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SIMPLICIAL MATRICES AND THE NERVES OF WEAK n-CATEGORIES I : NERVES OF BICATEGORIES

Dedicated to Jean Bénabou

JOHN W. DUSKIN

ABSTRACT. To a bicategory $\mathbb{B}$ (in the sense of Bénabou) we assign a simplicial set $\mathbf{Ner}(\mathbb{B})$, the *(geometric) nerve* of $\mathbb{B}$, which completely encodes the structure of $\mathbb{B}$ as a bicategory. As a simplicial set $\mathbf{Ner}(\mathbb{B})$ is a subcomplex of its 2-Coskeleton and itself isomorphic to its 3-Coskeleton, what we call a 2-dimensional *Postnikov complex.* We then give, somewhat more delicately, a complete characterization of those simplicial sets which are the nerves of bicategories as certain 2-dimensional Postnikov complexes which satisfy certain restricted "exact horn-lifting" conditions whose satisfaction is controlled by (and here defines) subsets of (abstractly) *invertible* 2 and 1-simplices. Those complexes which have, at minimum, their degenerate 2-simplices always invertible and have an invertible 2-simplex $\chi_2^1(x_{12}, x_{01})$ present for each "composable pair" $(x_{12}, \_, x_{01}) \in \bigwedge_2^1$ are exactly the nerves of bicategories. At the other extreme, where *all* 2 and 1-simplices are invertible, are those Kan complexes in which the Kan conditions are satisfied exactly in all dimensions > 2. These are exactly the nerves of *bigroupoids*–all 2-cells are isomorphisms and all 1-cells are equivalences.

Contents

This work began long ago as joint work on n-categories with Ross Street during two visits as a guest of Macquarie University, N.S.W., Australia. The work on bicategories presented in this paper is largely the outgrowth of research done while the author was a guest for an extended stay in the summer of 1999 at the Université Catholique de Louvain , Louvain-la-Neuve, Belgium. The hospitality shown the author and the superb work environment provided at these opposite poles of the earth made it possible. Two other people also deserve the author's special thanks: To one of the referees, who patiently read a very long manuscript and made several thoughtful suggestions on how the paper could be made more accessible to non-specialists in algebraic topology, and To the editor, Mike Barr, for having the patience to resurrect the hopelessly uncompilable LaTeX mess that was dumped on him by a hopelessly naive author. Thanks for the diagrams goes to XY-pic. They were almost the only thing that would run!

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1. Introduction

In the early 1960's, when to most people categories were large and hom-sets were small, Grothendieck made the observation that one can associate to any small category $\mathbb{C}$ (or not so small, since he put everything in terms of his universes) a simplicial set $\mathbf{Ner}(\mathbb{C})$ which, in analogy to the "nerve of a covering", he called the *nerve of the category* $\mathbb{C}$. In Grothendieck's original description the 0-simplices of $\mathbf{Ner}(\mathbb{C})$ are the objects of $\mathbb{C}$ and the n-simplices the set X_n of *composable sequences of length* n of the morphisms (or as he preferred in this context, the *arrows* of $\mathbb{C}$).

The face operators are given by the projections for the "*extremals*" 0 and n, with the inner ones d_i $(0 < i < n)$ given by composing out the i^{th} indexed object in the sequence. With the usual convention of drawing sequences of arrows from left to right

$$x_0 \xrightarrow{f_1} x_1 \cdots x_{n-1} \xrightarrow{f_n} x_n,$$

for $f_1 : x_0 \longrightarrow x_1 \in X_1$, this made $d_0(f_1) = x_1$, the *target* of f_1, and $d_1(f_1) = x_0$, the *source* of f_1, and is consistent with the conventional "face opposite vertex" numbering used in simplicial complexes. The degeneracy operators are obtained by expanding the i^{th} indexed object by its identity arrow. The simplicial identities are either consequences of the construction (the extremals as projections from iterated fiber products) or are *equivalent* to the associativity and identity axioms for a category.

Under this same correspondence, functors between categories correspond exactly to simplicial maps, *i.e.,* face and degeneracy preserving maps, between the nerves and natural transformations correspond exactly to homotopies of simplicial maps, so that the fundamental definitions of category theory are all captured within simplicial sets by the nerve. Now it easy to see that if one views simplicial sets as presheaves $X_\bullet$ on the skeletal category Δ of non-empty finite totally ordered sets and non-decreasing maps, then the nerves of categories are exactly the left-exact presheaves

$$\mathbf{Ner}(\mathbb{C})_n = \mathrm{Hom}_{\mathrm{Cat}}([n], \mathbb{C}),$$

where $[n] = \{0 < \cdots < n\}$ is considered to be a small category,and, in an observation first made by Ross Street, that these are the ones for which the canonical projections $\mathrm{pr}_{\hat{k}} : X_n \longrightarrow \bigwedge_n^k(X_\bullet)$ for $n > 1$ and $0 < k < n$ are all bijections, *i.e.,* the *weak* Kan complexes in which the *weak* Kan conditions $(0 < k < n)$ are satisfied *exactly* ($\mathrm{pr}_{\hat{k}}$ is a *bijection* rather than simply a surjection) in all dimensions > 1. If one includes the extremals ($k = 0$ and $k = n$) as well, one obtains the nerves of *groupoids* where every arrow in the category is an isomorphism.

In this paper we will recover this characterization of nerves of categories but within what we will call our "basic simplicial setting for $n = 1$" as certain subcomplexes of their 1-Coskeleta which are themselves isomorphic to their 2-Coskeleta ("1-dimensional Postnikov-complexes").[1] Street's horn conditions above will turn out to be the same, and non-trivial through dimension 3, but we will note that the complex will also satisfy additional restricted horn conditions, in this case for the extremals, which refer to and define a subset $I_1$of X_1 of what we will call the *invertible* 1-simplices. They will turn out, of course, to be the isomorphisms of the category and the category will be a groupoid precisely when $I_1 = X_1$. Although this diversion will add little to what is not already known in this particular case, it will be done in order to motivate with a simple and familiar example our concept of the "basic simplicial setting" of an n-dimensional Postnikov complex and the matrix methods used in this paper (and its sequents) which follow from it.

One would hope that one should be able capture in simplicial form a similar nerve for "higher dimensional" categories and groupoids, at least in so far as these concepts have found satisfactory definitions . In 1984 Street, based on his work on "orientals" and on an earlier conjecture in unpublished work of J.E. Roberts [Roberts 1978], made a more precise conjecture on a characterization of the nerve of a *strict* n-category, in fact, of a general ω-category (where strict associativity and strict unitary conditions are the only constraints and are not "weakened" in any way) by means of what he called simplicial sets with "hollowness".[2] The conjecture was published in [Street 1987] and, following work of Street [Street 1988] and Michael Zaks in 1988, was finally proved in 1993-94 by Martin Hyland's student, Dominic Verity, during a post-doctoral stay with Street. Unfortunately, his proof, although given in detail in seminars, remains unpublished [Street 2002].

With the exception of the work by Carlos Simpson's student Z. Tamsamani in his thesis [Tamsamani 1995][3] most recent work known to the author on weak n-categories, Gray categories, and similar "lax" higher dimensional category-like objects (where even precise definitions are not easily comparable and remain under debate[Leinster 2002]) has not attempted to describe them in simplicial terms...this in spite of the fact that much of the motivation for studying these objects, conjectured to exist by Grothendieck, would be to provide truly algebraic, *i.e.,* categorical, tools for studying in detail the homotopy n-type of a topological space, something which is directly supplied, at present only simplicially, by $X_\bullet^{(n)}$ or its limited minimalization, the so called "fundamental n-dimensional hypergroupoid" $\Pi_n(X)$ of the singular complex.[4]

[1] The n-Coskeleton is obtained by "n-truncating" the complex by forgetting all dimensions $> n$ and making this n-truncation back into a complex by iterating simplicial kernels starting with $K_{n+1}(X_\bullet) = \mathbf{Cosk}^n(X_\bullet)_{n+1}$. The n^{th} complex $X_\bullet^{(n)}$ in the natural Postnikov tower is just the image of the complex in its n-Coskeleton. The "co" terminology here is backwards: the n-coskeleton is the right adjoint to n-truncation; the left adjoint is the *n-skeleton*, but that's been around forever (*cf.* [Duskin 1975]). The basic properties of Coskeleta are reviewed in this paper in Section 2.2

[2] This has turned out to be closely related to our concepts of "commutativity" and "invertibility" and will be discussed in[Duskin 2002(c)].

[3] The relation of our nerve to Tamsamani and Simpson's concept of a *Segal Category* will be given in an appendix to the second paper in this series[Duskin 2002(a)].

[4] $\Pi_n(X) = \Pi_n(\mathrm{Sing}(X))$ is obtained by replacing $\mathrm{Sing}(X)_n$ with the set of homotopy classes of n-

Leaving aside, for the present, work on "tricategories" by [Gordon *et al.*, 1995], and on "trigroupoids" by [Leroy 1994], by common agreement, the weakest possible *useful* generalization of ordinary categories to the immediate next level of what is thought of as "higher category theory" are Jean Bénabou's *bicategories* [Bénabou 1967]. For Bénabou, in a bicategory [5] hom-sets become categories, composition becomes functorial rather than functional, but associativity and identity properties only hold "up to coherent specified natural isomorphisms". Of course, after 35 years one wants more, but in those 35 years, thanks principally to the efforts of the Australian, Italian, and Canadian Schools, the abstract theory of bicategories and its applications has been developed almost as fully (and widely) as category theory itself. Indeed, Bénabou, only half jokingly, could be said to have made possible a "policy of full employment" for an entire generation of young category theorists.

Now, like categories, bicategories do have a genuine simplicial set associated with them, their (geometric) nerve. Intuitively, it is also as genuinely 2-dimensional in nature as categories are 1-dimensional (as might be expected from the numerous "commutative planar surface 'pasting' diagrams" associated with them, *e.g.*, [Power 1990], [Johnson-Walters 1987], [Power 1991] , *et seq.*). Although the idea of such a nerve has been known, at least passingly, to the author for some years (*cf.* [Street 1996]), it has been only in a much more recent dawn that a full appreciation has come to him of just how direct and intimate a relation Bénabou's carefully chosen axioms bear to the simplicial identities. Since, I am told, these axioms were chosen only with a large number of distinctly "non-geometric" examples in mind, this has come as something of a surprise. Nevertheless, this newfound appreciation[6] has made it possible to intrinsically characterize those simplicial sets which are the nerves of bicategories. Fortunately, the techniques apparently generalize to tricategories and Gray-categories and give some hope of beyond as well.

Incidentally, as we shall see (and should keep as a cautionary note in our zeal to reach higher dimensions), the naive generalization of what we just noted holds for a category where $n = 1$ to where $n = 2$: "a weak Kan complex with the weak Kan condition satisfied exactly for $n > 2$", leads only to the (all-be-it interesting) special class of bicategories in which *all* 2-cells are isomorphisms. However, if one extends this to the extremals as well, by further demanding that all 1-simplices be invertible ($I_1 = X_1$) as well, which is then equivalent to the complex just being a 2-dimensional hypergroupoid (in the abstract

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simplices (by homotopies which leave their boundaries fixed), defining faces and degeneracies as in $\mathrm{Sing}(X)$ in dimensions below n, and then taking $\Pi_n(X)$ as the image of $\mathrm{Sing}(X)$ in the n-coskeleton of the just defined n-truncated complex . Although it is not entirely trivial to prove it, this construction gives an n-dimensional hypergroupoid $\Pi_n(X_\bullet)$ for any Kan complex $X_\bullet$, all of whose homotopy groups are identical to those of $X_\bullet$ in dimensions $\leq n$, but are trivial an all higher dimensions (for any basepoint). $\Pi_1(X)$ is just the (nerve of) the "fundamental groupoid" of the space X.

[5] Although not identical to those used in this paper, a very nice and quite concise detailed summary of definitions may be found in [Leinster 1998].

[6] (together with the development of some, I think neat, simplicial-matrix notation and techniques for using it (originally invented by Paul Glenn [Glenn 1982]) which made the proofs discoverable as well as comprehensible)

simplicial sense), one obtains those bicategories in which all 2-cells are isomorphisms and all 1-cells are equivalences ("bigroupoids"), so the term "2-dimensional hypergroupoid", chosen in the early 1980's seems, presciently, to indeed have been appropriate after all. Indeed, this is exactly the type of the complex $\Pi_2(X)$ which is obtained when one takes the singular complex of a topological space and replaces the set of 2-simplices with the set of (boundary-fixing) homotopy classes of 2-simplices. The corresponding bigroupoid is the "fundamental bigroupoid" of the space which has been extensively studied by I. Moerdijk [Moerdijk 1990], R. Kieboom and colleagues [Hardie *et al.* 2000], *et seq.*, and, in the more abstract context of the low dimensional classical exact sequences of homological algebra, by Enrico Vitale and colleagues [Vitale 1999], *et seq.*, as well as numerous other researchers cited in the *References* and elsewhere.

The author has attempted to write this paper so that parts of it can be read somewhat independently of the others. If the attentive reader finds certain passages overly repetitious, the author begs that reader's indulgence. In outline, it will proceed as follows:

Section 2 recalls the basic terminology of simplicial sets (classically, "complete semi-simplicial complexes") and the intuitive geometric underpinning for them which we make use of in low dimensions. Most of this is "well-known" and can be found in such classics as [May 1967] or, in a more modern treatment, in [Goerss-Jardine 1999]. However, we also define here certain not-so-well-known endo-functors on simplicial sets such as the n-Coskeleton ($\mathbf{Cosk}^n$), Shift ($\mathbf{Dec}_+$), and Path-Homotopy ($\mathbb{P}$), functors, which are extensively used in this paper and its sequents. In particular, we explain what we will be our "basic simplicial setting in dimension n". Namely, that we wish to restrict our attention to simplicial complexes which are (isomorphic to) subcomplexes of their n-Coskeleta and which are themselves $(n + 1)$-Coskeletal. Complexes which are in this setting will be called *n-dimensional Postnikov complexes* since it is this property which is satisfied by the n^{th}-complex $X_\bullet^{(n)}$ in the natural Postnikov tower of a Kan complex and by the n-dimensional (Kan) hypergroupoids of Glenn [Glenn 1982]. This property is also satisfied by non-Kan complexes such as the nerves of categories (where $n = 1$ and the 2-simplices are the "commutative triangles" of the category) and in the nerve of bicategory (where $n = 2$ and the 3-simplices are the "commutative tetrahedra" of the bicategory), but in this case satisfaction is a consequence of the definition of this nerve, and not in any immediate sense a consequence of the very restricted and partial horn conditions which are satisfied here. More importantly, this "basic setting" is the key which (after a slight modification) allows us to take advantage of Paul Glenn's simplicial matrix techniques [Glenn 1982]. As we hope to show in this paper and its sequents, these techniques take on an intuitive "life of their own" with their extraordinary utility lying in the fact that they allow us to conveniently construct proofs which, without them, would require virtually incomprehensible diagram chases.

Section 3 and Section 4 explore this "basic simplicial setting" for $n = 0$ (where sets, partially ordered sets, and equivalence relations appear) and for $n = 1$, to illustrate how these techniques may be used to simplicially capture the nerves of categories and groupoids. This sets the stage for $n = 2$ and beyond.

In Section 5, for completeness and avoidance of confusion, we will give the precise sets of axioms used *in this paper* as the definition of a bicategory.[7] All of the definitions used here may be viewed as slight, even sometimes redundant, expansions of the original ones of Bénabou [Bénabou 1967] but with an occasional apparently perverse reversal of source and target for the canonical isomorphisms which appear there. The changes are mainly for simplicial convenience and a desire to preserve consistency with an overall "odd to even" orientation of the interiors of the simplices which is already (inconspicuously) present in Grothendieck's definition of the nerve a category. In any case, these choices may be easily changed to the more familiar ones by the inclined reader who finds these changes counterintuitive.[8]

In Section 6 we *explicitly* define the sets of simplices (together with the face and degeneracy maps) of the 3-truncated complex whose coskeletal completion will be the simplicial set $\mathbf{Ner}(\mathbb{B})$ associated with a bicategory $\mathbb{B}$. At each appropriate level we explore the "restricted (and "partial") horn conditions" which $\mathbf{Ner}(\mathbb{B})$ satisfies and which turn out to be non-trivial through dimension 4. Although like Grothendieck's nerve of a category, this nerve could also be described in its entirety in a more elegant implicit fashion using the sets $[n]$ of Δ as trivial 2-categories, we have not done that here since we feel quite strongly that this would initially only hide the properties that are entirely evident in the explicit (but unsophisticated) description.

In Section 7 we give the most delicate part of the paper which provides the promised "internal" characterization of those simplicial sets which are the nerves of bicategories. The "basic simplicial setting for $n = 2$" (a 2-dimensional Postnikov complex) allows us to identify the 3-simplices of a complex in this setting with certain families (x_0, x_1, x_2, x_3) of 2-simplices which join together geometrically as the faces of a tetrahedron. We indicate those families which are in X_3 by using square brackets around the family and call them "commutative" to distinguish them from arbitrary elements of the simplicial kernel $(\mathbf{Cosk}^2(X_\bullet)_3)$ indicated by round brackets. Similarly, this setting allows us to identify the elements of X_4 with 5×4 simplicial matrices, each of whose five rows (its faces) are commutative. Using the canonical horn maps, we then define certain subsets $I_2(X_\bullet) \subseteq X_2$ and (later) $I_1(X_\bullet) \subseteq X_1$ of (formally) "invertible" 2 and 1-simplices as those which play the roles of the 2 and 1-simplices whose 2 and 1-cell interiors are, respectively, isomorphisms and (finally) equivalences in the nerve of a bicategory as previously defined in Section 6. We show that (a) If all degenerate 2-simplices are invertible, *i.e.,* $s_0(x_{01})$ and $s_1(x_{01}) \in I_2(X_\bullet)$, then the path-homotopy complex $\mathbb{P}(X_\bullet)$ of $X_\bullet$ is the nerve of a category and (b) If, in addition, the 1-horn map $\mathrm{pr}_{\hat{1}} : X_2 \longrightarrow \bigwedge_2^1(X_\bullet)$ is surjective with a section χ_2^1 whose image in X_2 consists of invertibles (*i.e.,* $\chi_2^1(x_{12}, x_{01}) \in I_2(X_\bullet)$), then $\mathbb{P}(X_\bullet)$ is

[7]The corresponding modified definitions of morphisms of bicategories, transformations of morphisms of bicategories, and modifications of transformations of morphisms of bicategories (as well as certain specializations of these such as homomorphisms and unitary morphisms,*etc.*) will be given in Part II of this paper [Duskin 2002(a)].

[8]Readers who are unfamiliar with the simplicial terminology which occurs throughout this paper will find many of the relevant definitions collected together in Section 2 (or in the references cited there) and may wish to look at this section before they read the remainder of the paper.

the underlying category of 2-cells of a bicategory $\mathbf{Bic}(X_\bullet)$ whose tensor product of 1-cells is given by the 1-face of χ_2^1: $d_1(\chi_2^1(x_{12}, x_{01})) =_{(Def)} x_{12} \otimes x_{01}$, and whose 3-simplices are exactly the "commutative tetrahedra" of the nerve of $\mathbf{Bic}(X_\bullet)$ as defined in Section 6.

The conclusions of Section 6 and Section 7 are summarized in the concluding Section 8 of this paper (Part I) where we show that the two constructions, **Ner** and **Bic** are mutually inverse in a very precise sense: Theorems 8.1 and 8.5. Finally, the relation of our notions of invertibility to Kan's horn-filling conditions is included in the characterizing Theorem 8.6, where we note, in particular, that $\mathbf{Bic}(X_\bullet)$ is a bigroupoid *iff* $I_2(X_\bullet) = X_2$ and $I_1(X_\bullet) = X_1$ (which, in turn, is equivalent to a $X_\bullet$'s being a Kan complex in which the horn maps are all bijections in dimensions > 2, *i.e.,* $X_\bullet$ is a 2-dimensional (Kan) hypergroupoid in the terminology of [Glenn 1982]).[9]

In Part II [Duskin 2002(a)] of this series of papers we extend the above characterization to include Bénabou's morphisms, transformations, and modifications. Morphisms of bicategories always define face maps between the nerves but these face maps preserve the degeneracies only up to a specified compatible 2-cell (which is the identity *iff* the morphism is strictly unitary). For such special face-preserving mappings of simplicial sets we revive an old terminology in a new guise and refer to them as being "*semi*-simplicial". Semi-simplicial maps between nerves always bijectively define morphisms of the corresponding bicategories with ordinary (face and degeneracy preserving) simplicial maps corresponding to strictly unitary morphisms of bicategories. Similarly, after making some conventions (which are unnecessary in the case of bigroupoids), transformations of morphisms correspond exactly to homotopies of semi-simplicial maps and modifications to special "level 2 homotopies of homotopies" which lead to a bicategorical structure which has the full (semi-)simplicial system as its nerve. Composition, however, is only well behaved for nerves of Bénabou's "*homo*morphisms" which preserve invertiblity. Given the nature of this "strong embedding" of bicategories into Simplicial Sets and Semi-Simplicial mappings, it would seem that this Nerve can play virtually the same role for bicategories as that of Grothendieck does for categories. We note also that the use of *semi*-simplicial mappings is unnecessary in the topological case with 2-dimensional hypergroupoids such as $\Pi_2(X)$. Since all 2 and 1-simplices are invertible there, *any* section of $\mathrm{pr}_{\hat{1}} : X_2 \longrightarrow \bigwedge_2^1(X_\bullet)$ will do, and it is trivial to even choose there a "normalized" one (and similarly for homotopies of semi-simplicial maps). Ordinary simplicial maps and homotopies will do. No restrictions or conventions are necessary: the embedding can be taken to be directly within the cartesian closed category of simplicial sets.

It is interesting to conjecture that most, if not all, of the technicalities that arise from having to use "semi"-simplicial rather than simplicial mappings (which arise from Bénabou's insistence on *not* requiring that his morphisms be strictly unitary) can be avoided in the non-topological case as well, at least indirectly : as a functor on simplicial sets, it may be possible that the semi-simplicial maps are representable by a bicategory

[9] $X_\bullet$ is the nerve of a 2-category *iff* the section χ_2^1 has two additional compatibility properties.

whose 0-cells are the (co-, in our orientation) monads of the given bicategory:

$$\mathbf{Mor}(\mathbb{A},\mathbb{B})\xrightarrow{\sim}\mathrm{SemiSimpl}(\mathbf{Ner}(\mathbb{A}),\mathbf{Ner}(\mathbb{B}))\xrightarrow{\sim}\mathrm{Simpl}(\mathbf{Ner}(\mathbb{A}),\mathbf{Ner}(\mathbf{CoMon}(\mathbb{B})).$$

Of course, one of the major motivations that caused Bénabou to insist that morphisms should not be *required* to be even unitary, much less, strictly unitary was to elegantly capture monads (as morphisms from [0]). In any case, the requirement is no more than a technical bother, and in the case of topological spaces with $\Pi_2(X)$, it can be avoided entirely.

In an *Appendix* to Part II we will relate what we have done for $n = 2$ to the multi-simplicial set approach of Simpson-Tamsamani ([Tamsamani 1995], [Simpson 1997] and Leinster [Leinster 2002]. There we will show that the simplicial set $\mathbf{Ner}(\mathbb{B})$ forms the simplicial set of objects of a simplicial category object $\mathbf{Ner}(\mathbb{B})_{\bullet\bullet}$ in simplicial sets which may be pictured by saying that each of the sets of n-simplices of $\mathbf{Ner}(\mathbb{B})$ forms the set of objects of a natural category structure in which the face and degeneracy maps become functorial: When $\mathbf{Ner}(\mathbb{B})$ is viewed as a 2-dimensional Postnikov complex $X_\bullet$, the 0-simplices become the nerve of the discrete category $X_{0\bullet} = K(X_0, 0)$; the 1-simplices become the objects of the category whose nerve $X_{1\bullet}$ is isomorphic to the path-homotopy complex $\mathbb{P}(X_\bullet)$; the 2-simplices the objects of the category whose arrows are the homotopies of 2-simplices which leave fixed the vertices of the 2-simplex source and target of the arrow and combinatorially just consist of $\Delta[2] \times \Delta[1]$ prisms decomposed into three commutative tetrahedra of $X_\bullet$, and the pattern is similar in all higher dimensions, just homotopies of n-simplices which leave fixed the vertices of the simplex.If one forms the iterated fiber product categories $\mathbb{P}(X_\bullet) \times_{K(X_0,0)} \mathbb{P}(X_\bullet) \times \ldots \times_{K(X_0,0)} \mathbb{P}(X_\bullet)$ (n times) one obtains the categories of sequences of n-fold *vertically* composable 2-cells of the bicategory $\mathbf{Bic}(X_\bullet)$ and the face functors define a canonical sequence of functors

$$X_{n\bullet} \longrightarrow \mathbb{P}(X_\bullet) \times_{K(X_0,0)} \mathbb{P}(X_\bullet) \times \ldots \times_{K(X_0,0)} \mathbb{P}(X_\bullet).$$

Each of these functors admits a right (in our orientation) adjoint section which is an equivalence (so that we have a *Segal category* in Simpson's terminology) precisely when every 2-cell of the bicategory is an isomorphism.Strikingly, each of these adjoint sections involves in its construction the defining 2 and 3-simplices which we have used in the constructions in Section 7 which lead to the proof of our characterization theorem. For example, the commutative 3-simplex which defines the 2-cell *interior* of a 2-simplex (7.15) is also the essential one in the prism which defines the counit for the adjoint pair for $X_{2\bullet} \longrightarrow \mathbb{P}(X_\bullet) \times_{K(X_0,0)} \mathbb{P}(X_\bullet)$ whose right adjoint is defined by the tensor product of 2-cells, and the Mac Lane-Stasheff pentagon occurs in the immediately succeeding pair. The sequence of simplicial categories used by Tamsamani is not the same sequence described here but are categorically equivalent ones, but only if one restricts entirely to groupoids. Nevertheless, the Simpson-Tamsamani approach would appear to be modifiable and offer an alternative to our approach to weak n-categories,although it is not clear that it would necessarily be any simpler than what we are proposing.

Part III [Duskin 2002(b)] will give our comparable simplicial set $\mathbf{Ner}(\mathbb{T})$ characterization for tricategories, essentially as they are defined in [Gordon *et al.*, 1995] and Part IV [Duskin 2002(c)] will give our proposed simplicial definitions for weak ω and n-categories.

In essence, our thesis in this entire series of papers is that the abstract characterization of those n-dimensional Postnikov complexes $X_\bullet$ which are the nerves of weak n-categories is made possible through the use of certain very restricted horn lifting conditions in dimensions $\leq n+1$, along with the uniqueness which occurs in dimension $n+1$, to *define* the algebraic structure involved, and then use the unique horn lifting conditions of dimension $n+2$ (in the form of a manipulation of simplicial matrices) to verify the equations satisfied by the so defined structure. (The nerve of the underlying weak $(n-1)$-category is automatically supplied by the path-homotopy complex $\mathbb{P}(X_\bullet)$, which just shifts everything down by one dimension by using only those simplices of $X_\bullet$ whose "last face" is totally degenerate.)

In contrast to the fact that the direct equational (enriched in $(n-1)$-Cat) description of these "weak higher dimensional categories" (even in "pasting diagram" shorthand) becomes increasingly complex and more difficult to grasp as n increases,[10] the *entirely equivalent* abstract "special n-dimensional Postnikov complex" description of their nerves does not. The simplicial matrices used for calculation grow larger and the list of abstractly invertible faces required for horn lifting grows more extensive, but the simplicial form and the nature of the "weakening" as one passes from dimension n to dimension $n+1$ is clear and remarkably regular. It is our hope that once we have described the nerve for $n = 0$, 1, and 2 both concretely and abstractly in this paper and have done the same for $n = 3$ (tricategories) in [Duskin 2002(b)], the reader will be able to consider the abstract simplicial characterization [Duskin 2002(c)] as a satisfactory *definition by total replacement* for whatever should be the proper definition of a "weak n-category". Apparently, all that such gadgets should be is encoded there and, in theory, could be *decoded* from there as well, given the time and the patience.

2. Simplicial Sets, their Coskeleta, and Simplicial Matrices

2.1. Simplicial Sets, Simplicial Kernels, and Horns. Recall that, by definition, a *simplicial set* or *(simplicial) complex*[11] is just a contravariant set-valued functor on the (skeletal) category Δ of (non-empty) finite totally ordered sets and non-decreasing mappings and that the *category of simplicial sets*, $\mathbf{Simpl}(\mathbf{Sets})$, is just the corresponding category of such functors and their natural transformations. If $X_\bullet$ is such a functor, its value X_n at the object $[n] = \{0 < 1 < \cdots < i < \cdots < n\}$ is called the set of (dimension) *n-simplices* of $X_\bullet$. The contravariant representable functor defined by the object $[n]$ is called the *standard n-simplex* and is usually denoted by $\Delta[n]$. The Yoneda embedding gives

$$\mathrm{Hom}_{\mathbf{Simpl(Sets)}}(\Delta[n], X_\bullet) \cong X_n,$$

[10](*cf.* [Gordon *et al.*, 1995])

[11]Originally called a *complete semi-simplicial complex*

$$\begin{tikzcd}[column sep=large]
\cdots\ X_{n+1} \arrow[r, shift left=3, "d_{n+1}"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=3, "d_0"'] & X_n \arrow[l, bend right=40, "s_0"'] \arrow[l, bend right=70, "s_n"'] \arrow[r, shift left=3, "d_n"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=3, "d_0"'] & X_{n-1} \arrow[l, bend right=40, "s_0"'] \arrow[l, bend right=70, "s_{n-1}"'] \ \cdots & X_2 \arrow[r, shift left=3, "d_2"] \arrow[r, "d_1"] \arrow[r, shift right=3, "d_0"'] & X_1 \arrow[l, bend right=40, "s_0"'] \arrow[l, bend right=60, "s_1"'] \arrow[r, shift left=2, "d_1"] \arrow[r, shift right=2, "d_0"'] & X_0 \arrow[l, bend right=40, "s_0"']
\end{tikzcd}$$

Figure 1: Generic Simplicial Complex as a Graded Set with Face and Degeneracy Operators

so that the elements of X_n can be depicted geometrically as points or "vertices" ($n = 0$), directed line segments ($n = 1$), solid triangles with directed edges ($n = 2$),. . . , *etc.*, each with $n + 1$, $(n - 1)$-dimensional *faces*, the images of the n-simplex under the *face maps* (or *face operators*) $d_i^n : X_n \longrightarrow X_{n-1}$, $(0 \le i \le n)$ each of which is determined by the injective map $\partial_i : [n - 1] \longrightarrow [n]$ of Δ which omits i in the image (and thus numbers the face $d_i^n(x)$ as being "opposite the vertex x_i"). In addition one has $n + 1$ *degeneracy maps* (or *degeneracy operators*) $s_i^n : X_n \longrightarrow X_{n+1}$, $(0 \le i \le n)$, each determined by the surjective map $\sigma_i : \Delta[n - 1] \longrightarrow \Delta[n]$ of Δ which repeats i . The $(n + 1)$-simplices of the form $s_i^n(x)$ for some n-simplex x are said to be *degenerate* with those of the form $s_0(s_0(\ldots s_0(x_0))\ldots)$ for some 0-simplex x_0 said to be *totally degenerate.*

Viewed as a graded set supplied with the above *generating* face and degeneracy operators, a simplicial complex in low dimensions is often diagrammatically pictured as in Figure 1. The superscript on these generating maps is usually omitted in the text (if the source and target are clear) and *the notation s_i^n and d_i^n is then reserved to indicate the n-fold composition of the successive operators all indexed by the same i.*

The *simplicial face identities*,

$$d_i^{n-1}(d_j^n(x)) = d_{j-1}^{n-1}(d_i^n(x)) \quad (0 \le i < j \le n),$$

which hold for any n-simplex x and follow from the corresponding properties of the generators of Δ, just express the geometry of how the numbered geometric faces of an n-simplex must fit together for it to be a directed line segment, triangle , tetrahedron or their higher dimensional analogs. This means that if we denote a typical n-simplex by $x_{012\ldots n}$, then we can use the geometric arrow notation $x_0 \xrightarrow{x_{01}} x_1$ or $x_{01} : x_0 \longrightarrow x_1$ to indicate that x_{01} is a 1-simplex with $d_0(x_{01}) = x_1$ and $d_1(x_{01}) = x_0$.[12]

Similarly, we can use the geometric notation displayed in Figure 2 to indicate that x_{012} is a 2-simplex with $d_0(x_{012}) = x_{12}$, $d_1(x_{012}) = x_{02}$, and $d_2(x_{012}) = x_{01}$. The simplicial face identities assert that $d_1(x_{02}) = x_0 = d_1(x_{01})$, $d_0(x_{01}) = x_1 = d_1(x_{12})$, and $d_0(x_{12}) = x_2 = d_0(x_{02})$, *i.e.,* that the directed edge faces of a 2-simplex and the vertex faces of these directed edges fit together as the boundary of a two dimensional solid triangle.

In dimension three, a typical 3-simplex may be geometrically indicated by a solid tetrahedron as in Figure 3. Its four 2-simplex faces $d_i(x_{0123}) = x_{01\hat{i}3}$ are each opposite the

[12]This is *not* a misprint since the numbering of faces is by "face d_i opposite vertex i". As a directed graph d_0^1 = Target Map and d_1^1 = Source Map. This is the only dimension where the notation seems counterintuitive to people who read from left to right. In any case, this "geometric" notation is of limited usefulness in dimensions much above 3.

$$\begin{tikzcd} x_0 \arrow[r, "x_{01}"] \arrow[dr, "x_{02}"'] \arrow[dr, phantom, "x_{012}", shift left=3ex] & x_1 \arrow[d, "x_{12}"] \\ & x_2 \end{tikzcd}$$

Figure 2: Geometric Notation for a Typical 2-Simplex $x_{012} \in X_2$

$$\begin{tikzcd} & x_0 \arrow[dl, "x_{03}"'] \arrow[dr, "x_{01}"] \arrow[dd] & \\ x_3 & & x_1 \arrow[ll] \arrow[dl, "x_{12}"] \\ & x_2 \arrow[ul, "x_{23}"] & \end{tikzcd}$$

Figure 3: Geometric Notation for a Typical 3-simplex $x_{0123} \in X_3$

vertex x_i, $(0 \leq i \leq 3)$ in the figure with the simplicial face identities forcing these 2-simplex faces to fit together to form the surface which is the boundary of the solid tetrahedron as shown. For any n-simplex x, the pattern of the face identities is that which when the k^{th} term in the vertex sequence is omitted. For instance, for any 3-simplex

$$d_0(d_0(0123)) = d_0(123) = (23) = d_0(023) = d_0(d_1(0123))$$

A possible model and notation for a generic 4-simplex x_{01234} is indicated in Figure 4. Think here of the vertex x_4 as sitting at the barycenter of the tetrahedron formed by the complementary vertices. The faces d_0, d_1,d_2, and d_3 are the four solid tetrahedra "inside" the face d_4, which is the solid "outside" tetrahedron. These are the five three dimensional solid faces projected from the solid four dimensional 4-simplex. Clearly, both as notation and model, such geometric pictures have little utility except in low dimensions, but do serve to give an intuitive idea of the concept of a simplicial complex and its geometric nature.

The *degeneracy maps* and their images in X_{n+1} $(n \geq 0)$, the degenerate $(n+1)$-simplices, satisfy the *simplicial degeneracy identities*:

$$\begin{aligned} s_i^{n+1} s_j^n &= s_{j+1}^{n+1} s_i^n \ (0 \leq i \leq j \leq n) \\ &\text{and} \\ d_i^{n+1} s_j^n &= s_{j-1}^{n+1} d_i^n \quad (i < j) \\ d_i^{n+1} s_j^n &= \mathrm{id}(X_n) \quad (i = j \text{ or } i = j+1) \\ d_i^{n+1} s_j^n &= s_j^{n-1} d_{i-1}^n \quad (i > j+1). \end{aligned}$$

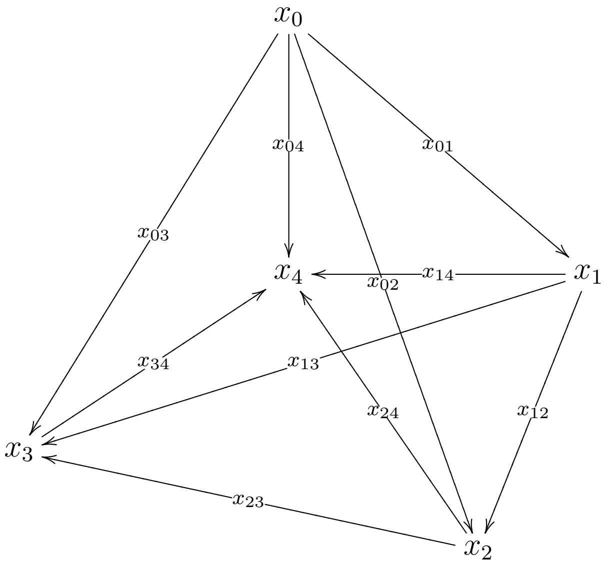

Figure 4: One possible 4-Simplex Model

$$
\begin{tikzcd}
x_0 \arrow[r, "s_0(x_0)"] \arrow[dr, "x_{01}"'] \arrow[dr, phantom, shift left=3, "s_0(x_{01})"] & x_0 \arrow[d, "x_{01}"] \\
& x_1
\end{tikzcd}
\quad
\begin{tikzcd}
x_0 \arrow[r, "x_{01}"] \arrow[dr, "x_{01}"'] \arrow[dr, phantom, shift left=3, "s_1(x_{01})"] & x_1 \arrow[d, "s_0(x_1)"] \\
& x_1
\end{tikzcd}
\quad
\begin{tikzcd}
x_0 \arrow[r, "s_0(x_0)"] \arrow[dr, "s_0(x_0)"'] \arrow[dr, phantom, shift left=3, "s_0^2(x_0)"] & x_0 \arrow[d, "s_0(x_0)"] \\
& x_0
\end{tikzcd}
$$

Figure 5: Geometric Notation for the Degeneracies $s_0(x_{01})$, $s_1(x_{01})$, and $s_0^2(x_0)$

In terms of the geometric notation above, for any 0-simplex x_0 $s_0(x_0)$ has the form of a "distinguished loop" $x_0 \xrightarrow{s_0(x_0)} x_0$. For any 1-simplex $x_0 \xrightarrow{x_{01}} x_1$ (including $s_0(x_0)$), $s_0(x_{01})$, $s_1(x_{01})$, and $s_1(s_0(x_0)) = s_0(s_0(x_0)) = s_0^2(x_0)$ have the form given in Figure 5.

For any n-simplex x, the degenerate $n{+}1$-simplex $s_i(x)$ has x as its i^{th} and $(i{+}1)^{st}$ face. All other faces are the appropriate degeneracies associated with the faces of x following the pattern $(01 \ldots ii \ldots n)$ which repeats the i^{th} vertex. For instance, for a 3-simplex:

$$
\begin{aligned}
d_0(s_2(0123)) &= d_0(01223) = (1223) = s_1(123) = s_1(d_0(0123)) \\
d_1(s_2(0123)) &= d_1(01223) = (0223) = s_1(023) = s_1(d_1(0123)) \\
d_2(s_2(0123)) &= d_2(01223) = (0123) \\
d_3(s_2(0123)) &= d_3(01223) = (0123) \\
d_4(s_2(0123)) &= d_4(01223) = (0122) = s_2(012) = s_2(d_3(0123))
\end{aligned}
$$

If $x \in X_n$ is an n-simplex, its $(n-1)$-*face boundary* $\partial_n(x)$ is just its sequence of

$(n-1)$-faces

$$\partial_n(x) = (d_0(x), d_1(x), \ldots, d_n(x))$$

and its *k-horn* (or "open simplicial k-box"), $\mathrm{pr}_{\hat{k}}(x)$, is just the image of the projection of its boundary to this same sequence of faces, but with the k^{th}-face omitted ,

$$\mathrm{pr}_{\hat{k}}(x) = (d_0(x), d_1(x), \ldots, d_{k-1}, \_\,, d_{k+1}(x), \ldots, d_n(x)).$$

The set $\mathrm{SimKer}(X]_0^{n-1})$, denoted by $K_n(X_\bullet)$ for brevity, of all possible sequences of $(n-1)$-simplices which could possibly be the boundary of any n-simplex is called the *simplicial kernel* of the complex in dimension n,

$$K_n(X_\bullet) =$$

$$\{(x_0, x_1, \ldots, x_{k-1}, x_k, x_{k+1}, \ldots, x_n) |\, d_i(x_j) = d_{j-1}(x_i),\ i < j\}$$
$$\subseteq X_{n-1}^{n+1}\ (CartesianProduct),$$

and the set $\bigwedge_n^k(X_\bullet)$ of all possible sequences of $(n-1)$-simplices which could possibly be the boundary of an n-simplex, except that the k^{th} one is missing, is called the *set of k-horns in dimension n.*

$$\textstyle\bigwedge_n^k(X_\bullet) =$$
$$\{(x_0, x_1, \ldots, x_{k-1}, \_\,, x_{k+1}, \ldots, x_n) |\, d_i(x_j) = d_{j-1}(x_i), i < j, i, j \neq k\}$$
$$\subseteq X_{n-1}^{n}\ (CartesianProduct).$$

For a complex $X_\bullet$ the *k-horn map in dimension n*,

$$\mathrm{pr}_{\hat{k}} : X_n \longrightarrow \textstyle\bigwedge_n^k(X_\bullet),$$

is the composition of the *boundary map*, $\partial_n : X_n \longrightarrow K_n$, and the projection

$$pr_{\hat{k}} : K_n \longrightarrow \textstyle\bigwedge_n^k$$

which omits the k^{th} $(n-1)$-simplex from the sequence. Given a k-horn in dimension n

$$(x_0, x_1, \ldots, x_{k-1}, \_\,, x_{k+1}, \ldots, x_n) \in \textstyle\bigwedge_n^k$$

if there exists an n-simplex x in $X_\bullet$ such that

$$\mathrm{pr}_{\hat{k}}(x) = (x_0, x_1, \ldots, x_{k-1}, \_\,, x_{k+1}, \ldots, x_n),$$

then the horn is said to *lift* to x. Such an n-simplex x said to be an n-simplex *filler* or *lift* for the k-horn. $d_k(x)$ then also "fills the missing k-face" of the horn. The k^{th}*-Kan (horn filling or horn lifting) condition in dimension n* is the requirement that the k-horn map in dimension n be surjective. The condition is satisfied *exactly* if $\mathrm{pr}_{\hat{k}}$ is injective as

$$\begin{tikzcd} \bigwedge^k[n] \arrow[r, "h"] \arrow[d, phantom, "\bigcap"] & X_\bullet \\ \Delta[n] \arrow[ur, dotted, "x"'] & \end{tikzcd}$$

Figure 6: Kan Condition as Injectivity of $X_\bullet$

well, *i.e.,* if $\mathrm{pr}_{\hat{k}}$ is a bijection.[13] Many people prefer to state this condition entirely in topos-theoretic terms using the fact that the set maps

$$X_n \xrightarrow{\partial_n} K_n \xrightarrow{pr_{\hat{k}}} \bigwedge_n^k$$

and the composite $X_n \xrightarrow{\mathrm{pr}_k} \bigwedge_n^k$ are all (co-)representable in the topos of simplicial sets by the sieves

$$\bigwedge^k[n] \subset \overset{\bullet}{\Delta}[n] \subset \Delta[n],$$

since "homing" this sequence into $X_\bullet$,

$$\begin{tikzcd} \mathrm{Hom}_{\mathbf{Simpl(Set)}}(\Delta[n], X_\bullet) \arrow[r] \arrow[dr] & \mathrm{Hom}_{\mathbf{Simpl(Set)}}(\overset{\bullet}{\Delta}[n], X_\bullet) \arrow[d] \\ & \mathrm{Hom}_{\mathbf{Simpl(Set)}}(\bigwedge^k[n], X_\bullet) \end{tikzcd}$$

is in natural bijection with the above sequence.

Here $\Delta[n]$ is the *standard n-simplex* (the contravariant representable on $[n]$) , $\overset{\bullet}{\Delta}[n]$ is the *boundary of the n-simplex* (the so called $(n-1)$-*Skeleton* of $\Delta[n]$ which is identical to $\Delta[n]$ in all dimensions $\leq n-1$ but has only degenerate simplices in higher dimensions), and $\bigwedge^k[n]$ is the *k-horn of the n-simplex* (identical to $\overset{\bullet}{\Delta}[n]$, except that $\delta_k : \Delta[n-1] \longrightarrow \Delta[n]$ is missing in $\Delta[n]_{n-1}$). Thus the Kan condition for the k-horn in dimension n becomes: For any simplicial map $h : \bigwedge^k[n] \longrightarrow X_\bullet$ there exists a simplicial map $x : \Delta[n] \longrightarrow X_\bullet$ which makes the diagram in Figure 6 commutative (hence the term "lifting" to an n-simplex x for the horn h). In categorical terms, this just says that the object $X_\bullet$ is *injective* with respect to the monomorphism $\bigwedge^k[n] \subseteq \Delta[n]$.

A complex is said to *satisfy*, respectively, *satisfy exactly*, the *Kan condition in dimension n* if all of the n+1 k-horn maps are surjective, respectively, bijective. A complex which satisfies the Kan condition in every dimension $n \geq 0$ is called a *Kan complex*.[14] In

[13]Note that if $X_\bullet$ satisfies the k^{th}-Kan condition exactly in dimension n (for any k, $0 \leq k \leq n$), the boundary map $\partial : X_n \longrightarrow K_n$ must be injective, so it is only with an injective boundary map that this condition could occur.

[14]This surjectivity is the case, for example, in the singular complex $\mathrm{Sing}(X)$ of a topological space. Quite remarkably, Kan showed that these "Kan conditions" are all that is needed to define all homotopy groups of the space in all dimensions at any base point.

categorical terms they are those objects of **Simpl**(**Set**) which are injective with respect to all of the monomorphisms $\bigwedge^k[n] \subseteq \Delta[n]$ for all $n \geq 0$ and all k for which $0 \leq k \leq n$.

