-complete if T^* is ζ-complete. An object is called a δ-object if is ζ-complete, ζ^*-complete and reflexive.

12.1. THEOREM. *The full subcategories of δ-objects are $*$-autonomous subcategories the categories of Mackey objects.*

PROOF. The proof uses one property that we will not verify. Namely that all compact objects are δ-objects. The duals of the compact objects are complete (in most cases discrete). For an object T, we define ζT as the intersection of all the ζ-complete subobjects of the completion of T. The crucial claim is that if T is ζ-complete, so is $(\zeta T^*)^*$. In fact, the adjunction arrow $T^* \longrightarrow \zeta T^*$ gives an arrow $(\zeta T^*)^* \longrightarrow T^{**} \cong T$. Now consider a diagram

$$\begin{tikzcd} C_0 \arrow[r] \arrow[d] & C \\ (\zeta T^*)^* \arrow[r] & T \end{tikzcd}$$

Since T is ζ-complete, there is an arrow $C \longrightarrow T$ that makes the square commute. This gives $T^* \longrightarrow C^*$ and, since C^* is complete, $\zeta T^* \longrightarrow C^*$, and then $C \cong C^{**} \longrightarrow (\zeta T^*)^*$, as required. We now invoke Theorem 2.3 of [Barr, 1996] to conclude that the δ-objects form a $*$-autonomous category. ∎

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