Following an elegant observation of Steven Schanuel, Kan complexes which satisfy the Kan conditions exactly for all $n > m$ have been called *m-dimensional (Kan) hypergroupoids*[15] by the author and Paul Glenn ([Glenn 1982]) who pioneered the matrix methods which we will make so much use of in this and subsequent papers.[16] Although we will reserve the proof to a later point (where it will appear as a corollary to a more general result), if one takes a Kan complex $X_\bullet$ and constructs a new complex in which its set of m-simplices is replaced with the set of equivalence classes of m-simplices under the equivalence relation which identifies m-simplices if they are homotopic by a homotopy which is constant on their boundaries,[17] the complex $\Pi_m(X_\bullet)$ so obtained is exactly an m-dimensional hypergroupoid. For example, $\Pi_0(X_\bullet)$ is the constant complex $K(\pi_0(X_\bullet), 0)$ of connected components of $X_\bullet$ and $\Pi_1(X_\bullet)$ is its *fundamental groupoid*. From the point of view of algebraic topology the interest of these hypergroupoids lies in the way that they model the "homotopy m-type" of any topological space. A corollary of the results of the work in this sequence of papers will show that these m-dimensional hypergroupoids are exactly the simplicial complexes that are associated with a certain essentially algebraic structure: For $0 \leq m \leq 3$ they are precisely the nerves of (weak) m-groupoids. It will be our contention that Kan complexes are the nerves of (weak) ω-groupoids.

Finally, a terminological caution about the term "weak". If the Kan condition in dimension n is satisfied for all k except possibly for the "extremal" maps $\mathrm{pr}_{\hat{0}}$ and $\mathrm{pr}_{\hat{n}}$, then the corresponding condition is called the *weak* Kan condition in dimension n. A complex which satisfies the weak Kan condition for all n is called a *weak Kan complex.*

What we seek to first establish,however, is a more general result which would characterize those simplicial sets which are the nerves of (weak) *n-categories*, at least in so far as that notion has been satisfactorily axiomatized in the literature ($0 \leq n \leq 3$). For this it is necessary that Kan's *horn lifting* conditions be considerably weakened beyond just leaving out the conditions for the "extremal horns". This will be done by allowing only certain "admissible" members of the full set of horns $\bigwedge^k_n(X_\bullet)$ to enjoy the lifting property. Moreover, the term *weak* as it appears here may be misleading: A weak Kan complex in which the weak Kan conditions are satisfied exactly in all dimensions > 1 is the nerve of a category, but a weak Kan complex in which the weak Kan conditions are satisfied exactly in all dimensions > 2 is the nerve of a bicategory in which *all of the 2-cells are*

[15]These complexes are always in our "basic simplicial setting" of m-dimensional Postnikov complexes (Definition 2.5).

[16]For categorical reasons Glenn did not *explicitly* require in his definition that the Kan conditions in dimensions $\leq m$ be satisfied, although it is clear that for sets this should be assumed.

[17]For n-simplices x and y in $X_\bullet$, a *(directed, boundary fixing) homotopy* $h_0 : x \rightharpoonup y$ is an $(n+1)$-simplex $h_0(y, x)$ in $X_\bullet$ whose boundary has the form

$$\partial(h_0(y, x)) = (y, x, s_0(d_1(x)), s_0(d_2(x)), \ldots, s_0(d_n(x))).$$

The existence of such a homotopy forces $\partial(x) = \partial(y)$. The degeneracy $s_0(x)$ is such a homotopy $s_0(x) : x \rightharpoonup x$.

isomorphisms, rather than the nerve of an arbitrary *weak* 2-*category* if, by that term, as is usual, we mean a bicategory in the sense of Bénabou.

2.2. Coskeleta of Simplicial Sets. If we consider the full subcategory of Δ whose objects are the totally ordered sets $\{[0],[1],\ldots,[n]\}$, then the restriction of a simplicial set to this full subcategory is the *truncation at level n functor* $tr^n(X_\bullet) = X_\bullet]_0^n$. It has adjoints on both sides: the left adjoint is called the *n-skeleton*, and the right adjoint the *n-coskeleton*.[18] The composite endofunctors on simplicial sets (denoted with capital letters: $\mathbf{Sk}^n(X_\bullet) = sk^n(tr^n(X_\bullet))$ and $\mathbf{Cosk}^n(X_\bullet) = sk^n(tr^n(X_\bullet)))$ are also adjoint with

$$\mathrm{Hom}_{\mathbf{Simpl(Sets)}}(\mathbf{Sk}^n(Y_\bullet), X_\bullet) \cong \mathrm{Hom}_{\mathbf{Simpl(Sets)}}(Y_\bullet, \mathbf{Cosk}^n(X_\bullet)).$$

This leads to

$$\mathrm{Hom}_{\mathbf{Simpl(Sets)}}(\mathbf{Sk}^n(\Delta[q]), X_\bullet) \cong \mathrm{Hom}_{\mathbf{Simpl(Sets)}}(\Delta[q], \mathbf{Cosk}^n(X_\bullet)).$$

Now $\mathbf{Sk}^n(Y_\bullet)$ is easily described: it is the subcomplex of $Y_\bullet$ which is identical to $Y_\bullet$ in all dimensions $\leq n$ but only has degenerate simplices ("degenerated" from Y_n) in all higher dimensions. Since $\Delta[q]$ has exactly one non-degenerate q-simplex,$\mathrm{id}(\Delta[q]) : \Delta[q] \longrightarrow \Delta[q]$, with all higher dimensions consisting only of degeneracies, *i.e.,* $\Delta[q]$ has *geometric dimension* q, $\mathbf{Sk}^n(\Delta[q]) = \Delta[q]$ if $n \geq q$. Consequently, by adjointness, $X_q = \mathbf{Cosk}^n(X_\bullet)_q$ for $q \leq n$ and since $\mathbf{Sk}^n(\Delta[n+1]) =_{Def} \overset{\bullet}{\Delta}[n+1]$ is just the boundary of the standard $(n+1)$-simplex,

$$\mathbf{Cosk}^n(X_\bullet)_{n+1} = \{(x_0, x_1, \ldots, x_{n+1}) | d_i(x_j) = d_{j-1}(x_i)\, 0 \leq i < j \leq n+1\}$$

in the cartesian product of n+2 copies of X_n. The canonical map just sends the $(n+1)$-simplices of $X_\bullet$ to their boundaries as a subset of $\mathbf{Cosk}^n(X)_{n+1}$. The set $\mathbf{Cosk}^n(X_\bullet)_{n+1}$ is precisely the *simplicial kernel* of the n-truncated complex,

$$\mathbf{Cosk}^n(X_\bullet)_{n+1} = \mathrm{SimKer}(X_\bullet]_0^n) = K_{n+1}(X_\bullet),$$

with faces given by the projections, $d_i^{n+1} = \mathrm{pr}_i \quad 0 \leq i \leq n+1$. All of the higher dimensional sets of simplices of $\mathbf{Cosk}^n(X_\bullet)$ are obtained just by iterating such simplicial kernels: $\mathbf{Cosk}^n(X_\bullet)_{n+2} = \mathrm{SimKer}(\mathbf{Cosk}^n(X_\bullet)]_0^{n+1})$, *etc.* The first non-identity map of the unit of the adjunction,

$$X_\bullet \longrightarrow \mathbf{Cosk}^n(X_\bullet) = cosk^n(tr_n(X_\bullet)),$$

is given by the $(n+1)$-boundary map

$$\partial_{n+1} : X_{n+1} \longrightarrow K_{n+1}(X_\bullet) = \mathbf{Cosk}^n(X_\bullet)_{n+1}.$$

The next sends an $(n+2)$-simplex to the family of the $(n+1)$ boundaries of its faces, and so on. If $\partial : X_\bullet \longrightarrow \mathbf{Cosk}^n(X_\bullet)$ is an isomorphism, then we will say that $X_\bullet$ is *n-Coskeletal.*

[18]The "co"-terminology is exactly backwards and is an historical accident.

Also note that the description of $\mathbf{Cosk}^n(X_\bullet)$ by iterated simplicial kernels makes evident the fact that the canonical map $\mathbf{Cosk}^n(X_\bullet) \longrightarrow \mathbf{Cosk}^m(\mathbf{Cosk}^n(X_\bullet))$ is an isomorphism if $m \geq n$. "Any n-coskeleton is m-coskeletal if $m \geq n$." However, if $m < n$, then $\mathbf{Cosk}^m(\mathbf{Cosk}^n(X_\bullet))$ is isomorphic to $\mathbf{Cosk}^m(X_\bullet)$, since then the truncations coincide: $\mathbf{Cosk}^n(X_\bullet)]_0^m = X_\bullet]_0^m$.

2.3. Simplicial Matrices. Following the author's version of Paul Glenn's enormously useful method ([Glenn 1982]), the first iterated simplicial kernel

$$\mathbf{Cosk}^n(X_\bullet)_{n+2} = \mathrm{SimKer}(\mathbf{Cosk}^n(X_\bullet)]_0^{n+1})$$

should have its $(n+2)$-simplex elements viewed as (n+3) $\times$ (n+2) *simplicial matrices* M with entries in X_n. Its n+3 rows, numbered from 0 to n+2, top to bottom, are the faces of the $(n+2)$-simplex M. Each of these rows R is an $(n+2)$-tuplet of n-simplices,$(\mathrm{pr}_k(\mathrm{R}))_{0\leq k\leq n+1}$ numbered left to right, which lies in the simplicial kernel $K_{n+1}(X_\bullet) = \mathbf{Cosk}^n(X_\bullet)_{n+1}$ and thus its faces, (*i.e.,* its length 1 columns) must satisfy the simplicial identities

$$d_i(\mathrm{pr}_j(\mathrm{R})) = d_{j-1}(\mathrm{pr}_i(\mathrm{R})) \quad 0 \leq i < j \leq n+1.$$

in X_{n-1}. Now the n-simplex entry $x_{j,i}$ in the j^{th}-row and i^{th}-column of the matrix $M \in \mathbf{Cosk}^n(X_\bullet)_{n+2}$ is $\mathrm{pr}_i(\mathrm{Row}_j) = d_i(d_j(M))$. The fact that the family of rows of this matrix

$$M = (\mathrm{Row}_0, \mathrm{Row}_1, \ldots, \mathrm{Row}_{n+2})$$

are the $(n+3)$-tuplets of the iterated simplicial kernel means that the rows and columns of M must themselves satisfy the simplicial identities, $d_i(d_j(M)) = d_{j-1}(d_i(M))$ for $(0 \leq i < j \leq n+2)$*i.e.,* , $\mathrm{pr}_i(\mathrm{Row}_j) = \mathrm{pr}_{j-1}(\mathrm{Row}_i)$ for $(0 \leq i < j \leq n+2)$. In other words, viewed directly in standard (row, col) indexing for these (n+3) $\times$ (n+2) simplicial matrices, the n-simplex entries $x_{j,i}$ in the j_{th}-row and i^{th}-column must satisfy the skewed symmetry,

$$x_{j,i} = x_{i,j-1} \quad 0 \leq i < j \leq n+2.$$

For $n = 0,\ 1$, and 2 , these matrices have the form

$$\begin{bmatrix} x_2 & x_1 \\ x_2 & x_0 \\ x_1 & x_0 \end{bmatrix}, \quad \begin{bmatrix} x_{23} & x_{13} & x_{12} \\ x_{23} & x_{03} & x_{02} \\ x_{13} & x_{03} & x_{01} \\ x_{12} & x_{02} & x_{01} \end{bmatrix}, \text{ and } \begin{bmatrix} x_{234} & x_{134} & x_{124} & x_{123} \\ x_{234} & x_{034} & x_{024} & x_{023} \\ x_{134} & x_{034} & x_{014} & x_{013} \\ x_{124} & x_{024} & x_{014} & x_{012} \\ x_{123} & x_{023} & x_{013} & x_{012} \end{bmatrix},$$

with each row of n-simplices assumed to be in the simplicial kernel $\mathbf{Cosk}^n(X_\bullet)_{n+1}$.

Note that any one row R_j of such a simplicial matrix is completely determined by the *other* rows R_i , $i \neq j$, in the matrix,

$$\mathrm{pr}_i(\mathrm{R}_j) = \begin{cases} \mathrm{pr}_{j-1}(\mathrm{R}_i) & 0 \leq i < j \\ \mathrm{pr}_j(\mathrm{R}_{i+1}) & j \leq i \leq n+1 \ , \end{cases}$$

(*e.g.*, the last row is uniquely determined by the last column and the first row is determined by the first column). But then, since $\bigwedge^j_{n+2}(\mathbf{Cosk}^n(X_\bullet))$ just consists of such matrices with row j omitted, and the j-horn map here is just the projection which omits the j^{th}-row, this fact is just another way of saying that the *Kan condition is satisfied exactly* in all dimensions $> n+1$ in $\mathbf{Cosk}^n(X_\bullet)$.

These same simplicial identities also reduce the maximum number of distinct, *i.e.,* non-identical, n-simplex entries to $\frac{(n+3)(n+2)}{2}$, exactly one-half of the naive possibilities. As will be seen, the form of these simplicial matrices provides an easily recognizable visual pattern which we will fully exploit. In particular, given a collection of $\mathbf{x}_i \in \mathbf{Cosk}^n(X_\bullet)_{n+1}$, there are only a very limited number of possible ways that they can be combinatorially fitted together as rows of such matrices. Specific low dimensional examples of this will occur throughout this work.

Since the k-horn map is just the boundary map followed by the projection map from the simplicial kernel to the k-horn, if $X_\bullet$ is a Kan complex, so is $\mathbf{Cosk}^n(X_\bullet)$, which is thus (trivially) an $n+1$-dimensional hypergroupoid. The same argument together with Kan exactness may be used to show that if $X_\bullet$ is an n-dimensional hypergroupoid, then $X_\bullet$ is a subcomplex of its n-Coskeleton and is isomorphic to its $(n+1)$-coskeleton. From the point of view of homotopy theory, the n-Coskeleton kills all of the homotopy groups of a complex in dimension n or higher (since any n-simplex all of whose faces are at a base point x_0, is homotopic to $s_0^n(x_0) = 0$ by a face fixing homotopy in $\mathbf{Cosk}^n(X_\bullet)_{n+1}$, the simplicial kernel). In contrast, an n-dimensional hypergroupoid has possible non-trivial homotopy groups up to *and including* dimension n, but only trivial groups in all strictly higher dimensions. These two properties of n-dimensional hypergroupoids (a subcomplex of its n-Coskeleton and itself $(n+1)$-Coskeletal) are shared with subcomplexes of the form $X_\bullet^{(n)} \subseteq \mathbf{Cosk}^n(X_\bullet)$, where $X_\bullet$ is a Kan complex and $X_\bullet^{(n)}$ is its image in its n-Coskeleton under the unit (n-boundary) $X_\bullet \longrightarrow \mathbf{Cosk}^n(X_\bullet)$. $X_\bullet^{(n)}$ is familiar to algebraic topologists as the n^{th} complex in the so called *Postnikov Tower of* $X_\bullet$ and it is the conjunction of these two shared properties that will form our "basic simplicial setting in dimension n" for our putative nerves of weak n-*categories* where the complexes under study are not, in general, Kan complexes, but only share with them certain very weak forms of horn lifting.[19]

2.4. The "Basic Simplicial Setting in Dimension n": n-Dimensional Postnikov Complexes .

2.5. Definition. **(n-dimensional Postnikov complex)** *The* "basic simplicial setting in dimension n" *will be that of a simplicial complex $X_\bullet$ which is a subcomplex of its n-*

[19]A note of caution for topologists: The *geometric realization* functor $|-| : \mathbf{Simpl}(\mathbf{Sets}) \longrightarrow \mathbf{Top}$ is *left* adjoint to the *singular complex* functor $\mathrm{Sing}(-)$. It thus behaves with geometric predictably when combined with the left adjoint $\mathbf{Sk}^n$, but *not* with the right adjoint $\mathbf{Cosk}^n$. The homotopy groups of arbitrary simplicial sets $X_\bullet$ are often *defined* using $|X_\bullet|$ so caution is advised when making predictions about complexes which are Coskeletal. For example, as we shall see, the nerves of both categories and groupoids are isomorphic to their 2-Coskeleta, yet it is well known that categories can model *any* homotopy type, while groupoids model only 1-types!

Coskeleton and is itself $(n+1)$-*Coskeletal.*

$$\mathbf{Cosk}^{n+1}(X_\bullet) = X_\bullet \subseteq \mathbf{Cosk}^n(X_\bullet)$$

For brevity, a complex $X_\bullet$ *in this "basic simplicial setting in dimension* n *" will be called an* n-dimensional Postnikov complex*.*

For reasons which we will make clear as we progress, in any of the n-dimensional Postnikov complexes which we consider, the set of $(n+1)$-simplices

$$X_{n+1} \subseteq \mathbf{Cosk}^n(X_\bullet)_{n+1} = K_{n+1}(X_\bullet) = \mathrm{SimKer}(X_\bullet]_0^n)$$

will be called the set *commutative simplices* of the complex $X_\bullet$. Since the simplices in dimension $n+1$ of the n-Coskeleton are just families $(x_0, x_1, x_2, \ldots, x_{n+1})$ of n-simplices of $X_\bullet$ whose faces satisfy the simplicial identities $d_i(x_j) = d_{j-1}(x_i)$ $(0 \leq i < j \leq n+1)$, families lying in the subset $X_{n+1} \subseteq K_{n+1}$ of commutative simplices of $X_\bullet$ will be said to be commutative and indicated, for convenience, by using square rather than round brackets, *i.e.,*

$$[x_0, x_1, x_2, \ldots, x_{n+1}]$$

will indicate that $(x_0, x_1, x_2, \ldots, x_{n+1})$ is an element of K_{n+1} which is *commutative*, *i.e.,* is just an element of X_{n+1}.[20]

The simplices of dimension $n+2$ in the n-Coskeleton of $X_\bullet$ are just elements of the simplicial kernel K_{n+2} of the truncated complex $\mathbf{Cosk}^n(X_\bullet)]_0^{n+1}$ and are, as we explained in the preceding Section 2.3 most conveniently thought of as a *simplicial matrix* whose rows, in our usage numbered from top to bottom, starting with 0 and continuing to n+2, are its faces and are just elements of K_{n+1}. The face maps on the individual rows are given by the projections, numbered left to right starting with 0 and continuing to n+1. As we noted there, in order that such a matrix define an element of K_{n+2}, it must be a *simplicial matrix*: the rows and their projections must satisfy the simplicial identities, and this forces the matrix to have, at most, only 1/2 of its $(n+3) \times (n+2)$ possible entries unequal and, overall, to have a distinctive pattern which may be easily identified visually.

Again as we noted in Section 2.3, in such a simplicial matrix, *any* particular row is completely determined by all of the other rows in the matrix. For instance the last row completely determines the last column; the first row, the first column,*etc.* Fortunately, and importantly for use in the proofs, given two or more elements of K_{n+1} which have certain entries (=faces) in common, *there are only a limited number of ways that they could naturally fit as rows in a simplicial matrix element of* K_{n+2}. Now an n-dimensional Postnikov-complex $X_\bullet$, our basic simplicial setting in dimension n, is also required to be

[20]This notation, while very convenient in the process of reasoning will occasionally lead to some linguistic/notational/syntactical redundancy, such as "$[x_0, x_1, \ldots, x_{n+1}]$ is commutative" or "$[x_0, x_1, \ldots, x_n] \in X_{n+1}$", when "$(x_0, x_1, \ldots, x_n) \in X_{n+1}$" or "$(x_0, x_1, \ldots, x_{n+1})$ is commutative" is what is meant, and, technically, "$[x_0, x_1, \ldots, x_{n+1}]$", all by itself would have sufficed. We hope that the reader will bear with us.

$(n+1)$-Coskeletal. Thus its set X_{n+2} of $(n+2)$-simplices is just the simplicial kernel of the subset of *commutative* $(n+1)$-simplices X_{n+1}. It is easy to see that, by the definition of a simplicial kernel, X_{n+2} just consists of those simplicial matrices in $\mathbf{Cosk}^n(X_\bullet)_{n+2}$ in which *every row is commutative, i.e.,* every row is in X_{n+1}. Similarly, each set $\bigwedge^k_{n+2}(X_\bullet)$ of k-horns in this dimension is just the subset of k-horns of simplicial matrices in which all of the "non-missing" rows are commutative. The "missing row" in any such horn is completely determined *as an element of* $\mathbf{Cosk}^n(X_\bullet)_{n+1}$ by the commutative non-missing rows, and thus *the canonical map from the* $(n+2)$*-simplices of our complex to any one of its sets of horns* $\bigwedge^k_{n+2}(X_\bullet)$ *is always injective*; it is only a question of whether the determined element of $\mathbf{Cosk}^n(X_\bullet)_{n+1}$ is commutative that is relevant for horn conditions in this dimension $(n+2)$ of an n-dimensional Postnikov complex $X_\bullet$.

The horn lifting conditions on such an $X_\bullet$ which we will state and which we will use in our characterization of the nerves of bicategories (where $n=2$) will always relate to the horn maps in our complex and its fibers. The crucial working parts in levels X_{n+1} and X_{n+2} of $X_\bullet$ will always concern

(a) the canonical maps to one or more of the sets of horns of the set X_{n+1} of commutative $(n+1)$-simplices and will be of the following form: given an element

$$(x_0, x_1, \ldots, x_{k-1}, \_\_, x_{k+1}, \ldots, x_{n+1}) \in \textstyle\bigwedge^k_{n+1}(X_\bullet),$$

under what circumstances does there exist a unique n-simplex x_k such that

$$(x_0, x_1, \ldots, x_{k-1}, x_k, x_{k+1}, \ldots, x_{n+1})$$

is commutative, which we will often abbreviate as

$$\text{"}\ldots \text{ such that } [x_0, x_1, \ldots, x_{k-1}, x_k, x_{k+1}, \ldots, x_{n+1}]\text{"}.$$

The answer, if conditional, will be determined by the properties of the n-simplex entries present in the horn (and possibly some of their faces).

(b) the canonical maps from the set X_{n+2} of simplicial matrices all of whose $(n+1)$-simplex rows are commutative to one or more of its sets of horns and will be of the following form: Given a matrix horn

$$[[\mathbf{x}_0], [\mathbf{x}_1], \ldots, [\mathbf{x}_{k-1}], (\mathbf{x}_k), [\mathbf{x}_{k+1}], \ldots, [\mathbf{x}_{n+2}]\ ]$$

in which all of the rows $\mathbf{x}_i$ but the k^{th} are commutative, under what circumstances is the uniquely determined row $\mathbf{x}_k$ commutative, which we will again often abbreviate as "... such that $[\mathbf{x}_k]$". The answer, again if conditional, will depend on the properties of some of the n-simplex entries (faces) in the commutative rows (and possibly some of their $n-1$ or lower dimensional faces).

The "properties in question" of the n-simplices present (and possibly of some their faces) in (a) and (b) will depend on their membership in certain subsets $I_j \subseteq X_j$ of j-simplices $1 \leq j \leq n$

$$I_n, I_{n-1}, \ldots, I_1$$

which we will call the set(s) of *(abstractly or formally) invertible* j-simplices. (In dimension $n+1$ and higher all of the simplices may be assumed to be invertible, while in lower dimensions the horn lifting conditions will be similar but will only ask for the not necessarily unique existence of an invertible simplex which fills the horn.)

In low dimensions, at least, the subsets $I_j \subseteq X_j$ of invertible j-simplices need not be viewed as an auxiliary structure on $X_\bullet$ but rather as having as members those j-simplices whose *defining property* is that of an affirmative answer in (a) and (b) with respect to the higher dimensional horns (in a "nested fashion" which will be made clear as we progress). However, this description becomes increasingly complicated and redundant as n increases and it is probably wiser to just start with the family $(I_j(X_\bullet))_{j\geq 1}$ of subsets as an additional structure on the complex $X_\bullet$.

In any case, for any of the Postnikov complexes that we consider as arising as nerves of (weak) n-categories, the sets I_j will never be empty since the degenerate j-simplices will always be required to be invertible.[21] The reader should be advised, however, that the terminology "commutative" and "invertible", when applied to simplices, may initially be misleading. In particular, as the dimension of the (weak) n-category increases, the defining properties of these sets of simplices in its nerve changes.

As we shall see, the particular Postnikov complex associated with the nerve of a (weak) n-category structure on its set of objects (its 0-cells) enfolds an enormous amount of algebraic structure which must somehow be recovered from the complex, in particular, going from the (weak) $(n-1)$-category structure on its 1-cells all the way down through a strict category structure on its $(n-1)$-cells to a 0-category, *i.e.,* discreet, structure on its n-cells. Each of these structures has itself a nerve and the easiest way to describe them simplicially is to note that each is obtained by repeated application of a particular endofunctor on simplicial sets, the *path-homotopy functor* $\mathbb{P}$. It is closely related to the "path space" and "loop space" construction familiar to algebraic topologists and serves an analogous purpose for complexes, like ours, which are not assumed to have a distinguished base point. $\mathbb{P}$ will be constructed "combinatorially" along with a short exploration of its basic properties in the next section.

2.6. The Shift, Path-Homotopy, and Loop Functors. ($\mathbf{Dec}_+$, $\mathbb{P}$, and $\mathbf{\Omega}$)

Recall that for any complex $X_\bullet$, the *shifted complex* $\mathbf{Dec}_+(X_\bullet)$ (the *décalage* of $X_\bullet$) has

$$(\mathbf{Dec}_+(X_\bullet))_n = X_{n+1}\,(n \geq 0)$$

with faces and degeneracies given by

$$d_i : (\mathbf{Dec}_+(X_\bullet))_n \longrightarrow (\mathbf{Dec}_+(X_\bullet))_{n-1} = d_i : X_{n+1} \longrightarrow X_n\,(0 \leq i \leq n)$$

$$s_i : (\mathbf{Dec}_+(X_\bullet))_n \longrightarrow (\mathbf{Dec}_+(X_\bullet))_{n+1} = s_i : X_{n+1} \longrightarrow X_{n+2}\,(0 \leq i \leq n).$$

[21]Although stated in"algebraic" terms, this hypothesis will always be related to an appropriate degree of minimality, *i.e.,* homotopic n-simplices are equal. Note also that since our simplicial complexes are always subcomplexes of their n-Coskeleta, the *degenerate* $(n+1)$*-simplices are always commutative*, by definition.

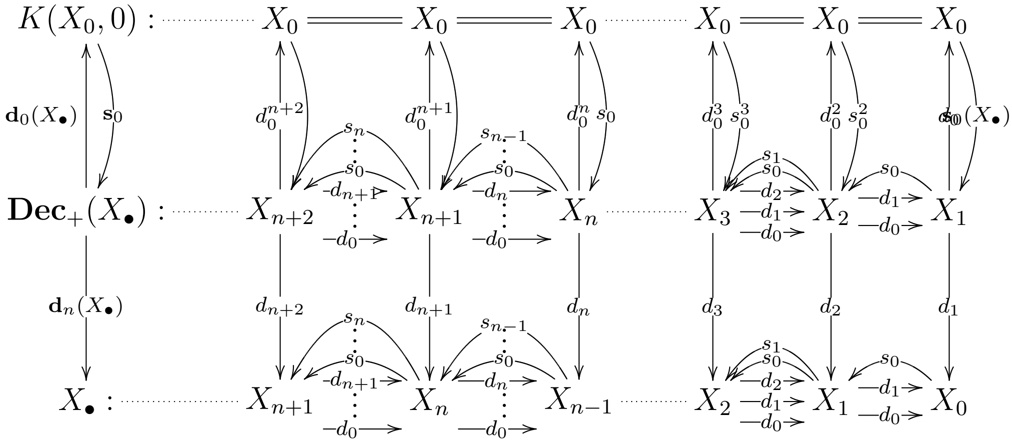

Figure 7: The Shifted Complex $\mathbf{Dec}_+(X_\bullet)$ and its Canonical Map $\mathbf{d}_n(X_\bullet)$ to $X_\bullet$ and its Retraction $\mathbf{d}_0(X_\bullet)$ to the Constant Complex $K(X_0, 0)$ as Graded Sets with Face and Degeneracy Operators.

The omitted "last face" map, $d_n : X_n \longrightarrow X_{n-1}$ defines a canonical simplicial map $\mathbf{d}_n(X_\bullet)$ back to $X_\bullet$ and the composition of $d_0 : X_1 \longrightarrow X_0$ with the retained face maps defines a canonical simplicial map $\mathbf{d}_0(X_\bullet)$ back to the constant complex $K(X_0, 0)$.[22] Composition with the degeneracy $s_0 : X_0 \longrightarrow X_1$ composed with the retained degeneracies defines a contracting homotopy back to $\mathbf{Dec}_+(X_\bullet)$ which makes $K(X_0, 0)$ a deformation retract of $\mathbf{Dec}_+(X_\bullet)$.[23]

As graded sets with face and degeneracy operators $X_\bullet$, $Dec_+(X_\bullet)$, and the constant complex $K(X_0, 0)$ is easily visualized as in Figure 7.

The *path-homotopy complex* $\mathbb{P}(X_\bullet)$ of $X_\bullet$ is the pull-back by the canonical simplicial map $\mathbf{d}_n(X_\bullet)$ (here just the inverse image in each dimension) of the 0-Skeleton of $X_\bullet$ (the subcomplex of $X_\bullet$ isomorphic to $K(X_0, 0)$ where all simplices are totally degenerate, *i.e.,* degenerated from the set of 0-simplices X_0). Projection and composition then define the 0-Target ($\mathbf{d_0}$) and 0-Source ($\mathbf{d_n}$) as simplicial maps back to the constant complex $K(X_0, 0)$

[22] If $X_\bullet$ is a Kan complex, then $\mathbf{d}_n(X_\bullet)$ is a Kan fibration. This has the effect of making all of complexes in Figure 8 Kan complexes whenever $X_\bullet$ is.

[23] If one "strips away the "last face" maps $d_n : X_n \longrightarrow X_{n-1}$ $n \geq 1$ of a complex" (*i.e.,* restricts the functor $X_\bullet$ to the subcategory which has the same objects as Δ but only those non-decreasing maps which preserve $n \in [n]$, which is generated by all of the generators of Δ except for the injections $\delta_n : [n-1] \longrightarrow [n]$, and renumbers, one obtains a complex which is augmented (to the constant complex $K(X_0, 0)$) and has the family $(s_n)_{n \geq 0}$ of last degeneracy operators defining a contracting homotopy. $\mathbf{Dec}(X_\bullet)$ is this process viewed as functor from simplicial sets to the category of such "contractible augmented complexes", which is just the category of sets from the point of view of homotopy. It has a trivial left-adjoint "+" : "just forget the augmentation and the contracting homotopy" and the category of simplicial sets is the category of Eilenberg-Moore algebras for this adjoint pair. $\mathbf{Dec}_+(X_\bullet)$ is the comonad on simplicial sets defined by this adjoint pair, which has $\mathbf{d}_n(X_\bullet) : \mathbf{Dec}_+(X_\bullet) \longrightarrow X_\bullet$ as its counit . The associated monadic ("triple") cohomology theory is classical "cohomology with local coefficient systems" and the theory of "**Dec**-split torsors" is the theory of "twisted cartesian products" (*cf.* [Duskin 1975]).

$$\begin{tikzcd}
\Omega(X_\bullet, x_0) \arrow[rr, "\mathbf{d_n}"] \arrow[dr, hook, "incl"'] \arrow[ddd, "\mathbf{d_0}"'] & & K(\{x_0\},0) \arrow[d, hook, "incl"] \\
& \mathbb{P}(X_\bullet) \arrow[r, "\mathbf{d_n}"] \arrow[d, hook, "incl"] \arrow[dr, phantom, "(pull\!-\!back)"] & \mathbf{Sk}^0(X_\bullet) \simeq K(X_0,0) \arrow[d, hook, "incl"] \\
& \mathbf{Dec}_+(X_\bullet) \arrow[r, "\mathbf{d_n}"] \arrow[d, "\mathbf{d_0}"] & X_\bullet \\
K(\{x_0\},0) \arrow[r, hook, "incl"] & K(X_0,0) \arrow[u, bend left, "\mathbf{s_0}"] \arrow[uu, bend left=60, "\mathbf{s_0}"] &
\end{tikzcd}$$

Figure 8: Loop, Path-Homotopy, and Shift Complexes of $X_\bullet$

with $\mathbf{s}_0$ as a common simplicial section, as in Figure 8.

Thus

$$\mathbb{P}(X_\bullet)_0 = X_1$$

and

$$\mathbb{P}(X_\bullet)_1 = \{\alpha \in X_2 | d_2(\alpha) = s_0(d_1(d_2(\alpha))) = s_0(d_1(d_1(\alpha)))\}, \textit{i.e.,}$$

in $X_\bullet$, such an α is a 2-simplex with its boundary given by

$$\partial(\alpha) = (d_0(\alpha), d_1(\alpha), d_2(\alpha)) = (y_{01}, x_{01}, s_0(x_0)).$$

In geometric notation α has both of the forms illustrated in Figure 9. By definition, in $\mathbb{P}(X_\bullet)_1$, $x_{01} = d_1(\alpha)$ and $y_{01} = d_0(\alpha)$. Thus in the geometric notation of the directed graph $\mathbb{P}(X_\bullet)]_0^1$, we shall refer to x_{01} as the 1-source of α and y_{01} as its 1-target and use a double arrow to indicate that α is a 2-simplex of $X_\bullet$: $\alpha : x_{01} \Longrightarrow y_{01}$. x_0 will be the 0-source of α and x_1 the 0-target both lying in X_0.(The directed double arrow could equally have been placed in the 2-simplex on the right.)[24] Similarly, a 2-simplex in $\mathbb{P}(X_\bullet)$ is a 3-simplex ξ in $X_\bullet$ whose 3-face is *totally degenerate* in $X_\bullet$, *i.e.,* $d_3(\xi) = s_0(s_0(x_0)) = s_0^2(x_0)$, for some $x_0 \in X_0$. But this forces $d_2(d_i(\xi)) = d_2(\alpha_i) = s_0(x_0)$ for $0 \le i \le 2$ so that $\xi \in \mathbb{P}(X_\bullet)_2$ has the geometric form shown in Figure 10.

In higher dimensions

$$\mathbb{P}(X_\bullet)_n = \{x \in X_{n+1} | d_{n+1}(x) = s_n(\ldots s_1(s_0(d_1(\ldots(d_{n-1}(d_n(x)))\ldots)\}$$

and the visualization is similar.

[24] If $X_\bullet$ is the singular complex of a topological space the 0-simplices of $X_\bullet$ are the points of the space and the 1-simplices x are the parameterized paths of the space from $d_1(x) = x_0$ to $d_0(x) = x_1$; the degenerate path $s_0(x_0)$ is the one which is constantly at x_0 for all values of the parameter. A 1- simplex in $\mathbb{P}(X_\bullet)$ (Figure 9) is a directed homotopy of the path x_{01} to the path y_{01} which leaves the end points of the paths (their boundary) fixed. Thus $\mathbb{P}(X_\bullet)$ is the "paths–homotopies of paths (which are constant on the boundaries)" complex.

$$
x_{01} \overset{\alpha}{\Longrightarrow} y_{01}
\qquad
\begin{tikzcd}
x_0 \arrow[r, "s_0(x_0)"] \arrow[dr, "x_{01}"'] & x_0 \arrow[d, "y_{01}"] \arrow[dl, phantom, "\alpha"] \\
 & x_1
\end{tikzcd}
$$

Figure 9: 1-simplex $\alpha : x_{01} \Longrightarrow y_{01} \in \mathbb{P}(X_\bullet)_1$ and as a 2-simplex in $X_\bullet$ with $d_0(\alpha) = y_{01}$, $d_1(\alpha) = x_{01}$, and $d_2(\alpha) = s_0(x_0)$ in X_2

$$
\begin{tikzcd}
x_{01} \arrow[r, Rightarrow, "\alpha_2"] \arrow[dr, Rightarrow, "\alpha_1"'] & y_{01} \arrow[d, Rightarrow, "\alpha_0"] \arrow[dl, phantom, "\xi"] \\
 & z_{01}
\end{tikzcd}
\qquad
\begin{tikzcd}
 & x_0 \arrow[dl, "x_{01}"'] \arrow[dr, "s_0(x_0)"] \arrow[dd] & \\
x_1 & & x_0 \arrow[ll] \arrow[dl, "s_0(x_0)"] \\
 & x_0 \arrow[ul, "z_{01}"] &
\end{tikzcd}
$$

Figure 10: Geometric Notation for a Typical 2-simplex $\xi \in \mathbb{P}(X_\bullet)_2$ and as $\xi \in X_3$ with $d_3(\xi) = s_0^2(x_0)$

If one iterates the construction of $\mathbb{P}(X_\bullet)$ to obtain $\mathbb{P}(\mathbb{P}(X_\bullet)) = \mathbb{P}^2(X_\bullet)$, the 0-simplices there become homotopies of paths. The 1-simplices there are the 3-simplices ξ of $X_\bullet$, which have not only their 3-face totally degenerate, $s_0^2(x_0) = d_3(\xi)$, but also must have $d_2(\xi) = s_0(x_{01})$ degenerate as well.[25] Thus a 1-simplex in $\mathbb{P}^2(X_\bullet)$ can be visualized in three ways as in Figure 11. The successive iterations of $\mathbb{P}(X_\bullet)$ with the corresponding 0-sources and 0-targets is the canonical *globular complex with degeneracies of directed homotopies of homotopies of homotopies* ... associated with any complex $X_\bullet$.

$$
\begin{tikzcd}
\cdots\cdots \mathbb{P}^3(X_\bullet)_0 \arrow[r, shift left, "d_1"] \arrow[r, shift right, "d_0"'] & \mathbb{P}^2(X_\bullet)_0 \arrow[l, bend right, "s_0"'] \arrow[r, shift left, "d_1"] \arrow[r, shift right, "d_0"'] & \mathbb{P}^1(X_\bullet)_0 \arrow[l, bend right, "s_0"'] \arrow[r, shift left, "d_1"] \arrow[r, shift right, "d_0"'] & \mathbb{P}^0(X_\bullet)_0 = X_0 \arrow[l, bend right, "s_0"']
\end{tikzcd}
$$

Most important for our purposes here is the following

2.7. Theorem. ***Commutation of*** $\mathbb{P}$ ***and*** **Cosk** ***with a Dimension Shift.*** *For any simplicial set* $X_\bullet$*, and* $n \geq 2$*,*

[25] In the singular complex of a space X, this is thus a "homotopy of homotopies which leaves the boundary *paths* of the two homotopies fixed". Geometrically, the continuous image of the standard 3-simplex whose restriction to the 3-face is the constant map $\equiv x_0$,to the 2-face the constant homotopy of the path x_{01} with itself, to the 0-face the "target" homotopy (of the path x_{01} to the path y_{01}) and, the restriction finally to the 1-face the other "source" homotopy. If one prefers, this can be imagined all at once as the continuous image in the space X of an appropriately parameterized solid ball whose north pole is always carried to x_0 and whose south pole to x_1, with one N-S half-meridian always to the path y_{01}, the other N-S one (on the same great circle) always to the path x_{01}, initially carrying the eastern hemisphere to one of the homotopies, the finally, the western hemisphere to the other. And similarly in higher dimensions ....

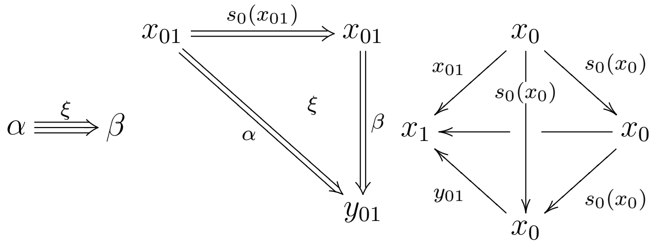

Figure 11: Geometric Notation for a Typical 1-simplex $\xi \in \mathbb{P}^2(X_\bullet)$ and as $\xi \in \mathbb{P}(X_\bullet)_2$ with $d_2(\xi) = s_0(x_{01})$ and as $\xi \in X_3$ with $d_3(\xi) = s_0^2(x_0)$ and $d_2(\xi) = s_0(x_{01})$

- $\mathbf{Dec}_+(\mathbf{Cosk}^n(X_\bullet)) \cong \mathbf{Cosk}^n(\mathbf{Dec}_+(X_\bullet))$.
- $\mathbb{P}(\mathbf{Cosk}^n(X_\bullet)) \cong \mathbf{Cosk}^{n-1}(\mathbb{P}(X_\bullet))$.
- *If $X_\bullet$ is an n-dimensional Postnikov-complex, then $\mathbb{P}(X_\bullet)$ is an $(n-1)$-dimensional Postnikov-complex.*

The proofs of these commutations are an immediate consequence of the simplicial identities and the definitions. For the first observation, note that for any complex $X_\bullet$ and any $m \geq 1$, the shifted complex has $\mathbf{Dec}_+(X_\bullet)_{m-1} = X_m$ and has exactly the same faces as $X_\bullet$ except that each of the "last face maps" $d_m : X_m \longrightarrow X_{m-1}$ has been omitted (along with X_0). Consequently, if $m \geq 2$, the simplicial face identities which define an m-horn in dimension m in $X_\bullet$ are exactly the same as those which define an element of the corresponding simplicial kernel in $\mathbf{Dec}_+(X_\bullet)$ and the m-horn map $\mathrm{pr}_{\hat{m}} : X_m \longrightarrow \bigwedge^m_m$ in $X_\bullet$ becomes the $(m-1)$-boundary map $\partial_{m-1} : \mathbf{Dec}_+(X_\bullet)_{m-1} \longrightarrow \mathbf{Cosk}^{m-1}(\mathbf{Dec}_+(X_\bullet))_m$. Thus if $X_\bullet$ is n-coskeletal, *i.e.,* $X_\bullet \xrightarrow{\sim} \mathbf{Cosk}^n(X_\bullet)$, our description of $\mathbf{Cosk}^n(X_\bullet)$ makes $X_m = \mathbf{Cosk}^n(X_\bullet)_m$, for $m \leq n$, and $X_{n+1} \xrightarrow{\sim} \mathbf{Cosk}^n(X_\bullet)_{n+1}$, the simplicial kernel of the n-truncation of $X_\bullet$, with $X_{n+2} \xrightarrow{\sim} \mathbf{Cosk}^n(X_\bullet)_{n+2}$, the corresponding simplicial kernel visualized as the appropriate set of simplicial matrices. But our discussion of simplicial matrices made it evident that the all of the $(n+2)$-horn maps here and in dimensions $\geq n+2$ are bijections, in particular, $\mathrm{pr}_{\widehat{n+2}} : X_{n+2} \xrightarrow{\sim} \mathbf{Cosk}^n(X_\bullet)_{n+2} \longrightarrow \bigwedge^{n+2}_{n+2}(X_\bullet)$ which is just the boundary map $\partial_{n+1} : \mathbf{Dec}_+(X_\bullet)_{n+1} \longrightarrow \mathbf{Cosk}^n(\mathbf{Dec}_+(X_\bullet))_{n+1}$. Thus we have the first item.

For the second, if

$$(\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n, \mathbf{x}_{n+1}) \in \mathrm{SimKer}(X_\bullet]^n_0) = \mathbf{Cosk}^n(X_\bullet)_{n+1}$$

with $\mathbf{x}_{n+1}$ totally degenerate, say $\mathbf{x}_{n+1} = s_0^n(x_0)$ for $x_0 \in X_0$, then

$$(\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n, s_0^n(x_0)) \in \mathbb{P}(\mathbf{Cosk}^n(X_\bullet))_n.$$

But

$$d_n(\mathbf{x}_i) = d_i(\mathbf{x}_{n+1}) = d_i(s_0^n(x_0)) = s_0^{n-1}(x_0)\,, (0 \leq i \leq n),$$

and thus

$$\mathbf{x}_i \in \mathbb{P}(X_\bullet)_{n-1}\,, (0 \leq i \leq n) \text{ and}$$

$$(\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n) \in \mathrm{SimKer}(\mathbb{P}(X_\bullet)]_0^{n-1}) = \mathbf{Cosk}^{n-1}(\mathbb{P}(X_\bullet))_n$$

The third part follows immediately from the second, since $\mathbb{P}$ is functorial and preserves inclusions.

Thus the path-homotopy complex $\mathbb{P}$ shifts our "basic simplicial setting in dimension n" down by one dimension to our "basic simplicial setting in dimension $n-1$ ".

The *loop complex* $\mathbf{\Omega}(X_\bullet, x_0)$ of $X_\bullet$ at the *base point* $x_0 \in X_0$ is the subcomplex of $\mathbb{P}(X_\bullet)$ where both the 0-Source and 0-Target are degenerated from the same point x_0. Thus in $\mathbf{\Omega}(X_\bullet, x_0)$ the 0-simplices are "loops", 1-simplices in $X_\bullet$ of the form $l : x_0 \longrightarrow x_0$, 2-simplices are directed homotopies of such loops. In $\mathbf{\Omega}(\mathbf{\Omega}(X_\bullet, x_0))$ the base point is $s_0(x_0)$ and the 0-simplices are the 2-simplices of $X_\bullet$ "all of whose faces are at the base point", *i.e.,* their boundaries have the form $(s_0(x_0), s_0(x_0), s_0(x_0))$. Thus for Kan complexes, $\pi_0(\mathbf{\Omega}(X_\bullet, x_0)) = \pi_1(X_\bullet, x_0)$; $\pi_0(\mathbf{\Omega}(\mathbf{\Omega}(X_\bullet, x_0)) = \pi_2(X_\bullet, x_0)$, as usual. Clearly, all of these constructions are functorial on simplicial maps (on $\mathbf{\Omega}$ only for basepoint preserving simplicial maps). Figure 8 gives the relations between these functors (evaluated at $X_\bullet$) in the category of simplicial sets.

2.8. Remark. As with Ω, many of the classical constructions of simplicial algebraic topology are directly related the constructions used in this paper diagramed in Figure 8, where, to avoid confusion, we have suppressed the additional four pull-backs that naturally occur. The contractible (to x_0) inverse image subcomplex

$$\mathcal{P}(X_\bullet) = \mathbb{P}(X_\bullet \downarrow x_0) = \mathbf{d_0}^{-1}(K(x_0, 0)) \subseteq \mathbf{Dec}_+(X_\bullet)$$

of *paths whose target is based at* x_0 is commonly called "*the path complex of* $X_\bullet$" when $X_\bullet$ is considered a pointed (by $x_0 \in X_0$) complex. The composition of its inclusion with $\mathbf{d_n}$, denoted by

$$\wp : \mathcal{P}(X_\bullet) \longrightarrow X_\bullet$$

is the *canonical path space fibration* over the pointed complex $X_\bullet$ which has $\mathcal{P}(X_\bullet)$ as contractible total space and, again pointed by x_0, $\Omega(X_\bullet)$ as its fiber. For pointed Kan complexes the fiber sequence

$$\mathbf{\Omega}(X_\bullet) \subseteq \mathcal{P}(X_\bullet) \longrightarrow X_\bullet$$

gives rise to the canonical identification

$$\pi_{n-1}(\Omega(X_\bullet)) \cong \pi_n(X_\bullet)$$

fundamental to the use of the loop space in base pointed homotopy theory.

Similarly, for a complex $X_\bullet$, its *universal covering space* $\tilde{X}_\bullet$ is just the pull-back of Figure 12 in which $\Pi_1(X_\bullet)$ is the fundamental groupoid of $X_\bullet$.

We now briefly observe these basic simplicial settings in dimensions 0 and 1.

$$\begin{tikzcd} \tilde{X}_\bullet \arrow[r, "\mathrm{pr}"] \arrow[d, "\mathrm{pr}"'] & \mathbf{Dec}_+(\Pi_1(X_\bullet)) \arrow[d, "\mathbf{d_n}"] \\ X_\bullet \arrow[r, "\pi"'] & \Pi_1(X_\bullet) \end{tikzcd}$$

Figure 12: Universal Covering Space $\tilde{X}_\bullet$

3. 0-Dimensional Postnikov Complexes: Nerves of Partially Ordered Sets, Equivalence Relations, and Discrete Sets.

Now suppose that $n = 0$, so that

$$\mathbf{Cosk}^1(X_\bullet) = X_\bullet \subseteq \mathbf{Cosk}^0(X_\bullet),$$

so that $X_\bullet$ is a 0-dimensional Postnikov-complex.

Now, up to isomorphism, the 0-Coskeleton of any complex is defined by nothing more than the successive (cartesian product) powers of X_0 with faces defined by the projections and degeneracies defined by the diagonal. For our purposes though, it is better to imagine the relevant parts of the sequence of simplices of $\mathbf{Cosk}^0(X_\bullet)$ as follows: in dimension 0 its vertices coincide with those of X_0; its simplices in dimension 1 are just ordered pairs (x_1, x_0) of elements of X_0 with

$$d_0(x_1, x_0) = x_1, \quad d_1(x_1, x_0) = x_0 \ (sic).$$

The simplices $X_0 \times X_0 \times X_0$ of dimension 2, in contrast, will be thought of as 3×2 simplicial matrices M with entries in X_0 :

$$M = \begin{bmatrix} (\, x_2 & x_1 \,) \\ (\, x_2 & x_0 \,) \\ (\, x_1 & x_0 \,) \end{bmatrix}$$

with

$$d_0(M) = (x_2, x_1), \quad d_1(M) = (x_2, x_0), \quad \text{and} \quad d_2(M) = (x_1, x_0).$$

The set of "commutative" 1-simplices of our complex ($=_{Def} X_1$) is a subset of $X_0 \times X_0$, which contains the diagonal, since $X_\bullet$ is a subcomplex of $\mathbf{Cosk}^0(X_\bullet)$ and the degeneracy map s_0 is given by $s_0(x_0) = (x_0, x_0)$ in the 0-Coskeleton. Following our convention, we will denote an element of X_1 by $[x_1, x_0]$ using square brackets. Thus what we have at this point $(X_\bullet]_0^1)$ is just the graph of any reflexive relation. Since $X_\bullet$ is also 1-Coskeletal, the elements of X_2 are just the matrices in $\mathbf{Cosk}^0(X_\bullet)_2$ of the form

$$\begin{bmatrix} [\, x_2 & x_1 \,] \\ [\, x_2 & x_0 \,] \\ [\, x_1 & x_0 \,] \end{bmatrix},$$

that is, those which have all of their faces in X_1.

The two horn maps in dimension 1 are just the projections to $X_0 = \bigwedge_1^0 = \bigwedge_1^1$. They are always surjective by our assumption that the degeneracy (here the diagonal) is in X_1.

The three horn sets in dimension 2 consist of matrices of the form

$$\bigwedge_2^0 : \begin{bmatrix} (\,x_2 & x_1\,) \\ [\,x_2 & x_0\,] \\ [\,x_1 & x_0\,] \end{bmatrix},\ \bigwedge_2^1 : \begin{bmatrix} [\,x_2 & x_1\,] \\ (\,x_2 & x_0\,) \\ [\,x_1 & x_0\,] \end{bmatrix},\ \text{and } \bigwedge_2^2 : \begin{bmatrix} [\,x_2 & x_0\,] \\ [\,x_2 & x_1\,] \\ (\,x_1 & x_0\,) \end{bmatrix},$$

where the "missing" face is indicated by round brackets since it is uniquely determined *as a* 1*-simplex in* $X_0 \times X_0 = \mathbf{Cosk}^0(X_\bullet)_1$ by the "non-missing" faces of the horn. As we have noted before, and is obvious here, this uniqueness forces the projection mappings from X_2 to each of these horns to be injective. Moreover, again because of the fact that $X_\bullet$ is 1-Coskeletal, *in all higher dimensions of* $X_\bullet$ *all of the horn maps are bijections* and thus the only interesting horn conditions for 0-dimensional Postnikov complexes must occur in dimensions ≤ 2. Thus if we require here the *weak Kan condition* (surjectivity on all horn maps in every dimension m except for the extremal ones, 0 and m) we must have surjectivity (and hence bijectivity) of the map of X_2 to the set of 1-horns, but this just means that if $[x_2, x_1]$, and $[x_1, x_0]$ then $[x_2, x_0]$. In short, $X_1 \subseteq X_0 \times X_0$ is just the *graph of a reflexive transitive relation, i.e., a partially ordered set* and $X_\bullet$ is the *nerve of* (*i.e.,* simplicial complex associated with) *a partially ordered set.* Each of the standard n-simplex complexes $\Delta[n]$ $(n \geq 0)$ furnishes a non trivial example.

Further, in this case, call a 1-simplex $[\mathbf{x_2}] = [x_1, x_0]$ *invertible* provided

$$((\mathbf{x_0}), [\mathbf{x_1}], [\mathbf{x_2}]) \in \bigwedge_2^0 \implies [\mathbf{x_0}].$$

Then it is easy to see (take $\mathbf{x_1} = [x_0, x_0]$) that $[\mathbf{x_2}] = [x_1, x_0]$ is just a symmetric pair, $[x_1, x_0] \iff [x_0, x_1]$ (so that for all $x_0 \in X_0$, the degenerate 1-simplices $s_0(x_0) = [x_0, x_0]$ are invertible). If all of X_1 is invertible, so that $X_\bullet$ is a Kan complex, then the 0-horn and 2-horn maps, as well as the 1-horn map, are bijections and X_1 is just the *graph of an equivalence relation.* This is the case for the image $X_\bullet^{(0)} \subseteq \mathbf{Cosk}^0(X_\bullet)$ of a Kan complex in its 0-coskeleton, the 0^{th}-complex in its canonical Postnikov tower , where it is just the equivalence relation whose quotient defines the set $\pi_0(X_\bullet)$ of connected components of $X_\bullet$. Note that $X_\bullet$ *is connected* iff *the canonical simplicial* 0*-boundary map* $X_\bullet \longrightarrow \mathbf{Cosk}^0(X_\bullet)$ *is surjective, i.e., so that* $X_\bullet^{(0)} = \mathbf{Cosk}^0(X_\bullet)$.

Finally, call a 0-simplex x_0 *isolated* if there exists a *unique* 0-simplex x_1 such that $[x_1, x_0]$. But since we have required that $[x_0, x_0] = s_0(x_0)$, uniqueness alone forces $x_1 = x_0$, *i.e.,* an isolated 0-simplex here is just an *isolated point.* If all of the 0-simplices are isolated in a 0-dimensional Postnikov-complex, *i.e.,* we have minimality for the set of 0-simplices, the complex is discrete: *all higher dimensional simplices are degenerate* and $\mathbf{Sk}^0(X_\bullet) \xrightarrow{\sim} X_\bullet \xrightarrow{\sim} K(X_0, 0)$, the constant complex. In Glenn's terminology, *a 0-dimensional hypergroupoid* is a constant complex since, by definition, all of the horn maps in dimensions > 0 are bijections. Clearly, *a* 0*-dimensional category* (or groupoid) should just be a set, and the *nerve of a* 0*-dimensional category* (or 0-dimensional groupoid) should just be a constant complex. That is the terminology we will adopt here. Note also that $\mathbb{P}(K(X_0,0)) = K(X_0,0)$, so that any number of iterations of $\mathbb{P}$ that results in a constant complex stabilizes at that point.

As we shall see, in spite of its triviality, for $n = 0$, this last case where the subset $X_1 \subseteq \mathbf{Cosk}^0(X_\bullet)_1 = X_0 \times X_0$ of "commutative" 1-simplices is the subset $\Delta \subseteq X_0 \times X_0$ of degenerate 1-simplices, is the one case which is most relevant to us in this paper. However, note for *any* simplicial complex $X_\bullet$, if we only require that for all $x_0 \in X_0 = \bigwedge_1^1(X_\bullet) = \bigwedge_1^0(X_\bullet)$, there exists a (not necessarily unique) invertible 1-simplex x_{01} such that $d_1(x_{01}) = x_0$ (or $d_1(x_{01}) = x_0$), then, as long as degenerate 1-simplices are invertible (in any sense given to the term) this requirement will always be satisfied by $s_0(x_0) : x_0 \longrightarrow x_0$. This will be the case in any of the complexes which we consider and where it will not, in general, be the case that all of the simplices of X_1 will be required to be invertible.

4. 1-Dimensional Postnikov Complexes: Nerves of Categories and Groupoids.

Now for a quick look at our basic context for $n = 1$ where $X_\bullet$ is a 1-dimensional Postnikov-complex:

$$\mathbf{Cosk}^2(X_\bullet) \simeq X_\bullet \subseteq \mathbf{Cosk}^1(X_\bullet).$$

The truncated complex $X_\bullet]_0^1$ is just a *directed graph* with *a distinguished set of loops*, $s_0(X_0) \subseteq X_1$. The 2-simplices of the 1-Coskeleton of $X_\bullet$ are just the triplets $(x_{12}\ ,\ x_{02}\ ,\ x_{01})$ of 1-simplices whose faces satisfy the simplicial identities, here $d_0(x_{12}) = d_0(x_{02}) = x_2$, $d_1(x_{02}) = d_1(x_{01}) = x_0$, and $d_0(x_{01}) = d_1(x_{12}) = x_1$, so that the simplicial matrix of boundaries becomes

$$[\partial(x_{12}), \partial(x_{02}), \partial(x_{01})] = \begin{bmatrix} x_2 & x_1 \\ x_2 & x_0 \\ x_1 & x_0 \end{bmatrix}.$$

If we direct x_{ij} with $d_0(x_{ij}) = x_j$ and $d_1(x_{ij}) = x_i$ as $x_{ij} : x_i \longrightarrow x_j$, making $d_0(x_{ij})$ the "target" and $d_1(x_{ij})$ the "source" of x_{ij} following the usual "face opposite vertex" simplicial conventions we are just considering the set $\mathbf{Cosk}^1(X_\bullet)_2 = K_2(X_\bullet) = \mathrm{SimKer}(X_\bullet]_0^1)$ of 2-simplices of $\mathbf{Cosk}^1(X_\bullet)$ as triplets of the form

$$(x_{12}\ ,\ x_{02}\ ,\ x_{01}) = (x_2 \leftarrow x_1, x_2 \leftarrow x_0, x_1 \leftarrow x_0),$$

which are, of course, geometrically nothing more than triangles of 1-simplices whose faces $d_i = \mathrm{pr}_i$ $(0 \leq i \leq 2)$ are the directed sides $d_i = x_{0\hat{i}2}$ $(0 \leq i \leq 2)$ as in Figure 13.

The set X_2 of 2-simplices of $X_\bullet$ is a subset of this set of 2-simplex triangles which we call the *commutative 2-simplices* ("the set of commutative triangles") of $X_\bullet$ and whose 2-simplex members we again denote for convenience by using *square brackets*, $[x_{12}\ ,\ x_{02}\ ,\ x_{01}]$. Since $X_\bullet$ is a subcomplex, for any 1-simplex $x_{01} : x_0 \longrightarrow x_1$, the degenerate simplices $s_0(x_{01}) = (x_{01}, x_{01}, s_0(x_0))$ and $s_1(x_{01}) = (s_0(x_1), x_{01}, x_{01})$ of $K_2(X_\bullet)$

$$
\begin{tikzcd}
x_0 \arrow[r, "x_{01}"] \arrow[dr, "x_{02}"'] & x_1 \arrow[d, "x_{12}"] \\
& x_2
\end{tikzcd}
$$

Figure 13: Typical element of the simplicial kernel $K_2(X_\bullet)$ of the directed graph $X_\bullet]_0^1$

$$
\begin{tikzcd}
& x_0 \arrow[dl, "x_{03}"'] \arrow[dr, "x_{01}"] \arrow[dd] & \\
x_3 & & x_1 \arrow[ll] \arrow[dl, "x_{12}"] \\
& x_2 \arrow[ul, "x_{23}"] &
\end{tikzcd}
$$

Figure 14: 3-simplex as a tetrahedron of open triangles

are both commutative, which we indicate by $s_0(x_{01}) = [x_{01}, x_{01}, s_0(x_0)]$ and $s_1(x_{01}) = [s_0(x_1), x_{01}, x_{01}]$.

Now the 3-simplices of $\mathbf{Cosk}^1(X_\bullet)$ are just tetrahedra of triangles, Figure 14, which we view as simplicial matrices

$$
\begin{bmatrix}
(x_{23} & x_{13} & x_{12}) \\
(x_{23} & x_{03} & x_{02}) \\
(x_{13} & x_{03} & x_{01}) \\
(x_{12} & x_{02} & x_{01})
\end{bmatrix},
$$

and since $X_\bullet$ is 2-coskeletal, the set of 3-simplices of $X_\bullet$ is the set of tetrahedra of commutative triangles

$$
\begin{bmatrix}
[x_{23} & x_{13} & x_{12}] \\
[x_{23} & x_{03} & x_{02}] \\
[x_{13} & x_{03} & x_{01}] \\
[x_{12} & x_{02} & x_{01}]
\end{bmatrix}.
$$

The horns in dimension 2 of $X_\bullet$ have the form

$$
(\,\_\_\, , x_{02} \,,\, x_{01}) \in \bigwedge\nolimits_2^0, (x_{12} \,, \_\_ \,,\, x_{01}) \in \bigwedge\nolimits_2^1, \text{ and } (x_{12} \,,\, x_{02} \,, \_\_\,) \in \bigwedge\nolimits_2^2,
$$

and geometrically appear as in Figure 15 (which is probably why these "simplicial open boxes" are called "horns").

Notice that the boundary of the "missing face" in any of these horns is uniquely determined by the boundaries of the simplices in the horn, for instance, in $\bigwedge_2^1$ by $(x_2, x_0) = (d_0(x_{12}), d_1(x_{01}))$.

$$
\begin{tikzcd}
x_0 \arrow[r, "x_{01}"] \arrow[dr, "x_{02}"'] & x_1 \\
& x_2
\end{tikzcd}
\qquad
\begin{tikzcd}
x_0 \arrow[r, "x_{01}"] & x_1 \arrow[d, "x_{12}"] \\
& x_2
\end{tikzcd}
\qquad
\begin{tikzcd}
x_0 \arrow[dr, "x_{02}"'] & x_1 \arrow[d, "x_{12}"] \\
& x_2
\end{tikzcd}
$$

Figure 15: Typical 0 , 1, and 2 horns of a directed graph

The assertion that here the 1-horn map $\mathrm{pr}_{\hat{1}} : X_2 \longrightarrow \bigwedge^1_2$ is surjective (resp., *bijective*) is equivalent to the assertion:

Given any $(x_{12}, \_, x_{01}) \in \bigwedge^1_2$, there exists a (resp. *unique*) 1-simplex x_{02} such that $[x_{12}, x_{02}, x_{01}]$, *i.e.,* such that $(x_{12}, x_{02}, x_{01}) \in X_2 \subseteq Cosk^1(X_\bullet)_2$.

Consequently, if $\mathrm{pr}_{\hat{1}} : X_2 \longrightarrow \bigwedge^1_2$ is a bijection, then X_2 is bijectively equivalent the graph of a "categorical composition law" ("categorical" at least with respect to when it is defined and what the source and target of the composition is) $(\_ \otimes \_) : \bigwedge^1_2 \longrightarrow X_1$, $(\_ \otimes \_) = d_1 \mathrm{pr}_{\hat{1}}^{-1}$ and

$$[x_{12}, x_{02}, x_{01}] \quad \Leftrightarrow \quad x_{02} = x_{12} \otimes x_{01}.$$

Note that this means that in geometric notation our use of the term "commutative" is fully equivalent to the standard mathematical (diagrammatic) use of the term: the triangle (element of the simplicial kernel $Cosk^1(X_\bullet)_2$) in Figure 13 is *commutative* iff $x_{02} = x_{12} \otimes x_{01}$.

Moreover, since for all 1-simplices x_{01}, the degenerate 2-simplices

$$s_0(x_{01}) = [x_{01}, x_{01}, s_0(x_0)] \quad \text{and} \quad s_1(x_{01}) = [s_0(x_1), x_{01}, x_{01}]$$

are commutative (because $X_\bullet \subseteq Cosk^1(X_\bullet)$ as a simplicial complex), we have that

$$s_0(x_1) \otimes x_{01} = x_{01} \quad \text{and} \quad x_{01} \otimes s_0(x_0) = x_{01},$$

so that $\mathrm{id}(x) = s_0(x) : x \longrightarrow x$ furnishes every 1-simplex with a source and target "identity" arrow for this categorical law of composition.

In dimension 3 the horns of a 1-dimensional Postnikov complex have the following typical simplicial matrix form:

$$\begin{bmatrix} (\, x_{23} & x_{13} & x_{12} \,) \\ [\, x_{23} & x_{03} & x_{02} \,] \\ [\, x_{13} & x_{03} & x_{01} \,] \\ [\, x_{12} & x_{02} & x_{01} \,] \end{bmatrix} \in \bigwedge^0_3,$$

$$\begin{bmatrix} [\, x_{23} & x_{13} & x_{12} \,] \\ (\, x_{23} & x_{03} & x_{02} \,) \\ [\, x_{13} & x_{03} & x_{01} \,] \\ [\, x_{12} & x_{02} & x_{01} \,] \end{bmatrix} \in \bigwedge^1_3, \qquad \begin{bmatrix} [\, x_{23} & x_{13} & x_{12} \,] \\ [\, x_{23} & x_{03} & x_{02} \,] \\ (\, x_{13} & x_{03} & x_{01} \,) \\ [\, x_{12} & x_{02} & x_{01} \,] \end{bmatrix} \in \bigwedge^2_3,$$

and

$$\begin{bmatrix} [\,x_{23} & x_{13} & x_{12}\,] \\ [\,x_{23} & x_{03} & x_{02}\,] \\ [\,x_{13} & x_{03} & x_{01}\,] \\ (\,x_{12} & x_{02} & x_{01}\,) \end{bmatrix} \in \bigwedge\nolimits_3^3 .$$

Each of the four horn maps $\mathrm{pr}_{\hat{k}} : X_3 \longrightarrow \bigwedge_3^k(X_\bullet)$ $(0 \leq k \leq 3)$ here is clearly injective and is thus a bijection provided it is surjective. For instance, if $X_\bullet$ is a weak Kan complex, $\mathrm{pr}_{\hat{1}}$ and $\mathrm{pr}_{\hat{2}}$ are both bijections which is thus here just the truth of the two implications

$$\begin{bmatrix} [\,x_{23} & x_{13} & x_{12}\,] \\ (\,x_{23} & x_{03} & x_{02}\,) \\ [\,x_{13} & x_{03} & x_{01}\,] \\ [\,x_{12} & x_{02} & x_{01}\,] \end{bmatrix} \in \bigwedge\nolimits_3^1(X_\bullet) \Longrightarrow [x_{23}, x_{03}, x_{02}]$$

and

$$\begin{bmatrix} [\,x_{23} & x_{13} & x_{12}\,] \\ [\,x_{23} & x_{03} & x_{02}\,] \\ (\,x_{13} & x_{03} & x_{01}\,) \\ [\,x_{12} & x_{02} & x_{01}\,] \end{bmatrix} \in \bigwedge\nolimits_3^2(X_\bullet) \Longrightarrow [x_{13}, x_{03}, x_{01}].$$

Now suppose that $X_\bullet$ verifies the hypothesis:

(a) Given any $(x_{12}, \_\,, x_{01}) \in \bigwedge_2^1(X_\bullet)$, there exists a *unique* $x_{02} \in X_1$ such that $[x_{12}, x_{02}, x_{01}]$ so that, equivalently, $\mathrm{pr}_{\hat{1}} : X_2 \longrightarrow \bigwedge_2^1$ is a bijection.

Then the unitary law of composition $\_ \otimes \_$ provided by (a) is associative if, and any if, $\mathrm{pr}_{\hat{1}} : X_3 \longrightarrow \bigwedge_3^1$ or $\mathrm{pr}_{\hat{2}} : X_3 \longrightarrow \bigwedge_3^2$, and hence both, are bijections.

If (a) is verified, then we have

$$\begin{bmatrix} [\,x_{23} & x_{23} \otimes x_{12} & x_{12}\,] \\ (\,x_{23} & (x_{23} \otimes x_{12}) \otimes x_{01} & x_{12} \otimes x_{01}\,) \\ [\,x_{23} \otimes x_{12} & (x_{23} \otimes x_{12}) \otimes x_{01} & x_{01}\,] \\ [\,x_{12} & x_{12} \otimes x_{01} & x_{01}\,] \end{bmatrix} \in \bigwedge\nolimits_3^1(X_\bullet).$$

But if one has that $\mathrm{pr}_{\hat{1}}$ is a bijection, then Row 1 is commutative, *i.e.,*

$$[\ x_{23}\ , (x_{23} \otimes x_{12}) \otimes x_{01},\ x_{12} \otimes x_{01}\ ],$$

which is just then equivalent to the equality

$$(x_{23} \otimes x_{12}) \otimes x_{01} = x_{23} \otimes (x_{12} \otimes x_{01}).$$

The same equality is reflexively given by the assumption that $\mathrm{pr}_{\hat{2}}$ is a bijection.

Thus if we have a 1-dimensional Postnikov complex $X_\bullet$ in which the horn maps $\mathrm{pr}_{\hat{1}} : X_2 \longrightarrow \bigwedge_2^1(X_\bullet)$ and $\mathrm{pr}_{\hat{i}} : X_3 \longrightarrow \bigwedge_3^i(X_\bullet)$ $\;1 \leq i \leq 2$ (and because $X_\bullet$ is 2-Coskeletal *all* higher dimensional ones as well) are bijections, then the unitary law of composition provided by $\mathrm{pr}_{\hat{1}}$ is associative and we have the structure of a category $\mathbb{C} =_{Def} \mathbf{Cat}(X_\bullet)$ whose objects are the 0-simplices of $X_\bullet$ and whose hom sets $\mathrm{Hom}_{\mathbb{C}}(x_0, x_1)$ are the fibers of the mapping $\langle S, T\rangle = \langle d_1, d_0\rangle : X_1 \longrightarrow X_0 \times X_0$. $X_1 = \coprod_{(x_0,x_1)\in X_0\times X_0} \mathrm{Hom}_{\mathbb{C}}(x_0, x_1)$ is then the set of arrows of $\mathbb{C}$. In fact, it is not difficult to see that $X_\bullet \xrightarrow{\sim} \mathrm{Ner}(\mathbb{C})$, the Grothendieck Nerve of $\mathbb{C}$, as described in the *Introduction.* For example, another way to look at this is to observe that the elements of the simplicial kernel $Cosk^2(X_\bullet)_3$ are in bijective correspondence with the set of composable triplets

$$x_0 \xrightarrow{x_{01}} x_1 \xrightarrow{x_{12}} x_2 \xrightarrow{x_3} x_3$$

of arrows in X_1 *whose composition is associative* and this is always a subset of Grothendieck's $\mathrm{Ner}(\mathbb{C})_3$ which is the set of *all* composable triplets, and which, in turn, is in bijective correspondence with each of the sets of horns $\bigwedge_2^1(X_\bullet)$ and $\bigwedge_2^1(X_\bullet)$. But to say that these two sets are equal is just to say that the law of composition is associative.

Finally note here that in describing the nerve of a category as a 1-dimensional Postnikov complex $X_\bullet$ in which $\mathrm{pr}_{\hat{1}} : X_2 \longrightarrow \bigwedge_2^1(X_\bullet)$ and $\mathrm{pr}_{\hat{i}} : X_3 \longrightarrow \bigwedge_3^i(X_\bullet)\quad (1 \le i \le 2)$ are all bijections, we have a *weak* Kan complex $X_\bullet$ in which the (weak) Kan conditions are satisfied *exactly* in all dimensions > 1, and it was just such complexes that Ross Street identified as being the nerves of categories. Thus to close the equivalence of these characterizations we only need to note that Street's "*weak* 1-dimensional hypergroupoids" are always 1-dimensional Postnikov complexes. The proof of this is an inductive diagram chase which we leave to the reader.

We now explore the notion of "invertibility", at first for complexes which are not yet assumed to be nerves of categories.

If $X_\bullet$ is a weak Kan complex which is a 1-dimensional Postnikov-complex, let us call a 1-simplex $x_{01} : x_0 \longrightarrow x_1$ *invertible* if given any $(\_, x_{02}, x_{01}) \in \bigwedge_2^0(X_\bullet)$ there exists a *unique* 1-simplex $x_{12} \in X_1$ such that $[x_{12}, x_{02}, x_{01}]$

Let $I_1 \subseteq X_1$ be the set of invertible 1-simplices.

Now if $X_\bullet$ is a weak Kan complex, $\mathrm{pr}_{\hat{1}} : X_2 \longrightarrow \bigwedge_2^1(X_\bullet)$ is surjective: for any $(x_{12}, \_, x_{01}) \in \bigwedge_2^1(X_\bullet)$ there exists a 1-simplex $x_{02} \in X_1$ such that $[x_{12}, x_{02}, x_{01}] \in X_2$.

Now suppose that $X_\bullet$ further satisfies the following hypothesis: *For any* 0*-simplex* $x_0 \in X_0$*, the degenerate simplex* $s_0(x_0) : x_0 \longrightarrow x_0$ *is invertible.* Then since $X_\bullet$ is a subcomplex of $\mathbf{Cosk}^1(X_\bullet)$, $s_0(x_{01}) = [x_{01}, x_{01}, s_0(x_0)]$ and x_{01} is the unique filler of $(\_, x_{01}, s_0(x_0)) \in \bigwedge_2^0(X_\bullet)$.

$$[x'_{01}, x_{01}, s_0(x_0)] \in X_2 \Longrightarrow x'_{01} = x_{01}.$$

4.1. Remark. Thus since $(x'_{01}, x_{01}, s_0(x_0)) \in K_2^1 = Cosk^1(X_\bullet)_2$, x'_{01} and x_{01} necessarily have the same faces and the 2-simplex $[x'_{01}, x_{01}, s_0(x_0)]$ is just, by definition, a directed simplicial homotopy of x_{01} with x'_{01} in $X_\bullet$. *The hypothesis that degenerate* 1*-simplices are invertible is here equivalent to requiring minimality for the* 1*-simplices of* $X_\bullet$*: homotopic* 1*-simplices are equal.*

Looking again at the 1-horn in dimension 2: we claim that this hypothesis of invertibility, $s_0(X_0) \subseteq I_1$, forces the 1-horn map to be injective.

In effect, for $(x_{12}, \_\_ , x_{01}) \in \bigwedge^1_2(X_\bullet)$ suppose that we have $[x_{12}, x_{02}, x_{01}]$ and $[x_{12}, x'_{02}, x_{01}]$ in X_2. Then the simplicial matrix

$$\begin{bmatrix} [x_{12} & x_{02} & x_{01}] \\ [x_{12} & x'_{02} & x_{01}] \\ (x_{02} & x'_{02} & s_0(x_0)) \\ [x_{01} & x_{01} & s_0(x_0)] \end{bmatrix} \in \textstyle\bigwedge^2_3(X_\bullet).$$

But then $[x_{02}, x'_{02}, s_0(x_0)]$ and the hypothesis forces $x_{02} = x'_{02}$, as asserted. But then $\mathrm{pr}_{\hat{1}} : X_2 \longrightarrow \bigwedge^1_2(X_\bullet)$ is a bijection and $X_\bullet$ is the nerve of a category. ($\mathrm{pr}_{\hat{1}} : X_3 \longrightarrow \bigwedge^1_3$ is already a bijection since we assumed that $X_\bullet$ was a weak Kan complex.)

Note that if $X_\bullet$ is the nerve of a category, then this minimality of the set of 1-simplices is equivalent to the statement that the path-homotopy complex $\mathbb{P}(X_\bullet)$ (Section 2.6) is discrete,

$$\mathbb{P}(X_\bullet) \simeq K(X_1, 0),$$

where X_1 is the set of arrows of the category. In effect, $\mathbb{P}(X_\bullet)_0 = X_1$ and $\mathbb{P}(X_\bullet)_2$ is the subset of the set of commutative triangles whose 2-face is degenerate. But then, in any simplicial complex, such a 2-simplex $\mathbf{t}$ is just a directed homotopy of the 1-simplex $d_1(\mathbf{t}) = x_{01}$ to $d_0(\mathbf{t}) = y_{01}$ and in the nerve of a category must be a *commutative* triangle of the form $\mathbf{t} = [y_{01}, x_{01}, s_0(x_0))]$:

$$\begin{tikzcd} x_0 \arrow[r, "s_0(x_0)=\mathrm{id}(x_0)"] \arrow[dr, "x_{01}"'] & x_0 \arrow[d, "y_{01}"] \\ & x_1 \end{tikzcd}$$

But in a category this is equivalent to $x_{01} = y_{01} \otimes s_0(x_0) = y_{01}$ since $s_0(x_0) = \mathrm{id}(x_0)$. Simplicially, $\mathbf{t} = s_0(x_{01})$ and similarly in all higher dimensions.[26]

Now, as an exercise to show how "possible horn conditions" interact with each other in this dimension let us, in addition to any of the above abstract requirements which make $X_\bullet$ into the nerve of a category, define a 1-simplex x_{01} to be **invertible* provided that it satisfies the following two properties:

(a) given any 0-horn of the form $(\_, x_{02}, x_{01}) \in \bigwedge^0_2(X_\bullet)$ (*i.e.*, with x_{01} as its 2-face), there exists a 1-simplex x_{12} such that $[x_{12}, x_{02}, x_{01}]$ ($\Longleftrightarrow x_{12} \otimes x_{01} = x_{02}$) , and
(b) given any 0-horn in $\bigwedge^0_3(X_\bullet)$ with $d_2([\mathbf{x_2}]) = d_2([\mathbf{x_3}]) = x_{01}$, one has $[x_{23}, x_{13}, x_{12}]$, *i.e.*,

$$\begin{bmatrix} (x_{23} & x_{13} & x_{12}) \\ [x_{23} & x_{03} & x_{02}] \\ [x_{13} & x_{03} & x_{01}] \\ [x_{12} & x_{02} & x_{01}] \end{bmatrix} \Longrightarrow [x_{23}, x_{13}, x_{12}].$$

[26] In the next dimension of our basic simplicial setting ($n = 2$), where $X_\bullet$ is the nerve of a bicategory, $\mathbb{P}(X_\bullet)$ will be the nerve of a category and $\mathbb{P}^2(X_\bullet)$ will be discrete, and if $n = 3$, where $X_\bullet$ is the nerve of a tricategory, $\mathbb{P}(X_\bullet)$ will be the nerve of a bicategory and $\mathbb{P}^3(X_\bullet)$ will be discrete.

If x_{01} is *invertible, then the 1-simplex x_{12} which fills $(\_, x_{02}, x_{01})$ is unique, for if $[x'_{12}, x_{02}, x_{01}]$, then since $s_1(x_{02}) = [s_0(x_2), x_{02}, x_{02}]$ we may form the simplicial matrix

$$\begin{bmatrix} (s_0(x_2) & x'_{12} & x_{12}) \\ [s_0(x_2) & x_{02} & x_{02}] \\ [\ \ x'_{12} & x_{02} & x_{01}] \\ [\ \ x_{12} & x_{02} & x_{01}] \end{bmatrix} \in \bigwedge\nolimits_3^0(X_\bullet),$$

and use (b) to conclude that Row 0 is commutative, but then

$$[s_0(x_2), x'_{12}, x_{12}] \iff x_{12} = s_0(x_2) \otimes x_{12} = x'_{12},$$

since we already have

$$s_1(x_{12}) = [s_0(x_2), x_{12}, x_{12}] \iff x_{12} = s_0(x_2) \otimes x_{12}.$$

If x_{01} is *invertible, let x^*_{01} be a 1-simplex such that

$$[x^*_{01}, s_0(x_0), x_{01}]\ (\iff\ x^*_{01} \otimes x_{01} = s_0(x_0)\,)$$

and using the two commutative degeneracies of x_{01} form the simplicial matrix

$$\begin{bmatrix} (x_{01} & s_0(x_1) & x^*_{01}) \\ [x_{01} & x_{01} & s_0(x_0)] \\ [s_0(x_1) & x_{01} & x_{01}] \\ [x^*_{01} & s_0(x_0) & x_{01}] \end{bmatrix} \in \bigwedge\nolimits_3^0(X_\bullet)$$

From (b) it follows that Row 0 is commutative, but then

$$[x_{01}, s_0(x_1), x^*_{01}] \iff x_{01} \otimes x^*_{01} = s_0(x_1)$$

and thus x_{01} is an *isomorphism* in the usual categorical sense.[27] If x_{01} is an isomorphism in a category, it is easy to see that x_{01} satisfies (a) and (b) and hence that x_{01} is invertible. The same result would have occurred if we had defined "*invertibility" using instead the other two extremal horns, $\bigwedge_2^2$ and $\bigwedge_3^3$ with x_{23} in place of x_{01}.

Since we *defined* a 1-simplex x_{01} to be *invertible iff* it satisfied the condition:

Given any 0*-horn of the form*

$$(\_, x_{02}, x_{01}) \in \bigwedge\nolimits_2^0(X_\bullet),$$

there exists a unique 1*-simplex* x_{12} *such that* $[x_{12}, x_{02}, x_{01}]$,

then this property *alone* is necessary and sufficient for x_{01} to be an isomorphism in a category and the two definitions are equivalent. If every 1-simplex is invertible, the category is a groupoid (*i.e.,* every arrow is an isomorphism) and then the 0-horn maps in dimensions 2 and 3 are bijections, as the above shows, and the same is true of the 2 and 3-horn maps in these two dimensions.

To summarize, we have the following

---

[27]In more familiar categorical terms, Property (b) in the definition of *invertibility is equivalent to right cancellability for x_{01} in the presence of associativity and a left identity, *i.e.,* x_{01} is an *epimorphism.* Property (a) applied with $x_{02} = s_0(x_0)$ says that x_{01} is a *section* with x^*_{01} a retraction for it. (a) and (b) together then say that x_{01} is an epimorphic section, hence an isomorphism.

4.2. Theorem. **(Simplicial Characterization of the Nerve of a Category)** *The following are equivalent for a simplicial complex $X_\bullet$:*

- $X_\bullet$ *is the Grothendieck nerve of a category* *(i.e.,* $X_n \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{CAT}}([n], \mathbb{C})$*).*

- $X_\bullet$ *is a* 1*-dimensional Postnikov-complex* *(i.e.,* $Cosk^2(X_\bullet) \xrightarrow{\sim} X_\bullet \subseteq Cosk^1(X_\bullet)$*) in which the inner-horn maps* $\mathrm{pr}_{\hat{1}} : X_3 \longrightarrow \bigwedge_3^1(X_\bullet)$, $\mathrm{pr}_{\hat{2}} : X_3 \longrightarrow \bigwedge_3^2(X_\bullet)$ *and* $\mathrm{pr}_{\hat{1}} : X_2 \longrightarrow \bigwedge_2^1(X_\bullet)$ *are bijections.*

- $X_\bullet$ *is a* 1*-dimensional Postnikov-complex in which the inner-horn maps* $\mathrm{pr}_{\hat{1}} : X_2 \longrightarrow \bigwedge_2^1(X_\bullet)$ *and* $\mathrm{pr}_{\hat{1}} : X_3 \longrightarrow \bigwedge_3^1(X_\bullet)$ *are surjective and in which every degenerate* 1*-simplex is invertible.*

- $X_\bullet$ *is a weak Kan complex which is a* 1*-dimensional Postnikov-complex in which every degenerate* 1*-simplex is invertible.*

- $X_\bullet$ *is a weak Kan complex which is a* 1*-dimensional Postnikov-complex in which the set of* 1*-simplices is homotopicly minimal (i.e.,* $[x_{01}, x'_{01}, s_0(x_0)] \Longrightarrow x_{01} = x'_{01}$*).*

- $X_\bullet$ *is a weak Kan complex which is a* 1*-dimensional Postnikov-complex for which the homotopy-path complex* $\mathbb{P}(X_\bullet)$ *is discreet (i.e.,* $\mathbb{P}(X_\bullet) \xrightarrow{\sim} K(X_1, 0)$*).*

- $X_\bullet$ *is a weak Kan complex in which the weak Kan conditions are satisfied exactly (i.e.,* $\mathrm{pr}_{\hat{i}} : X_n \xrightarrow{\sim} \bigwedge_n^k(X_\bullet) \quad (0 < k < n)$*) in all dimensions* $n > 1$ *(Ross Street's criterion).*

as well as the

4.3. Theorem. **Simplicial Characterization of the Nerve of a Groupoid** *The following are equivalent for a simplicial complex $X_\bullet$:*

- $X_\bullet$ *is the Grothendieck nerve of a groupoid,*

- $X_\bullet$ *is the Grothendieck nerve of a category in which every* 1*-simplex is invertible, i.e.,* $I(X_\bullet)_1 = X_1$

- $X_\bullet$ *is a Kan complex in which the Kan condition is satisfied exactly (i.e.,* $\mathrm{pr}_{\hat{i}} : X_n \xrightarrow{\sim} \bigwedge_n^k(X_\bullet) \quad (0 \leq k \leq n)$*) in all dimensions* $n > 1$*. (*$X_\bullet$ *is a* 1*-dimensional (Kan) hypergroupoid in the terminology of [Glenn 1982]).*

$$\begin{tikzcd} x_0 \arrow[r, bend left, "x_{01}"] \arrow[r, bend right, "y_{01}"'] \arrow[r, phantom, "\Downarrow"] & x_1 \end{tikzcd}$$

Figure 16: Typical 2-cell of $\mathbb{B}$

4.4. Remark. The game played above with the simplicial matrices will be typical of those which we will use in our characterization of the nerve of a bicategory where invertibility constraints are unavoidable and the simplicial matrix proofs will be the most efficient ones possible. If one looks at condition (b) above in the definition of *invertible and the two desired conclusions, that of uniqueness and that x^*_{01} was a right inverse as well as a (by definition) left inverse, then the simplicial matrices used in the proofs are essentially determined. They are among the very few possible ones that could be constructed using the data and the requirement that the matrix be simplicial, as the reader will discover on playing with the conditions.

5. Bicategories in the Sense of Bénabou

For convenience in this exposition, in particular for the simplicial part, the definition of a ("small") bicategory that will be used here will be a slightly modified and expanded (and hence somewhat redundant) version of Bénabou's [Bénabou 1967]. To avoid confusion and to make the differences clear, we will now give the explicit definition which we will use in this paper.

5.1. The Definition of a Bicategory Used in this Paper. A *bicategory* $\mathbb{B}$ will have as **Data**

1. A set $\mathbf{Ob}(\mathbb{B})$ of *objects* (or 0-cells)

2. For each ordered pair (x_0, x_1) of objects of $\mathbb{B}$, a *(small) category* $\mathbb{B}(x_0, x_1)$ whose objects $x_{01} : x_0 \longrightarrow x_1$ are called 1-*cells* with 0-*source* x_0 and 0-*target* x_1 and whose arrows are called 2-*cells* and denoted by $\alpha : x_{01} \Longrightarrow y_{01}$ (the 1-*source* of α is x_{01} and the 1-*target* of α is y_{01}). Thus a typical 2-cell of $\mathbb{B}$ is depicted in Figure 16. Composition of 2-cells in the category $\mathbb{B}(x_0, x_1)$ is called *vertical composition* and will be denoted by simple juxtaposition $\beta\alpha$ or, if emphasis is needed, by $\beta \circ \alpha$.

3. For every ordered triplet (x_0, x_1, x_2) of objects of $\mathbb{B}$ a law of composition, called *horizontal or tensor composition*

$$\otimes : \mathbb{B}(x_0, x_1) \times \mathbb{B}(x_1, x_2) \longrightarrow \mathbb{B}(x_0, x_2)$$

$$(\alpha, \beta) \mapsto \beta \otimes \alpha$$

4. For every object x_0 of $\mathbb{B}$, a distinguished 1-cell $I(x_0) : x_0 \longrightarrow x_0$, called the *pseudo-identity* of x_0.

5. For each 1-cell $x_0 \xrightarrow{x_{01}} x_1$ of $\mathbb{B}$, 2-cell *left and right pseudo-identity isomorphisms*

$$\lambda(x_{01}) : x_{01} \overset{\sim}{\Longrightarrow} I(x_1) \otimes x_{01} \text{ and } \rho(x_{01}) \overset{\sim}{\Longrightarrow} x_{01} \otimes I(x_0).$$

6. For every composable triplet of 1-cells,

$$x_o \xrightarrow{x_{01}} x_1 \xrightarrow{x_{12}} x_2 \xrightarrow{x_{23}} x_3,$$

a 2-cell *associativity isomorphism*

$$\mathrm{A}(x_{23}, x_{12}, x_{01}) : x_{23} \otimes (x_{12} \otimes x_{01}) \overset{\sim}{\Longrightarrow} (x_{23} \otimes x_{12}) \otimes x_{01}.$$

These Data are required to satisfy the following **Axioms**:

- **Full Interchange Law:** Horizontal (tensor) composition is *functorial*, *i.e.*, for all vertically composable 2-cells,

$$\mathrm{id}(x_{12}) \otimes \mathrm{id}(x_{01}) = \mathrm{id}(x_{12} \otimes x_{01})$$

and

$$(\beta_2 \circ \beta_1) \otimes (\alpha_2 \circ \alpha_1) = (\beta_2 \otimes \alpha_2) \circ (\beta_1 \otimes \alpha_1)$$

hold.

We break this axiom into the equivalent conjunction of three more easily digested parts:

- **Right and Left Functorial Actions of 1-cells on 2-cells :** For any 1-cells $x_{12} : x_1 \longrightarrow x_2$, and $x_{01} : x_0 \longrightarrow x_1$ we have defined *functorial* actions

$$\mathbb{B}(x_{01}, x_2) : \mathbb{B}(x_1, x_2) \longrightarrow \mathbb{B}(x_0, x_2),$$

denoted by

$$\mathbb{B}(x_{01}, x_2)(\beta) = \beta \otimes x_{01} : x_{12} \otimes x_{01} \Longrightarrow y_{12} \otimes x_{01}$$

and

$$\mathbb{B}(x_0, x_{12}) : \mathbb{B}(x_0, x_1) \longrightarrow \mathbb{B}(x_0, x_2),$$

denoted by

$$\mathbb{B}(x_0, x_{12})(\alpha) = x_{12} \otimes \alpha : x_{12} \otimes x_{01} \Longrightarrow x_{12} \otimes y_{01}$$

- **Godement Interchange Law:** If 2-cells β and α are such that $T_0(\alpha) = S_0(\beta)$, for example $(\alpha, \beta) \in \mathbb{B}(x_0, x_1) \times \mathbb{B}(x_1, x_2)$, the diagram in Figure 17 is commutative, *i.e.*,

$$(\beta \otimes T_1(\alpha)) \circ (S_1(\beta) \otimes \alpha) = (T_1(\beta) \otimes \alpha) \circ (\beta \otimes S_1(\alpha)).$$

The common diagonal in Figure 17 then defines the tensor composition $\beta \otimes \alpha$. 1-cells are identified with their 2-cell identities.

Naturality and Coherence of the 2-cell Associativity Isomorphism: The associativity isomorphism is *natural* in each of its three variables:

$$\begin{tikzcd}
S_1(\beta)\otimes S_1(\alpha) \arrow[r, Rightarrow, "\beta\otimes S_1(\alpha)"] \arrow[d, Rightarrow, "S_1(\beta)\otimes\alpha"'] \arrow[dr, Rightarrow, "\beta\otimes\alpha"] & T_1(\beta)\otimes S_1(\alpha) \arrow[d, Rightarrow, "T_1(\beta)\otimes\alpha"] \\
S_1(\beta)\otimes T_1(\alpha) \arrow[r, Rightarrow, "\beta\otimes T_1(\alpha)"'] & T_1(\beta)\otimes T_1(\alpha)
\end{tikzcd}$$

Figure 17: Godement Interchange Law

$$\begin{tikzcd}
(x_{23}\otimes x_{i-1i})\otimes x_{01} \arrow[r, Rightarrow, "\mathbf{A}(x_{23},x_{i-1i},x_{01})"] \arrow[d, Rightarrow, "(x_{23}\otimes\alpha_{i-1i})\otimes x_{01}"'] & x_{23}\otimes(x_{i-1i}\otimes x_{01}) \arrow[d, Rightarrow, "x_{23}\otimes(\alpha_{i-1i}\otimes x_{01})"] \\
(x_{23}\otimes y_{i-1i})\otimes x_{01} \arrow[r, Rightarrow, "\mathbf{A}(x_{23},y_{i-1i},x_{01})"'] & x_{23}\otimes(y_{i-1i}\otimes x_{01})
\end{tikzcd}$$

Figure 18: Naturality of $\mathbf{A}$ $(1 \leq i \leq 3)$

- For any composable triplet $x_o \xrightarrow{x_{01}} x_1 \xrightarrow{x_{12}} x_2 \xrightarrow{x_{23}} x_3$, of 1-cells, and for any 2-cell $\alpha_i : x_{i-1\ i} : x_{i-1\ i} \Longrightarrow y_{i-1\ i}$ $(1 \leq i \leq 3)$, the diagram in Figure 18 is commutative.
- The associativity isomorphism is *self-coherent*: For every composable quadruplet of 1-cells
$$x_o \xrightarrow{x_{01}} x_1 \xrightarrow{x_{12}} x_2 \xrightarrow{x_{23}} x_3 \xrightarrow{x_{34}} x_4\,,$$
the pentagonal diagram of 2-cells in Figure 19 (*Mac Lane–Stasheff Pentagon*) is commutative.

Naturality and Coherence of the 2-cell Pseudo-Identity Isomorphisms:

- The right and left pseudo-identity isomorphisms are *natural*, *i.,e.*: For any 2-cell $\alpha : x_{01} \Longrightarrow y_{01}$ the diagrams in Figure 20 are commutative.
- The pseudo-identity isomorphisms λ and ρ are *compatible* with the pseudo-identity I and with the associativity isomorphism A :

  For any object x_0 in $\mathbb{B}$, the 2-cell isomorphisms
$$\lambda(I(x_0)) : I(x_0) \overset{\sim}{\Longrightarrow} I(x_0) \otimes I(x_0) \text{and}$$
$$\rho(I(x_0)) : I(x_0) \overset{\sim}{\Longrightarrow} I(x_0) \otimes I(x_0)$$
  are equal. Moreover, the diagrams in Figure 21 are commutative.

Using the definitions of Section 5 we will now explicitly define the simplices, faces and degeneracies of the simplicial set $\mathbf{Ner}(\mathbb{B})$ which will be our *(geometric) nerve of the bicategory* $\mathbb{B}$.[28]

[28]In the second paper of this series, we will show that the complex defined here can equally well described as the simplicial set $\mathbf{Ner}(\mathbb{B})_n = \text{StrictlyUnitaryMorphisms}([n], \mathbb{B})$. This description, however

$$
\begin{tikzcd}
& x_{34}\otimes((x_{23}\otimes x_{12})\otimes x_{01}) \arrow[rr, Rightarrow, "\mathbf{A}(x_{34},x_{23}\otimes x_{12},x_{01})"] & & (x_{34}\otimes(x_{23}\otimes x_{12}))\otimes x_{01} \arrow[dr, Rightarrow, "\mathbf{A}(x_{34},x_{23},x_{12})\otimes x_{01}"] & \\
x_{34}\otimes(x_{23}\otimes(x_{12}\otimes x_{01})) \arrow[ur, Rightarrow, "x_{34}\otimes\mathbf{A}(x_{23},x_{12},x_{01})"] \arrow[drr, Rightarrow, "\mathbf{A}(x_{34},x_{23},x_{12}\otimes x_{01})"'] & & & & ((x_{34}\otimes x_{23})\otimes x_{12})\otimes x_{01} \\
& & (x_{34}\otimes x_{23})\otimes(x_{12}\otimes x_{01}) \arrow[urr, Rightarrow, "\mathbf{A}(x_{34}\otimes x_{23},\ x_{12},\ x_{01})"'] & &
\end{tikzcd}
$$

Figure 19: Mac Lane–Stasheff Pentagon

$$
\begin{tikzcd}
x_{01} \arrow[r, Rightarrow, "\alpha"] \arrow[d, Rightarrow, "\rho(x_{01})"'] & y_{01} \arrow[d, Rightarrow, "\rho(y_{01})"] \\
x_{01}\otimes I(x_0) \arrow[r, Rightarrow, "\alpha\otimes I(x_0)"'] & y_{01}\otimes I(x_0)
\end{tikzcd}
\qquad
\begin{tikzcd}
x_{01} \arrow[r, Rightarrow, "\alpha"] \arrow[d, Rightarrow, "\lambda(x_{01})"'] & y_{01} \arrow[d, Rightarrow, "\lambda(y_{01})"] \\
I(x_1)\otimes x_{01} \arrow[r, Rightarrow, "I(x_1)\otimes\alpha"'] & I(x_1)\otimes y_{01}
\end{tikzcd}
$$

Figure 20: Naturality of ρ and λ

$$
\begin{tikzcd}
x_{12}\otimes x_{01} \arrow[d, Rightarrow, "x_{12}\otimes\rho(x_{01})"'] \arrow[dr, Rightarrow, "\rho(x_{12}\otimes x_{01})"] & \\
x_{12}\otimes(x_{01}\otimes I(x_0)) \arrow[r, Rightarrow, "\mathbf{A}"'] & (x_{12}\otimes x_{01})\otimes I(x_0)
\end{tikzcd}
\qquad
\begin{tikzcd}
& x_{12}\otimes x_{01} \arrow[dl, Rightarrow, "\lambda(x_{12}\otimes x_{01})"'] \arrow[d, Rightarrow, "\lambda(x_{12})\otimes x_{01}"] \\
I(x_2)\otimes(x_{12}\otimes x_{01}) \arrow[r, Rightarrow, "\mathbf{A}"'] & (I(x_2)\otimes x_{12})\otimes x_{01}
\end{tikzcd}
$$

$$
\begin{tikzcd}
& x_{12}\otimes x_{01} \arrow[dl, Rightarrow, "x_{12}\otimes\lambda(x_{01})"'] \arrow[dr, Rightarrow, "\rho(x_{12})\otimes x_{01}"] & \\
x_{12}\otimes(I(x_1)\otimes x_{01}) \arrow[rr, Rightarrow, "\mathbf{A}"'] & & (x_{12}\otimes I(x_1))\otimes x_{01}
\end{tikzcd}
$$

Figure 21: Compatibility of ρ, λ & ρ, and λ with $\mathbf{A}$

$$
\begin{tikzcd}
x_0 \arrow[r, "x_{01}"] \arrow[dr, "x_{02}"'] \arrow[dr, phantom, shift left=3ex, "\Nearrow \tilde{x}_{012}"] & x_1 \arrow[d, "x_{12}"] \\
& x_2
\end{tikzcd}
$$

Figure 22: 2-simplex x_{012}

6. The Explicit Description of the Simplicial Set $\mathbf{Ner}(\mathbb{B})$ associated with a Bicategory

6.1. The 0, 1, and 2-simplices of $\mathbf{Ner}(\mathbb{B})$. As a simplicial set:

The 0-*simplices* of $\mathbf{Ner}(\mathbb{B})$ are the objects (0-cells) of the bicategory $\mathbb{B}$.

The 1-*simplices* of $\mathbf{Ner}(\mathbb{B})$ are the 1-cells with the face operator d_0 defined as the 0-target of the 1-cell and d_1 defined as the 0-source of the 1-cell.

$$x_{01} : x_0 \longrightarrow x_1 \iff d_0(x_{01}) = x_1 \text{ and } d_1(x_{01}) = x_0$$

If x_0 is a 0-cell of $\mathbb{B}$ and $I(x_0)$ is the corresponding pseudo-identity, then we define the corresponding *degenerate* 1-*simplex*

$$s_0(x_0) : x_0 \longrightarrow x_0 \text{ by } s_0(x_0) = I(x_0),$$

considered as a 1-simplex of $\mathbf{Ner}(\mathbb{B})$.

The 2-*simplices* of $\mathbf{Ner}(\mathbb{B})$ are ordered pairs

$$x_{012} = (\partial(x_{012}), \mathrm{Int}(x_{012}))$$

which have a triangle of 1-simplices in their usual simplicial "face opposite vertex" numbering,

$$\partial(x_{012}) = (d_0(x_{012}), d_1(x_{012}), d_2(x_{012})) = (x_{12}, x_{02}, x_{01}) \ ,$$

as their *boundary* together with an *interior*, $\mathrm{Int}(x_{012})$, often abbreviated as $\tilde{x}_{012}$, which is, by definition in the orientation [29] we have chosen, a 2-cell of $\mathbb{B}$ of the form

$$\mathrm{Int}(x_{012}) = \tilde{x}_{012} : x_{02} \Longrightarrow x_{12} \otimes x_{01} \ ,$$

elegant, obscures much of the detail which we need in this paper and which is made almost trivially evident in the explicit description.

[29] There are two possible "orientations" which can be used. The one we have chosen "odd faces to even ones" is consistent with that made in the definition of the nerve of a category where d_0 was chosen to represent the target of the arrow and d_1 the source. The opposite simplicial set then represents the opposite of the category. Here the other orientation, $x_{012} : x_{02} \Longleftarrow x_{12} \otimes x_{01}$ "even to odd", corresponds to the particular bicategorical dual which has the opposite of the category of 2-cells.

$$
\begin{tikzcd}
x_0 \arrow[r, "s_0(x_0)=I(x_0)"] \arrow[dr, "x_{01}"'] \arrow[dr, phantom, shift left=3, "\tilde{s}_0(x_{01})=\rho(x_{01}) \Nearrow"] & x_0 \arrow[d, "x_{01}"] \\
& x_1
\end{tikzcd}
\qquad
\begin{tikzcd}
x_0 \arrow[r, "x_{01}"] \arrow[dr, "x_{01}"'] \arrow[dr, phantom, shift left=3, "\tilde{s}_1(x_{01})=\lambda(x_{01}) \Nearrow"] & x_1 \arrow[d, "s_0(x_1)=I(x_1)"] \\
& x_1
\end{tikzcd}
$$

Figure 23: The degeneracies $s_0(x_{01})$ and $s_1(x_{01})$

where the tensor represents the composition of composable 1-cells. (Thus the 2-simplices are really the quadruplets consisting of this data.)

If $x_{01} : x_0 \to x_1$ is a 1-cell (with $d_0(x_{01}) = x_1$ and $d_1(x_{01}) = x_0$) and $s_0(x_0) : x_0 \to x_0$ and $s_0(x_1) : x_1 \to x_1$ designate the pseudo-identities (which define the first degeneracy operator) and we designate by $\mathrm{Int}(s_1(x_{01}) : x_{01} \overset{\sim}{\Longrightarrow} s_0(x_0) \otimes x_{01}$ the left identity 2-cell isomorphism $\lambda(x_{01}) : x_{01} \overset{\sim}{\Longrightarrow} I(x_0) \otimes x_{01}$ and by $\mathrm{Int}(s_0(x_{01}) : x_{01} \overset{\sim}{\Longrightarrow} x_{01} \otimes s_0(x_1)$ the right identity 2-cell isomorphism $\rho(x_{01}) : x_{01} \overset{\sim}{\Longrightarrow} x_{01} \otimes I(x_1)$, then the degeneracy operator $s_1(x_{01})$ has $(s_0(x_1), x_{01}, x_{01})$ as boundary and $\rho(x_{01})$ for interior, while $s_0(x_{01})$ has $(x_{01}, x_{01}, s_0(x_0))$ for boundary and $\lambda(x_{01})$ for interior.

The compatibility of ρ and λ with I, $\rho(I(x_0)) = \lambda(I(x_0))$, ensures that the simplicial identity $s_0(s_0(x_0)) = s_1(s_0(x_0))$ is always satisfied.

6.2. Horn Lifting Criteria for the Set of 2-simplices.. Notice that contained in the set of 2-simplices, for any composable pair of 1-cells,$(x_{12}, \_, x_{01})$ there is also the 2-simplex $\chi(x_{12}, x_{01})$ which has

$$\partial(\chi(x_{12}, x_{01})) = (x_{12}, x_{12} \otimes x_{01}, x_{01})$$

for boundary and the identity 2-cell identity isomorphism

$$\mathrm{Int}(\chi(x_{12}, x_{01})) = \mathrm{id}(x_{12} \otimes x_{01}) : x_{12} \otimes x_{01} \overset{\sim}{\Longrightarrow} x_{12} \otimes x_{01}$$

for interior. Since the set $\bigwedge_2^1$ of 1-horns in this dimension is just the set of composable pairs of 1-simplices, $(x_{12}, \_, x_{01}) \in \bigwedge_2^1(\mathbf{Ner}(\mathbb{B}))$, the 1-horn mapping

$$\mathrm{pr}_{\hat{1}} : \mathbf{Ner}(\mathbb{B})_2 \to \bigwedge_2^1(\mathbf{Ner}(\mathbb{B}))$$

is always surjective, with χ defining a distinguished section.

For an arbitrary bicategory, the partial surjectivity of the projections to the extremal horns $\bigwedge_2^0$ and $\bigwedge_2^2$ is more subtle and depends on specific properties of the 1-cells of $\mathbb{B}$. For our purposes here we note that if for a 1-cell $x_{01} : x_0 \longrightarrow x_1$, the right action functor $\_\_ \otimes x_{01} = \mathbb{B}(x_{01}, x_2) : \mathbb{B}(x_1, x_2) \longrightarrow \mathbb{B}(x_0, x_2)$ is *essentially surjective* for any 0-cell x_2, then

$$\begin{tikzcd} x_0 \arrow[r, "x_{01}"] \arrow[dr, "x_{02}\otimes x_{01}"'] & x_1 \arrow[d, "x_{12}"] \\ & x_2 \arrow[u, phantom, shift right=6, "\Uparrow \mathrm{id}"] \end{tikzcd}$$

Figure 24: The 2-simplex $\chi(x_{12}, x_{01})$

given any $(\_\,, x_{02}, x_{01}) \in \bigwedge_2^0(\mathbf{Ner}(\mathbb{B}))$, there exists a 1-cell $x_{12} : x_1 \longrightarrow x_2$ and a 2-simplex $x_{012} \in \mathbf{Ner}(\mathbb{B})_2$ whose boundary is (x_{12}, x_{02}, x_{01}) and whose interior is an isomorphism $\tilde{x}_{012} : x_{02} \overset{\sim}{\Longrightarrow} x_{12} \otimes x_{01}$ so that $\mathrm{pr}_{\hat{0}}(x_{012}) = (\_\,, x_{02}, x_{01})$, and similarly, that if for a 1-cell $x_1 \longrightarrow x_2$ the left action functor $x_{12}\otimes\_ = \mathbb{B}(x_0, x_{12}) : \mathbb{B}(x_0, x_1) \longrightarrow \mathbb{B}(x_0, x_2)$ is *essentially surjective* for any 0-cell x_0, then for any $(x_{12}, x_{02}, \_\,) \in \bigwedge_2^2(\mathbf{Ner}(\mathbb{B}))$, there exists a 1-cell $x_{01} : x_0 \longrightarrow x_1$ and a 2-simplex $x_{012} \in \mathbf{Ner}(\mathbb{B})_2$ whose boundary is (x_{12}, x_{02}, x_{01}) and whose interior is an isomorphism $\tilde{x}_{012} : x_{02} \overset{\sim}{\Longrightarrow} x_{12} \otimes x_{01}$ so that $\mathrm{pr}_{\hat{2}}(x_{012}) = (x_{12}, x_{02}, \_\,)$.

In particular, if all of the 1-cells of $\mathbb{B}$ are equivalences, then the horn maps $\mathrm{pr}_{\hat{0}} : \mathbf{Ner}(\mathbb{B})_2 \longrightarrow \bigwedge_2^0(\mathbf{Ner}(\mathbb{B}))$ and $\mathrm{pr}_{\hat{2}} : \mathbf{Ner}(\mathbb{B})_2 \longrightarrow \bigwedge_2^2(\mathbf{Ner}(\mathbb{B}))$, as well as $\mathrm{pr}_{\hat{1}} : \mathbf{Ner}(\mathbb{B})_2 \to \bigwedge_2^1(\mathbf{Ner}(\mathbb{B}))$, are surjective.

To summarize the "**horn lifting criteria in dimension** 2": For

$$(x_0, \hat{x}_k, x_2) \in \bigwedge_2^k(\mathbf{Ner}(\mathbb{B})),$$

- $(\mathbf{k = 0})$ If $x_2 = x_{01}$ is essentially surjective as a right functorial action, then there exists a 2-simplex x_{012} whose interior is an isomorphism and fills $(\_\,, x_{12}, x_{01})$.

- $(\mathbf{k = 1})$ *No conditions.* $\chi(x_{12}, x_{01})$ has the identity map for interior and fills $(x_{12}, \_\,, x_{01})$. The mapping χ provides a distinguished section for $\mathrm{pr}_{\hat{1}}$.

- $(\mathbf{k = 2})$ If $x_0 = x_{12}$ is essentially surjective as a left functorial action, then there exists a 2-simplex x_{012} whose interior is an isomorphism and fills $(x_{12}, x_{02}, \_\,)$.

6.3. The 3-simplices of $\mathbf{Ner}(\mathbb{B})$. The set of 3-*simplices* of $\mathbf{Ner}(\mathbb{B})$ is the subset of the simplicial kernel $\mathrm{cosk}^2(\mathbf{Ner}(\mathbb{B})]_0^2)_3$ of the just constructed 2-truncated complex consisting of those tetrahedra

$$(x_{123}, x_{023}, x_{013}, x_{012}) \in \mathrm{cosk}^2(\mathbf{Ner}(\mathbb{B})]_0^2)_3$$

of simplicially matching 2-simplices (Figure 25) whose interiors after tensoring and composing in the bicategory make the diagram in Figure 26 commutative,*i.e.*, they satisfy the equation

$$\boxed{(\,\mathbf{T}\,) \quad \mathbf{A}(x_{23}, x_{12}, x_{01}) \circ (x_{23} \otimes \tilde{x}_{012}) \circ \tilde{x}_{023} = (\tilde{x}_{123} \otimes x_{01}) \circ \tilde{x}_{013}}$$

$$
\begin{tikzcd}
x_0 \arrow[r, "x_{01}"] \arrow[d, "x_{03}"'] \arrow[dr, "x_{02}" description] & x_1 \arrow[d, "x_{12}"] \\
x_3 \arrow[ur, phantom, "\overset{\tilde{x}_{023}}{\Rightarrow}\ d_1\ \ d_3\ \overset{\tilde{x}_{012}}{\Rightarrow}"] & x_2 \arrow[l, "x_{23}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
x_0 \arrow[r, "x_{01}"] \arrow[d, "x_{03}"'] \arrow[dr, phantom, "\overset{\tilde{x}_{013}}{\Rightarrow}\ d_2\ \ d_0\ \overset{\tilde{x}_{123}}{\Rightarrow}"] & x_1 \arrow[d, "x_{12}"] \arrow[dl, "x_{13}" description] \\
x_3 & x_2 \arrow[l, "x_{23}"]
\end{tikzcd}
$$

Figure 25: Odd faces and even faces of the 3-simplex

$$
\begin{tikzcd}[column sep=small]
 & & x_{03} \arrow[dl, Rightarrow, "\tilde{x}_{023}=\mathrm{Int}(d_1(x_{0123}))"'] \arrow[dr, Rightarrow, "\tilde{x}_{013}=\mathrm{Int}(d_2(x_{0123}))"] & & \\
 & x_{23}\otimes x_{02} \arrow[d, Rightarrow, "x_{23}\otimes\tilde{x}_{012}=x_{23}\otimes\mathrm{Int}(d_3(x_{0123}))"'] & \mathrm{Int}(x_{0123}) & x_{13}\otimes x_{01} \arrow[d, Rightarrow, "\tilde{x}_{123}\otimes x_{01}=\mathrm{Int}(d_0(x_{0123}))\otimes x_{01}"] & \\
 & x_{23}\otimes(x_{12}\otimes x_{01}) \arrow[rr, Rightarrow, "\mathbf{A}(x_{12},x_{23},x_{01})"'] & & (x_{23}\otimes x_{12})\otimes x_{01} &
\end{tikzcd}
$$

Figure 26: 2-cell interiors of odd and even faces of the 3-simplex

where $x_{01} = d_2(x_{013}), x_{23} = d_0(x_{023}),\ x_{12} = d_2(x_{123})$, and

$$\mathbf{A}(x_{23}, x_{12}, x_{01}) : x_{23} \otimes (x_{12} \otimes x_{01}) \stackrel{\sim}{\Longrightarrow} (x_{23} \otimes x_{12}) \otimes x_{01}$$

is the associativity isomorphism, which in our rendering of the Bénabou axioms *shifts parentheses to the left.*

In other words, the set of the 3-simplices of $\mathbf{Ner}(\mathbb{B})$ is the subset of the simplicial kernel of the 2-truncated complex $\mathbf{Ner}(\mathbb{B})]_0^2$ consisting of the "*commutative tetrahedra*". [30] We will often indicate that

$$(x_{123}, x_{023}, x_{013}, x_{012}) \in cosk^2(\mathbf{Ner}(\mathbb{B})]_0^2)_3$$

is a 3-simplex of $\mathbf{Ner}(\mathbb{B})$ by enclosing it in square rather than round brackets:

$$[x_{123}, x_{023}, x_{013}, x_{012}] \in \mathbf{Ner}(\mathbb{B})_3.$$

6.4. Remark. Equation $(\mathbf{T})$ above came from an element of the simplicial kernel of the 2-truncated complex $\mathbf{Ner}(\mathbb{B})]_0^2$. The only simplicial matching conditions apply to the boundaries of the 2-simplices where they just require that they form the surface of a tetrahedron (Figure 25). In order to make composable the interiors of the odd and even faces of this tetrahedron (Figure 26) one needs a two-sided source and target preserving tensor action of 1-cells on 2-cells and in order that the two sides close, one also needs the associativity isomorphism. Finally, one uses composition of 2-cells to form the two sides of the equation (and associativity of 2-cell composition to make the left hand side unambiguous).

6.5. Horn Lifting Criteria for the Set of 3-Simplices.. If we consider any one of the four horn sets $\bigwedge_3^k(\mathbf{Ner}(\mathbb{B}))\,(0 \le k \le 3)$ of the set of commutative tetrahedra $\mathbf{Ner}(\mathbb{B})_3$, and the canonical maps $\mathrm{pr}_{\hat{i}}$ from $\mathbf{Ner}(\mathbb{B})_3$ to it, we can see that for any element $(x_0, \ldots, \hat{x}_k, \ldots, x_3) \in \bigwedge_3^k(\mathbf{Ner}(\mathbb{B}))$, the bounding 1-simplex faces of the "missing" 2-simplex x_k are completely determined simplicially by the 1-simplex faces of the $x_i\,(i \neq k)$ present in the horn. In order to have a unique 3-simplex filler for the horn, it is only necessary to find a unique interior for the "missing" face x_k of the horn,and for this it is only necessary to be able to solve equation $(\mathbf{T})$ *uniquely* for the interior $\tilde{x}_k$ of x_k. A glance at Figure 26 and equation $(\mathbf{T})$, remembering that the associativity 2-cell A is an isomorphism, leads immediately to the following criteria:

$$\text{For } (x_0, \ldots, \hat{x}_k, \ldots, x_3) \in \bigwedge_3^k(\mathbf{Ner}(\mathbb{B})) \subseteq (\mathbf{Ner}(\mathbb{B})_2)^3\ {\scriptstyle (CartesianProduct)}$$

[30]Equation $(\mathbf{T})$ is just our bicategorical interpretation of the assertion of the "equality of the two pasting diagrams" (the odd sides and the even sides) which appear in Figure 25. In the nerve of a *tricategory*, the associativity isomorphism will be replaced by an equivalence and incorporated into the 2-simplex interior of the 3-face and the equality of $(\mathbf{T})$ will be replaced by a 3-cell which will become the interior of the 3-simplex. True equality will first appear there in the next dimension with the definition of the 4-simplices.

- $(\mathbf{k=0})$ If $\tilde{x}_2 = \tilde{x}_{013}$ is an *isomorphism*, and $d_2(x_2) = x_{01}$ is *fully faithful* as a right functorial action, there exists a unique 2-cell $\tilde{x}_0 = \tilde{x}_{123} : x_{13} \Longrightarrow x_{23} \otimes x_{12}$ which makes $(\mathbf{T})$ commutative. Consequently there then exists a unique 2-simplex x_0 with boundary
$$\partial(x_0) = (d_0(x_1), d_0(x_2), d_0(x_3)) = (x_{23}, x_{13}, x_{12})$$
and interior $\tilde{x}_0$ such that (x_0, x_1, x_2, x_3) is commutative and uniquely fills the horn.

- $(\mathbf{k=1})$ If $\tilde{x}_3 = \tilde{x}_{012}$ is an *isomorphism*, there exists a unique 2-cell $\tilde{x}_1 = \tilde{x}_{023} : x_{03} \Longrightarrow x_{23} \otimes x_{02}$ which makes $(\mathbf{T})$ commutative. Consequently there exists a unique 2-simplex x_1 with boundary
$$\partial(x_1) = (d_2(x_0), d_1(x_2), d_1(x_3)) = (x_{23}, x_{03}, x_{02})$$
and interior $\tilde{x}_1$, such that (x_0, x_1, x_2, x_3) is commutative and uniquely fills the horn.

- $(\mathbf{k=2})$ If $\tilde{x}_0 = x_{123}$ is an *isomorphism*, there exists a unique 2-cell $\tilde{x}_2 = \tilde{x}_{013} : x_{03} \Longrightarrow x_{13} \otimes x_{01}$ which makes $(\mathbf{T})$ commutative. Consequently there exists a unique 2-simplex x_2 with boundary
$$\partial(x_2) = (d_1(x_0), d_1(x_1), d_2(x_3)) = (x_{13}, x_{03}, x_{01})$$
and interior $\tilde{x}_2$, such that (x_0, x_1, x_2, x_3) is commutative and uniquely fills the horn.

- $(\mathbf{k=3})$ If $\tilde{x}_1 = \tilde{x}_{023}$ is an *isomorphism and* $d_0(x_1) = x_{23}$ is *fully faithful* as a left functorial action, there exists a unique 2-cell $\tilde{x}_3 = \tilde{x}_{012} : x_{02} \Longrightarrow x_{12} \otimes x_{01}$ which makes $(\mathbf{T})$ commutative. Consequently there then exists a unique 2-simplex x_3 with boundary
$$\partial(x_3) = (d_2(x_0), d_2(x_1), d_2(x_2)) = (x_{12}, x_{02}, x_{01})$$
and interior $\tilde{x}_3$ such that (x_0, x_1, x_2, x_3) is commutative and uniquely fills the horn.

Note that we may add the fact that if a horn
$$(x_0, \ldots, \hat{x}_k, \ldots, x_3) \in \bigwedge\nolimits^k_3(\mathbf{Ner}(\mathbb{B}))$$
satisfies the above criterion for k and the 2-cell interiors of the 2-simplex faces present in the horn are *all* isomorphisms, then the 2-cell interior of the "filler" 2-simplex x_k is an isomorphism as well. Also note that as in the previous dimension, *no* conditions on 1-cells are required for the non-extremal horns $\bigwedge^1_3(\mathbf{Ner}(\mathbb{B}))$ and $\bigwedge^1_3(\mathbf{Ner}(\mathbb{B}))$. Consequently, if all of the 2-cells of $\mathbb{B}$ are isomorphisms, then the non-extremal horn maps $\mathrm{pr}_{\hat{1}} : \mathbf{Ner}(\mathbb{B}) \longrightarrow \bigwedge^1_3$ and $\mathrm{pr}_{\hat{2}} : \mathbf{Ner}(\mathbb{B}) \longrightarrow \bigwedge^2_3$ are both bijections and that if in addition, all of the 1-cells of $\mathbb{B}$ are fully faithful as left and right functorial actions, then the extremal horn maps $\mathrm{pr}_{\hat{0}} : \mathbf{Ner}(\mathbb{B}) \longrightarrow \bigwedge^0_3$ and $\mathrm{pr}_{\hat{3}} : \mathbf{Ner}(\mathbb{B}) \longrightarrow \bigwedge^3_3$ are bijections as well. In particular, if all of the 2-cells of $\mathbb{B}$ are isomorphisms and all of the 1-cells are equivalences, then $\mathbf{Ner}(\mathbb{B})]^3_0$ is at least a truncated Kan complex, all of the horn maps defined through this level are surjective.[31]

[31] We will shortly show that this condition will guarantee that $\mathbf{Ner}(\mathbb{B})$ is, in fact, a Kan complex.

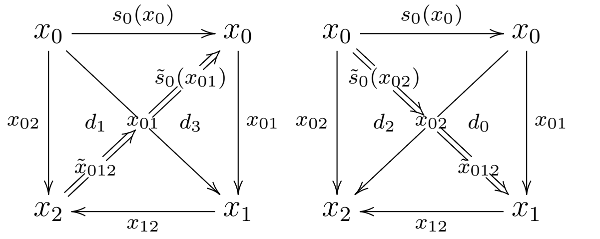

Figure 27: The odd and even faces of $s_0(x_{012})$

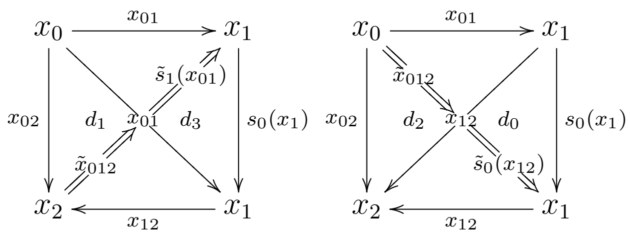

Figure 28: The odd and even faces of $s_1(x_{012})$

6.6. The Degeneracies $\mathbf{s_0}$, $\mathbf{s_1}$, and $\mathbf{s_2}$. If x_{012} is a 2-simplex, then each of the degeneracies

$$s_0(x_{012}) = (x_{012}, x_{012}, s_0(x_{12}), s_0(x_{01})),$$

$$s_1(x_{012}) = (s_0(x_{12}), x_{012}, x_{012}, s_1(x_{01})),$$

and

$$s_2(x_{012}) = (s_1(x_{12}), s_1(x_{02}), x_{012}, x_{012})$$

is in the simplicial kernel (Figure 27, 28, and 29).

That each of them is also a 3-simplex of $\mathbf{Ner}(\mathbb{B})$, *i.e.,* that each of the three pentagonal interior diagrams is commutative, is an immediate consequence of the naturality of the pseudo-identity isomorphisms ρ and λ used in the definition of the interiors of their faces,

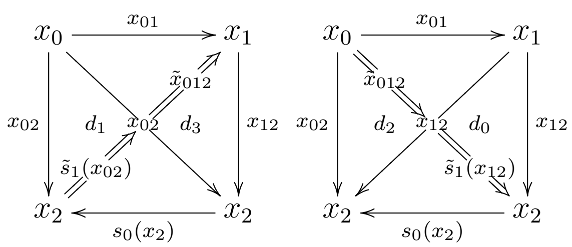

Figure 29: The odd and even faces of $s_2(x_{012})$

$$
\begin{tikzcd}[column sep=small]
 & & x_{02} \arrow[dll, Rightarrow, "\tilde{x}_{012}"'] \arrow[drr, Rightarrow, "\tilde{s}_0(x_{02})"] \arrow[d, phantom, "\text{Naturality of } \tilde{s}_0=\rho"] & & \\
x_{12}\otimes x_{01} \arrow[dr, Rightarrow, "x_{12}\otimes\tilde{s}_0(x_{01})"'] \arrow[drrr, Rightarrow, "\tilde{s}_0(x_{12}\otimes x_{01})"] \arrow[drr, phantom, "\text{Ax.BCI}_0"] & & {} & & x_{02}\otimes s_0(x_0) \arrow[dl, Rightarrow, "\tilde{x}_{012}\otimes s_0(x_0)"] \\
 & x_{12}\otimes(x_{01}\otimes s_0(x_0)) \arrow[rr, Rightarrow, "\mathbf{A}"'] & & (x_{12}\otimes x_{01})\otimes s_0(x_0) &
\end{tikzcd}
$$

Figure 30: The commutative interior of $s_0(x_{012})$

together with their axiomatic compatibility with $\mathbf{A}$. (Figures 20 and 21) , as the diagrams in Figures 30, 31, and 32 make evident:

It is perhaps worth noting at this point that the three Bénabou axioms for the pseudo-identities are simply the assertion that for any composable pair of 1-simplices, the three tetrahedra $s_i(\chi(x_{12}, x_{01}))$, $\quad 0 \leq i \leq 2$, are commutative.[32]

6.7. The full complex $\mathbf{Ner}(\mathbb{B})$ and its 4-simplices. We complete the 3-truncated complex $\mathbf{Ner}(\mathbb{B})]_0^3$ just defined up to this point to the full simplicial complex $\mathbf{Ner}(\mathbb{B})$ by iterating simplicial kernels: Thus by definition,

$$\mathbf{Ner}(\mathbb{B}) = \mathbf{cosk}^3(\mathbf{Ner}(\mathbb{B})]_0^3).$$

The 4-simplex elements of the simplicial kernel $\mathbf{Ner}(\mathbb{B})_4$ thus consist of five simplicially matching *commutative* tetrahedra. Their 2-cell interiors fit into a more complicated geometric diagram, broken as usual, into odd and even sides (Figures 33 and 34 below) in which there appear

- The three commutative pentagons of 2-cells which come from the interiors of the faces d_1,d_2, and d_3 which (after composition) correspond to the odd and even sides of the corresponding equations $(\mathbf{T})$ above.
- The two pentagons $x_{34} \otimes \mathrm{Int}(d_4)$ and $\mathrm{Int}(d_0) \otimes x_{01}$, whose commutativity is guaranteed by the functoriality of the left and right actions of 1-cells on 2-cells.

[32]Bénabou's original *single* one is equivalent to the assertion that $s_1(\chi(x_{12}, x_{01}))$ is commutative.

$$
\begin{tikzcd}[column sep=small]
 & & x_{02} \arrow[dll, Rightarrow, "\tilde{x}_{012}"'] \arrow[drr, Rightarrow, "\tilde{x}_{012}"] \arrow[d, phantom, "\text{Tautology}"] & & \\
x_{12}\otimes x_{01} \arrow[dr, Rightarrow, "x_{12}\otimes\tilde{s}_1(x_{01})"'] \arrow[drrr, Rightarrow, "\tilde{s}_0(x_{12})\otimes x_{01}"] \arrow[drr, phantom, "\text{Axiom BCI}_1"] & & {} & & x_{12}\otimes x_{01} \arrow[dl, Rightarrow, "s_0(x_{12})\otimes x_{01}"] \\
 & x_{12}\otimes(s_0(x_1)\otimes x_{01}) \arrow[rr, Rightarrow, "\mathbf{A}"'] & & (x_{12}\otimes\tilde{s}_0(x_1))\otimes x_{01} &
\end{tikzcd}
$$

Figure 31: Commutative interior of $s_1(x_{012})$

$$
\begin{tikzcd}[column sep=small]
 & & x_{02} \arrow[dll, Rightarrow, "s_1(x_{02})"'] \arrow[drr, Rightarrow, "\tilde{x}_{012}"] \arrow[d, phantom, "\text{Naturality of } \tilde{s}_1=\lambda"] & & \\
s_0(x_2)\otimes x_{02} \arrow[dr, Rightarrow, "\tilde{s}_0(x_2)\otimes\tilde{x}_{012}"'] & & {} & & x_{12}\otimes x_{01} \arrow[dlll, Rightarrow, "\tilde{s}_1(x_{12}\otimes x_{01})"'] \arrow[dl, Rightarrow, "\tilde{s}_1(x_{12})\otimes x_{01}"] \arrow[dll, phantom, "\text{Axiom BIC}_2"] \\
 & \tilde{s}_0(x_2)\otimes(x_{12}\otimes x_{01}) \arrow[rr, Rightarrow, "\mathbf{A}"'] & & (s_0(x_2)\otimes x_{12})\otimes x_{01} &
\end{tikzcd}
$$

Figure 32: The commutative interior of $s_2(x_{012})$

- The square whose commutativity is guaranteed by the *Godement Interchange Law* and has, as its common value after composition of its sides, the tensor product $\tilde{x}_{234} \otimes \tilde{x}_{012}$ of the interior 2-cells of $d_0(x_{0234}) = x_{234}$ and $d_3(x_{0124}) = x_{012}$.

- The edges (in the correct order) of the *Mac Lane-Stasheff Associativity Pentagon* (Stasheff's K-4), with edges coming from the corresponding associativity isomorphisms included in the interiors of the five tetrahedral faces, and whose commutativity is guaranteed by axiom.

- The three squares each of whose commutativity is equivalent to the naturality of the associativity isomorphism in one of its three variables, again guaranteed by axiom.

6.8. Horn Lifting Criteria for the Set of 4-Simplices.. The diagrams of the 2-cell interiors of a 4-simplex, Figures 33 and 34[33] should be viewed as if they formed a partition of a 2-sphere into "upper and lower hemispheres" with their common octagonal boundary forming the equator. The Mac Lane–Stasheff Pentagon, the Interchange Square, and the three Naturality Squares are always, by axiom, commutative. Since this diagram came from an element $\mathbf{x}_{01234}$ of the simplicial kernel of the set of commutative tetrahedra, $\mathbf{Ner}(\mathbb{B})_4 = \mathbf{Cosk}^3(\mathbf{Ner}(\mathbb{B})]_0^3)_4$, the interior pentagons of the faces, $\mathrm{Int}(d_i(\mathbf{x}))$ for $0 \leq i \leq 4$, are all commutative. Thus the pentagons $\mathrm{Int}(d_i(\mathbf{x}))$ for $0 < i < 4$ which appear directly in the diagram are commutative, as are $x_{34} \otimes \mathrm{Int}(d_4(\mathbf{x}))$ and $\mathrm{Int}(d_0(\mathbf{x})) \otimes x_{01}$, but here by functoriality of the left and right tensor product actions.

Now consider the canonical map from this simplicial kernel $\mathbf{Ner}(\mathbb{B})_4$ to any one of its five horn sets $\bigwedge_4^k(\mathbf{Ner}(\mathbb{B}))$ $(0 \leq k \leq 4)$, each element of which consists of four simplicially matching *commutative* tetrahedra $x_i (i \neq k)$. This mapping is always injective and the 2-simplex faces of the "missing tetrahedron" are uniquely determined by the faces of those present in the horn. Consequently, the diagrams of the 2-cell interiors of $\bigwedge_4^k(\mathbf{Ner}(\mathbb{B}))$ are identical to those of Figures 33 and 34 but where the pentagon associated with the k^{th} tetrahedron is no longer assumed commutative. Note, however, that all of the individual 2-cell paths of maximal length that appear here have x_{04} as origin (**S**) and $((x_{24} \otimes x_{23}) \otimes x_{12})) \otimes x_{01}$ as terminus (**F**), and that the odd and even sides of the k^{th} pentagon are each subpaths of exactly two of these which coincide outside of the k^{th} pentagon. Thus if all of the pentagons except for the k^{th} are assumed commutative, it immediately follows that the 2-cell (associative) compositions of the two paths from **(S)**

[33] In the corresponding description of the nerve of a tricategory, the pentagonal interior of the 3-simplex (Figure 26) will be redrawn as a square with the 2-cell **A** becoming a 2-cell *equivalence* and part of the 3-face of the square. Its commutative interior will be replaced by a 3-cell connecting the non-associative composition of the odd and even sides. The two diagrams here (Figures 33 and 34) of the interiors of the 4-simplex's odd and even faces will similarly be redrawn as the "odd and even sides" of a *cube* with 3-cells replacing commutativity in the interiors. The 2^{nd} variable naturality square will be placed on the other side, along with the Mac Lane-Stasheff Pentagon (similarly squared and supplied with a 3-cell isomorphism) becoming part of the 4-face of the cube. The "Interchange Axiom" square (with a 3-cell isomorphism replacing its commutative interior) along with the 1^{st}-variable square will form the "bottom face" of the odd-numbered sides of this cube.

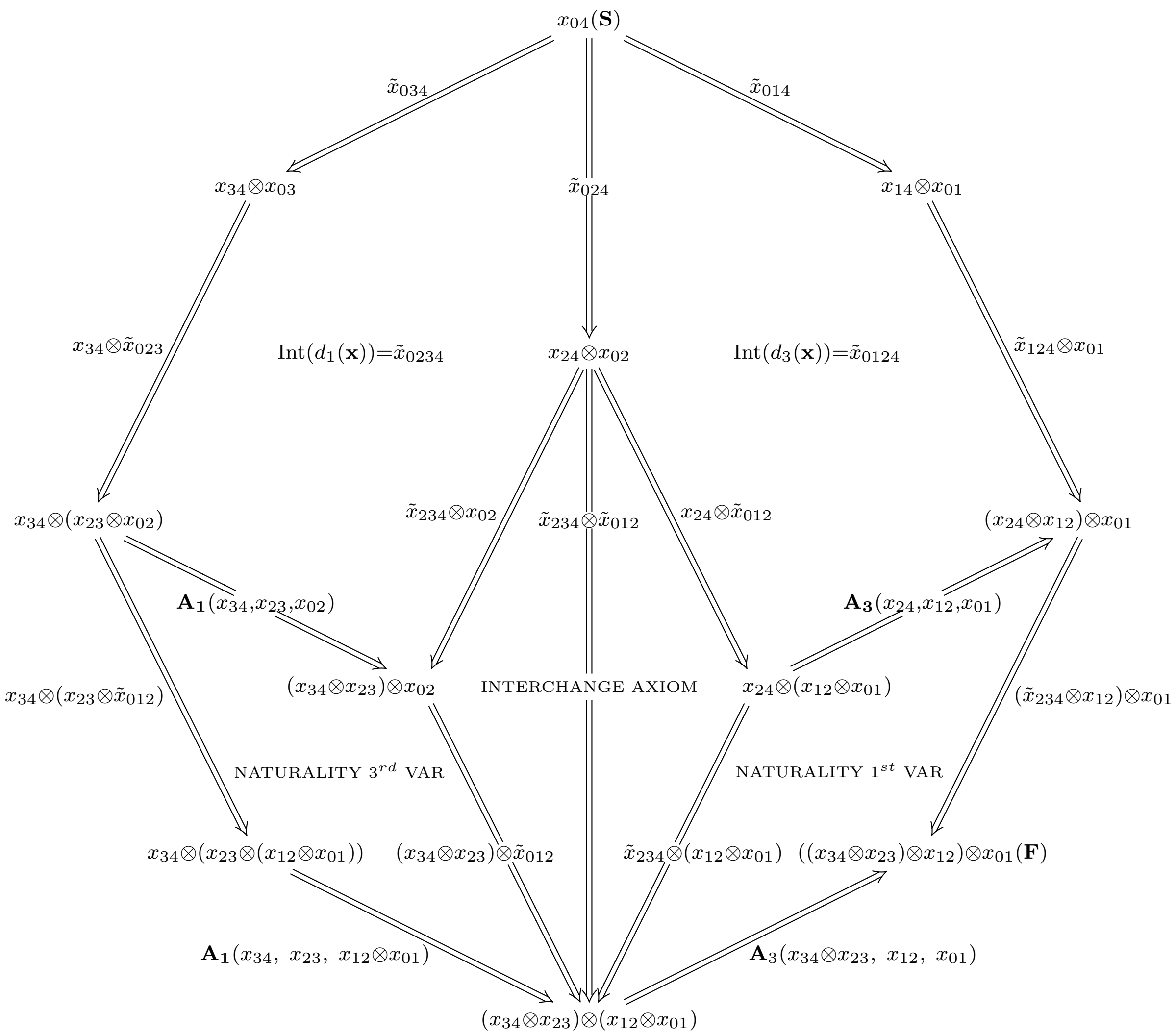

Figure 33: 2-cell interior of the odd faces of the 4-simplex

$$
\begin{tikzcd}[column sep=tiny]
 & & & & x_{04}\,(\mathbf{S}) \arrow[dlll, Rightarrow, "\tilde{x}_{034}"'] \arrow[drrr, Rightarrow, "\tilde{x}_{014}"] & & & & \\
 & x_{34}{\otimes}x_{03} \arrow[drr, Rightarrow, "x_{34}{\otimes}\tilde{x}_{013}"] \arrow[ddl, Rightarrow, "x_{34}{\otimes}\tilde{x}_{023}"'] & & & \mathrm{Int}(d_2(\mathbf{x}))=\tilde{x}_{0134} & & & x_{14}{\otimes}x_{01} \arrow[dll, Rightarrow, "\tilde{x}_{134}{\otimes}x_{01}"'] \arrow[ddr, Rightarrow, "\tilde{x}_{124}{\otimes}x_{01}"] & \\
 & & & x_{34}{\otimes}(x_{13}{\otimes}x_{01}) \arrow[rr, Rightarrow, "\mathbf{A}_2(x_{34},x_{13},x_{01})"] \arrow[dd, Rightarrow, "x_{34}{\otimes}(\tilde{x}_{123}{\otimes}x_{01})"'] & & (x_{34}{\otimes}x_{13}){\otimes}x_{01} \arrow[dd, Rightarrow, "(x_{34}{\otimes}\tilde{x}_{123}){\otimes}x_{01}"] & & & \\
x_{34}{\otimes}(x_{23}{\otimes}x_{02}) \arrow[ddr, Rightarrow, "x_{34}{\otimes}(x_{23}{\otimes}\tilde{x}_{012})"'] & x_{34}{\otimes}\mathrm{Int}(d_4(\mathbf{x}))=x_{34}{\otimes}\tilde{x}_{0123} & & & \text{NATURALITY } 2^{nd} \text{ VAR} & & \mathrm{Int}(d_0(\mathbf{x})){\otimes}x_{01}=\tilde{x}_{1234}{\otimes}x_{01} & & (x_{24}{\otimes}x_{12}){\otimes}x_{01} \arrow[ddl, Rightarrow, "(\tilde{x}_{234}{\otimes}x_{12}){\otimes}x_{01}"] \\
 & & & x_{34}{\otimes}((x_{23}{\otimes}x_{12}){\otimes}x_{01}) \arrow[rr, Rightarrow, "\mathbf{A}_2(x_{34},x_{23}{\otimes}x_{12},x_{01})"'] & & (x_{34}{\otimes}(x_{23}{\otimes}x_{12})){\otimes}x_{01} \arrow[drr, Rightarrow, "\mathbf{A}_0(x_{34},x_{23},x_{12}){\otimes}x_{01}"] & & & \\
 & x_{34}{\otimes}(x_{23}{\otimes}(x_{12}{\otimes}x_{01})) \arrow[urr, Rightarrow, "x_{34}{\otimes}\mathbf{A}_4(x_{23},x_{12},x_{01})"] \arrow[drrr, Rightarrow, "\mathbf{A_1}(x_{34},x_{23},x_{12}{\otimes}x_{01})"'] & & & \text{MAC LANE–STASHEFF PENTAGON} & & & ((x_{34}{\otimes}x_{23}){\otimes}x_{12}){\otimes}x_{01}(\mathbf{F}) & \\
 & & & & (x_{34}{\otimes}x_{23}){\otimes}(x_{12}{\otimes}x_{01}) \arrow[urrr, Rightarrow, "\mathbf{A_3}(x_{34}{\otimes}x_{23},\ x_{12},\ x_{01})"'] & & & &
\end{tikzcd}
$$

Figure 34: 2-cell interior of the even faces of the 4-simplex

to **(F)** which contain the odd and even sides of the k^{th} pentagon must be equal: Just use the commutativity of complementary pentagons to successively exchange the composition of the paths containing one side of the k^{th} pentagon until one reaches one of the **S-T** equatorial edges of the hemisphere containing the k^{th}, then continue the exchange around the other hemisphere until one reaches the other **S-T** equatorial edge. Then pass back over original hemisphere until one finally reaches the composition of the path which contains the other side of the k^{th} pentagon. Given that the associativity 2-cells **A** are all isomorphisms, we obtain the following criteria for the commutativity of the "missing" k^{th} tetrahedron:

For

$$(x_0, x_1, \ldots, \hat{x}_k, \ldots, x_4) \in \bigwedge_4^k(\mathbf{Ner}(\mathbb{B})),$$

where all x_i $(i \neq k)$ are commutative.

- $(\mathbf{k = 0})$ If $d_2(x_2) = \tilde{x}_{014}$ is an *isomorphism* and $d_2(d_2(x_2)) = x_{01}$ is *fully faithful* as a right functorial action, then x_0 is commutative.
- $(\mathbf{k = 1})$ If $d_3(x_3) = \tilde{x}_{012}$ is an *isomorphism*, then x_1 is commutative.
- $(\mathbf{k = 2})$ If $d_0(x_4) = \tilde{x}_{123}$ is an *isomorphism*, then x_2 is commutative.
- $(\mathbf{k = 3})$ If $d_0(x_1) = \tilde{x}_{234}$ is an *isomorphism*, then x_3 is commutative.
- $(\mathbf{k = 4})$ If $d_1(x_2) = \tilde{x}_{034}$ is an *isomorphism* and $d_0(d_1(x_2)) = x_{34}$ is *fully faithful* as a left functorial action, then x_4 is commutative.

Thus in each of these cases there exists a *unique* 4-simplex in $\mathbf{Ner}(\mathbb{B})$ which "fills the missing face of the horn". Notice again in this dimension as well, that for the sets of non-extremal horns, $\bigwedge_4^k(\mathbf{Ner}(\mathbb{B}))$, $1 \leq k \leq 3$, *no* conditions on 1-cells appear in these criteria. Since $\mathbf{Ner}(\mathbb{B})]_0^3 \subseteq \mathbf{Cosk}^2(\mathbf{Ner}(\mathbb{B}))]_0^3$ and we have defined $\mathbf{Ner}(\mathbb{B})$ as $\mathbf{cosk}^3(\mathbf{Ner}(\mathbb{B})]_0^3$, $\mathbf{Ner}(\mathbb{B})$ is a subcomplex of its 2-Coskeleton and is isomorphic to its 3-Coskeleton. Thus $\mathbf{Ner}(\mathbb{B})$ is a 2-dimensional Postnikov complex. Consequently, in the dimensions > 4 of $\mathbf{Ner}(\mathbb{B})$, we are always dealing with sets of simplices which are two-fold iterates of simplicial kernels where *all* horn maps (including the extremals) are strictly *bijective.* Thus if $\mathbb{B}$ is bicategory in which all 2-cells are isomorphisms and all 1-cells are equivalences, *i.e.,* $\mathbb{B}$ is a bi*groupoid*, then what we have now shown is that $\mathbf{Ner}(\mathbb{B})$ is a 2-*dimensional (Kan)hypergroupoid* (in the simplicial sense of Paul Glenn: a Kan complex in which the Kan conditions are satisfied exactly — all horn maps are bijections — in all dimensions > 2). Section 7 of this paper will show that *all* 2-dimensional (Kan)hypergroupoids are the nerves of bigroupoids. However, if $\mathbb{B}$ is a bicategory in which it is the case only that all 2-cells are isomorphisms, our analysis at this point allows us to conclude that the nerve of $\mathbb{B}$ is a weak Kan complex[34] in which the weak Kan conditions are satisfied exactly in dimensions > 2. Consequently, as we noted in the Introduction, this

[34]See Section 2 for a review of the terms used here.

$$\begin{tikzcd}
 & x_0 \arrow[dl, "x_{03}"'] \arrow[dr, "x_{01}"] \arrow[dd] & \\
x_3 & & x_1 \arrow[ll] \arrow[dl, "x_{12}"] \\
 & x_2 \arrow[ul, "x_{23}"] &
\end{tikzcd}$$

Figure 35: Geometric model of 3-simplex as a tetrahedron

property (terminologically: a 2-dimensional hyper*category*?), taken as an immediate naive generalization of the property characteristic of the nerves of categories, is already much too strong to characterize the nerves of arbitrary bi*categories*.

Nevertheless, it should be apparent to the reader that we have indeed made essential use of all the data and all of the axioms required in Bénabou's definition of a bicategory $\mathbb{B}$ just in order to *define* $\mathbf{Ner}(\mathbb{B})$ as a simplicial set which is a 2-dimensional Postnikov complex whose various (non-trivially bijective) horn maps satisfy the above sets of rather "restricted Kan conditions" in dimensions ≤ 4. Thus it is not at all unreasonable to certainly expect to be able to recover the bicategory from its nerve and to characterize in appropriate terms those simplicial sets which are the nerves of arbitrary bicategories. We will give this characterization in the next section.

7. 2-Dimensional Postnikov Complexes : Nerves of Bicategories and Bigroupoids

7.1. The Basic Simplicial Setting for $n = 2$. Here the basic simplicial setting is the assumption that the complex $X_\bullet$ is a 2-dimensional Postnikov complex:

$$\mathbf{Cosk}^3(X_\bullet) = X_\bullet \subseteq \mathbf{Cosk}^2(X_\bullet).$$

Thus the set X_3 of formally commutative 3-simplices is a subset of the set of 3-simplices of the 2-Coskeleton, the simplicial kernel $K_3(X_\bullet) = \mathbf{Cosk}^2(X_\bullet)_3$ of the 2-truncated complex $X_\bullet]_0^2 = \mathbf{Cosk}^2(X_\bullet)]_0^2$. As before, any element of the simplicial kernel is a 4-tuplet

$$\mathbf{x} = (x_0, x_1, x_2, x_3)$$

with

$$x_i \in X_2 \ (0 \leq i \leq 3)\text{and } d_i(x_j) = d_{j-1}(x_i) \ (0 \leq i < j \leq 3).$$

The simplicial identities here just mean that it may be imagined geometrically as a tetrahedron of solid triangular 2-simplices, each of which has a triangle of 1-simplices as its boundary and a 2-dimensional surface as its interior (Figure 35).

$$\begin{tikzcd}
x_0 \arrow[r, "x_{01}"] \arrow[d, "x_{03}"'] \arrow[dr, "x_{02}" description] \arrow[d, phantom, shift left=4ex, "d_1(\mathbf{x})"] & x_1 \arrow[d, "x_{12}"] \arrow[d, phantom, shift right=4ex, "d_3(\mathbf{x})"] \\
x_3 & x_2 \arrow[l, "x_{23}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
x_0 \arrow[r, "x_{01}"] \arrow[d, "x_{03}"'] \arrow[d, phantom, shift left=4ex, "d_2(\mathbf{x})"] & x_1 \arrow[d, "x_{12}"] \arrow[dl, "x_{13}" description] \arrow[d, phantom, shift right=4ex, "d_0(\mathbf{x})"] \\
x_3 & x_2 \arrow[l, "x_{23}"]
\end{tikzcd}$$

Figure 36: Odd Faces and Even Faces of the 3-Simplex

$$\begin{bmatrix} x_{23} & x_{13} & x_{12} \\ x_{23} & x_{03} & x_{02} \\ x_{13} & x_{03} & x_{01} \\ x_{12} & x_{02} & x_{01} \end{bmatrix}.$$

Figure 37: 4×3 Simplicial Matrix With Entries in X_1

7.2. Remark. In Figure 35 the 2-simplex face $d_i(\mathbf{x}) = \mathrm{pr}_i(\mathbf{x}) = x_i$ is the triangular surface "opposite" the vertex labeled x_i in the figure (the "standard simplicial numbering"); its boundary is the triangle of 1-simplices which is opposite that same vertex. The simplicial identities $d_i(x_j) = d_{j-1}(x_i)$ $(0 \leq i < j \leq 3)$ just assert that the four triangular boundaries match as shown to form the tetrahedron. Since this an element of the simplicial kernel, it is made entirely of 2-simplices forming the surface of the tetrahedron and itself should be imagined just as a "possible boundary" of some 3-simplex which we could imagine as looking like the same Figure 35 but with a "solid 3-dimensional interior".

Broken into its odd and even triangular faces we can imagine such a 3-simplex as in Figure 36.

Notice here as well that the information that

$$\mathbf{x} = (d_0(\mathbf{x}), d_1(\mathbf{x}), d_2(\mathbf{x}), d_3(\mathbf{x}))) \in K_3 \subseteq X_2 \times X_2 \times X_2 \times X_2,$$

i.e., $d_i(d_j(\mathbf{x})) = d_{j-1}(d_i(\mathbf{x}))$ for $0 \leq i < j \leq 3$, which is also the geometric content of Figures 35 and 36, may also be conveniently conveyed by the equivalent statement that the 4×3 matrix of "faces of faces" of $\mathbf{x}$,

$$\partial_2(\mathbf{x}) =_{(Def)} \begin{bmatrix} \partial(d_0(\mathbf{x})) \\ \partial(d_1(\mathbf{x})) \\ \partial(d_2(\mathbf{x})) \\ \partial(d_3(\mathbf{x}))) \end{bmatrix} =_{(Def)} \begin{bmatrix} d_0(d_0(\mathbf{x})) & d_1(d_0(\mathbf{x})) & d_2(d_0(\mathbf{x})) \\ d_0(d_1(\mathbf{x})) & d_1(d_1(\mathbf{x})) & d_2(d_1(\mathbf{x})) \\ d_0(d_2(\mathbf{x})) & d_1(d_2(\mathbf{x})) & d_2(d_2(\mathbf{x})) \\ d_0(d_3(\mathbf{x})) & d_1(d_3(\mathbf{x})) & d_2(d_3(\mathbf{x})) \end{bmatrix},$$

is *simplicial*, *i.e.*, its form has the "affine reflective symmetry" expressed by the identities among the (1-simplex) entries of Figure 37. This is, of course, nothing more than the statement that the 1-simplex entries of $\partial_2(\mathbf{x})$ satisfy the simplicial identities.

$$\left[\begin{array}{ccccccc}
(\,x_{234} & & x_{134} & & x_{124} & & x_{123}\,) \\
\Vert & & & & & & \\
(\,x_{234} & = & x_{034} & & x_{024} & & x_{023}\,) \\
 & & \Vert & & & & \\
(\,x_{134} & & x_{034} & = & x_{014} & & x_{013}\,) \\
 & & & & \Vert & & \\
(\,x_{124} & & x_{024} & & x_{014} & = & x_{012}\,) \\
 & & & & & & \Vert \\
(\,x_{123} & & x_{023} & & x_{013} & & x_{012}\,)
\end{array}\right]$$

Figure 38: 5×4 Simplicial Matrix with Entries in X_2

Again as before, we will use square brackets to indicate that an element of this same simplicial kernel $K_3(X_\bullet)$ is a member of the set X_3 of the complex, the set of "commutative 3-simplices". Thus $[x_0, x_1, x_2, x_3]$ means that $(x_0, x_1, x_2, x_3) \in X_3 \subseteq K_3$.[35]

The elements of $K_4 = \mathbf{Cosk}^2(X_\bullet)_4$, the simplicial kernel of the truncated complex $\mathbf{Cosk}^2(X_\bullet)]_0^3$, will be identified with 5×4 simplicial matrices with 2-simplex entries in X_2. Each of the rows $\mathbf{x}_j = (\ \mathrm{pr}_0(\mathbf{x}_j), \mathrm{pr}_1(\mathbf{x}_j), \mathrm{pr}_2(\mathbf{x}_j), \mathrm{pr}_3(\mathbf{x}_j)\ )$ $(0 \le j \le 4)$ is in the simplicial kernel K_3 and, numbered from 0=top to 5=bottom, are its faces as an element of K_4. The columns, numbered from 0=left to 3=right, are the 2-simplex faces of each of the rows as 3-simplex elements of K_3. Thus the entry in the j^{th}-row and i^{th}-column is $d_i(\mathbf{x}_j) = \mathrm{pr}_i(\mathbf{x}_j)$. In order that the matrix represent an element of the simplicial kernel K_4, the simplicial identities must be satisfied, *i.e.*,

$$\mathrm{pr}_i(\mathbf{x}_j) = d_i(\mathbf{x}_j) = d_{j-1}(\mathbf{x}_i) = \mathrm{pr}_{j-1}(\mathbf{x}_i)\ (\ 0 \le i < j \le 4\ ).$$

Such matrices have the form shown in Figure 38.

The dotted double arrows in Figure 38 indicate the equalities which are required by the simplicial identities and also make obvious the fact that any one row of a simplicial matrix is completely determined by its complementary set of rows. This again is just the statement that each of the canonical projection mappings $\mathrm{pr}_{\hat{k}}$ $(0 \le k \le 4)$ from this "simplicial kernel of a simplicial kernel" to any of its five sets of horns is bijective.

7.3. Remark. The same simplicial information can be conveyed "geometrically" by imagining a 4-simplex as projected from 4-space as in Figure 39, say, with x_4 at the barycenter of the tetrahedron (x_0, x_1, x_2, x_3). Each of the five simplicially matching tetrahedral faces can be seen by deleting a vertex and all of the 1-simplices which directly

[35] By abuse of language, we will often, redundantly, say "$[x_0, x_1, x_2, x_3]$ is commutative" when it more properly would be "(x_0, x_1, x_2, x_3) is commutative". In any case, the term "commutative" will be justified in the course of this section.

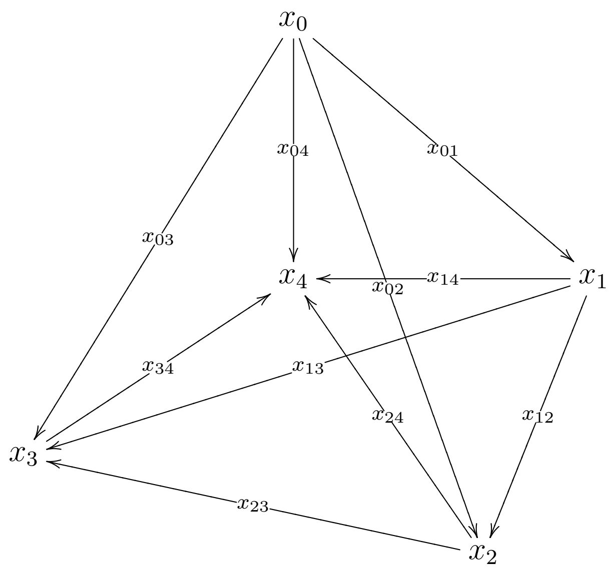

Figure 39: 4-Simplex Model

connect it to the others (making the 0-face the "inside" tetrahedron and the 4-face the "outside" one). Unfortunately, the complications inherent in mentally manipulating numbers of such diagrams seem to severely limit their usefulness. As we hope to show in this paper, sequences of appropriate simplicial matrices carry essentially the same desired matching information in an entirely different "geometric" form and have their very own intuitive "geometric" logic about them.

7.4. Simplicial and Glenn Matrix Notation for Membership in X_4 and its Horns. Since the complex $X_\bullet$ is 3-coskeletal, its set X_4 of 4-simplices is the simplicial kernel of the truncated complex $X_\bullet]_0^3$ and is thus just the subset of the set of simplicial matrices K_4 (Figure 38) consisting of those in which all of the five rows are commutative, we will denote elements of X_4 by simplicial matrices, each of whose rows is individually enclosed in square, "[...]", rather than round, "(...)" brackets.

Similarly, elements of the corresponding sets of horns will be denoted by simplicial matrices in which all but one of the rows will be enclosed by square brackets, with that one "missing row" (uniquely determined as an element of $K_3 = \mathbf{Cosk}^2(X_\bullet)_3$ by the others) enclosed in round brackets. Row and face numbering, we emphasize, will be from 0 to 4, top to bottom, and 0 to 3, left to right, respectively.

For typographical convenience in the proofs that will follow, we will usually place the simplicial matrix ("S-matrix") in a table which will leave room for a reference number for the matrix, the "state of a row " (in X_3 or K_3) and a short reason for the commutativity of the row, if needed. Figure 40 gives a format for a 1-horn example.

7.5. Notation. In the proofs we will be taking an element of some horn of our complex, *e.g.*, the S-matrix in Figure 40, and using some property of the facial entries in the

S-Matrix $\langle\#\rangle$	$\langle S\rangle$	0	1	2	3	$\langle S\rangle$	$\langle Reason\rangle$
0	[	x_{234}	x_{134}	x_{124}	x_{123}	]	$\langle Reason\rangle$
1	(	x_{234}	x_{034}	x_{024}	x_{023}	)	
2	[	x_{134}	x_{034}	x_{014}	x_{013}	]	$\langle Reason\rangle$
3	[	x_{124}	x_{024}	x_{014}	x_{012}	]	$\langle Reason\rangle$
4	[	x_{123}	x_{023}	x_{013}	x_{012}	]	$\langle Reason\rangle$

Figure 40: S-Matrix in $\bigwedge^1_4(X_\bullet)$ in Tabular Form

GM-$\langle\#\rangle$	$\langle S\rangle$	0	1	2	3	$\langle S\rangle$	$\langle Reason\rangle$
0	[	x_{234}	x_{134}	x_{124}	x_{123}	]	$\langle Reason\rangle$
(1)	[	x_{234}	x_{034}	x_{024}	x_{023}	]	$x_{012} \in I_2$
2	[	x_{134}	x_{034}	x_{014}	x_{013}	]	$\langle Reason\rangle$
3	[	x_{124}	x_{024}	x_{014}	x_{012}	]	$\langle Reason\rangle$
4	[	x_{123}	x_{023}	x_{013}	x_{012}	]	$\langle Reason\rangle$

Figure 41: Glenn Matrix in X_4 obtained from $\bigwedge^1_4(X_\bullet)$

commutative rows, conclude that the uniquely determined (as an element of K_3) "missing row" is commutative. The resulting matrix is then an element of X_4, and will be presented in tabular form as what we will call a *Glenn matrix* as in Figure 41. The "formerly missing" row will have its number indicated by being enclosed in parentheses, with the reason for the commutativity of the row appearing to the right. Thus the Glenn matrix of Figure 41 can then completely replace the one in Figure 40 which it still encodes and present an entire chain of reasoning in a compact immediately comprehensible form.

If the full matrix display is inconvenient, the rows as elements of K_3 will appear in boldface $\mathbf{x}_j$ $(0 \le j \le 4)$ with $[\mathbf{x}_j]$ indicating membership in X_3 and $\mathbf{x}_j$ (or redundantly, $(\mathbf{x}_j)$) indicating membership in K_3 but not necessarily in X_3. Thus, for example, $[[\mathbf{x}_0], \mathbf{x}_1, [\mathbf{x}_2], [\mathbf{x}_3], [\mathbf{x}_4]]$ or $[[\mathbf{x}_0], (\mathbf{x}_1), [\mathbf{x}_2], [\mathbf{x}_3], [\mathbf{x}_4]]$ will be shorthand for a 5×4 simplicial matrix element of $\bigwedge^1_4(X_\bullet)$.

7.6. The Definition of the Invertible 2 and 1-Simplices. The notion of invertibility plays a crucial role in what follows. We will first define the set $I_2 \subseteq X_2$ of *invertible 2-simplices* which will only involve fiber-bijectivity on the non-extremal "inner horn sets". For this we consider the following sets of 2-simplices in X_2:

$$\begin{aligned}\hat{I}^1_3 &= \left\{\, x \;\middle|\; \hat{\mathbf{x}} \in \textstyle\bigwedge^1_3(X_\bullet) \cdot d_3(\hat{\mathbf{x}}) = x \Longrightarrow \exists!\, \mathbf{x} \in X_3 \cdot \mathrm{pr}_{\hat{1}}(\mathbf{x}) = \hat{\mathbf{x}} \right\} \\ &= \left\{\, x \;\middle|\; \forall\, (x_0, \_, x_2, x) \in \textstyle\bigwedge^1_3(X_\bullet) \cdot \exists!\, x_1 \cdot \ni \cdot [x_0, x_1, x_2, x] \,\right\}\end{aligned}$$

$$\hat{I}^2_3 = \left\{\, x \;\middle|\; \hat{\mathbf{x}} \in \textstyle\bigwedge^2_3(X_\bullet) \cdot d_0(\hat{\mathbf{x}}) = x \Longrightarrow \exists!\, \mathbf{x} \in X_3 \cdot \mathrm{pr}_{\hat{2}}(\mathbf{x}) = \hat{\mathbf{x}} \right\}$$

$$= \left\{ \; x \;\middle|\; \forall\,(x, x_1, \_\,, x_3) \in \bigwedge_3^2(X_\bullet) \cdot \exists!\, x_2 \cdot \ni \cdot [x, x_1, x_2, x_3] \right\}$$

$$\hat{I}_4^1 = \left\{ \; x \;\middle|\; \hat{\mathbf{x}} \in \bigwedge_4^1(X_\bullet) \cdot d_3(d_3(\hat{\mathbf{x}})) = x \Longrightarrow \exists!\, \mathbf{x} \in X_4 \cdot \mathrm{pr}_{\hat{1}}(\mathbf{x}) = \hat{\mathbf{x}} \right\}$$

$$= \left\{ x \middle| [\,[\mathbf{x}_0], (\mathbf{x}_1), [\mathbf{x}_2], [\mathbf{x}_3], [\mathbf{x}_4]\,] \in \bigwedge_4^1(X_\bullet) \cdot d_3(\mathbf{x}_3) = x \cdot \Longrightarrow \; [\mathbf{x}_1] \right\}$$

$$\hat{I}_4^2 = \left\{ \; x \;\middle|\; \hat{\mathbf{x}} \in \bigwedge_4^2(X_\bullet) \cdot d_0(d_4(\hat{\mathbf{x}})) = x \Longrightarrow \exists!\, \mathbf{x} \in X_4 \cdot \mathrm{pr}_{\hat{2}}(\mathbf{x}) = \hat{\mathbf{x}} \right\}$$

$$= \left\{ x \middle| [\,[\mathbf{x}_0], [\mathbf{x}_1], (\mathbf{x}_2), [\mathbf{x}_3], [\mathbf{x}_4]\,] \in \bigwedge_4^2(X_\bullet) \cdot d_0(\mathbf{x}_4) = x \cdot \Longrightarrow [\mathbf{x}_2] \right\}$$

$$\hat{I}_4^3 = \left\{ \; x \;\middle|\; \hat{\mathbf{x}} \in \bigwedge_4^3(X_\bullet) \cdot d_0(d_1(\hat{\mathbf{x}})) = x \Longrightarrow \exists!\, \mathbf{x} \in X_4 \cdot \mathrm{pr}_{\hat{2}}(\mathbf{x}) = \hat{\mathbf{x}} \right\}$$

$$= \left\{ x \middle| [\,[\mathbf{x}_0], [\mathbf{x}_1], [\mathbf{x}_2], (\mathbf{x}_3), [\mathbf{x}_4]\,] \in \bigwedge_4^3(X_\bullet) \cdot d_0(\mathbf{x}_1) = x \cdot \Longrightarrow [\mathbf{x}_3] \right\}$$

If $\mathbf{x} = (x_{123}, x_{023}, x_{013}, x_{012}) \in K_3$, then the 2-simplex x in $\hat{I}_3^1$ is $x = x_{012}$; in $\hat{I}_3^2$, $x = x_{123}$. In the S-matrix of Figure 38 and in the Glenn matrix of Figure 41, the position of the 2-simplex x in $\hat{I}_4^1$ is $x = x_{012}$ and in ; in $\hat{I}_4^2$, $x = x_{123}$; in $\hat{I}_4^3$, $x = x_{234}$ (see below).

7.7. Definition. ***(Invertible* 2*-simplex)*** *A* 2*-simplex x will be said to be* invertible *if x satisfies the defining properties of all five of the above sets. The set of* invertible 2-simplices *is thus given by*

$$I_2(X_\bullet) = \hat{I}_3^1 \cap \hat{I}_3^2 \cap \hat{I}_4^1 \cap \hat{I}_4^2 \cap \hat{I}_4^3.$$

Thus in what follows **the working inner horn lifting conditions will be:**[36] For 3-simplices,

- $(\mathbf{k=1})$ If $(x_{123}, \_\,, x_{023}, x_{012}) \in \bigwedge_3^1(X_\bullet)$ with $x_{012} \in I_2(X_\bullet)$, then there exists a *unique* 2-simplex x_{023}[37] such that

$$[x_{123}, x_{023}, x_{023}, x_{012}] \in X_3.$$

[36]Readers who wish to consider the sets of formally invertible 1 and 2-simplices as an additional structure on the Postnikov complex, may consider these *working conditions* as horn conditions for each of each of the bulleted horns which hold in the complex whenever the invertibility membership conditions on the appropriate simplices are satisfied, in particular, those marked in the ⟨*Reason*⟩ columns for the 4-simplices .

[37]Note that the boundary of the "missing face" is uniquely determined by the boundaries of the 2-simplices present in the horn.

- $(\mathbf{k=2})$ If $(x_{123}, x_{023}, \_, x_{012}) \in \bigwedge_3^2(X_\bullet)$ with $x_{123} \in I_2(X_\bullet)$, then there exists a *unique* 2-simplex x_{013} such that

$$[x_{123}, x_{023}, x_{023}, x_{012}] \in X_3.$$

and for 4-simplices, when viewed as Glenn matrices (7.5), the inner horn lifting conditions become

- $(\mathbf{k=1})$

GM-$\langle\#\rangle$	$\langle S\rangle$	0	1	2	3	$\langle S\rangle$	$\langle Reason\rangle$
0	[	x_{234}	x_{134}	x_{124}	x_{123}	]	$\langle Reason\rangle$
(1)	[	x_{234}	x_{034}	x_{024}	x_{023}	]	$x_{012} \in I_2$
2	[	x_{134}	x_{034}	x_{014}	x_{013}	]	$\langle Reason\rangle$
3	[	x_{124}	x_{024}	x_{014}	x_{012}	]	$\langle Reason\rangle$
4	[	x_{123}	x_{023}	x_{013}	x_{012}	]	$\langle Reason\rangle$

- $(\mathbf{k=2})$

GM-$\langle\#\rangle$	$\langle S\rangle$	0	1	2	3	$\langle S\rangle$	$\langle Reason\rangle$
0	[	x_{234}	x_{134}	x_{124}	x_{123}	]	$\langle Reason\rangle$
1	[	x_{234}	x_{034}	x_{024}	x_{023}	]	$\langle Reason\rangle$
(2)	[	x_{134}	x_{034}	x_{014}	x_{013}	]	$x_{123} \in I_2$
3	[	x_{124}	x_{024}	x_{014}	x_{012}	]	$\langle Reason\rangle$
4	[	x_{123}	x_{023}	x_{013}	x_{012}	]	$\langle Reason\rangle$

- $(\mathbf{k=3})$

GM-$\langle\#\rangle$	$\langle S\rangle$	0	1	2	3	$\langle S\rangle$	$\langle Reason\rangle$
0	[	x_{234}	x_{134}	x_{124}	x_{123}	]	$\langle Reason\rangle$
1	[	x_{234}	x_{034}	x_{024}	x_{023}	]	$\langle Reason\rangle$
2	[	x_{134}	x_{034}	x_{014}	x_{013}	]	$\langle Reason\rangle$
(3)	[	x_{124}	x_{024}	x_{014}	x_{012}	]	$x_{234} \in I_2$
4	[	x_{123}	x_{023}	x_{013}	x_{012}	]	$\langle Reason\rangle$

We now use the preceding definition of $I_2(X_\bullet)$ and look at four more sets, this time of 1-simplices:

$$\hat{I}_3^0 = \left\{ x \,\middle|\, \hat{\mathbf{x}} \in \bigwedge_3^0 \cdot d_2(\hat{\mathbf{x}}) \in I_2 \cdot \wedge \cdot d_2(d_2(\hat{\mathbf{x}})) = x \cdot \Longrightarrow \exists!\, \mathbf{x} \in X_3 \cdot \ \mathrm{pr}_{\hat{0}}(\mathbf{x}) = \hat{\mathbf{x}} \right\}$$

$$= \left\{ x \,\middle|\, (\_, x_1, x_2, x_3) \in \bigwedge_3^0 \cdot x_2 \in I_2 \cdot \wedge \cdot d_2(x_2) = x \cdot \Longrightarrow \exists!\, x_0 \cdot \ni \cdot [x_0, x_1, x_2, x_3] \right\}$$

$$\hat{I}_3^3 = \left\{ x \,\middle|\, \hat{\mathbf{x}} \in \bigwedge_3^3 \cdot d_1(\hat{\mathbf{x}}) \in I_2 \cdot \wedge \cdot d_0(d_1(\hat{\mathbf{x}})) = x \cdot \Longrightarrow \exists!\, \mathbf{x} \in X_3 \cdot \mathrm{pr}_{\hat{3}}(\mathbf{x}) = \hat{\mathbf{x}} \right\}$$

$$= \left\{ x \,\middle|\, (x_0, x_1, x_2, \_) \in \bigwedge_3^3 \cdot x_1 \in I_2 \cdot \wedge \cdot d_0(x_1) = x \cdot \Longrightarrow \exists!\, x_3 \cdot \ni \cdot [x_0, x_1, x_2, x_3] \right\}$$

$$\hat{I}_4^0 = \left\{ x \,\middle|\, \hat{\mathbf{x}} \in \bigwedge_4^0 \cdot d_2(d_2(\hat{\mathbf{x}})) \in I_2 \cdot \wedge \cdot d_2(d_2(d_2(\hat{\mathbf{x}}))) = x \cdot \Longrightarrow \exists!\, \mathbf{x} \in X_4 \cdot \mathrm{pr}_{\hat{0}}(\mathbf{x}) = \hat{\mathbf{x}} \right\}$$

$$= \left\{ x \middle| [(\mathbf{x}_0), [\mathbf{x}_1], [\mathbf{x}_2], [\mathbf{x}_3], [\mathbf{x}_4]] \in \bigwedge_4^0 \cdot d_2(\mathbf{x}_2) \in I_2 \cdot \wedge \cdot d_2(d_2(x_2)) = x \cdot \Longrightarrow [\mathbf{x}_0] \right\}$$

$$\hat{I}_4^4 = \left\{ x \,\middle|\, \hat{\mathbf{x}} \in \bigwedge_4^4 \cdot d_1(d_2(\hat{\mathbf{x}})) \in I_2 \cdot \wedge \cdot d_0(d_1(d_2(\hat{\mathbf{x}}))) = x \cdot \Longrightarrow \exists!\, \mathbf{x} \in X_4 \cdot \mathrm{pr}_{\hat{4}}(\mathbf{x}) = \hat{\mathbf{x}} \right\}$$

$$= \left\{ x \middle| [[\mathbf{x}_0], [\mathbf{x}_1], [\mathbf{x}_2], [\mathbf{x}_3], (\mathbf{x}_4)] \in \bigwedge_4^4 \cdot d_1(\mathbf{x}_2) \in I_2 \cdot \wedge \cdot d_2(d_2(x_2)) = x \cdot \Longrightarrow [\mathbf{x}_4] \right\}$$

If $\mathbf{x} = (x_{123}, x_{023}, x_{013}, x_{012}) \in K_3$, then the 2-simplex x_2 in $\hat{I}_3^0$ is $x = x_{013}$, the 1-simplex x is $x = x_{01}$; in $\hat{I}_3^3$, $x_1 = x_{023}$, $x = x_{23}$.In the S-matrix of Figure 38, the position of the invertible 2-simplex $d_2(\mathbf{x}_2))$ in $\hat{I}_4^0$ is x_{014}, the 1-simplex x is x_{01}; in $\hat{I}_4^4$, the invertible $d_1(\mathbf{x}_2)$is x_{034} and $x = x_{34}$.

7.8. Definition. ***(Weakly Invertible*** **1*-Simplex)*** *A* 1*-simplex is said to be* weakly invertible *if it satisfies the defining properties of the sets of* 1*-simplices associated with the extremal horn sets above. The set of weakly invertible* 1*-simplices is thus*

$$I_1^w = \hat{I}_3^0 \cap \hat{I}_3^3 \cap \hat{I}_4^0 \cap \hat{I}_4^4.$$

Finally, look at the following sets of 1-simplices:

$$\hat{I}_2^0 = \left\{ x \,\middle|\, \hat{\mathbf{x}} \in \bigwedge_2^0 \cdot d_1(\hat{\mathbf{x}}) = x \cdot \Longrightarrow \exists\, t \in I_2 \cdot \mathrm{pr}_{\hat{0}}(t) = \hat{\mathbf{x}} \right\}$$

$$= \left\{ x \,\middle|\, (\_, x, x_2) \in \bigwedge_2^0 \cdot \Longrightarrow \exists\, x_0 \in X_1 \cdot \wedge \cdot t \in I_2 \cdot \ni \cdot \partial(t) = (x_0, x, x_2) \right\}$$

$$\hat{I}_2^2 = \left\{ x \,\middle|\, \hat{\mathbf{x}} \in \bigwedge_2^2 \cdot d_1(\hat{\mathbf{x}}) = x \cdot \Longrightarrow \exists\, t \in I_2 \cdot \mathrm{pr}_{\hat{2}}(t) = \hat{\mathbf{x}} \right\}$$

$$= \left\{ x \,\middle|\, (x_0, x, \_) \in \bigwedge_2^2 \cdot \Longrightarrow \exists\, x_2 \in X_1 \cdot \wedge \cdot t \in I_2 \cdot \ni \cdot \partial(t) = (x_0, x, x_2) \right\}$$

7.9. Definition. ***Invertible* 1*-Simplex.*** *We say that a* 1*-simplex is* invertible *if it satisfies both the defining properties of* I_1^w *as well as that of these last two sets. Thus the set* I_1 *of* invertible 1-simplices *is given by*

$$I_1 = I_1^w \cap \hat{I}_2^0 \cap \hat{I}_2^2.$$

Thus in what follows **the working extremal horn lifting conditions will be:**
For 2-simplices:

- $(\mathbf{k=0})$: For $(\_, x_{02}, x_{01}) \in \bigwedge_2^0(X_\bullet)$ with $x_{01} \in I_1(X_\bullet)$, there exists a 2-simplex $x_{012} \in I_2(X_\bullet)$ which fills the horn. x_{012} thus defines through $d_0(x_{012}) = x_{12}$ *a pseudo-extension* of x_{02} through the invertible 1-simplex x_{01}, since the boundary of x_{012} here has the form

$$\begin{tikzcd} x_1 \arrow[dr, dotted, "x_{12}"] & \\ x_0 \arrow[u, "x_{01}"] \arrow[r, "x_{02}"'] & x_2. \end{tikzcd}$$

- $(\mathbf{k=2})$: For $(x_{12}, x_{02}, \_) \in \bigwedge_2^2(X_\bullet)$ with $x_{12} \in I_1(X_\bullet)$, there exists an 2-simplex $x_{012} \in I_2(X_\bullet)$ which fills the horn. x_{012} thus defines through $d_2(x_{012}) = x_{01}$ *a pseudo-lifting* of x_{02} through the invertible 1-simplex x_{12}, since the boundary of x_{012} here has the form

$$\begin{tikzcd} & x_1 \arrow[d, "x_{12}"] \\ x_0 \arrow[ur, dotted, "x_{01}"] \arrow[r, "x_{02}"'] & x_2. \end{tikzcd}$$

For 3-simplices:

- $(\mathbf{k=0})$: For $(\_, x_{023}, x_{013}, x_{012}) \in \bigwedge_3^0(X_\bullet)$ with $x_{013} \in I_2(X_\bullet)$ and $d_2(x_{013}) = x_{01} \in I_1(X_\bullet)$ there exists a unique 2-simplex x_{123} such that

$$[x_{123}, x_{023}, x_{013}, x_{012}] \in X_3.$$

- $(\mathbf{k=3})$: For $(x_{123}, x_{023}, x_{013}, \_) \in \bigwedge_3^3(X_\bullet)$ with $x_{023} \in I_2(X_\bullet)$ and $d_0(x_{023}) = x_{23} \in I_1(X_\bullet)$, there exists a unique 2-simplex x_{012} such that

$$[x_{123}, x_{023}, x_{013}, x_{012}] \in X_3.$$

For 4-simplices, the corresponding Glen matrices are

- $(\mathbf{k=0})$:

GM-⟨#⟩	⟨S⟩	0	1	2	3	⟨S⟩	⟨Reason⟩
(0)	[	x_{234}	x_{134}	x_{124}	x_{123}	]	$x_{014} \in I_2$ and $d_2(x_{014}) = x_{01} \in I_1$
1	[	x_{234}	x_{034}	x_{024}	x_{023}	]	⟨*Reason*⟩
2	[	x_{134}	x_{034}	x_{014}	x_{013}	]	⟨*Reason*⟩
3	[	x_{124}	x_{024}	x_{014}	x_{012}	]	⟨*Reason*⟩
4	[	x_{123}	x_{023}	x_{013}	x_{012}	]	⟨*Reason*⟩

- $(\mathbf{k} = \mathbf{4})$:

GM-⟨#⟩	⟨S⟩	0	1	2	3	⟨S⟩	⟨Reason⟩
0	[	x_{234}	x_{134}	x_{124}	x_{123}	]	⟨*Reason*⟩
1	[	x_{234}	x_{034}	x_{024}	x_{023}	]	⟨*Reason*⟩
2	[	x_{134}	x_{034}	x_{014}	x_{013}	]	⟨*Reason*⟩
3	[	x_{124}	x_{024}	x_{014}	x_{012}	]	⟨*Reason*⟩
(4)	[	x_{123}	x_{023}	x_{013}	x_{012}	]	$x_{034} \in I_2$ and $d_0(x_{034}) = x_{34} \in I_1$

The above definitions are more redundant than they need to be and will be streamlined at a later point.[38] If $X_\bullet$=$\mathbf{Ner}(\mathbb{B})$ is the nerve of a bicategory, then the we have shown in Section 6 that the following hold :

- A 2-simplex is invertible if its 2-cell interior is an isomorphism;
- a 1-simplex is weakly invertible if the corresponding tensor functors are fully faithful and is invertible if the corresponding tensor functors are equivalences.

Interestingly, no use of the foregoing *extremal* horn lifting conditions need be made in the next section. The conditions imposed only make use of invertibility applied to $\bigwedge_n^k$ for $0 < k < n$ as bulleted above.

7.10. Characterization of Nerves of Bicategories.. Let $X_\bullet$ be a 2-dimensional Postnikov-complex:

$$\mathbf{Cosk}^3(X_\bullet) = X_\bullet \subseteq \mathbf{Cosk}^2(X_\bullet),$$

and let the set of invertible 2-simplices, $I_2(X_\bullet) \subseteq X_2$ be defined as above (Definition 7.7).

We now make our

First Basic Assumption in the Case $n = 2$***: For all*** $x_{01} \in X_1$ ***the degenerate*** 2***-simplices*** $s_0(x_{01})$ ***and*** $s_1(x_{01})$ ***are invertible.***

We make it the hypothesis of the following theorem.

[38] Again: The reader who finds the forgoing unsatisfactory as *definitions* of invertibility can just consider that we have a 2-dimensional Postnikov complex $X_\bullet$ which is supplied with the additional structure of distinguished subsets, $I_2(X_\bullet) \subseteq X_2$ and $I_1(X_\bullet) \subseteq X_1$, of *(formally) invertible* simplices, for which the above bulleted horn lifting properties are satisfied.

$$\begin{tikzcd}
x_0 \arrow[r, "s_0(x_0)"] \arrow[dr, "x_{01}"'] \arrow[dr, phantom, shift left=3, "\alpha"] & x_0 \arrow[d, "y_{01}"] \\
& x_1
\end{tikzcd}$$

Figure 42: $\alpha \in \mathbb{P}(X_\bullet)_1$ as 2-cell $\alpha : x_{01} \Longrightarrow y_{01}$ in $\mathbb{B}(X_\bullet)$

$$\begin{tikzcd}
(0)\, x_{01} \arrow[r, Rightarrow, "\alpha_2"] \arrow[dr, Rightarrow, "\alpha_1"'] & y_{01}\, (1) \arrow[d, Rightarrow, "\alpha_0"] \\
& z_{01}\, (2)
\end{tikzcd}
\qquad
\begin{tikzcd}
& x_0\,(0) \arrow[dl, "x_{01}"'] \arrow[dr, "s_0(x_0)"] \arrow[dd] & \\
(3)\, x_1 & & x_0\,(1) \arrow[ll, "s_0(x_0)"'] \arrow[dl, "s_0(x_0)"] \\
& x_0\,(2) \arrow[ul, "z_{01}"] &
\end{tikzcd}$$

Figure 43: $\mathbf{x}$ in $\mathbb{P}(X_\bullet)_2$ as $\mathbf{x}$ in X_3 with $d_3(\mathbf{x}) = s_0^2(x_0)$.

7.11. THEOREM. **($\mathbb{P}(X_\bullet)$ is the Nerve of a Category.)** *Suppose that for all $x_{01} \in X_1$ the degenerate 2-simplices $s_0(x_{01})$ and $s_1(x_{01})$ are invertible, then the path-homotopy complex $\mathbb{P}(X_\bullet)$ (Section 2.6) is the nerve of a category $\mathbb{B}(X_\bullet)$ whose objects are the 1-simplices of $X_\bullet$ and whose arrows are the 2-simplices α of $X_\bullet$ whose boundary is of the form*

$$\partial(\alpha) = (y_{01}, x_{01}, s_0(x_0)).$$

Theorem 2.7 immediately gives that $\mathbb{P}(X_\bullet)$ is a 1-dimensional Postnikov complex. Such an $\alpha \in \mathbb{P}(X_\bullet)_1$ when viewed as an arrow of $\mathbb{B}(X_\bullet)$ will be called a 2-cell with source $x_{01} = d_1(\alpha)$ and target $y_{01} = d_0(\alpha)$ and be denoted by $\alpha : x_{01} \Longrightarrow y_{01}$ (Figure 42).

If $\mathbf{x}$ is an element of $\mathbb{P}(X_\bullet)_2$, then $\mathbf{x}$ has the form

$$\mathbf{x} = [\alpha_0, \alpha_1, \alpha_2, s_0^2(x_0)] \in X_3$$

and is thus commutative. The boundary matrix of $\mathbf{x}$ immediately shows that α_0, α_1, and α_2 are 2-cells of $\mathbb{B}(X_\bullet)$ and that $\mathbf{x}$ has the form shown "geometrically" in Figure 43.

Since $s_0^2(x_0) = s_0(s_0(x_0))$ is invertible, $d_1(\mathbf{x}) = \alpha_1$ is uniquely determined by

$$(\alpha_0, \_\,, \alpha_2, s_0^2(x_0)) \in \textstyle\bigwedge_3^1(X_\bullet) \simeq \bigwedge_2^1(\mathbb{P}(X_\bullet)),$$

(which we may with justification call the set of composable pairs of 2-cells in $\mathbb{B}(X_\bullet)$) and call the uniquely determined α_1 the *composition of the 2-cells α_0 and α_2 in* $\mathbb{B}(X_\bullet)$ and denote it by

$$\alpha_1 = \alpha_0 \circ \alpha_2 \quad \text{(or more simply, } \alpha_1 = \alpha_0\alpha_2\,\text{)}.$$

$$\begin{bmatrix} [& \alpha_{23} & \alpha_{13} & \alpha_{12} & s_0^2(x_0) &] \\ (& \alpha_{23} & \alpha_{03} & \alpha_{02} & s_0^2(x_0) &) \\ [& \alpha_{13} & \alpha_{03} & \alpha_{01} & s_0^2(x_0) &] \\ [& \alpha_{12} & \alpha_{02} & \alpha_{01} & s_0^2(x_0) &] \\ [& s_0^2(x_0) & s_0^2(x_0) & s_0^2(x_0) & s_0^2(x_0) &] \end{bmatrix}$$

Figure 44: $\mathbf{x} \in \bigwedge_3^1(\mathbb{P}(X_\bullet))$

$$\begin{bmatrix} [& \alpha_{23} & \alpha_{23} \circ \alpha_{12} & \alpha_{12} & s_0^2(x_0) &] \\ (& \alpha_{23} & (\alpha_{23} \circ \alpha_{12}) \circ \alpha_{01} & x_{124} \circ \alpha_{01} & s_0^2(x_0) &) \\ [& \alpha_{23} \circ \alpha_{12} & (\alpha_{23} \circ \alpha_{12}) \circ \alpha_{01} & \alpha_{01} & s_0^2(x_0) &] \\ [& \alpha_{12} & \alpha_{12} \circ \alpha_{01} & \alpha_{01} & s_0^2(x_0) &] \\ [& s_0^2(x_0) & s_0^2(x_0) & s_0^2(x_0) & s_0^2(x_0) &] \end{bmatrix}$$

Figure 45: $\mathbf{x} \in \bigwedge_3^1(\mathbb{P}(X_\bullet))$

Thus

$$[\alpha_0, \alpha_1, \alpha_2, s_0^2(x_0)] \in X_3,\ s_0^2(x_0) \in I_2(X_\bullet) \iff \alpha_1 = \alpha_0 \circ \alpha_2 \text{ in } \mathbb{B}(X_\bullet).$$

Since degenerates in X_3 are always commutative, for any 2-cell

$$\alpha : x_{01} \Longrightarrow y_{01}$$

in $\mathbb{B}(X_\bullet)$,

$$s_0(\alpha) = [\alpha, \alpha, s_0(x_{01}), s_0^2(x_0)] \iff \alpha = \alpha \circ s_0(x_{01}), \quad \text{and}$$
$$s_1(\alpha) = [s_0(y_{01}), \alpha, \alpha, s_0^2(x_0)] \iff s_0(y_{01}) \circ \alpha = \alpha,$$

so that $s_0(d_0(\alpha))$ and $s_0(d_1(\alpha))$ furnishes the right and left identity 2-cells for any 2-cell α in $\mathbb{B}(X_\bullet)$.

Now we look at any element of $\bigwedge_3^k(\mathbb{P}(X_\bullet))$. As an element of $\bigwedge_4^k(X_\bullet)$ it has the form of a simplicial matrix 4-simplex $\mathbf{x}$ whose last row $d_4(\mathbf{x})$ is totally degenerate,

$$d_4(\mathbf{x}) = s_0^3(x_0) = s_0(s_0^2(x_0)) = [s_0^2(x_0), s_0^2(x_0), s_0^2(x_0), s_0^2(x_0)],$$

for some $x_0 \in X_0$. But this means that the last column consists of the degeneracies $s_0^2(x_0)$, or equivalently, that for each row $\mathbf{x}_i$, $d_3(\mathbf{x}_i) = s_0^2(x_0)$ and thus that each row is an element of $K_3(\mathbb{P}(X_\bullet))$, commutative or not, depending on k.

For associativity we take $k = 1$, so that $\mathbf{x}$ has the form of the matrix in Figure 44.

Using commutativity and the definition of the composition, the matrix is identical that of Figure 45.

But $d_3(\text{Row } 3) = s_0^2(x_0)$ is invertible, so by the second property of invertibility, Row $1 = d_1(\mathbf{x})$ is commutative, or equivalently, that

$$(\alpha_{23} \circ \alpha_{12}) \circ \alpha_{01} = \alpha_{23} \circ (\alpha_{12} \circ \alpha_{01})$$

$$\begin{bmatrix} [& \alpha & s_0(y_{01}) & \alpha^* & s_0^2(x_0)\,] \\ [& \alpha & \alpha & s_0(x_{01}) & s_0^2(x_0)\,] \\ [& s_0(y_{01}) & \alpha & \alpha & s_0^2(x_0)\,] \\ (& \alpha^* & s_0(x_{01}) & \alpha & s_0^2(x_0)\,) \\ [& s_0^2(x_0) & s_0^2(x_0) & s_0^2(x_0) & s_0^2(x_0)\,] \end{bmatrix}$$

Figure 46: α^* Matrix

and we have associativity. This completes the proof of Theorem 7.11 that $\mathbb{B}(X_\bullet)$ is a category with nerve

$$\mathbf{Ner}(\mathbb{B}(X_\bullet)) = \mathbb{P}(X_\bullet).$$

7.12. Proposition. *Let* $\alpha : x_{01} \Longrightarrow y_{01}$ *be a* 2*-cell in* $\mathbb{B}(X_\bullet)$*. If* α *as a* 2*-simplex in* $X_\bullet$ *is* invertible*,i.e.,* $\alpha \in I_2(X_\bullet)$*, then* α *is an* isomorphism *in* $\mathbb{B}(X_\bullet)$*.*

In effect,

$$(\alpha, s_0(y_{01}), \_\,, s_0^2(x_0)) \in \bigwedge\nolimits_3^2(X_\bullet).$$

If α is invertible, then there exists a unique 2-simplex (necessarily a 2-cell) α^* in X_2 such that

$$[\alpha, s_0(y_{01}), \alpha^*, s_0^2(x_0)] \Longleftrightarrow \alpha \circ \alpha^* = s_0(y_{01}).$$

Now consider the simplicial matrix in Figure 46.

By hypothesis, the top row, Row 0, is commutative. Row 1, Row 2, and Row 4 are, respectively, the degenerates $s_0(\alpha)$, $s_1(\alpha)$, and $s_0(s_0^2(x_0)))$, and are thus commutative. But $\alpha = d_0$(Row 1) is invertible, so that we have that Row 3 is commutative, but then

$$[\,\alpha^*, s_0(x_{01}), \alpha, s_0^2(x_0)\,] \Longleftrightarrow \alpha^* \circ \alpha = s_0(x_{01}),$$

and $\alpha^* = \alpha^{-1}$ as asserted.

The converse is also true, but we will establish it as part of a more general statement which will characterize all of the invertible 2-simplices of $X_\bullet$ in terms of the 2-cell isomorphisms of the category $\mathbb{B}(X_\bullet)$.

7.13. Remark. The basic proof just given above that $\mathbb{B}(X_\bullet)$ is a category could equally well be obtained as an application of the characterization of nerves of categories given in Section 4, as can the above proposition using the characterization of invertibles given there. $\mathbb{P}(X_\bullet)$ just shifts down the properties by one dimension, provided that the degeneracies are invertible. $\mathbb{P}(\mathbb{P}(X_\bullet)) = \mathbb{P}^2(X_\bullet)$ here ($n = 2$) is discrete. This will be put to advantage when we look at dimension 3 (and above). For dimension $n = 3$, $\mathbb{P}(X_\bullet)$ will immediately be seen to be the nerve of a bicategory, and consequently that $\mathbb{P}(\mathbb{P}(X_\bullet)) = \mathbb{P}^2(X_\bullet)$ is the nerve of a category whose objects are 2-cells and whose arrows (3-simplices of $X_\bullet$ whose 2-face is a degenerate 2-simplex and whose 3-face is totally degenerate) will be a 3-cell of the tricategory with d_0 and d_1 being the target and source 2-cell.

$$\begin{tikzcd}
x_0 \arrow[r, "x_{01}"] \arrow[dr, "x_{12}\otimes x_{01}"'] & x_1 \arrow[d, "x_{12}"] \arrow[dl, phantom, "\chi(x_{12},x_{01})"] \\
 & x_2
\end{tikzcd}$$

Figure 47: Invertible 2-simplex $\chi(x_{12}, x_{01})$

We now add a ***Second Basic Assumption in the Case*** $n = 2$ to that of invertibility of all degenerate 2-simplices:

With $X_\bullet$ as above, consider the canonical projection $\mathrm{pr}_{\hat{1}} : X_2 \longrightarrow \bigwedge^1_2(X_\bullet)$. We will assume that $\mathrm{pr}_{\hat{1}}$ is surjective and has a chosen section $\chi^1_2 : \bigwedge^1_2(X_\bullet) \longrightarrow X_2$ $(\mathrm{pr}_{\hat{1}\,\chi^1_2} = \mathrm{id}(\bigwedge^1_2(X_\bullet))$ whose image in X_2 consists of invertible 2-simplices (an *invertible section*).[39] In other words:

For every $(x_{12}, \_, x_{01}) \in \bigwedge^1_2(X_\bullet)$ ***there exists an invertible*** 2***-simplex*** $\chi^1_2(x_{12}, \_, x_{01}) \in I_2(X_\bullet)$ ***such that*** $\mathrm{pr}_{\hat{1}}(\chi^1_2(x_{12}, \_, x_{01})) = (x_{12}, \_, x_{01})$**.**

7.14. Definition. **(Tensor Product of 1-Simplices).** *For the chosen invertible section, we will define the* tensor product *of the (now)* 1*-composable pair* $(x_{12}, \_, x_{01}) \in \bigwedge^1_2(X_\bullet)$ *by*

$$d_1(\chi^1_2(x_{12}, \_, x_{01})) =_{Def} x_{12} \otimes x_{01}$$

and often abbreviate $\chi^1_2(x_{12}, \_, x_{01}) \in X_2$ *by* $\chi^1_2(x_{12}, x_{01})$*, or even more simply, by* $\chi(x_{12}, x_{01})$*.*

Thus, by definition, the boundary of the invertible 2-simplex $\chi(x_{12}, x_{01})$ is given by

$$\partial(\chi(x_{12}, x_{01})) = (x_{12}, x_{12} \otimes x_{01}, x_{01}),$$

and may be geometrically pictured by Figure 47.

We now show in what follows that with this definition of tensor product, the category $\mathbb{B}(X_\bullet)$ becomes the category of 2-cells of a bicategory (Section 5.1), denoted by $\mathbf{Bic}(X_\bullet)$, whose set of 0-cells is the set X_0 of 0-simplices of $X_\bullet$ and whose set of 1-cells is the set X_1 of 1-simplices of $X_\bullet$. This will justify our calling the set of arrows of $\mathbb{B}(X_\bullet)$ $(= \mathbb{P}(X_\bullet)_1)$ the 2-cells of $\mathbf{Bic}(X_\bullet)$ and using a double arrow notation for them.

We define the 0-cell source and 0-cell target of a 2-cell using the canonical simplicial maps of $\mathbb{P}(X_\bullet)$ to the constant complex $K(X_0, 0)$, so that if $\alpha : x_{01} \Longrightarrow y_{01}$ is a 2-cell of $\mathbf{Bic}(X_\bullet)$, then its 0-source is $d_1(x_{01}) = x_0 = d_1(y_{01})$ and its 0-target is $d_0(x_{01}) = x_1 =$

[39]In the case of a topological space X and its fundamental 2-dimensional hypergroupoid $\Pi_2(X)$, *all* 2-simplices are invertible since all of the horn maps in dimensions > 2 are bijective and thus all 2-simplices satisfy the defining conditions for $I_2(\Pi_2(X))$. Since $\mathrm{pr}_{\hat{1}} : \Pi_2(X)_2 \longrightarrow \bigwedge^1_2(\Pi_2(X))$ is surjective *any* section for $\mathrm{pr}_{\hat{1}}$ will do, although it is simpler to choose one which is a least *normalized*, *i.e.,* such that $\chi(s_0(x_1), x_{01}) = s_1(x_{01})$ and $\chi(x_{01}, s_0(x_0)) = s_0(x_{01})$.

$d_0(y_{01})$. Since, from the definition of $\mathbb{P}(X_\bullet)$, for 1-cells x_{01} and y_{01},

$$(\alpha : x_{01} \Rightarrow y_{01}) \in \mathbb{P}(X_\bullet)_1 \quad \Longrightarrow \quad \partial(x_{01}) = (x_0, x_1) = \partial(y_{01}),$$

1-cells must have the same 0-source and 0-target before they can be connected by a 2-cell. If $\mathbb{B}(x_0, x_1)$ denotes the category whose objects are those 1-cells of the form $x_{01} : x_0 \to x_1$, then

$$\mathbb{P}(X_\bullet) \simeq \coprod_{(x_0,x_1)\in X_0\times X_0} \mathbf{Ner}(\mathbb{B}(x_0, x_1)),$$

or equivalently,

$$\mathbb{B}(X_\bullet) \cong \coprod_{(x_0,x_1)\in X_0\times X_0} \mathbb{B}(x_0, x_1).$$

Before we define the tensor actions, let us show that the requirement that $\chi(x_{12}, x_{01})$ and $s_0(x_{01})$ be invertible for all $(x_{12}, x_{01}) \in \bigwedge^1_2(X_\bullet)$ and $x_{01} \in X_1$ allows us to make a *fundamental correspondence between* 2*-simplices and* 2*-cells.*

Let $x_{012} \in X_2$ be a 2-simplex in $X_\bullet$. Consider the 2-horn

$$(\chi(x_{12}, x_{01}), x_{012}, \text{—}, s_0(x_{01})) \in \textstyle\bigwedge^2_3(X_\bullet),$$

where $d_0(x_{012}) = x_{12}$ and $d_2(x_{012}) = x_{01}$. Since $\chi(x_{12}, x_{01})$ is invertible, there exists a unique 2-simplex $\tilde{x}_{012} \in X_2$ such that

$$\begin{aligned} &[\chi(x_{12}, x_{01}), x_{012}, \tilde{x}_{012}, s_0(x_{01})] \\ &= [\chi(d_0(x_{012}), d_2(x_{012})), x_{012}, \tilde{x}_{012}, s_0(d_2(x_{012}))] \in X_3. \end{aligned}$$

The boundary of $\tilde{x}_{012}$ is

$$\partial(\tilde{x}_{012}) = (x_{12} \otimes x_{01}, x_{02}, s_0(x_0)),$$

so that $\tilde{x}_{012}$ is a 2-cell of the form

$$\tilde{x}_{012} : x_{02} \Longrightarrow x_{12} \otimes x_{01}.$$

7.15. Definition. **(Interior of a 2-simplex)** *The just defined unique* 2*-cell* $\tilde{x}_{012}$ *will be called the* interior of the 2-simplex x_{012} *and be denoted by* $\mathrm{Int}(x_{012})$ *or, more simply, by* $\tilde{x}_{012}$ *if the context is clear. (Figure 48) The commutative* 3*-simplex*

$$[\chi(x_{12}, x_{01}), x_{012}, \mathrm{Int}(x_{012}), s_0(x_{01})]$$

will be called the defining 3-simplex for $\tilde{x}_{012}$.

Now given $\chi(x_{12}, x_{01})$ and any 2-cell of $\mathbb{B}(X_\bullet)$ of the form $\alpha : x_{02} \Longrightarrow x_{12} \otimes x_{01}$, the invertibility of $s_0(x_{01})$ guarantees that there exists a unique 2-simplex x_{012} with boundary $\partial(x_{012}) = (x_{12}, x_{02}, x_{01})$ such that

$$[\chi(x_{12}, x_{01}), x_{012}, \alpha, s_0(x_{01})] \in X_3,$$

and thus $\alpha = \tilde{x}_{012}$. We thus have

$$\begin{tikzcd}
x_0\,(0) \arrow[rr, "s_0(x_0))"] \arrow[dr, "x_{01}"] \arrow[ddr, "x_{02}"'] & & x_0\,(1) \arrow[dl, "x_{01}"'] \arrow[ddl, "x_{12}\otimes x_{01}"] \\
& x_1\,(2) \arrow[d, "x_{12}"] & \\
& x_2\,(3) &
\end{tikzcd}$$

Figure 48: $[\chi(x_{12}, x_{01}), x_{012}, \tilde{x}_{012}, s_0(x_{01})] \in X_3$

7.16. THEOREM. *If the 2-simplices $\chi(x_{12}, x_{01})$ and $s_0(x_{01})$ are invertible for all $(x_{12}, x_{01}) \in \bigwedge_2^1(X_\bullet)$ and all $x_{01} \in X_1$, then the correspondence*

$$x_{012} \mapsto (\partial(x_{012}), \tilde{x}_{012}),$$

where $\partial(x_{012}) = (d_0(x_{012}), d_1(x_{012}), d_2(x_{012}))$ and $\tilde{x}_{012}$ is the 2-cell $\mathrm{Int}(x_{012}) : d_1(x_{012}) \Longrightarrow d_1(\ \chi(d_0(x_{012}), d_2(x_{012}))\ )$ of Definition 7.15, defines a bijection

$$X_2 \xrightarrow{\sim} K_2(X_\bullet) \times_{X_1} \mathbb{P}(X_\bullet)_1$$

of the set of 2-simplices of $X_\bullet$ with the subset of those ordered pairs of boundaries and 2-cells which have the form

$$((x_{12}, x_{02}, x_{01})\ ,\ \alpha : x_{02} \Longrightarrow x_{12} \otimes x_{01}).$$

7.17. PROPOSITION. *Under the above correspondence, for any $(x_{12}, \_\,, x_{01}) \in \bigwedge_2^1(X_\bullet)$,*

$$\mathrm{Int}(\chi(x_{12}, x_{01})) = \tilde{\chi}(x_{12}, x_{01}) = s_0(x_{12} \otimes x_{01})$$

$$= \mathrm{id}(x_{12} \otimes x_{01}) : x_{12} \otimes x_{01} \Longrightarrow x_{12} \otimes x_{01}.$$

In effect, the defining 3-simplex for $\tilde{\chi}(x_{12}, x_{01})$ is

$$[\chi(x_{12}, x_{01}), \chi(x_{12}, x_{01}), \tilde{\chi}(x_{12}, x_{01}), s_0(x_{01})],$$

But

$$s_0(\chi(x_{12}, x_{01})) = [\chi(x_{12}, x_{01}), \chi(x_{12}, x_{01}), s_0(x_{12} \otimes x_{01}), s_0(x_{01})]$$

is commutative and has the same image under $\mathrm{pr}_{\hat{1}} : X_3 \longrightarrow \bigwedge_3^2(X_\bullet)$, therefore $\tilde{\chi}(x_{12}, x_{01}) = s_0(x_{12} \otimes x_{01})$, which has already been identified as the identity 2-cell for the object $x_{12} \otimes x_{01}$ in the category $\mathbb{B}(X_\bullet)$.

S-M$\langle 1\rangle$	S	0	1	2	3	S	$\langle Reason\rangle$
0	[	$\chi(x_{12}, x_{01})$	x_{012}	$\tilde{x}_{012}$	$s_0(x_{01})$	]	Def of $\tilde{x}_{012}$
1	[	$\chi(x_{12}, x_{01})$	$\chi(x_{12}, x_{01})$	$s_0(x_{12} \otimes x_{01})$	$s_0(x_{01})$	]	$s_0(\chi(x_{12}, x_{01}))$
2	[	x_{012}	$\chi(x_{12}, x_{01})$	$\tilde{x}^*_{012}$	$s_0(x_{01})$	]	Def of $\tilde{x}^*_{012}$
3	(	$\tilde{x}_{012}$	$s_0(x_{12} \otimes x_{01})$	$\tilde{x}^*_{012}$	$s_0^2(x_0)$	)	
4	[	$s_0(x_{01})$	$s_0(x_{01})$	$s_0(x_{01})$	$s_0^2(x_0)$	]	$s_0(s_0(x_{01}))$

Figure 49: S-Matrix $\langle 1\rangle$ in $\bigwedge_4^3$ in Tabular Form

7.18. Proposition. *If x_{012} is an invertible 2-simplex in $X_\bullet$, then $\operatorname{Int}(x_{012}) = \tilde{x}_{012}$ is an isomorphism in the category $\mathbb{B}(X_\bullet)$ (and conversely, as we will establish later).*

For the proof, we will use a pair of simplicial matrices and take advantage of the tabular notation of Figure 40 for them. Both of these S-matrices are in $\bigwedge_4^3(X_\bullet)$; the reasons for the commutativity (State=[xxxx]) of the row faces appear in the last column.

Now if x_{012} is invertible, then there exists a unique 2-simplex $\tilde{x}^*_{012}$ such that

$$[x_{012}, \chi(x_{12}, x_{01}), \tilde{x}^*_{012}, s_0(x_{01})] \in X_3.$$

A check of the boundary shows that the 2-simplex is a 2-cell

$$\tilde{x}^*_{012} : x_{12} \otimes x_{01} \Longrightarrow x_{02}.$$

We now form the simplicial matrix ($\langle 1\rangle$) shown in Figure 49.

But since $d_0(\text{Row } 1) = \chi(x_{12}, x_{01})$ is invertible, we have that Row 3 is commutative,

$$[\tilde{x}_{012}, s_0(x_{12} \otimes x_{01}), \tilde{x}^*_{012}, s_0^2(x_0)] \in X_3,$$

which is equivalent to

$$\tilde{x}_{012} \circ \tilde{x}^*_{012} = s_0(x_{12} \otimes x_{01}) = \operatorname{id}(x_{12} \otimes x_{01}),$$

using the definition of the composition in the category $\mathbb{B}(X_\bullet)$.We now show that $\tilde{x}^*_{012}$ is a left inverse as well. For this we form a new simplicial matrix (S-matrix $\langle 2\rangle$) shown in Figure 50. But $d_0(\text{Row } 1) = x_{012}$ is invertible and thus Row 3 is commutative, which is equivalent to

$$\tilde{x}^*_{012} \circ \tilde{x}_{012} = s_0(x_{02}) = \operatorname{id}(x_{02}).$$

Thus $\tilde{x}^*_{012} = \tilde{x}^{-1}_{012}$, and $\tilde{x}_{012}$ is an isomorphism as asserted.

7.19. Remark. Once we have shown that $\mathbf{Bic}(X_\bullet)$ with the chosen tensor product is indeed a bicategory, Proposition 7.16 will show that

$$\mathbf{Ner}(\mathbf{Bic}(X_\bullet))_2 \xrightarrow{\sim} X_2.$$

S-M$\langle 2\rangle$	S	0	1	2	3	S	$\langle Reason\rangle$
0	[	x_{012}	$\chi(x_{12}, x_{01})$	$\tilde{x}^*_{012}$	$s_0(x_{01})$	]	Def of $\tilde{x}^*_{012}$
1	[	x_{012}	x_{012}	$s_0(x_{02})$	$s_0(x_{01})$	]	$s_0(x_{012})$
2	[	$\chi(x_{12}, x_{01})$	x_{012}	$\tilde{x}_{012}$	$s_0(x_{01})$	]	Def of $\tilde{x}_{012}$
3	(	$\tilde{x}^*_{012}$	$s_0(x_{02})$	$\tilde{x}_{012}$	$s_0^2(x_0)$	)	
4	[	$s_0(x_{01})$	$s_0(x_{01})$	$s_0(x_{01})$	$s_0^2(x_0)$	]	$s_0^2(x_{01})$

Figure 50: S-Matrix $\langle 2\rangle$ in $\bigwedge^4_3$ in Tabular Form

Since it identifies the 2-simplices of $X_\bullet$ as consisting of a *boundary* ($\in K_2(X_\bullet) = \mathrm{SimKer}(X_\bullet]^1_0)$) and an *interior* (a 2-cell of $\mathbf{Bic}(X_\bullet)$) just as we have defined the 2-simplices in the nerve of a bicategory in Section 6. This, of course, is the justification for calling the 2-cell $\tilde{x}_{012}$ the *interior* of the 2-simplex x_{012} and using the notation "$\mathrm{Int}(x_{012})$" for the 2-cell $\tilde{x}_{012}$. We have stated the Theorem 7.16 here because the correspondence $x_{012} \mapsto \mathrm{Int}(x_{012}) = \tilde{x}_{012}$ plays a fundamental role in the definitions which we are about to give of the structural elements of $\mathbf{Bic}(X_\bullet)$ as a bicategory. Theorem 7.16 is immediate on the hypothesis of the invertibility of the degenerate 2-simplices, which we have already used to show that $\mathbf{Ner}(\mathbb{B}(X_\bullet)) = \mathbb{P}(X_\bullet)$ is then at least, a *category.*

7.20. The Definition of the Structural Elements of $\mathbf{Bic}(X_\bullet)$ as a Bicategory. :

- *Left Action of* 1-*Cells on* 2-*Cells*

7.21. Definition. **(Left Action)** *Let $\alpha : x_{01} \Longrightarrow y_{01}$ be a 2-cell with 0-target x_1 and $x_{12} : x_1 \longrightarrow x_2$ be a 1-cell with $d_1(x_{01}) = x_1$, so that*

$$(\chi(x_{12}, y_{01}), \chi(x_{12}, x_{01}), \_\,, \alpha) \in \bigwedge\nolimits^2_3(X_\bullet).$$

Since $\chi(x_{01}, y_{01})$ is invertible, there exists a unique 2-simplex $x_{12} \otimes \alpha$ such that

$$[\chi(x_{12}, y_{01}), \chi(x_{12}, x_{01}), x_{12} \otimes \alpha, \alpha] \in X_3,$$

and since the boundary of $x_{12} \otimes \alpha$ is necessarily

$$(d_1(\chi(x_{12}, y_{01}), d_1(\chi(x_{12}, x_{01}), d_2(\alpha)) = (x_{12} \otimes y_{01}, x_{12} \otimes x_{01}, s_0(x_0)),$$

this 2-simplex is a 2-cell $x_{12} \otimes \alpha : x_{12} \otimes x_{01} \Longrightarrow x_{12} \otimes y_{01}$. We take this as the definition of the left action of the 1-cell x_{12} on the 2-cell α. *(Figure 51)*

- *Right Action of* 1-*Cells on* 2-*Cells* :

7.22. Definition. **(Right Action)** *Let $\beta : x_{12} \Longrightarrow y_{12}$ be a 2-cell with 0-source x_1 and $x_{01} : x_0 \longrightarrow x_1$ be a 1-cell with $d_0(x_{01}) = x_1$. Now*

$$(\beta, \_\,, \chi(x_{12}, x_{01}), s_1(x_{01})) \in \bigwedge\nolimits^1_3(X_\bullet),$$

$$\begin{tikzcd}
x_0\,(0) \arrow[rr, "s_0(x_0)"] \arrow[dr, "x_{01}"] \arrow[ddr, "x_{12}\otimes x_{01}"'] & & x_0\,(1) \arrow[dl, "y_{01}"'] \arrow[ddl, "x_{12}\otimes y_{01}"] \\
& x_1\,(2) \arrow[d, "x_{12}"] & \\
& x_2\,(3) &
\end{tikzcd}$$

Figure 51: $[\chi(x_{12}, y_{01}), \chi(x_{12}, x_{01}), x_{12} \otimes \alpha, \alpha]$

and since $s_1(x_{01})$ is invertible, there exists a unique 2-simplex $\{\beta \otimes x_{01}\}$ such that

$$[\beta, \{\beta \otimes x_{01}\}, \chi(x_{12}, x_{01}), s_1(x_{01})] \in X_3,$$

but then

$$(\chi(y_{12}, x_{01}), \{\beta \otimes x_{01}\}, \_\,, s_0(x_{01})) \in \textstyle\bigwedge^2_3(X_\bullet)$$

and since $\chi(y_{12}, x_{01})$ is invertible, there exists a unique 2-simplex $\beta \otimes x_{01}$ such that

$$[\chi(y_{12}, x_{01}), \{\beta \otimes x_{01}\}, \beta \otimes x_{01}, s_0(x_{01})] \in X_3.$$

Calculating the boundary of $\beta \otimes x_{01}$ shows that it is a 2-cell of the form

$$\beta \otimes x_{01} : x_{12} \otimes x_{01} \Longrightarrow y_{12} \otimes x_{01},$$

as required. In fact

$$\beta \otimes x_{01} = \mathrm{Int}(\{\beta \otimes x_{01}\}),$$

and we take this as the definition of the right action of the 1-cell x_{01} on the 2-cell β.

We will show that these actions are functorial.

7.23. Remark. Geometrically, the defining tetrahedra appear in Figure 52 , which properly should be viewed as the decomposition of a prism $\Delta[2] \times \Delta[1]$ into three commutative tetrahedra,

$$\begin{bmatrix} [& \chi(y_{12}, x_{01}) & \{\beta \otimes x_{01}\} & \beta \otimes x_{01} & s_0(x_{01}) &] \\ [& \beta & \{\beta \otimes x_{01}\} & \chi(x_{12}, x_{01}) & s_1(x_{01}) &] \\ [& s_0(x_{12}) & \chi(x_{12}, x_{01}) & \chi(x_{12}, x_{01}) & s_1(\chi(x_{12}, x_{01})) &] \end{bmatrix}$$

but where we have suppressed drawing the most leftward one,

$$s_1(\chi(x_{12}, x_{01})) = [s_0(x_{12}), \chi(x_{12}, x_{01}), \chi(x_{12}, x_{01}), s_1(\chi(x_{12}, x_{01}))].$$

In the full prism Figure 53, β is the upper triangle in the nearest outside square face and $\beta \otimes x_{01}$ is the upper triangle on the farthermost outside square face. The intermediate $\{\beta \otimes x_{01}\}$ is then hidden in the interior of the prism.)

$$
\begin{tikzcd}
 & x_0\,(0) \arrow[dl, "x_{01}"'] \arrow[dr, "x_{01}"] \arrow[ddr] & \\
x_1\,(1) \arrow[rr, "s_0(x_1)"] \arrow[drr, "x_{12}"'] & & x_1\,(2) \arrow[d, "y_{12}"] \\
 & & x_2\,(3)
\end{tikzcd}
\qquad
\begin{tikzcd}
x_0\,(0) \arrow[rr, "s_0(x_0)"] \arrow[dr, "x_{01}"] \arrow[ddr, "x_{12}\otimes x_{01}"'] & & x_0\,(1) \arrow[dl, "x_{01}"'] \arrow[ddl, "y_{12}\otimes x_{01}"] \\
 & x_1\,(2) \arrow[d, "y_{12}"] & \\
 & x_2\,(3) &
\end{tikzcd}
$$

Figure 52: $[\beta, \{\beta \otimes x_{01}\}, \chi(x_{12}, x_{01}), s_1(x_{01})]$ and $[\chi(y_{12}, x_{01}), \{\beta \otimes x_{01}\}, \beta \otimes x_{01}, s_0(x_{01})]$

$$
\begin{tikzcd}
 & x_0\,(0) \arrow[rr, "s_0(x_0)"] \arrow[dl, "x_{01}"'] \arrow[dr, "x_{01}"] \arrow[ddl, "x_{01}"] \arrow[ddr, "x_{01}\otimes x_{12}"] & & x_0\,(1) \arrow[dl, "x_{01}"'] \arrow[ddl, "x_{01}\otimes y_{12}"] \\
x_1\,(1) \arrow[rr, "s_0(x_1)"] \arrow[d, "s_0(x_1)"] \arrow[drr, "x_{12}"] & & x_1\,(2) \arrow[d, "y_{12}"] & \\
x_1\,(2) \arrow[rr, "x_{12}"] & & x_2\,(3) &
\end{tikzcd}
$$

Figure 53: Defining Prism for $\beta \otimes x_{01}$

• *Associativity Isomorphism:* The associativity map (which we will show to be a natural isomorphism) is defined in a similar two step fashion. Let

$$x_0 \xrightarrow{x_{01}} x_1 \xrightarrow{x_{12}} x_2 \xrightarrow{x_{23}} x_3$$

be a composable triplet of 1-cells. Then

$$(\chi(x_{23}, x_{12}), \chi(x_{23}, x_{12} \otimes x_{01}), \_, \chi(x_{12}, x_{01})) \in \bigwedge\nolimits_3^2(X_\bullet).$$

7.24. Definition. **(Associativity Isomorphism)** *Since* $\chi(x_{23}, x_{12})$ *is invertible, there exists a unique* 2*-simplex* $\langle A(x_{23}, x_{12}, x_{01})\rangle$ *such that*

$$[\chi(x_{23}, x_{12}), \chi(x_{23}, x_{12} \otimes x_{01}), \langle A(x_{23}, x_{12}, x_{01})\rangle, \chi(x_{12}, x_{01})] \in X_3,$$

but then

$$(\chi(x_{23} \otimes x_{12}, x_{01}), \langle A(x_{23}, x_{12}, x_{01})\rangle, \_, s_0(x_{01})) \in \bigwedge\nolimits_3^2(X_\bullet).$$

We define the associativity isomorphism $A(x_{23}, x_{12}, x_{01})$ *as the unique* 2*-simplex which makes*

$$[\chi(x_{23} \otimes x_{12}, x_{01}), \langle A(x_{23}, x_{12}, x_{01})\rangle, A(x_{23}, x_{12}, x_{01}), s_0(x_{01})] \in X_3.$$

Again, that is

$$A(x_{23}, x_{12}, x_{01}) = \mathrm{Int}(\langle A(x_{23}, x_{12}, x_{01})\rangle).$$

A computation of the boundary shows that $A(x_{23}, x_{12}, x_{01})$ *is a* 2*-cell*

$$A(x_{23}, x_{12}, x_{01}) : x_{23} \otimes (x_{12} \otimes x_{01}) \Longrightarrow (x_{23} \otimes x_{12}) \otimes x_{01},$$

which in our formulation, shifts parentheses to the left.*(Figure 54)*

If for all composable triplets

$$\langle A(x_{23}, x_{12}, x_{01})\rangle = \chi(x_{23} \otimes x_{12}, x_{01}),$$

that is

$$[\chi(x_{23}, x_{12}), \chi(x_{23}, x_{12} \otimes x_{01}), \chi(x_{23} \otimes x_{12}, x_{01}), \chi(x_{12}, x_{01})] \in X_3,$$

then

$$x_{23} \otimes (x_{12} \otimes x_{01}) = (x_{23} \otimes x_{12}) \otimes x_{01}$$

and

$$\mathrm{Int}(\langle A\rangle) = s_0(x_{23} \otimes x_{12} \otimes x_{01}) = \mathrm{id}(x_{23} \otimes x_{12} \otimes x_{01})$$

(and conversely). In this case the tensor product defined by χ *is said to be* associative.

We will show that A is indeed a natural isomorphism in Proposition 7.34.

• *Right and Left Pseudo-identity Isomorphisms.*

$$\begin{tikzcd}
 & x_0 \arrow[rr, "s_0(x_0)"] \arrow[dl, "x_{01}"'] \arrow[dr, "x_{12}\otimes x_{01}"] \arrow[ddr, "x_{23}\otimes(x_{12}\otimes x_{01})"] & & x_0 \arrow[dl, "x_{12}\otimes x_{01}"'] \arrow[ddl, "(x_{23}\otimes x_{12})\otimes x_{01}"] \\
x_1 \arrow[rr, "x_{12}"] \arrow[drr, "x_{23}\otimes x_{12}"'] & & x_2 \arrow[d, "x_{23}"] & \\
 & & x_3 &
\end{tikzcd}$$

Figure 54: The Defining Prism for A

7.25. Definition. **(Pseudo-identity 1-cell)** *If x_0 is a 0-cell, we define its* pseudo-identity $Id(x_0)$ *as the 1-simplex $s_0(x_0)$ considered as a 1-cell,*

$$Id(x_0) = s_0(x_0) : x_0 \longrightarrow x_0.$$

Since for any 1-simplex (= 1-cell), $x_{01} : x_0 \longrightarrow x_1$, the boundary of $s_0(x_{01})$ is $(x_{01}, x_{01}, s_0(x_0))$, $s_0(x_{01}) \in \mathbb{P}(X_\bullet)_1$, and is necessarily a 2-cell of the form $s_0(x_{01}) : x_{01} \Longrightarrow x_{01}$ which we have already seen to be the identity arrow of the object x_{01} in the category $\mathbb{B}(X_\bullet)$. However, if we consider the 2-cell *interior* of $s_0(x_{01})$, $\mathrm{Int}(s_0(x_{01})) = \tilde{s}_0(x_{01})$, defined as the unique 2-simplex which makes

$$[\chi(x_{01}, s_0(x_0)), s_0(x_{01}), \tilde{s}_0(x_{01}), s_0(s_0(x_0))] \in X_3,$$

then $\tilde{s}_0(x_{01})$ is also a 2-cell and is of the form

$$\tilde{s}_0(x_{01}) : x_{01} \Longrightarrow x_{01} \otimes s_0(x_0)$$

in $\mathbb{B}(X_\bullet)$.

7.26. Definition. **(Right Pseudo-identity Isomorphism)** *We define the* right pseudo-identity isomorphism *by*

$$\rho(x_{01}) = \mathrm{Int}(s_0(x_{01})) = \tilde{s}_0(x_{01}),$$

so that

$$[\chi(x_{01}, s_0(x_0)), s_0(x_{01}), \tilde{s}_0(x_{01}), s_0(s_0(x_0))] \in X_3.$$

We will show that it is natural and is indeed an isomorphism (Hint: for the latter, look at its defining 3-simplex or use Proposition 7.12).

7.27. DEFINITION. **(Left Pseudo-Identity Isomorphism)** *Similarly, we define the* left pseudo-identity isomorphism *for a* 1*-simplex* $x_{01} : x_0 \longrightarrow x_1$ *to be the interior the degenerate* 2*-simplex* $s_1(x_{01})$*. By definition, it is the unique* 2*-simplex* $\tilde{s}_1(x_{01})$ *which makes*

$$[\chi(s_0(x_1), x_{01}), s_1(x_{01}), \tilde{s}_1(x_{01}), s_0(x_{01})] \in X_3.$$

As a 2-cell,

$$\lambda(x_{01}) = \mathrm{Int}(s_1(x_{01})) = \tilde{s}_1(x_{01}) : x_{01} \Longrightarrow s_0(x_1) \otimes x_{01}.$$

Since $s_1(x_{01})$ is invertible in $X_\bullet$, Proposition 7.18 shows that for all 1-cells x_{01} in $\mathbf{Bic}(X_\bullet)$, $\tilde{s}_1(x_{01})$ is an isomorphism in $\mathbb{B}(X_\bullet)$. We will show that it too is natural.

7.28. DEFINITION. **(Normalized Tensor Product)** *If the choice of the section* χ *can be made so that for all* 1*-simplices* $x_{01} : x_0 \longrightarrow x_1$*,*

$$\chi(x_{01}, s_0(x_0)) = s_0(x_{01}) \quad \textit{and} \quad \chi(s_0(x_1), x_{01}) = s_1(x_{01}),$$

i.e., if χ *coincides on the degeneracies, then*

$$x_{01} \otimes s_0(x_0) = x_{01} = s_0(x_1) \otimes x_{01},$$

and the defining 3*-simplices for* $\tilde{s}_0$ *and* $\tilde{s}_1$ *become*

$$[\chi(s_0(x_1), x_{01}), s_1(x_{01}), \tilde{s}_1(x_{01}), s_0(x_{01}))]$$

$$= [s_1(x_{01}), s_1(x_{01}), s_0(x_{01}), s_0(x_{01})] = s_0(s_1(x_{01})),$$

and

$$[\chi(x_{01}, s_0(x_0)), s_0(x_{01}), \tilde{s}_1(x_{01}), s_0^2(x_0)]$$
$$= [s_0(x_{01}), s_0(x_{01}), s_0(x_{01}), s_0^2(x_0)] = s_0(s_0(x_{01})),$$

so that the left and right pseudo-identity isomorphisms are the identity isomorphisms. If such a choice of χ *can be made then the resulting tensor product will be said to be a* normalized tensor product.[40]

The choice of a normalized tensor product greatly simplifies or even eliminates numerous problems of a purely technical nature that arise without it. In many applications its assumption is either a very natural one to make or, at worst, entirely anodyne. However, we will not make this assumption in this paper, but will rather content ourselves with pointing out the consequences as they arise.

This completes the definition of the structural elements of $\mathbf{Bic}(X_\bullet)$ as a bicategory. We will now show that our version of the Bénabou Axioms (Section 5.1) are satisfied.

[40]This clearly the case for $\Pi_2(X)$ of a topological space.

G-M 1	S	0	1	2	3	S	$\langle Reason \rangle$
0	[	$\chi(x_{12}, z_{01})$	$\chi(x_{12}, y_{01})$	$x_{01} \otimes \alpha$	α	]	Def: $x_{01} \otimes \alpha$
1	[	$\chi(x_{12}, z_{01})$	$\chi(x_{12}, x_{01})$	$x_{12} \otimes (\alpha \circ \beta)$	$\alpha \circ \beta$	]	Def: $x_{12} \otimes (\alpha \circ \beta)$
2	[	$\chi(x_{12}, y_{01})$	$\chi(x_{12}, x_{01})$	$x_{01} \otimes \beta$	β	]	Def: $x_{01} \otimes \beta$
(3)	[	$x_{01} \otimes \alpha$	$x_{12} \otimes (\alpha \circ \beta)$	$x_{01} \otimes \beta$	$s_0^2(x_0)$	]	$\chi(x_{12}, z_{01}) \in I_2$
4	[	α	$\alpha \circ \beta$	β	$s_0^2(x_0)$	]	Def: $\alpha \circ \beta$

Figure 55: Glenn Matrix 1 for the Left Action

7.29. Verification of the Bénabou Axioms for $\mathbf{Bic}(X_\bullet)$. ***Throughout the following set of propositions, we will assume that for all*** $(x_{12}, \_, x_{01}) \in \bigwedge_2^1(X_\bullet)$ ***and all*** $x_{01} \in X_2$***, the*** 2***-simplices*** $\chi(x_{12}, x_{01})$***,*** $s_0(x_{01})$ ***and*** $s_1(x_{01})$ ***which define the tensor product and the degeneracies are all invertible (***$\in I_2(X_\bullet) \subseteq X_2$***)*** and will take advantage in the proofs of the compact tabular Glenn matrix notation (Figure 41) as given in the Definition 7.7 of the Invertible 2-simplices of $X_\bullet$.

7.30. Proposition. **(Functoriality of the Left Action)**

Let $x_{01} \overset{\beta}{\Longrightarrow} y_{01} \overset{\alpha}{\Longrightarrow} z_{01}$ *be a composable pair of* 2*-cells and* $x_{12} : x_1 \longrightarrow x_2$ *a* 1*-cell in* $\mathbf{Bic}(X_\bullet)$. *Then*

$$x_{12} \otimes (\alpha \circ \beta) = (x_{12} \otimes \alpha) \circ (x_{12} \otimes \beta) \text{ and}$$

$$x_{12} \otimes s_0(x_{01}) = s_0(x_{12} \otimes x_{01}).$$

Preservation of identities is immediate: If $\alpha = s_0(x_{01}) : x_{01} \Longrightarrow x_{01}$ is the identity 2-cell for x_{01}, then the defining commutative 3-simplex for $x_{12} \otimes \mathrm{id}(x_{01}) = x_{12} \otimes s_0(x_{01})$ is

$$[\chi(x_{12}, x_{01}), \chi(x_{12}, x_{01}), x_{12} \otimes s_0(x_{01}), s_0(x_{01})],$$

but

$$s_0(\chi(x_{12}, x_{01})) = [\chi(x_{12}, x_{01}), \chi(x_{12}, x_{01}), s_0(x_{12} \otimes x_{01}), s_0(x_{01})]$$

is commutative, and hence

$$x_{12} \otimes \mathrm{id}(x_{01}) = x_{12} \otimes s_0(x_{01}) = s_0(x_{12} \otimes x_{01}) = \mathrm{id}(x_{12} \otimes x_{01}).$$

For the functoriality of the 2-cell composition, consider the Glenn Matrix in Figure 55

Using the definition of composition of 2-cells in $\mathbb{B}(X_\bullet)$, the commutativity of Row 3 is equivalent to

$$x_{12} \otimes (\alpha \circ \beta) = (x_{12} \otimes \alpha) \circ (x_{12} \otimes \beta),$$

and we have the functoriality of the left action.

G-M 1	S	0	1	2	3	S	⟨*Reason*⟩
0	[	α	$\alpha\circ\beta$	β	$s_0^2(x_1)$	]	Def: $\alpha\circ\beta$
(1)	[	α	$\{(\alpha\circ\beta)\otimes x_{01}\}$	$\{\beta\otimes x_{01}\}$	$s_1(x_{01}$	]	$s_1(x_{01})\in I_2$
2	[	$\alpha\circ\beta$	$\{(\alpha\circ\beta)\otimes x_{01}\}$	$\chi(x_{12},x_{01})$	$s_1(x_{01})$	]	Def: $\{(\alpha\circ\beta)\otimes x_{01}\}$
3	[	β	$\{\beta\otimes x_{01}\}$	$\chi(x_{12},x_{01})$	$s_1(x_{01})$	]	Def: $\{\beta\otimes x_{01}\}$
4	[	$s_0^2(x_1)$	$s_1(x_{01})$	$s_1(x_{01})$	$s_1(x_{01})$	]	$s_1^2(x_{01})$

Figure 56: Glenn Matrix 1 for the Right Action

7.31. Proposition. **(Functoriality of the Right Action)** *Let* $x_{12}\overset{\beta}{\Longrightarrow} y_{12}\overset{\alpha}{\Longrightarrow} z_{12}$ *be a composable pair of* 2*-cells and*
$x_{01}: x_0 \longrightarrow x_1$ *a* 1*-cell in* $\mathbb{B}(X_\bullet)$. *Then*

$$(\alpha\circ\beta)\otimes x_{01} = (\alpha\otimes x_{01})\circ(\beta\otimes x_{01}) \text{ and}$$

$$x_{01}\otimes s_0(x_{12}) = s_0(x_{01}\otimes x_{01}).$$

In effect $\{s_0(x_{12})\otimes x_{01}\}$ is defined by the commutative 3-simplex

$$[s_0(x_{12}), \{s_0(x_{12})\otimes x_{01}\}, \chi(x_{12},x_{01}), s_1(x_{01})],$$

but

$$s_1(\chi(x_{12},x_{01})) = [s_0(x_{12}), \chi(x_{12},x_{01}), \chi(x_{12},x_{01}), s_1(x_{01})],$$

so that

$$\{s_0(x_{12})\otimes x_{01}\} = \chi(x_{12},x_{01}).$$

Consequently,

$$s_0(x_{12})\otimes x_{01} = \mathrm{Int}(\{s_0(x_{12})\otimes x_{01}\} = \mathrm{Int}(\chi(x_{12},x_{01})) = s_0(x_{12}\otimes x_{01}),$$

where the last equality is just Proposition 7.17.

For the functoriality of composition, we consider a sequence of three Glenn matrices, successively extracting a commutative row from one and inserting it into a later one, until we finally arrive at the desired conclusion.

We first form the Glen Matrix 1 of Figure 56. We extract the commutative Row 1 derived from Matrix 1 and insert it as Row 1 of the Glenn Matrix 2 of Figure 57. Extracting the commutative Row 2 derived from Matrix 2, we insert it as Row 2 in the Glenn Matrix 3 of Figure 58. As before, using the definition of the composition in $\mathbb{B}(X_\bullet)$, the derived commutativity of Row 3 of Matrix 3 in Figure 58 is equivalent to

$$(\alpha\circ\beta)\otimes x_{01} = (\alpha\otimes x_{01})\circ(\beta\otimes x_{01}),$$

and we have the functoriality of the right action.

G-M 2	S	0	1	2	3	S	⟨*Reason*⟩
0	[	α	$\{\alpha \otimes x_{01}\}$	$\chi(y_{12}, x_{01})$	$s_1(x_{01})$	]	Def: $\{\alpha \otimes x_{01}\}$
1	[	α	$\{(\alpha \circ \beta) \otimes x_{01}\}$	$\{\beta \otimes x_{01}\}$	$s_1(x_{01}$	]	Row 1 of Matrix 1
(2)	[	$\{\alpha \otimes x_{01}\}$	$\{(\alpha \circ \beta) \otimes x_{01}\}$	$\beta \otimes x_{01}$	$s_0(x_{01})$	]	$s_1(x_{01}) \in I_2$
3	[	$\chi(y_{12}, x_{01})$	$\{\beta \otimes x_{01}\}$	$\beta \otimes x_{01}$	$s_0(x_{01})$	]	Def: $\beta \otimes x_{01}$
4	[	$s_1(x_{01})$	$s_1(x_{01})$	$s_0(x_{01})$	$s_0(x_{01})$	]	$s_0(s_1(x_{01}))$

Figure 57: Glenn Matrix 2 for the Right Action

G-M 3	S	0	1	2	3	S	⟨*Reason*⟩
0	[	$\chi(z_{12}, x_{01})$	$\{\alpha \otimes x_{01}\}$	$\alpha \otimes x_{01}$	$s_0(x_{01})$	]	Def: $\alpha \otimes x_{01}$
1	[	$\chi(z_{12}, x_{01})$	$\{(\alpha \circ \beta) \otimes x_{01}\}$	$(\alpha \circ \beta) \otimes x_{01}$	$s_0(x_{01})$	]	Def: $(\alpha \circ \beta) \otimes x_{01}$
2	[	$\{\alpha \otimes x_{01}\}$	$\{(\alpha \circ \beta) \otimes x_{01}\}$	$\beta \otimes x_{01}$	$s_0(x_{01})$	]	Row 2 of Matrix 2
(3)	[	$\alpha \otimes x_{01}$	$(\alpha \circ \beta) \otimes x_{01}$	$\beta \otimes x_{01}$	$s_0^2(x_0)$	]	$\chi(z_{12}, x_{01}) \in I_2$
4	[	$s_0(x_{01})$	$s_0(x_{01})$	$s_0(x_{01})$	$s_0^2(x_0)$	]	$s_0^2(x_{01})$

Figure 58: Glenn Matrix 3 for the Right Action

7.32. PROPOSITION. **(Godement Interchange Law)** *Let* $\alpha : x_{01} \Longrightarrow y_{01}$ *and* $\beta : x_{12} \Longrightarrow y_{12}$ *be* 2*-cells in* $\mathbf{Bic}(X_\bullet)$ *with the* 0*-target of* α *equal to the* 0*-source of* β*, i.e.,* $d_0^2(\alpha) = x_1 = d_1^2(\beta)$*. Then with the left and right actions of* 1*-cells on* 2*-cells defined as above,*

$$(y_{12} \otimes \alpha) \circ (\beta \otimes x_{01}) = (\beta \otimes y_{01}) \circ (x_{12} \otimes \alpha).$$

We again define a sequence of Glenn matrices, each one allowing the extraction of a commutative row which may be fed into subsequent ones until the conclusion is reached. We begin with the Glen matrix in Figure 59. We extract the commutative Row 2 derived from Matrix 1 and insert it as Row 2 of Matrix 2 of Figure 60. Extracting the commutative Row 1 derived from Matrix 2, we insert it as Row 1 of Matrix 3 of Figure 61. The derived

G-M 1	S	0	1	2	3	S	⟨*Reason*⟩
0	[	β	$\{\beta \otimes y_{01}\}$	$\chi(x_{12}, y_{01})$	$s_1(y_{01})$	]	Def: $\{\beta \otimes y_{01}\}$
1	[	β	$\{\beta \otimes x_{01}\}$	$\chi(x_{12}, x_{01})$	$s_1(x_{01})$	]	Def: $\{\beta \otimes x_{01}\}$
(2)	[	$\{\beta \otimes y_{01}\}$	$\{\beta \otimes x_{01}\}$	$x_{12} \otimes \alpha$	α	]	$s_1(y_{01}) \in I_2$
3	[	$\chi(x_{12}, y_{01})$	$\chi(y_{12}, x_{01})$	$x_{12} \otimes \alpha$	α	]	Def: $x_{12} \otimes \alpha$
4	[	$s_1(y_{01})$	$s_1(x_{01})$	α	α	]	$s_2(\alpha)$

Figure 59: Glenn Matrix 1 for Godement Interchange

2	S	0	1	2	3	S	$\langle Reason\rangle$
0	[	$\chi(y_{12}, y_{01})$	$\{\beta \otimes y_{01}\}$	$\beta \otimes y_{01}$	$s_0(y_{01})$	]	Def: $\beta \otimes y_{01}$
(1)	[	$\chi(y_{12}, y_{01})$	$\{\beta \otimes x_{01}\}$	$(\beta \otimes y_{01}) \circ (x_{12} \otimes \alpha)$	α	]	$s_0^2(x_0) \in I_2$
2	[	$\{\beta \otimes y_{01}\}$	$\{\beta \otimes x_{01}\}$	$x_{12} \otimes \alpha$	α	]	Row 2:G-M 1
3	[	$\beta \otimes y_{01}$	$(\beta \otimes y_{01}) \circ (x_{12} \otimes \alpha)$	$x_{12} \otimes \alpha$	$s_0^2(x_0)$	]	Def: comp.
4	[	$s_0(y_{01})$	α	α	$s_0^2(x_0)$	]	$s_1(\alpha)$

Figure 60: Glenn Matrix 2 for Godement Interchange

3	S	0	1	2	3	S	$\langle Reason\rangle$
0	[	$\chi(y_{12}, y_{01})$	$\chi(y_{12}, x_{01})$	$y_{12} \otimes \alpha$	α	]	Def: $y_{12} \otimes \alpha$
1	[	$\chi(y_{12}, y_{01})$	$\{\beta \otimes x_{01}\}$	$(\beta \otimes y_{01}) \circ (x_{12} \otimes \alpha)$	α	]	Row 1: G-M 2
2	[	$\chi(y_{12}, x_{01})$	$\{\beta \otimes x_{01}\}$	$\beta \otimes x_{01}$	$s_0(x_{01})$	]	Def: $\beta \otimes x_{01}$
(3)	[	$y_{12} \otimes \alpha$	$(\beta \otimes y_{01}) \circ (x_{12} \otimes \alpha)$	$\beta \otimes x_{01}$	$s_0^2(x_0)$	]	$\chi(y_{12}, y_{01}) \in I_2$
4	[	α	α	$s_0(x_{01})$	$s_0^2(x_0)$	]	$s_0(\alpha)$

Figure 61: Glenn Matrix 3 for Godement Interchange

commutativity of Row 3 of Matrix 3 is the Interchange Law:

$$(y_{12} \otimes \alpha) \circ (\beta \otimes x_{01}) = (\beta \otimes y_{01}) \circ (x_{12} \otimes \alpha).$$

7.33. Remark. (Toward Dimension 3) An alternate method (unnecessarily cumbersome in this 2-dimensional case) for this same result would be to take, instead of Matrix 3 (Figure 61), the matrix of Figure 62, and extract Row 1 from it. Then Row 1 of Matrix 3^* may be placed along with Row 2 extracted from Matrix 2 into the Matrix 4 of Figure 63.

The commutativity of Row 3 of Matrix 4 now also gives the Interchange Law:

$$(y_{12} \otimes \alpha) \circ (\beta \otimes x_{01}) = ((y_{12} \otimes \alpha) \circ (\beta \otimes x_{01})) \circ s_0(x_{12} \otimes x_{01}) = (\beta \otimes y_{01}) \circ (x_{12} \otimes \alpha).$$

This is, of course, a "long-winded" way to obtain this result in this Case $n = 2$. However, looking further along, toward $\Pi_3(X)$ for instance, in contrast to the case here (which is homotopicly minimal— α, β 2-cells, $\partial(\alpha) = \partial(\beta)$ and α homotopic to β implies $\alpha = \beta$ — in dimension 3 the basic simplicial setting (Case $n = 3$) has $X_4 \subseteq \mathrm{SimKer}(X_\bullet]_0^3)$ as its set of commutative 4-simplices and the Glenn matrices that we have been looking

3^*	S	0	1	2	3	S	$\langle Reason\rangle$
0	[	$\chi(y_{12}, y_{01})$	$\chi(y_{12}, x_{01})$	$y_{12} \otimes \alpha$	α	]	Def: $y_{12} \otimes \alpha$
(1)	[	$\chi(y_{12}, y_{01})$	$\{\beta \otimes x_{01}\}$	$(y_{12} \otimes \alpha) \circ (\beta \otimes x_{01})$	α	]	$s_0^2(x_0) \in I_2$
2	[	$\chi(y_{12}, x_{01})$	$\{\beta \otimes x_{01}\}$	$\beta \otimes x_{01}$	$s_0(x_{01})$	]	Def: $\beta \otimes x_{01}$
3	[	$y_{12} \otimes \alpha$	$(y_{12} \otimes \alpha) \circ (\beta \otimes x_{01})$	$\beta \otimes x_{01}$	$s_0^2(x_0)$	]	Def: comp.
4	[	α	α	$s_0(x_{01})$	$s_0^2(x_0)$	]	$s_0(\alpha)$

Figure 62: Alternate Glenn Matrix 3^* for Godement Interchange

GM4	S	0	1	2	3	S	$\langle Reason\rangle$
0	[	$\chi(y_{12}, y_{01})$	$\{\beta \otimes x_{01}\}$	$(y_{12}{\otimes}\alpha){\circledcirc}(\beta{\otimes}x_{01})$	α	]	Row 1:GM3*
1	[	$\chi(y_{12}, y_{01})$	$\{\beta \otimes x_{01}\}$	$(\beta{\otimes}y_{01}){\circledcirc}(x_{12}{\otimes}\alpha)$	α	]	Row 1:GM2
2	[	$\{\beta \otimes x_{01}\}$	$\{\beta \otimes x_{01}\}$	$s_0(x_{12} \otimes x_{01})$	$s_0(x_{01})$	]	$s_0(\{\beta \otimes x_{01}\}$
(3)	[	$(y_{12}{\otimes}\alpha){\circledcirc}(\beta{\otimes}x_{01})$	$(\beta \otimes y_{01}){\circledcirc}(x_{12} \otimes \alpha)$	$s_0(x_{12} \otimes x_{01})$	$s_0^2 x_0)$	]	$\chi(y_{12}, y_{01}){\in}I_2$
4	[	α	α	$s_0(x_{01})$	$s_0^2(x_0)$	]	$s_0(\alpha)$

Figure 63: Glenn Matrix 4 for Godement Interchange

at here are the *2-boundary* matrices of such commutative 4-simplices. Each row is the boundary of "solid" 3-simplex face of the commutative 4-simplex. The path complex $\mathbb{P}(X_\bullet)$ here is the nerve of a bicategory and $\mathbb{P}(\mathbb{P}(X_\bullet)) = \mathbb{P}^2(X_\bullet)$ is the nerve of a category, whose arrows are $\mathbb{P}^2(X_\bullet)_1 \subseteq X_3$ and whose objects are $\mathbb{P}^2(X_\bullet)_0 = \mathbb{P}(X_\bullet)_1$. They are the 3-cells and 2-cells, respectively, of a "tricategory" $\mathbb{T}(X_\bullet)$ whose nerve is $X_\bullet$. The 2-cells are precisely those under study here, the set of $\alpha \in X_2$ such that $d_2(\alpha)$ is degenerate, *i.e.,* homotopies of 1-simplex "paths". If $\beth$ is a 3-cell with 2-source α and 2-target β, then α and β must have the same 1-source and the same 1-target, say $\alpha : x_{01} \Longrightarrow y_{01}$ and $\beta : x_{01} \Longrightarrow y_{01}$. Then $\beth : \alpha \Longrightarrow \beta$ is a 3-simplex whose boundary is given by

$$\partial(\beth) = (\beta, \alpha, s_0(x_{01}), s_0^2(x_0)).$$

Row 3 of Matrix 4 of Figure 63, is an example of such a 3-cell, *precisely a homotopy of its* 0*-face with its* 1*-face*, here mediating the non-equality of two possible ways to define the $\otimes$-compositions of the 2-cell sides of the Godement Interchange Square. This "Chronic Defect of the Interchange Law" becomes the natural 3-cell isomorphism,

$$\beta \otimes \alpha : (y_{12} \otimes \alpha) \circledcirc (\beta \otimes x_{01}) \Rrightarrow (\beta \otimes y_{01}) \circledcirc (x_{12} \otimes \alpha),$$

that is always present in a general tricategory and in the topological case leads to the definition of the non-trivial Whitehead product

$$[-,-] : \pi_2(X_\bullet, x_0) \times \pi_2(X_\bullet, x_0) \longrightarrow \pi_3(X_\bullet, x_0).$$

This "defect" prevents $X_\bullet$ from defining at any $x_0 \in X_0$ an H-space, *i.e.,* here the would-be nerve of a 3-category with a single object.

7.34. Proposition. *The associativity* 2*-cell (Definition 7.24) is an isomorphism in* $\mathbb{B}(X_\bullet)$ *and is natural in each of its three variables.*

Proposition 7.34 will be consequence of the following four lemmas which will prove that it is an isomorphism and then, separately, natural in each of its three variables.

7.35. Lemma. *For all composable triplets* (x_{23}, x_{12}, x_{01}) *of* 1*-cells,*

$$A(x_{23}, x_{12}, x_{01}) : (x_{23} \otimes (x_{12} \otimes x_{01}) \Longrightarrow (x_{23} \otimes x_{12}) \otimes x_{01}),$$

is an isomorphism in $\mathbb{B}(X_\bullet)$*.*

1	S	0	1	2	3	S	$\langle Reason \rangle$
0	[	$\chi(x_{23}, x_{12})$	$\chi(x_{23}, x_{12} \otimes x_{01})$	$\langle A \rangle$	$\chi(x_{12}, x_{01})$	]	Def: $\langle A \rangle$
1	[	$\chi(x_{23}, x_{12})$	$\langle A^* \rangle$	$\chi(x_{23} \otimes x_{12}, x_{01})$	$\chi(x_{12}, x_{01})$	]	Def: $\langle A^* \rangle$
2	[	$\chi(x_{23}, x_{12} \otimes x_{01})$	$\langle A^* \rangle$	A^*	$s_0(x_{12} \otimes x_{01})$	]	Def: A^*
(3)	[	$\langle A \rangle$	$\chi(x_{23} \otimes x_{12}, x_{01})$	A^*	$s_0(x_{01})$	]	$\chi(x_{23}, x_{12}) \in I_2$
4	[	$\chi(x_{12}, x_{01})$	$\chi(x_{12}, x_{01})$	$s_0(x_{12} \otimes x_{01})$	$s_0(x_{01})$	]	$s_0(\chi(x_{12}, x_{01}))$

Figure 64: Glenn Matrix 1 for $A \circ A^* = \mathrm{id}$

2	S	0	1	2	3	S	$\langle Reason \rangle$
0	[	$\chi(x_{23} \otimes x_{12}, x_{01})$	$\langle A \rangle$	A	$s_0(x_{01})$	]	Def: A
1	[	$\chi(x_{23} \otimes x_{12}, x_{01})$	$\chi(x_{23} \otimes x_{12}, x_{01})$	$s_0((x_{23} \otimes x_{12}) \otimes x_{01})$	$s_0(x_{01})$	]	$s_0(\chi)$
2	[	$\langle A \rangle$	$\chi(x_{23} \otimes x_{12}, x_{01})$	A^*	$s_0(x_{01})$	]	Row 3: GM-1
(3)	[	A	$s_0((x_{23} \otimes x_{12}) \otimes x_{01})$	A^*	$s_0^2(x_0)$	]	$d_0(\text{Row } 1) \in I_2$
4	[	$s_0(x_{01})$	$s_0(x_{01})$	$s_0(x_{01})$	$s_0^2(x_0)$	]	$s_0^2(x_{01})$

Figure 65: Glenn Matrix 2 for $A \circ A^* = \mathrm{id}$

For the inverse to A, first define $\langle A^* \rangle$ as the unique 2-simplex which makes

$$[\chi(x_{23}, x_{12}), \langle A^* \rangle, \chi(x_{23} \otimes x_{12}, x_{01}), \chi(x_{12}, x_{01})] \in X_3.$$

Then define A^* as $\mathrm{Int}(\langle A^* \rangle)$, so that

$$[\chi(x_{23}, x_{12} \otimes x_{01}), \langle A^* \rangle), A^*, s_0(x_{12} \otimes x_{01})] \in X_3.$$

Now form the Glenn Matrix 1 of Figure 64. We then extract the commutative Row 3 derived from GM-1 and insert it as Row 2 of the Matrix GM-2 in Figure 65. The derived commutativity of Row 3 of GM-2 is equivalent to

$$A \circ A^* = s_0((x_{23} \otimes x_{12}) \otimes x_{01}).$$

For $A^* \circ A = \mathrm{id}$, we form the Glenn Matrix in Figure 66. Row 2 is now inserted as Row 2 in GM-2 of Figure 67. The derived commutativity of Row 3 of GM-2 is equivalent to

$$A^* A = s_0(x_{12} \otimes x_{01}).$$

Thus A is an isomorphism, $A^{-1} = A^*$.

1	S	0	1	2	3	S	$\langle Reason \rangle$
0	[	$\chi(x_{23}, x_{12})$	$\langle A^* \rangle$	$\chi(x_{23} \otimes x_{12}, x_{01})$	$\chi(x_{12}, x_{01})$	]	Def: $\langle A^* \rangle$
1	[	$\chi(x_{23}, x_{12})$	$\chi(x_{23}, x_{12} \otimes x_{01})$	$\langle A \rangle$	$\chi(x_{12}, x_{01})$	]	Def: $\langle A \rangle$
(2)	[	$\chi(x_{23} \otimes x_{12}, x_{01})$	$\chi(x_{23}, x_{12} \otimes x_{01})$	A	$s_0(x_{01})$	]	$d_0(\text{Row } 4) \in I_2$
3	[	$\chi(x_{23} \otimes x_{12}, x_{01})$	$\langle A \rangle$	A	$s_0(x_{01})$	]	Def: A
4	[	$\chi(x_{12}, x_{01})$	$\chi(x_{12}, x_{01})$	$s_0(x_{12} \otimes x_{01})$	$s_0(x_{01})$	]	$s_0(\chi(x_{12}, x_{01}))$

Figure 66: Glenn Matrix 1 for $A^* \circ A = \mathrm{id}$

2	S	0	1	2	3	S	$\langle Reason \rangle$
0	[	$\chi(x_{23}, x_{12} \otimes x_{01})$	$\langle A^* \rangle$	A^*	$s_0(x_{12} \otimes x_{01})$	]	Def: A^*
1	[	$\chi(x_{23}, x_{12} \otimes x_{01})$	$\chi(x_{23}, x_{12} \otimes x_{01})$	$s_0(x_{23} \otimes (x_{12} \otimes x_{01}))$	$s_0(x_{12} \otimes x_{01})$	]	$s_0(\chi)$
2	[	$\langle A \rangle$	$\chi(x_{23}, x_{12} \otimes x_{01})$	A	$s_0(x_{01})$	]	Row 2 : $GM-2$
(3)	[	A^*	$s_0(x_{12} \otimes x_{01})$	A	$s_0^2(x_0)$	]	d_0(Row 1) $\in I_2$
4	[	$s_0(x_{12} \otimes x_{01})$	$s_0(x_{12} \otimes x_{01})$	$s_0(x_{12} \otimes x_{01})$	$s_0^2(x_0)$	]	$s_0^2(x_{12} \otimes x_{01})$

Figure 67: Glenn Matrix 2 for $A^* \circ A = \mathrm{id}$

$$\begin{tikzcd}
x_{23} \otimes (x_{12} \otimes x_{01}) \arrow[r, Rightarrow, "A=A(x_{23},x_{12},x_{01})"] \arrow[d, Rightarrow, "\theta_1 \otimes (x_{12} \otimes x_{01})"'] & (x_{23} \otimes x_{12}) \otimes x_{01} \arrow[d, Rightarrow, "(\theta_1 \otimes x_{12}) \otimes x_{01}"] \\
y_{23} \otimes (x_{12} \otimes x_{01}) \arrow[r, Rightarrow, "A'=A(y_{23},x_{12},x_{01})"'] & (y_{23} \otimes x_{12}) \otimes x_{01}
\end{tikzcd}$$

Figure 68: First Variable: Naturality Square for A

7.36. Lemma. *A is natural in its first variable: For all 2-cells $\theta_1 : x_{23} \Longrightarrow y_{23}$, the diagram of Figure 68 is commutative in $\mathbb{B}(X_\bullet)$.*

We form a sequence of Glenn Matrices, the first in Figure 69.

We place the commutative Row 2 of GM-1 as Row 2 in GM-2 of Figure 70. Now

$$(\theta_1 \otimes x_{12}, \text{---}, \langle A \rangle, s_1(x_{01})) \in \bigwedge\nolimits_3^2(X_\bullet),$$

and $s_1(x_{01})$ is invertible, so that we may take X as the unique 2-simplex which makes

$$[\theta_1 \otimes x_{12}, X, \langle A \rangle, s_1(x_{01})] \in X_3.$$

We place the resulting commutative 3-simplex as Row 3 in GM-2. The commutative Row 1 of GM-2 will become Row 1 of GM-3 in Figure 71. We now extract the derived Row 3 of GM-3 and insert it as Row 2 in GM-4 of Figure 72. Row 1 of GM-4 will now become Row 1 of GM-6 in Figure 74. But first we form GM-5 of Figure 73.

We now combine Row 1 of GM-4 and Row 2 of GM-5 in GM-6 of Figure 74.

The derived commutativity of Row 3 of GM-6 is equivalent to

$$((\theta_1 \otimes x_{12}) \otimes x_{01}) \circ A = A' \circ (\theta_1 \otimes (x_{12} \otimes x_{01})),$$

which is naturality in the first variable for A.

1	S	0	1	2	3	S	$\langle Reason \rangle$
0	[	θ_1	$\{\theta_1 \otimes x_{12}\}$	$\chi(x_{23}, x_{12})$	$s_0(x_{12})$	]	Def:$\{\theta_1 \otimes x_{12}\}$
1	[	θ_1	$\{\theta_1 \otimes (x_{12} \otimes x_{01})\}$	$\chi(x_{23}, x_{12} \otimes x_{01})$	$s_1(x_{12} \otimes x_{01})$	]	Def: $\{\theta_1 \otimes (x_{12} \otimes x_{01})\}$
(2)	[	$\{\theta_1 \otimes x_{12}\}$	$\{\theta_1 \otimes (x_{12} \otimes x_{01})\}$	$\langle A \rangle$	$\chi(x_{12}, x_{01})$	]	$s_1(x_{12}) \in I_2$
3	[	$\chi(x_{23}, x_{12})$	$\chi(x_{23}, x_{12} \otimes x_{01})$	$\langle A \rangle$	$\chi(x_{12}, x_{01})$	]	Def: $\langle A \rangle$
4	[	$s_0(x_{12})$	$s_1(x_{12} \otimes x_{01})$	$\chi(x_{12}, x_{01})$	$\chi(x_{12}, x_{01})$	]	$s_2(\chi(x_{12}, x_{01}))$

Figure 69: Glenn Matrix 1 for First Variable

2	S	0	1	2	3	S	$\langle Reason \rangle$
0	[	$\chi(y_{23}, x_{12})$	$\{\theta_1 \otimes x_{12}\}$	$\theta_1 \otimes x_{12}$	$s_0(x_{12})$	]	Def: $\theta_1 \otimes x_{12}$
(1)	[	$\chi(y_{23}, x_{12})$	$\{\theta_1 \otimes (x_{12} \otimes x_{01})\}$	X	$\chi(x_{12}, x_{01})$	]	$s_1(x_{01}) \in I_2$
2	[	$\{\theta_1 \otimes x_{12}\}$	$\{\theta_1 \otimes (x_{12} \otimes x_{01})\}$	$\langle A \rangle$	$\chi(x_{12}, x_{01})$	]	Row 2: GM-1
3	[	$\theta_1 \otimes x_{12}$	X	$\langle A \rangle$	$s_0(x_{01})$	]	Def: X
4	[	$s_0(x_{12})$	$\chi(x_{12}, x_{01})$	$\chi(x_{12}, x_{01})$	$s_1(x_{01})$	]	$s_1(\chi(x_{12}, x_{01}))$

Figure 70: Glenn Matrix 2 for First Variable

3	S	0	1	2	3	S	$\langle Reason \rangle$
0	[	$\chi(y_{23} \otimes x_{12}, x_{01})$	$\langle A' \rangle$	A'	$\chi(x_{12}, x_{01})$	]	Def: A'
1	[	$\chi(y_{23}, x_{12})$	$\{\theta_1 \otimes (x_{12} \otimes x_{01})\}$	X	$\chi(x_{12}, x_{01})$	]	Row 2: GM-2
2	[	$\chi(y_{23}, x_{12} \otimes x_{01})$	$\{\theta_1 \otimes (x_{12} \otimes x_{01})\}$	$\theta_1 \otimes (x_{12} \otimes x_{01})$	$s_0(x_{12} \otimes x_{01})$	]	Def:$\theta_1 \otimes (x_{12} \otimes x_{01})$
(3)	[	$\langle A' \rangle$	X	$\theta_1 \otimes (x_{12} \otimes x_{01})$	$s_0(x_{01})$	]	$\chi(y_{23}, x_{12}) \in I_2$
4	[	$\chi(x_{12}, x_{01})$	$\chi(x_{12}, x_{01})$	$s_0(x_{12} \otimes x_{01})$	$s_0(x_{01})$	]	$s_0(\chi(x_{12}, x_{01}))$

Figure 71: Glenn Matrix 3 for First Variable

4	S	0	1	2	3	S	$\langle Reason \rangle$
0	[	$\chi(y_{23}, \otimes x_{12}, x_{01})$	$\langle A' \rangle$	A'	$s_0(x_{01})$	]	Def: A'
(1)	[	$\chi(y_{23}, \otimes x_{12}, x_{01})$	X	$A' \circ (\theta_1 \otimes (x_{12} \otimes x_{01}))$	$s_0(x_{01})$	]	$s_0^2(x_0) \in I_2$
2	[	$\langle A' \rangle$	X	$\theta_1 \otimes (x_{12} \otimes x_{01})$	$s_0(x_{01})$	]	Row 3: GM-3
3	[	A'	$A' \circ (\theta_1 \otimes (x_{12} \otimes x_{01}))$	$\theta_1 \otimes (x_{12} \otimes x_{01})$	$s_0^2(x_0)$	]	Def: comp.
4	[	$s_0(x_{01})$	$s_0(x_{01})$	$s_0(x_{01})$	$s_0^2(x_0)$	]	$s_0^2(x_{01})$

Figure 72: Glenn Matrix 4 for First Variable

5	S	0	1	2	3	S	$\langle Reason \rangle$
0	[	$\theta_1 \otimes x_{12}$	$\{(\theta_1 \otimes x_{12}) \otimes x_{01}\}$	$\chi(x_{23} \otimes x_{12}, x_{01})$	$s_1(x_{01})$	]	Def:$\{(\theta_1 \otimes x_{12}) \otimes x_{01}\}$
1	[	$\theta_1 \otimes x_{12}$	X	$\langle A \rangle$	$s_1(x_{01})$	]	Def: X
(2)	[	$\{(\theta_1 \otimes x_{12}) \otimes x_{01}\}$	X	A	$s_0(x_{01}$	]	$s_1(x_{01}) \in I_2$
3	[	$\chi(x_{23} \otimes x_{12}, x_{01})$	$\langle A \rangle$	A	$s_0(x_{01})$	]	Def: A
4	[	$s_1(x_{01})$	$s_1(x_{01})$	$s_0(x_{01})$	$s_0(x_{01})$	]	$s_0(s_1(x_{01}))$

Figure 73: Glenn Matrix 5 for First Variable

6	S	0	1	2	3	S	$\langle Reason \rangle$
0	[	$\chi(y_{23}, \otimes x_{12}, x_{01})$	$\{(\theta_1 \otimes x_{12}) \otimes x_{01}\}$	$(\theta_1 \otimes x_{12}) \otimes x_{01}$	$s_0(x_{01})$	]	Def: d_2(Row 0)
1	[	$\chi(y_{23}, \otimes x_{12}, x_{01})$	X	$A' \circ (\theta_1 \otimes (x_{12} \otimes x_{01}))$	$s_0(x_{01})$	]	Row 1: GM-4
2	[	$\{(\theta_1 \otimes x_{12}) \otimes x_{01}\}$	X	A	$s_0(x_{01}$	]	Row 2: GM-5
(3)	[	$(\theta_1 \otimes x_{12}) \otimes x_{01}$	$A' \circ (\theta_1 \otimes (x_{12} \otimes x_{01}))$	A	$s_0^2(x_0)$	]	d_0(Row 1) $\in I_2$
4	[	$s_0(x_{01})$	$s_0(x_{01})$	$s_0(x_{01})$	$s_0^2(x_0)$	]	$s_0^2(x_{01})$

Figure 74: Glenn Matrix 6 for First Variable

$$\begin{tikzcd}
x_{23} \otimes (x_{12} \otimes x_{01}) \arrow[r, Rightarrow, "A=A(x_{23},x_{12},x_{01})"] \arrow[d, Rightarrow, "x_{23}\otimes(\theta_2\otimes x_{01})"'] & (x_{23} \otimes x_{12}) \otimes x_{01} \arrow[d, Rightarrow, "(x_{23}\otimes\theta_2)\otimes x_{01}"] \\
x_{23} \otimes (y_{12} \otimes x_{01}) \arrow[r, Rightarrow, "A'=A(x_{23},y_{12},x_{01})"'] & (x_{23} \otimes y_{12}) \otimes x_{01}
\end{tikzcd}$$

Figure 75: Second Variable: Naturality Square for A

GM-1	S	0	1	2	3	S	$\langle Reason\rangle$
0	[	$\chi(x_{23}, y_{12})$	$\chi(x_{23}, x_{12})$	$x_{23} \otimes \theta_2$	θ_2	]	Def: $x_{23} \otimes \theta_2$
(1)	[	$\chi(x_{23}, y_{12})$	$\chi(x_{23}, x_{12} \otimes x_{01})$	X	$\{\theta_2 \otimes x_{01}\}$	]	$s_1(x_{01}) \in I_2$
2	[	$\chi(x_{23}, x_{12})$	$\chi(x_{23}, x_{12} \otimes x_{01})$	$\langle A\rangle$	$\chi(x_{12}, x_{01})$	]	Def: $\langle A\rangle$
3	[	$x_{23} \otimes \theta_2$	X	$\langle A\rangle$	$s_1(x_{01})$	]	Def: X
4	[	θ_2	$\{\theta_2 \otimes x_{01}\}$	$\chi(x_{12}, x_{01})$	$s_1(x_{01})$	]	Def: $\{\theta_2 \otimes x_{01}\}$

Figure 76: Glenn Matrix 1 for Second Variable

7.37. Lemma. *The associativity isomorphism is natural in the second variable. For all 2-cells $\theta_2 : x_{12} \Longrightarrow y_{12}$, the diagram in Figure 75 is commutative.*

For the second variable we note that

$$(x_{23} \otimes \theta_2, \_, \langle A\rangle, s_1(x_{01})) \in \textstyle\bigwedge_3^1(X_\bullet)$$

and since $s_1(x_{01})$ is invertible there exists a unique 2-simplex X such that

$$[x_{23} \otimes \theta_2, X, \langle A\rangle, s_1(x_{01}] \in X_3.$$

We insert it as Row 3 of GM-1 in Figure 76. Row 1 of GM-1 now becomes Row 1 of GM-2 in Figure 77.

Row 3 of GM-2 becomes Row 2 of GM-3 in Figure 78.

Now form GM-4 in Figure 79

We now insert Row 2 of GM-4 and Row 1 of GM-3 into GM-5 of Figure 80.

The extracted commutative Row 3 of GM-5 is equivalent to

$$((x_{23} \otimes \theta_2) \otimes x_{01}) \circ A = A' \circ (x_{23} \otimes (\theta_2 \otimes x_{01})),$$

which is naturality in the second variable.

2	S	0	1	2	3	S	$\langle Reason\rangle$
0	[	$\chi(x_{23}, y_{12})$	$\chi(x_{23}, y_{12} \otimes x_{01})$	$\langle A'\rangle$	x_{123}	]	Def: $\langle A'\rangle$
1	[	$\chi(x_{23}, y_{12})$	$\chi(x_{23}, x_{12} \otimes x_{01})$	X	$\{\theta_2 \otimes x_{01}\}$	]	Row 1: GM-1
2	[	$\chi(x_{23}, y_{12} \otimes x_{01})$	$\chi(x_{23}, x_{12} \otimes x_{01})$	$x_{23} \otimes (\theta_2 \otimes x_{01})$	$\theta_2 \otimes x_{01}$	]	Def: $x_{23} \otimes (\theta_2 \otimes x_{01})$
(3)	[	$\langle A'\rangle$	X	$x_{23} \otimes (\theta_2 \otimes x_{01})$	$s_0(x_{01})$	]	$\chi(x_{23}, y_{12}) \in I_2$
4	[	x_{123}	$\{\theta_2 \otimes x_{01}\}$	$\theta_2 \otimes x_{01}$	$s_0(x_{01})$	]	$\langle Reason\rangle$

Figure 77: Glenn Matrix 2 for Second Variable

3	S	0	1	2	3	S	$\langle$*Reason*$\rangle$
0	[	$\chi(x_{23} \otimes y_{12}, x_{01})$	$\langle A' \rangle$	A'	$s_0(x_{01})$	]	Def:A'
(1)	[	$\chi(x_{23} \otimes y_{12}, x_{01})$	X	$A' \circ x_{23} \otimes (\theta_2 \otimes x_{01})$	$s_0(x_{01})$	]	$s_0^2(x_0) \in I_2$
2	[	$\langle A' \rangle$	X	$x_{23} \otimes (\theta_2 \otimes x_{01})$	$s_0(x_{01})$	]	Row 3: GM-2
3	[	A'	$A' \circ x_{23} \otimes (\theta_2 \otimes x_{01})$	$x_{23} \otimes (\theta_2 \otimes x_{01})$	$s_0^2(x_0)$	]	Def: comp.
4	[	$s_0(x_{01})$	$s_0(x_{01})$	$s_0(x_{01})$	$s_0^2(x_0)$	]	$s_0^2(x_{01})$

Figure 78: Glenn Matrix 3 for Second Variable

4	S	0	1	2	3	S	$\langle$*Reason*$\rangle$
0	[	$x_{23} \otimes \theta_2$	$\{(x_{23} \otimes \theta_2) \otimes x_{01}\}$	$\chi(x_{23} \otimes x_{12}, x_{01})$	$s_1(x_{01})$	]	Def: $\{(x_{23} \otimes \theta_2) \otimes x_{01}\}$
1	[	$x_{23} \otimes \theta_2$	X	$\langle A \rangle$	$s_1(x_{01})$	]	Def: X
(2)	[	$\{(x_{23} \otimes \theta_2) \otimes x_{01}\}$	X	A	$s_0(x_{01})$	]	$s_1(x_{01}) \in I_2$
3	[	$\chi(x_{23} \otimes x_{12}, x_{01})$	$\langle A \rangle$	A	$s_0(x_{01})$	]	Def: A
4	[	$s_1(x_{01})$	$s_1(x_{01})$	$s_0(x_{01})$	$s_0(x_{01})$	]	$s_0(s_1(x_{01}))$

Figure 79: Glenn Matrix 4 for Second Variable

7.38. Lemma. *The associativity isomorphism is natural in the third variable: For every 2-cell $\theta_3 : x_{01} \Longrightarrow y_{01}$, the diagram in Figure 81 is commutative.*

For the third variable, we first form the Glenn Matrix GM-1 of Figure 82.

We extract the commutative Row 3 of GM-1 and insert it as Row 2 of GM-2 of Figure 83.

Finally, we extract Row 1 of GM-2 and insert it as Row 1 of GM-3 of Figure 84.

The derived commutativity of Row 3 of GM-3 is equivalent to

$$(x_{23} \otimes x_{12}) \otimes \theta_3) \circ A = A' \circ (x_{23} \otimes (x_{12} \otimes \theta_3)),$$

which is the naturality of the third variable.

This completes the proof of Proposition 7.34.

7.39. Proposition. **(Mac Lane-Stasheff Pentagon)** *The associativity isomorphism A is coherent in $\mathbf{Bic}(X_\bullet)$: For any composable quadruplet of 1-cells in $\mathbf{Bic}(X_\bullet)$,*

$$x_0 \xrightarrow{x_{01}} x_1 \xrightarrow{x_{12}} x_2 \xrightarrow{x_{23}} x_3 \xrightarrow{x_{34}} x_4,$$

the pentagonal diagram (Mac Lane–Stasheff Pentagon) in Figure 85 is commutative in $\mathbb{B}(X_\bullet)$.

5	S	0	1	2	3	S	$\langle$*Reason*$\rangle$
0	[	$\chi(x_{23} \otimes y_{12}, x_{01})$	$\{(x_{23} \otimes \theta_2) \otimes x_{01}\}$	$(x_{23} \otimes \theta_2) \otimes x_{01}$	$s_0(x_{01})$	]	Def: $\langle A' \rangle$
1	[	$\chi(x_{23} \otimes y_{12}, x_{01})$	X	$A' \circ x_{23} \otimes (\theta_2 \otimes x_{01})$	$s_0(x_{01})$	]	Row 1: GM-3
2	[	$\{(x_{23} \otimes \theta_2) \otimes x_{01}\}$	X	A	$s_0(x_{01})$	]	Row 2: GM-4
(3)	[	$(x_{23} \otimes \theta_2) \otimes x_{01}$	$A' \circ x_{23} \otimes (\theta_2 \otimes x_{01})$	A	$s_0^2(x_{01})$	]	$d_0(\text{Row 1}) \in I_2$
4	[	$s_0(x_{01})$	$s_0(x_{01})$	$s_0(x_{01})$	$s_0^2(x_{01})$	]	$s_0^2(x_{01})$

Figure 80: Glenn Matrix 5 for Second Variable

$$\begin{tikzcd}
x_{23} \otimes (x_{12} \otimes x_{01}) \arrow[r, Rightarrow, "A=A(x_{23},x_{12},x_{01})"] \arrow[d, Rightarrow, "x_{23}\otimes(x_{12}\otimes\theta_3)"'] & (x_{23} \otimes x_{12}) \otimes x_{01} \arrow[d, Rightarrow, "(x_{23}\otimes x_{12})\otimes\theta_3"] \\
x_{23} \otimes (x_{12} \otimes y_{01}) \arrow[r, Rightarrow, "A'=A(x_{23},x_{12},y_{01})"'] & (x_{23} \otimes x_{12}) \otimes y_{01}
\end{tikzcd}$$

Figure 81: Third Variable: Naturality Square for A

1	S	0	1	2	3	S	$\langle Reason \rangle$
0	[	$\chi(x_{23}, x_{12})$	$\chi(x_{23}, x_{12} \otimes y_{01})$	$\langle A' \rangle$	$\chi(x_{12}, y_{01})$	]	Def: $\langle A' \rangle$
1	[	$\chi(x_{23}, x_{12})$	$\chi(x_{23}, x_{12} \otimes x_{01})$	$\langle A \rangle$	$\chi(x_{12}, x_{01})$	]	Def: $\langle A \rangle$
2	[	$\chi(x_{23}, x_{12} \otimes y_{01})$	$\chi(x_{23}, x_{12} \otimes x_{01})$	$x_{23} \otimes (x_{12} \otimes \theta_3)$	$x_{12} \otimes \theta_3$	]	Def: $x_{23} \otimes (x_{12} \otimes \theta_3)$
(3)	[	$\langle A' \rangle$	$\langle A \rangle$	$x_{23} \otimes (x_{12} \otimes \theta_3)$	θ_3	]	$\chi(x_{23}, x_{12}) \in I_2$
4	[	$\chi(x_{12}, y_{01})$	$\chi(x_{12}, x_{01})$	$x_{12} \otimes \theta_3$	θ_3	]	Def: $x_{12} \otimes \theta_3$

Figure 82: Glenn Matrix 1 for Third Variable

2	S	0	1	2	3	S	$\langle Reason \rangle$
0	[	$\chi(x_{23} \otimes x_{12}, y_{01})$	$\langle A' \rangle$	A'	$s_0(y_{01})$	]	Def: A'
(1)	[	$\chi(x_{23} \otimes x_{12}, y_{01})$	$\langle A \rangle$	$A' \circ (x_{23} \otimes (x_{12} \otimes \theta_3))$	x_{023}	]	$s_0^2(x_0) \in I_2$
2	[	$\langle A' \rangle$	$\langle A \rangle$	$x_{23} \otimes (x_{12} \otimes \theta_3)$	θ_3	]	Row 3: GM-1
3	[	A'	$A' \circ (x_{23} \otimes (x_{12} \otimes \theta_3))$	$x_{23} \otimes (x_{12} \otimes \theta_3)$	$s_0^2(x_0)$	]	Def: comp.
4	[	$s_0(y_{01})$	θ_3	θ_3	$s_0^2(x_0)$	]	$s_1(\theta_3)$

Figure 83: Glenn Matrix 2 for Third Variable

3	S	0	1	2	3	S	$\langle Reason \rangle$
0	[	$\chi(x_{23} \otimes x_{12}, y_{01})$	$\chi(x_{23} \otimes x_{12}, x_{01})$	$(x_{23} \otimes x_{12}) \otimes \theta_3$	θ_3	]	Def: d_2(Row 0)
1	[	$\chi(x_{23} \otimes x_{12}, y_{01})$	$\langle A \rangle$	$A' \circ (x_{23} \otimes (x_{12} \otimes \theta_3))$	θ_3	]	Row 1: GM-2
2	[	$\chi(x_{23} \otimes x_{12}, x_{01})$	$\langle A \rangle$	A	$s_0(x_{01})$	]	Def: A
(3)	[	$(x_{23} \otimes x_{12}) \otimes \theta_3$	$A' \circ (x_{23} \otimes (x_{12} \otimes \theta_3))$	A	$s_0^2(x_0)$	]	d_0(Row 1)$\in I_2$
4	[	θ_3	θ_3	$s_0(x_{01})$	$s_0^2(x_0)$	]	$s_0(\theta_3)$

Figure 84: Glenn Matrix 3 for Third Variable

$$\begin{tikzcd}
& x_{34}\otimes(x_{23}\otimes(x_{12}\otimes x_{01})) \arrow[dl, Rightarrow, "\mathbf{A}(x_{34},x_{23},x_{12}\otimes x_{01})=A_1"'] \arrow[dr, Rightarrow, "x_{34}\otimes\mathbf{A}(x_{23},x_{12},x_{01})=x_{34}\otimes A_4"] & \\
(x_{34}\otimes x_{23})\otimes(x_{12}\otimes x_{01}) \arrow[ddr, Rightarrow, "\mathbf{A}(x_{34}\otimes x_{23},\ x_{12},\ x_{01})=A_3"'] & & x_{34}\otimes((x_{23}\otimes x_{12})\otimes x_{01}) \arrow[d, Rightarrow, "\mathbf{A}(x_{34},x_{23}\otimes x_{12},x_{01})=A_2"] \\
& & (x_{34}\otimes(x_{23}\otimes x_{12}))\otimes x_{01} \arrow[dl, Rightarrow, "\mathbf{A}(x_{34},x_{23},x_{12})\otimes x_{01}=A_0\otimes x_{01}"] \\
& ((x_{34}\otimes x_{23})\otimes x_{12})\otimes x_{01} &
\end{tikzcd}$$

Figure 85: Mac Lane–Stasheff Coherence Pentagon

The associativity isomorphisms in Figure 85 have been numbered as they correspond to the Grothendieck numbering in the nerve of a category given in the *Introduction*:

$$\begin{array}{rcl}
A_i &=& A(d_i(x_0 \xrightarrow{x_{01}} x_1 \xrightarrow{x_{12}} x_2 \xrightarrow{x_{23}} x_3 \xrightarrow{x_{34}} x_4)) \quad (0 \leq i \leq 4) \\
S &=& (x_0 \xrightarrow{x_{01}} x_1 \xrightarrow{x_{12}} x_2 \xrightarrow{x_{23}} x_3 \xrightarrow{x_{34}} x_4) \\
A_0 &=& A(d_0(S)) = A(x_1 \xrightarrow{x_{12}} x_2 \xrightarrow{x_{23}} x_3 \xrightarrow{x_{34}} x_4) = A(x_{34}, x_{23}, x_{12}) \\
A_1 &=& A(d_1(S)) = A(x_0 \xrightarrow{x_{12}\otimes x_{01}} x_2 \xrightarrow{x_{23}} x_3 \xrightarrow{x_{34}} x_4) = A(x_{34}, x_{23}, x_{12} \otimes x_{01}) \\
A_2 &=& A(d_2(S)) = A(x_0 \xrightarrow{x_{01}} x_1 \xrightarrow{x_{23}\otimes x_{12}} x_3 \xrightarrow{x_{34}} x_4) = A(x_{34}, x_{23} \otimes x_{12}, x_{01}) \\
A_3 &=& A(d_3(S)) = A(x_0 \xrightarrow{x_{01}} x_1 \xrightarrow{x_{12}} x_2 \xrightarrow{x_{34}\otimes x_{23}} x_4) = A(x_{34} \otimes x_{23}, x_{12}, x_{01}) \\
A_4 &=& A(d_4(S)) = A(x_0 \xrightarrow{x_{01}} x_1 \xrightarrow{x_{12}} x_2 \xrightarrow{x_{23}} x_3) = A(x_{23}, x_{12}, x_{01}).
\end{array}$$

The Mac Lane–Stasheff Coherence Equation is thus

$$\boxed{\textbf{(P)} \quad A_3 \circ A_1 = (A_0 \otimes x_{01}) \circ A_2 \circ (x_{34} \otimes A_4).}$$

The proof of Proposition 7.39 will proceed by first obtaining a sequence of Glenn matrices which will define the composition of the odd side of **(P)**. We will then do the same for the composition of the even side and finally, place the compositions together. First let X be the unique 2-simplex which makes

$$[\chi(x_{34} \otimes x_{23}, x_{12}), \langle A_1 \rangle, X, \chi(x_{12}, x_{01})] \in X_3,$$

so that the boundary of X is

$$\partial(X) = ((x_{34} \otimes x_{23}) \otimes x_{12}, x_{34} \otimes (x_{23} \otimes x_{01})), x_{01}),$$

1	S	0	1	2	3	S	$\langle Reason \rangle$
0	[	$\chi(x_{34}\otimes x_{23}, x_{12})$	$\chi(x_{34}\otimes x_{23}, x_{12}\otimes x_{01})$	$\langle A_3\rangle$	$\chi(x_{12}, x_{01})$	]	Def: $\langle A_3\rangle$
1	[	$\chi(x_{34}\otimes x_{23}, x_{12})$	$\langle A_1\rangle$	X	$\chi(x_{12}, x_{01})$	]	Def: X
2	[	$\chi(x_{34}\otimes x_{23}, x_{12}\otimes x_{01})$	$\langle A_1\rangle$	A_1	$s_0(x_{12}\otimes x_{01})$	]	Def: A_1
(3)	[	$\langle A_3\rangle$	X	A_1	$s_0(x_{01})$	]	d_0(Row 1)$\in I_2$
4	[	$\chi(x_{12}, x_{01})$	$\chi(x_{12}, x_{01})$	$s_0(x_{12}\otimes x_{01})$	$s_0(x_{01})$	]	$s_0(\chi(x_{12}, x_{01}))$

Figure 86: Glenn Matrix 1 for Coherence Pentagon

2	S	0	1	2	3	S	$\langle Reason \rangle$
0	[	$\chi((x_{34}\otimes x_{23})\otimes x_{12}, x_{01})$	$\langle A_3\rangle$	A_3	$s_0(x_{01})$	]	Def: A_3
(1)	[	$\chi((x_{34}\otimes x_{23})\otimes x_{12}, x_{01})$	X	$A_3\circ A_1$	$s_0(x_{01})$	]	$s_0^2(x_0)\in I_2$
2	[	$\langle A_3\rangle$	X	A_1	$s_0(x_{01})$	]	Row 3: GM-1
3	[	A_3	$A_3\circ A_1$	A_1	$s_0^2(x_0)$	]	Def: composition
4	[	$s_0(x_{01})$	$s_0(x_{01})$	$s_0(x_{01})$	$s_0^2(x_0)$	]	$s_0^2(x_{01})$

Figure 87: Glenn Matrix 2 for Coherence Pentagon

and form GM-1 of Figure 86.

Extracting the derived commutativeRow 3 of GM-1, we fit it as Row 2 of GM-3 in Figure 87. Row 1 of GM-1 identifies X, X is the unique 2-simplex which has $\mathrm{Int}(X) = A_1\circ A_3$. We now look as the even side of the pentagon. First define Y as the unique 2-simplex which makes

$$[\chi(x_{34}, x_{23}\otimes x_{12}), \chi(x_{34}, x_{23}\otimes(x_{12}\otimes x_{01})), Y, \langle A_4\rangle] \in X_3.$$

The boundary of Y is then

$$\partial(Y) = (x_{34}\otimes(x_{23}\otimes x_{12}), x_{34}\otimes(x_{23}\otimes(x_{12}\otimes x_{01})), x_{01}).$$

Form GM-3 of Figure 88 using the defining 3-simplex for Y as Row 1.

We use the defining 3-simplex for Y once again in GM-4 of Figure 89, this time as Row 2. We can now identify Y in GM-5 of Figure 90. The derived commutativity of Row 1 of GM-5 is equivalent to

$$\mathrm{Int}(Y) = A_2\circ(x_{34}\otimes A_4).$$

We now link X and Y in GM-6 of Figure 91. The derived commutative Row 3 of GM-6 becomes the link between X and Y. We place it in GM-7 of Figure 92. The final Glenn

3	S	0	1	2	3	S	$\langle Reason \rangle$
0	[	$\chi(x_{34}, x_{23}\otimes x_{12})$	$\chi(x_{34}, (x_{23}\otimes x_{12})\otimes x_{01})$	$\langle A_2\rangle$	$\chi(x_{23}\otimes x_{12}, x_{01})$	]	Def: $\langle A_2\rangle$
1	[	$\chi(x_{34}, x_{23}\otimes x_{12})$	$\chi(x_{34}, x_{23}\otimes(x_{12}\otimes x_{01}))$	Y	$\langle A_4\rangle$	]	Def: Y
2	[	$\chi(x_{34}, (x_{23}\otimes x_{12})\otimes x_{01})$	$\chi(x_{34}, x_{23}\otimes(x_{12}\otimes x_{01}))$	$x_{34}\otimes A_4$	A_4	]	Def:$x_{34}\otimes A_4$
(3)	[	$\langle A_2\rangle$	Y	$x_{34}\otimes A_4$	$s_0(x_{01})$	]	d_0(Row 1)$\in I_2$
4	[	$\chi(x_{23}\otimes x_{12}, x_{01})$	$\langle A_4\rangle$	A_4	$s_0(x_{01})$	]	Def: A_4

Figure 88: Glenn Matrix 3 for Coherence Pentagon

4	S	0	1	2	3	S	$\langle Reason \rangle$
0	[	$\chi(x_{34}, x_{23})$	$\chi(x_{34}, x_{23} \otimes x_{12})$	$\langle A_0 \rangle$	$\chi(x_{23}, x_{12})$	]	Def: $\langle A_0 \rangle$
1	[	$\chi(x_{34}, x_{23})$	$\chi(x_{34}, x_{23} \otimes (x_{12} \otimes x_{01}))$	$\langle A_1 \rangle$	$\chi(x_{23}, x_{12} \otimes x_{01})$	]	Def: $\langle A_1 \rangle$
2	[	$\chi(x_{34}, x_{23} \otimes x_{12})$	$\chi(x_{34}, x_{23} \otimes (x_{12} \otimes x_{01}))$	Y	$\langle A_4 \rangle$	]	Def: Y
(3)	[	$\langle A_0 \rangle$	$\langle A_1 \rangle$	Y	$\chi(x_{12}, x_{01})$	]	$\chi(x_{34}, x_{23}) \in I_2$
4	[	$\chi(x_{23}, x_{12})$	$\chi(x_{23}, x_{12} \otimes x_{01})$	$\langle A_4 \rangle$	$\chi(x_{12}, x_{01})$	]	Def: $\langle A_4 \rangle$

Figure 89: Glenn Matrix 4 for Coherence Pentagon

5	S	0	1	2	3	S	$\langle Reason \rangle$
0	[	$\chi(x_{34} \otimes (x_{23} \otimes x_{12}), x_{01})$	$\langle A_2 \rangle$	A_2	$s_0(x_{01})$	]	Def: A_0
(1)	[	$\chi(x_{34} \otimes (x_{23} \otimes x_{12}), x_{01})$	Y	$A_2 \circ (x_{34} \otimes A_4)$	$s_0(x_{01})$	]	$s_0^2(x_0) \in I_2$
2	[	$\langle A_2 \rangle$	Y	$x_{34} \otimes A_4$	$s_0(x_{01})$	]	Row 3: GM-3
3	[	A_2	$A_2 \circ (x_{34} \otimes A_4)$	$x_{34} \otimes A_4$	$s_0^2(x_0)$	]	Def: composition
4	[	$s_0(x_{01})$	$s_0(x_{01})$	$s_0(x_{01})$	$s_0^2(x_0)$	]	$s_0^2(x_{01})$

Figure 90: Glenn Matrix 5 for Coherence Pentagon

6	S	0	1	2	3	S	$\langle Reason \rangle$
0	[	$\chi(x_{34} \otimes x_{23}, x_{12})$	$\langle A_0 \rangle$	A_0	$s_0(x_{12})$	]	Def: A_0
1	[	$\chi(x_{34} \otimes x_{23}, x_{12})$	$\langle A_1 \rangle$	X	$\chi(x_{12}, x_{01})$	]	Def: X
2	[	$\langle A_0 \rangle$	$\langle A_1 \rangle$	Y	$\chi(x_{12}, x_{01})$	]	Row 3: GM-4
(3)	[	A_0	X	Y	$s_1(x_{01})$	]	$\chi(x_{34} \otimes x_{23}, x_{12}) \in I_2$
4	[	$s_0(x_{12})$	$\chi(x_{12}, x_{01})$	$\chi(x_{12}, x_{01})$	$s_1(x_{01})$	]	$s_1(\chi(x_{12}, x_{01}))$

Figure 91: Glenn Matrix 6 for Coherence Pentagon

7	S	0	1	2	3	S	$\langle Reason \rangle$
0	[	A_0	$\{A_0 \otimes x_{01}\}$	$\chi(x_{34} \otimes (x_{23} \otimes x_{12}), x_{01})$	$s_0(x_{01})$	]	Def: $\{A_0 \otimes x_{01}\}$
1	[	A_0	X	Y	$s_1(x_{01})$	]	Row 3: GM-6
(2)	[	$\{A_0 \otimes x_{01}\}$	X	$A_2 \circ (x_{34} \otimes A_4)$	$s_0(x_{01})$	]	$s_1(x_{01}) \in I_2$
3	[	$\chi(x_{34} \otimes (x_{23} \otimes x_{12}), x_{01})$	Y	$A_2 \circ (x_{34} \otimes A_4)$	$s_0(x_{01})$	]	Row 1: GM-5
4	[	$s_1(x_{01})$	$s_1(x_{01})$	$s_0(x_{01})$	$s_0(x_{01})$	]	$s_0(s_1(x_{01}))$

Figure 92: Glenn Matrix 7 for Coherence Pentagon

8	S	0	1	2	3	S	$\langle Reason \rangle$
0	[	$\chi((x_{34} \otimes x_{23}) \otimes x_{12}, x_{01})$	$\{A_0 \otimes x_{01}\}$	$A_0 \otimes x_{01}$	$s_0(x_{01})$	]	Def: $A_0 \otimes x_{01}$
1	[	$\chi((x_{34} \otimes x_{23}) \otimes x_{12}, x_{01})$	X	$A_3 \circ A_1$	$s_0(x_{01})$	]	Row 1: GM-2
2	[	$\{A_0 \otimes x_{01}\}$	X	$A_2 \circ (x_{34} \otimes A_4)$	$s_0(x_{01})$	]	Row 2: GM-7
(3)	[	$A_0 \otimes x_{01}$	$A_3 \circ A_1$	$A_2 \circ (x_{34} \otimes A_4)$	$s_0^2(x_0)$	]	$d_0(\text{Row } 1) \in I_2$
4	[	$s_0(x_{01})$	$s_0(x_{01})$	$s_0(x_{01})$	$s_0^2(x_0)$	]	$s_0^2(x_{01})$

Figure 93: Glenn Matrix 8 for Coherence Pentagon

8*	S	0	1	2	3	S	⟨*Reason*⟩
0	[	$\chi((x_{34} \otimes x_{23}) \otimes x_{12}, x_{01})$	$\{A_0 \otimes x_{01}\}$	$A_0 \otimes x_{01}$	$s_0(x_{01})$	]	Def: $A_0 \otimes x_{01}$
(1)	[	$\chi((x_{34} \otimes x_{23}) \otimes x_{12}, x_{01})$	X	C^{even}	$s_0(x_{01})$	]	$s_0^2(x_0) \in I_2$
2	[	$\{A_0 \otimes x_{01}\}$	X	$A_2 \circ (x_{34} \otimes A_4)$	$s_0(x_{01})$	]	Row 2: GM-7
3	[	$A_0 \otimes x_{01}$	C^{even}	$A_2 \circ (x_{34} \otimes A_4)$	$s_0^2(x_0)$	]	Def: comp.
4	[	$s_0(x_{01})$	$s_0(x_{01})$	$s_0(x_{01})$	$s_0^2(x_0)$	]	$s_0^2(x_{01})$

Figure 94: Glenn Matrix 8* for Coherence Pentagon

9	S	0	1	2	3	S	⟨*Reason*⟩
0	[	$\chi((x_{34} \otimes x_{23}) \otimes x_{12}, x_{01})$	X	$(A_0 \otimes x_{01}) \circ (A_2 \circ (x_{34} \otimes A_4))$	$s_0(x_{01})$	]	Row 1: GM-8*
1	[	$\chi((x_{34} \otimes x_{23}) \otimes x_{12}, x_{01})$	X	$A_3 \circ A_1$	$s_0(x_{01})$	]	Row 1: GM-2
2	[	X	X	$s_0(x_{34} \otimes (x_{23} \otimes x_{01})))$	$s_0(x_{01})$	]	$s_0(X)$
(3)	[	$(A_0 \otimes x_{01}) \circ (A_2 \circ (x_{34} \otimes A_4))$	$A_3 \circ A_1$	$s_0(x_{34} \otimes (x_{23} \otimes x_{01})))$	$s_0^2(x_0)$	]	$d_0(\text{Row } 1) \in I_2$
4	[	$s_0(x_{01})$	$s_0(x_{01})$	$s_0(x_{01})$	$s_0^2(x_0)$	]	$s_0^2(x_{01})$

Figure 95: Glenn Matrix 9 for Coherence Pentagon

Matrix bringing these together is GM-8 of Figure 93. The derived commutativity of Row 3 of GM-8 is just the coherence equation

$$A_3 \circ A_1 = (A_0 \otimes x_{01}) \circ A_2 \circ (x_{34} \otimes A_4)$$

and Proposition 7.39 is established.

7.40. Remark. Again looking to the next dimension 3, we note that instead of using GM-8 as we did, we could have used GM-8* in Figure 94, where

$$C^{\text{even}} = (A_0 \otimes x_{01}) \circ (A_2 \circ (x_{34} \otimes A_4)).$$

Then Row 1 of GM8* could have been combined with Row 1 of GM-2 in the matrix of Figure 95. The resulting commutative Row 3 of GM-9 then gives the same coherence equation **(P)** using the definition of composition in $\mathbb{B}(X_\bullet)$.

However, as we remarked in the case of the Interchange Law (Remark 7.33), if we are in dimension 3, these same matrices are the matrices of the 2-boundaries of *commutative* 4-*simplices*, the 3-simplices here are "solid" tetrahedra, and the 3-simplex whose boundary is Row 3 of GM-9 is in $\mathbb{P}^2(X_\bullet)_1$ and represents a natural 3-cell isomorphism (Figure 96):

$$\wp(x_{34}, x_{23}, x_{12}, x_{01}) : A_3 \circledcirc A_1 \Longrightarrow (A_0 \otimes x_{01}) \circledcirc (A_2 \circledcirc (x_{34} \otimes A_4)).$$

Here "$\circledcirc$" is a bicategorical tensor product of 2-cells, chosen on $\mathbb{P}(X_\bullet)$ which is (automatically) the nerve of a bicategory, as is $\mathbb{P}^2(X_\bullet)$ (automatically) the nerve of a category.

We now look at the left and right pseudo-identities and their coherence with associativity. We will prove here that they are natural isomorphisms in $\mathbb{B}(X_\bullet)$. Their coherence with associativity will be Corollary 7.46 to Theorem 7.45.

$$\begin{tikzcd}
 & & x_{34}\otimes(x_{23}\otimes(x_{12}\otimes x_{01})) \arrow[ddll, Rightarrow, "A_1"'] \arrow[drr, Rightarrow, "x_{34}\otimes A_4"] \arrow[dddrr, Rightarrow, "A_2\circledcirc(x_{34}\otimes A_4)" description] \arrow[dddd, Rightarrow, "(A_0\otimes x_{01})\circledcirc(A_2\circledcirc(x_{34}\otimes A_4))"] & & \\
 & & & & x_{34}\otimes((x_{23}\otimes x_{12})\otimes x_{01}) \arrow[dd, Rightarrow, "A_2"] \\
(x_{34}\otimes x_{23})\otimes(x_{12}\otimes x_{01}) \arrow[ddrr, Rightarrow, "A_3"'] \arrow[rr, phantom, "\overset{\wp}{\Rrightarrow}"] & & {} & & \\
 & & & & (x_{34}\otimes(x_{23}\otimes x_{12}))\otimes x_{01} \arrow[dll, Rightarrow, "A_0\otimes x_{01}"] \\
 & & ((x_{34}\otimes x_{23})\otimes x_{12})\otimes x_{01} & &
\end{tikzcd}$$

Figure 96: Pentagon 3-Cell Isomorphism $\wp$

$$\begin{tikzcd}
x_{01} \arrow[r, Rightarrow, "\tilde{s}_0(x_{01})"] \arrow[d, Rightarrow, "\theta"'] & x_{01}\otimes s_0(x_0) \arrow[d, Rightarrow, "\theta\otimes s_0(x_0)"] \\
y_{01} \arrow[r, Rightarrow, "\tilde{s}_0(y_{01})"'] & y_{01}\otimes s_0(x_0)
\end{tikzcd}$$

Figure 97: Naturality Square for $\rho = \mathrm{Int}(s_0)$

7.41. Proposition. *For any 1-simplex (=1-cell) $x_{01} : x_0 \longrightarrow x_1$, the right pseudo-identity*

$$\rho(x_{01}) = \mathrm{Int}(s_0(x_{01})) = \tilde{s}_0(x_{01}) : x_{01} \Longrightarrow x_{01} \otimes s_0(x_0)$$

is an isomorphism in $\mathbb{B}(X_\bullet)$.

Moreover, $\tilde{s}_0(x_{01})$ is natural: For any 2-cell $\theta : x_{01} \Longrightarrow y_{01}$, the diagram in Figure 97 is commutative.

Consider the Glenn matrix of Figure 98. From GM-1 we conclude that Row 3 is

GM-1	S	0	1	2	3	S	⟨*Reason*⟩
0	[	$\chi(x_{01}, s_0(x_0))$	$s_0(x_{01})$	$\tilde{s}_0(x_{01})$	$s_0^2(x_0)$	]	Def: $\rho(x_{01}) = \tilde{s}_0(x_{01})$
1	[	$\chi(x_{01}, s_0(x_0))$	$\chi(x_{01}, s_0(x_0))$	$s_0(x_{01} \otimes s_0(x_0))$	$s_0^2(x_0)$	]	$s_0(\chi(x_{01}, s_0(x_0)))$
2	[	$s_0(x_{01})$	$\chi(x_{01}, s_0(x_0))$	$\chi(x_{01}, s_0(x_0))$	$s_0^2(x_0)$	]	$s_1(\chi(x_{01}, s_0(x_0)))$
(3)	[	$\tilde{s}_0(x_{01})$	$s_0(x_{01} \otimes s_0(x_0))$	$\chi(x_{01}, s_0(x_0))$	$s_0^2(x_0)$	]	$\chi(x_{01}, s_0(x_0)) \in I_2$
4	[	$s_0^2(x_0)$	$s_0^2(x_0)$	$s_0^2(x_0)$	$s_0^2(x_0)$	]	$s_0^3(x_0)$

Figure 98: Glenn Matrix 1 for invertibility of $\mathrm{Int}(s_0)$ in $\mathbb{B}(X_\bullet)$

GM-1	S	0	1	2	3	S	$\langle Reason \rangle$
0	[	$\chi(y_{01}, s_0(x_0))$	$s_0(x_0)$	$\tilde{s}_0(y_{01})$	$s_0^2(x_0)$	]	Def: $\rho(y_{01}) = \tilde{s}_0(y_{01})$
(1)	[	$\chi(y_{01}, s_0(x_0))$	θ	$\tilde{s}_0(y_{01}) \circ \theta$	$s_0^2(x_0)$	]	$s_0^2(x_0) \in I^2$
2	[	$s_0(y_{01})$	θ	θ	$s_0^2(x_0)$	]	$s_1(\theta)$
3	[	$\tilde{s}_0(y_{01})$	$\tilde{s}_0(y_{01}) \circ \theta$	θ	$s_0^2(x_0)$	]	Def: composition
4	[	$s_0^2(x_0)$	$s_0^2(x_0)$	$s_0^2(x_0)$	$s_0^2(x_0)$	]	$s_0^3(x_0)$

Figure 99: Glenn Matrix 1 for Naturality of $\rho = \mathrm{Int}(s_0)$

commutative:

$$[\tilde{s}_0(x_{01}), s_0(x_{01} \otimes s_0(x_0)), \chi(x_{01}, s_0(x_0)), s_0^2(x_0)] \in X_3.$$

The defining 3-simplex for $\tilde{s}_0(x_{01})$ is

$$[\chi(x_{01}, s_0(x_0)), s_0(x_{01}), \tilde{s}_0(x_{01}), s_0^2(x_0)].$$

Using the definition of composition in $\mathbb{B}(X_\bullet)$, the commutativity of these two 3-simplices is equivalent to

$$\tilde{s}_0(x_{01}) \circ \chi(x_{01}, s_0(x_0)) = s_0(x_{01} \otimes s_0(x_0))$$

and

$$\chi(x_{01}, s_0(x_0)) \circ \tilde{s}_0(x_{01}) = s_0(x_{01}).$$

But this just says that $\tilde{s}_0(x_{01})$ is an isomorphism in $\mathbb{B}(X_\bullet)$, with

$$\tilde{s}_0(x_{01})^{-1} = \chi(x_{01}, s_0(x_0))$$

since $s_0(x_{01} \otimes s_0(x_0))$ and $s_0(x_{01})$ are identity 2-cells in $\mathbb{B}(X_\bullet)$.

An alternate proof could use the facts that $\chi(x_{01}, s_0(x_0)) : x_{01} \otimes s_0(x_0) \Longrightarrow x_{01}$ as a 2-cell in $\mathbb{B}(X_\bullet)$, and $\chi(x_{01}, s_0(x_0))$ is invertible as a 2-simplex in $X_\bullet$. By Proposition 7.12, $\chi(x_{01}, s_0(x_0))$ is thus an isomorphism in $\mathbb{B}(X_\bullet)$. But the defining 3-simplex for $\rho(x_{01}) = \tilde{s}_0(x_{01})$ is

$$[\chi(x_{01}, s_0(x_0)), s_0(x_{01}), \tilde{s}_0(x_{01}), s_0^2(x_0)],$$

and from the definition of composition in $\mathbb{B}(X_\bullet)$, this commutativity is equivalent to

$$\chi(x_{01}, s_0(x_0)) \circ \tilde{s}_0(x_{01}) = s_0(x_{01}).$$

Since $s_0(x_{01}) : x_{01} \Longrightarrow x_{01}$ is the identity 2-cell for the object x_{01} in the category $\mathbb{B}(X_\bullet)$, $\tilde{s}_0(x_{01})$ must also be an isomorphism with

$$\tilde{s}_0(x_{01})^{-1} = \chi(x_{01}, s_0(x_0)) : x_{01} \otimes s_0(x_0) \Longrightarrow x_{01}.$$

For naturality we first use the Glenn matrices of Figures 99 and 100. We then place the derived commutative Row 1 of GM-1 and Row 2 of GM-2 in GM-3 of Figure 101. The derived commutativity of Row 3 of GM-3 is equivalent to

$$(\theta \otimes s_0(x_0)) \circ \tilde{s}_0(x_{01}) = \tilde{s}_0(y_{01}) \circ \theta,$$

which gives the naturality of ρ.

GM-2	S	0	1	2	3	S	$\langle Reason \rangle$
0	[	θ	$\{\theta \otimes s_0(x_0)\}$	$\chi(x_{01}, s_0(x_0))$	$s_0^2(x_0)$	]	Def: $\{\theta \otimes s_0(x_0)\}$
1	[	θ	θ	$s_0(x_{01})$	$s_0^2(x_0)$	)	$s_0(\theta)$
(2)	[	$\{\theta \otimes s_0(x_0)\}$	θ	$\tilde{s}_0(x_{01})$	$s_0^2(x_0)$	]	$s_0^2(x_0) \in I_2$
3	[	$\chi(x_{01}, s_0(x_0)$	$s_0(x_{01})$	$\tilde{s}_0(x_{01})$	$s_0^2(x_0)$	]	Def: $\tilde{s}_0(x_{01})$
4	[	$s_0^2(x_0)$	$s_0^2(x_0)$	$s_0^2(x_0)$	$s_0^3(x_0)$	]	$s_0^2(x_0)$

Figure 100: Glenn Matrix 2 for Naturality of $\rho = \mathrm{Int}(s_0)$

GM-3	S	0	1	2	3	S	$\langle Reason \rangle$
0	[	$\chi(y_{01}, s_0(x_0))$	$\{\theta \otimes s_0(x_0)\}$	$\theta \otimes s_0(x_0)$	$s_0^2(x_0)$	]	Def: $\theta \otimes s_0(x_0)$
1	[	$\chi(y_{01}, s_0(x_0))$	θ	$\tilde{s}_0(y_{01}) \circ \theta$	$s_0^2(x_0)$	]	Row 1: GM-1
2	[	$\{\theta \otimes s_0(x_0)\}$	θ	$\tilde{s}_0(x_{01})$	$s_0^2(x_0)$	]	Row 2: GM-2
(3)	[	$\theta \otimes s_0(x_0)$	$\tilde{s}_0(y_{01}) \circ \theta$	$\tilde{s}_0(x_{01})$	$s_0^2(x_0)$	]	$\chi(y_{01}, s_0(x_0)) \in I_2$
4	[	$s_0^2(x_0)$	$s_0^2(x_0)$	$s_0^2(x_0)$	$s_0^2(x_0)$	]	$s_0^3(x_0)$

Figure 101: Glenn Matrix 3 for Naturality of $\rho = \mathrm{Int}(s_0)$

7.42. Remark. All of the commutative rows of GM-1,GM-2, and GM-3 above, as well as those of the immediately foregoing GM-1, are equivalent to equations, mostly trivial, in $\mathbb{B}(X_\bullet)$. The interested reader may wish to make the translations of each of these and see how exactly the same results of isomorphism and naturality can be obtained in a more familiar fashion using them.

7.43. Proposition. *Let $x_{01} : x_0 \longrightarrow x_1$ be a 1-simplex (=1-cell) in $\mathbb{B}(X_\bullet)$. Then the 2-cell $\lambda(x_{01}) = \mathrm{Int}(s_1(x_{01})) : x_{01} \Longrightarrow s_0(x_1) \otimes x_{01}$ is an isomorphism in $\mathbb{B}(X_\bullet)$. Moreover, it is natural: For any 2-cell $\theta : x_{01} \Longrightarrow y_{01}$ in $\mathbb{B}(X_\bullet)$, the diagram in Figure 102 is commutative.*

To show that $\tilde{s}_1(x_{01})$ is an isomorphism, define X to be the unique 2-simplex for which

$$[s_1(x_{01}), \chi(s_0(x_1), x_{01}), X, s_0(x_{01})] \in X_3\,,$$

which is possible since $s_1(x_{01}) \in I_2$. Then X is a 2-cell,

$$X : s_0(x_1) \otimes x_{01} \Longrightarrow x_{01},$$

$$\begin{tikzcd}
x_{01} \arrow[r, Rightarrow, "\tilde{s}_1(x_{01})"] \arrow[d, Rightarrow, "\theta"'] & s_0(x_1) \otimes x_{01} \arrow[d, Rightarrow, "s_0(x_1)\otimes\theta"] \\
y_{01} \arrow[r, Rightarrow, "\tilde{s}_1(y_{01})"'] & s_0(x_1) \otimes y_{01}
\end{tikzcd}$$

Figure 102: Naturality Square for $\lambda = \mathrm{Int}(s_1)$

GM-1	S	0	1	2	3	S	⟨*Reason*⟩
0	[	$\chi(s_0(x_1), x_{01})$	$s_1(x_{01})$	$\tilde{s}_1(x_{01})$	$s_0(x_{01})$	]	Def: $\tilde{s}_1(x_{01})$
1	[	$\chi(s_0(x_1), x_{01})$	$\chi(s_0(x_1), x_{01})$	$s_0(s_0(x_1) \otimes x_{01})$	$s_0(x_{01})$	]	$s_0(\chi(s_0(x_1), x_{01}))$
2	[	$s_1(x_{01})$	$\chi(s_0(x_1), x_{01})$	X	$s_0(x_{01})$	]	Def: X
(3)	[	$\tilde{s}_1(x_{01})$	$s_0(s_0(x_1) \otimes x_{01})$	X	$s_0^2(x_0)$	]	$\chi(s_0(x_1), x_{01}) \in I_2$
4	[	$s_0(x_{01})$	$s_0(x_{01})$	$s_0(x_{01})$	$s_0^2(x_0)$	]	$s_0^2(x_{01})$

Figure 103: Glenn Matrix 1 for invertibility of $\lambda = \mathrm{Int}(s_1)$ in $\mathbb{B}(X_\bullet)$

GM-2	S	0	1	2	3	S	⟨*Reason*⟩
0	[	$s_1(x_{01})$	$\chi(s_0(x_1), x_{01})$	X	$s_0(x_{01})$	]	Def: X
1	[	$s_1(x_{01})$	$s_1(x_{01})$	$s_0(x_{01})$	$s_0(x_{01})$	]	$s_0(s_1(x_{01}))$
2	[	$\chi(s_0(x_1), x_{01})$	$s_1(x_{01})$	$\tilde{s}_1(x_{01})$	$s_0(x_{01})$	]	Def: $\tilde{s}_1(x_{01})$
(3)	[	X	$s_0(x_{01})$	$\tilde{s}_1(x_{01})$	$s_0^2(x_0)$	]	$s_1(x_{01}) \in I_2$
4	[	$s_0(x_{01})$	$s_0(x_{01})$	$s_0(x_{01})$	$s_0^2(x_0)$	]	$s_0^2(x_{01})$

Figure 104: Glenn Matrix 2 for invertibility of $\lambda = \mathrm{Int}(s_1)$ in $\mathbb{B}(X_\bullet)$

and we may form the Glenn matrices GM-1 of Figure 103 and Figure 104. The derived commutativity of Row 3 of GM-1 and Row 3 of GM-2 are, respectively, equivalent to

$$\tilde{s}_1(x_{01}) \circ X = s_0(s_0(x_1) \otimes x_{01})$$

and

$$X \circ \tilde{s}_1(x_{01}) = s_0(x_{01}),$$

which together assert that $\tilde{s}_1(x_{01})$ is an isomorphism, with

$$\tilde{s}_1(x_{01})^{-1} = X.$$

For naturality, we first form GM-1 of Figure 105. We extract the commutative Row 1 of GM-1 and insert it as Row 1 of GM-2 in Figure 106.

The derived commutativity of Row 3 of GM-2 is equivalent to

$$\tilde{s}_1(y_{01}) \circ \theta = (s_0(x_1) \otimes \theta) \circ \tilde{s}_1(x_{01})),$$

1	S	0	1	2	3	S	⟨*Reason*⟩
0	[	$\chi(s_0(x_1), y_{01})$	$\chi(s_0(x_1), x_{01})$	$s_0(x_1) \otimes \theta$	θ	]	Def: $s_0(x_1) \otimes \theta$
(1)	[	$\chi(s_0(x_1), y_{01})$	$s_1(x_{01})$	$(s_0(x_1) \otimes \theta) \circ \tilde{s}_1(x_{01})$	θ	]	$s_0^2(x_0) \in I_2$
2	[	$\chi(s_0(x_1), x_{01})$	$s_1(x_{01})$	$\tilde{s}_1(x_{01})$	$s_0(x_{01})$	]	Def: $\tilde{s}_1(x_{01})$
3	[	$s_0(x_1) \otimes \theta$	$(s_0(x_1) \otimes \theta) \circ \tilde{s}_1(x_{01})$	$\tilde{s}_1(x_{01})$	$s_0^2(x_0)$	]	Def: composition
4	[	θ	θ	$s_0(x_{01})$	$s_0^2(x_0)$	]	$s_0(\theta)$

Figure 105: Glenn Matrix 1 for Naturality of $\lambda = \mathrm{Int}(s_1)$

2	S	0	1	2	3	S	⟨*Reason*⟩
0	[	$\chi(s_0(x_1), y_{01})$	$s_1(y_{01})$	$\tilde{s}_1(y_{01})$	$s_0(y_{01})$	]	Def: $\tilde{s}_1(y_{01})$
1	[	$\chi(s_0(x_1), y_{01})$	$s_1(x_{01})$	$(s_0(x_1) \otimes \theta) \circ \tilde{s}_1(x_{01})$	θ	]	Row 1: GM-1
2	[	$s_1(y_{01})$	$s_1(x_{01})$	θ	θ	]	$s_2(\theta)$
(3)	[	$\tilde{s}_1(y_{01})$	$(s_0(x_1) \otimes \theta) \circ \tilde{s}_1(x_{01})$	θ	$s_0^2(x_0)$	]	$\chi(s_0(x_1), y_{01}) \in I_2$
4	[	$s_0(x01)$	θ	θ	$s_0^2(x_0)$	]	$s_0(\theta)$

Figure 106: Glenn Matrix 2 for Naturality of $\lambda = \mathrm{Int}(s_1)$

which is the naturality of λ. Propositions 7.41 and 7.43 prove that both $\rho = \mathrm{Int}(s_0)$ and $\lambda = \mathrm{Int}(s_1)$ are natural isomorphisms.

That they are compatible with each other,*i.e.*, for all $x_0 \in X_0, \rho(s_0(x_0)) = \lambda(s_0(x_0))$, is an immediate consequence of the simplicial identity on the first three degeneracies which requires that for all $x_0 \in X_0,\quad s_0(s_0(x_0)) = s_1(s_0(x_0))$, together with the definitions, $\rho(s_0(x_0)) = \tilde{s}_0(s_0(x_0))$ and $\lambda(s_0(x_0)) = \tilde{s}_1(s_0(x_0))$. The latter are defined (using the invertibility of $\chi(s_0(x_0), s_0(x_0)))$ as the unique 2-simplices which make

$$[\chi(s_0(x_0), s_0(x_0)), s_0(s_0(x_0)), \tilde{s}_0(s_0(x_0)), s_0^2(x_0)] \in X_3$$

and

$$[\chi(s_0(x_0), s_0(x_0)), s_1(s_0(x_0)), \tilde{s}_1(s_0(x_0)), s_0^2(x_0)] \in X_3.$$

The simplicial identity together with the invertibility of $\chi(s_0(x_0), s_0(x_0))$ gives the result.

The compatibility of ρ and λ with the associativity A, although it could be proven separately, will be taken as the first corollary of Theorem 7.45 in the next section. This theorem is the second major link (after Theorem 7.16) between bicategories and those simplicial sets which are their nerves. As in the case for categories and the subset $X_2 \subseteq \mathrm{SimKer}(X_\bullet]_0^1)$ of commutative 2-simplices (= "commutative triangles") of their nerves, this theorem again gives full justification for using the terminology "commutative 3-simplices" (or "commutative tetrahedra") as the name of the subset $X_3 \subseteq \mathrm{SimKer}(X_\bullet]_0^2)$ which started out just as the set of 3-simplices of $X_\bullet$ in our basic simplicial setting for the $n = 2$ case of our theory.

Note also that once the compatibility of ρ and λ with the associativity A has been shown as a corollary to Theorem 7.45, we will have completed the proof that $\mathbf{Bic}(X_\bullet)$, with the above defined structural components is indeed a bicategory.

7.44. The Justification of the Term "Commutative" for the 3-Simplices. Recall that we are still assuming that all degeneracies in X_2 are invertible and that a tensor product has been chosen as
$d_1(\chi(x_{12}, x_{01})) = x_{12} \otimes x_{01}$ with the 2-simplex $\chi(x_{12}, x_{01})$ invertible.

Also recall that for any 2-simplex $x_{012} \in X_2$, with $\partial(x_{012}) = (x_{12}, x_{02}, x_{01})$, we have defined (Definition7.15) the *interior* of x_{012}, $\mathrm{Int}(x_{012}) = \tilde{x}_{012}$, to be the unique 2-simplex in X_2 which makes

$$[\chi(x_{12}, x_{01}), x_{012}, \tilde{x}_{012}, s_0(x_{01})] \in X_3.$$

As a 2-cell in $\mathbb{B}(X_\bullet)$, $\tilde{x}_{012} : x_{02} \Longrightarrow x_{12} \otimes x_{01}$.

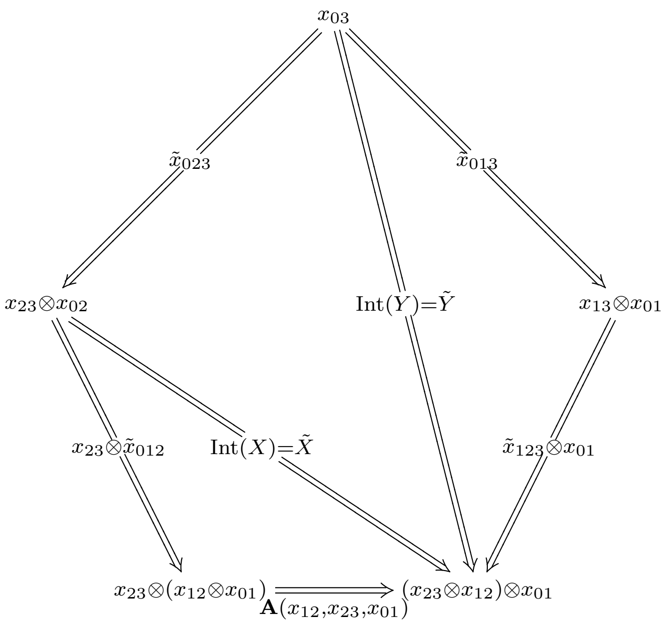

Figure 107: Interior 2-cells of Equation **(T)** in $\mathbb{B}(X_\bullet)$ for $\mathbf{x}_{0123} = (x_{123}, x_{023}, x_{013}, x_{012}) \in K_3(X_\bullet)$ and X , Y defined in Proof of Theorem 7.45

7.45. Theorem. **(Commutative 3-Simplex $\Leftrightarrow$ 2-Cell Interior is Commutative in $\mathbb{B}(X_\bullet)$).** *For any*

$$\mathbf{x}_{0123} = (x_{123}, x_{023}, x_{013}, x_{012}) \in K_3(X_\bullet) = \mathrm{SimKer}(X_\bullet]_0^2) = \mathbf{Cosk}^2(X_\bullet)_3,$$

$\mathbf{x}_{0123}$ *is commutative, i.e.,*

$$[x_{123}, x_{023}, x_{013}, x_{012}] = \mathbf{x}_{0123} \in X_3$$

if, and only if, in the category $\mathbb{B}(X_\bullet)$, the equation (Figure 107)

$$\boxed{(\,\mathbf{T}\,) \qquad A(x_{23}, x_{12}, x_{01}) \circ (x_{23} \otimes \tilde{x}_{012}) \circ \tilde{x}_{023} = (\tilde{x}_{123} \otimes x_{01}) \circ \tilde{x}_{013}}$$

is satisfied.

For the proof of one direction of Theorem 7.45, namely that the commutativity of $(x_{123}, x_{023}, x_{013}, x_{012}) \in K_3$ implies that the interior equation $(\,\mathbf{T}\,)$ is satisfied in $\mathbb{B}(X_\bullet)$,

$$[x_{123}, x_{023}, x_{013}, x_{012}] \Rightarrow (\,\mathbf{T}\,),$$

we define a sequence of Glenn matrices.

First, for Row 1 of GM-1 (Figure 108), we define X as the unique 2-simplex which makes

$$[\chi(x_{23}, x_{12}), \chi(x_{23}, x_{02}), X, x_{012}] \in X_3.$$

1	S	0	1	2	3	S	$\langle Reason \rangle$
0	[	$\chi(x_{23}, x_{12})$	$\chi(x_{23}, x_{12} \otimes x_{01})$	$\langle A \rangle$	$\chi(x_{12}, x_{01})$	]	Def: $\langle A \rangle$
1	[	$\chi(x_{23}, x_{12})$	$\chi(x_{23}, x_{02})$	X	x_{012}	]	Def: X
2	[	$\chi(x_{23}, x_{12} \otimes x_{01})$	$\chi(x_{23}, x_{02})$	$x_{23} \otimes x_{012}$	$\tilde{x}_{012}$	]	Def: $x_{23} \otimes x_{012}$
(3)	[	$\langle A \rangle$	X	$x_{23} \otimes x_{012}$	$s_0(x_{01})$	]	$\chi(x_{23}, x_{12}) \in I_2$
4	[	$\chi(x_{12}, x_{01})$	x_{012}	$\tilde{x}_{012}$	$s_0(x_{01})$	]	Def: $\tilde{x}_{012}$

Figure 108: Glenn Matrix 1 for $[x_0, x_1, x_2, x_3] \Rightarrow (\,\mathbf{T}\,)$

2	S	0	1	2	3	S	$\langle Reason \rangle$
0	[	$\chi(x_{23} \otimes x_{12}, x_{01})$	$\langle A \rangle$	A	$s_0(x_{01})$	]	Def: A
(1)	[	$\chi(x_{23} \otimes x_{12}, x_{01})$	X	$A \circ x_{23} \otimes \tilde{x}_{012}$	$s_0(x_{01})$	]	$s_0^2(x_0) \in I_2$
2	[	$\langle A \rangle$	X	$x_{23} \otimes x_{012}$	$s_0(x_{01})$	]	Row 3: GM-1
3	[	A	$A \circ x_{23} \otimes \tilde{x}_{012}$	$x_{23} \otimes \tilde{x}_{012}$	$s_0^2(x_0)$	]	Def: composition
4	[	$s_0(x_{01})$	$s_0(x_{01})$	$s_0(x_{01})$	$s_0^2(x_0)$	]	$s_0^2(x_{01}) \in I_3$

Figure 109: Glenn Matrix 2 for $[x_0, x_1, x_2, x_3] \Rightarrow (\,\mathbf{T}\,)$

This gives X the boundary $\partial(X) = (x_{23} \otimes x_{12}, x_{23} \otimes x_{02}, x_{01})$.

We extract the commutative Row 3 of GM-1 and place it into GM-2 of Figure 109.

The derived commutativity of Row 1 of GM-2 identifies X as the unique 2-simplex for which $\mathrm{Int}(X) = \tilde{X} = A \circ (x_{23} \otimes \tilde{x}_{012})$, the composition of the last two factors of the odd side of the pentagon in Figure 107.

We now define Y as the unique 2-simplex which makes

$$[\chi(x_{23}, x_{12}), x_{023}, Y, x_{012}] \in X_3,$$

so that $\partial(Y) = (x_{23} \otimes x_{12}, x_{03}, x_{01})$, and insert it as Row 1 of GM-3 in Figure 110. We then place the derived commutative Row 3 of GM-3 along with the derived commutative Row 1 of GM-2 into GM-4 of Figure 111. The derived commutativity of Row 1 of GM-4 identifies Y as the unique 2-simplex whose interior is the composition of the odd side of the pentagon (in the order specified) and is the left hand side of equation $(\,\mathbf{T}\,)$,

$$\mathrm{Int}(Y) = \tilde{Y} = (A \circ (x_{23} \otimes \tilde{x}_{012})) \circ \tilde{x}_{023}.$$

GM-3	S	0	1	2	3	S	$\langle Reason \rangle$
0	[	$\chi(x_{23}, x_{12})$	$\chi(x_{23}, x_{02})$	X	x_{012}	]	Def: X
1	[	$\chi(x_{23}, x_{12})$	x_{023}	Y	x_{012}	]	Def: Y
2	[	$\chi(x_{23}, x_{02}$	x_{023}	$\tilde{x}_{023}$	$s_0(x_{02})$	]	Def: $\tilde{x}_{023}$
(3)	[	X	Y	$\tilde{x}_{023}$	$s_0(x_{01})$	]	$\chi(x_{23}, x_{12}) \in I_2$
4	[	x_{012}	x_{012}	$s_0(x_{02})$	$s_0(x_{01})$	]	$s_0(x_{012})$

Figure 110: Glenn Matrix 3 for $[x_0, x_1, x_2, x_3] \Rightarrow (\,\mathbf{T}\,)$

4	S	0	1	2	3	S	⟨*Reason*⟩
0	[	$\chi(x_{23} \otimes x_{12}, x_{01})$	X	$A \circ (x_{23} \otimes \tilde{x}_{012})$	$s_0(x_{01})$	]	Row 1: GM-2
(1)	[	$\chi(x_{23} \otimes x_{12}, x_{01})$	Y	$(A \circ x_{23} \otimes \tilde{x}_{012}) \circ \tilde{x}_{023}$	$s_0(x_{01})$	]	$s_0^2(x_0) \in I_2$
2	[	X	Y	$\tilde{x}_{023}$	$s_0(x_{01})$	]	Row 3 : GM-3
3	[	$A \circ (x_{23} \otimes \tilde{x}_{012})$	$(A \circ x_{23} \otimes \tilde{x}_{012}) \circ \tilde{x}_{023}$	$\tilde{x}_{023}$	$s_0(x_{01})$	]	Def: composition
4	[	$s_0(x_{01})$	$s_0(x_{01})$	$s_0(x_{01})$	$s_0^2(x_0)$	]	$s_0^2(x_{01})$

Figure 111: Glenn Matrix 4 for $[x_0, x_1, x_2, x_3] \Rightarrow (\mathbf{T})$

GM-5	S	0	1	2	3	S	⟨*Reason*⟩
0	[	$\chi(x_{23}, x_{12})$	x_{123}	$\tilde{x}_{123}$	$s_0(x_{12})$	]	Def: $\tilde{x}_{123}$
1	[	$\chi(x_{23}, x_{12})$	x_{023}	Y	x_{012}	]	Def: Y
2	[	x_{123}	x_{023}	x_{013}	x_{012}	]	*Hypothesis*
(3)	[	$\tilde{x}_{123}$	Y	x_{013}	$s_1(x_{01})$	]	$\chi(x_{23}, x_{12}) \in I_2$
4	[	$s_0(x_{12})$	x_{012}	x_{012}	$s_1(x_{01})$	]	$s_1(x_{012})$

Figure 112: Glenn Matrix 5 for $[x_0, x_1, x_2, x_3] \Rightarrow (\mathbf{T})$

We now again use the original definition of Y and link it to the even numbered 2-simplex sides of the pentagon as well as to the hypothesized commutativity of the 3-simplex in GM-5 of Figure 112. The derived commutativity of Row 3 of GM-5 allows it to be placed in GM-6 of Figure 113. For the final step, we extract Row 2 of GM-6 and place it in GM-7 of Figure 114. The derived commutativity of Row 3 of GM-7 is equivalent to Equation $(\mathbf{T})$ in $\mathbb{B}(X_\bullet)$.

For the converse statement in Theorem 7.45, note that in GM-7, the commutativity of Row 1 is established only by using the definition of Y and the matrices GM-1,GM-2,GM-3, and GM-4. Thus the commutativity of Row 3 in GM-7 is equivalent to the Commutativity of Row 2 in GM-7 since both $s_0^2(x_0)$ and $\chi(x_{23} \otimes x_{12}, x_{01})$ are invertible. Similarly, in GM-6 the commutativity of Row 2 is equivalent to the commutativity of Row 1 since both $s_0(x_{01})$ and $s_1(x_{01})$ are invertible. Likewise, in GM-5 the commutativity of Row 2 is equivalent to the commutativity of Row 3, since both $\chi(x_{23}, x_{12})$ and $s_0(x_{12})$ are invertible. But Row 3 of GM-5 is identical to Row 1 of GM-6 and Row 2 of GM-6 is identical to Row 2 of GM-7. Row 3 of GM-7 is

$$(\tilde{x}_{123} \otimes x_{01}, (A \circ (x_{23} \otimes \tilde{x}_{012})) \circ \tilde{x}_{023}, \tilde{x}_{013}, s_0^2(x_0)),$$

GM-6	S	0	1	2	3	S	⟨*Reason*⟩
0	[	$\tilde{x}_{123}$	$\{\tilde{x}_{123} \otimes x_{01}\}$	$\chi(x_{13}, x_{01})$	$s_1(x_{01})$	]	Def: $\{\tilde{x}_{123} \otimes x_{01}\}$
1	[	$\tilde{x}_{123}$	Y	x_{013}	$s_1(x_{01})$	]	Row 3 GM-5
(2)	[	$\{\tilde{x}_{123} \otimes x_{01}\}$	Y	$\tilde{x}_{013}$	$s_0(x_{01})$	]	$s_1(x_{01}) \in I_2$
3	[	$\chi(x_{13}, x_{01})$	x_{013}	$\tilde{x}_{013}$	$s_0(x_{01})$	]	Def: $\tilde{x}_{013}$
4	[	$s_1(x_{01})$	$s_1(x_{01})$	$s_0(x_{01})$	$s_0(x_{01})$	]	$s_0(s_1(x_{01}))$

Figure 113: Glenn Matrix 6 for $[x_0, x_1, x_2, x_3] \Rightarrow (\mathbf{T})$

7	S	0	1	2	3	S	⟨*Reason*⟩
0	[	$\chi(x_{23} \otimes x_{12}, x_{01})$	$\{\tilde{x}_{123} \otimes x_{01}\}$	$\tilde{x}_{123} \otimes x_{01}$	$s_0(x_{01})$	]	Def: $\tilde{x}_{123} \otimes x_{01}$
1	[	$\chi(x_{23} \otimes x_{12}, x_{01})$	Y	$(A \circ (x_{23} \otimes \tilde{x}_{012})) \circ \tilde{x}_{023}$	$s_0(x_{01})$	]	Row 1: GM-4
2	[	$\{\tilde{x}_{123} \otimes x_{01}\}$	Y	$\tilde{x}_{013}$	$s_0(x_{01})$	]	Row 2: GM-6
(3)	[	$\tilde{x}_{123} \otimes x_{01}$	$(A \circ (x_{23} \otimes \tilde{x}_{012})) \circ \tilde{x}_{023}$	$\tilde{x}_{013}$	$s_0^2(x_0)$	]	$d_0(\text{ROW1}) \in I_2$
4	[	$s_0(x_{01})$	$s_0(x_{01})$	$s_0(x_{01})$	$s_0^2(x_0)$	]	$s_0^2(x_{01})$

Figure 114: Glenn Matrix 7 for $[x_0, x_1, x_2, x_3] \Rightarrow (\,\mathbf{T}\,)$

$$
\begin{array}{ccccc}
[\text{Row 3 of GM-7}] & \Longleftrightarrow & [\text{Row 2 of GM-7}] & & \\
\Updownarrow & & \| & & \\
(\,\mathbf{T}\,) & & [\text{Row 2 of GM-6}] & \Longleftrightarrow & [\text{Row 1 of GM-6}] \\
 & & & & \| \\
[x_0, x_1, x_2, x_3] & \Longleftrightarrow & [\text{Row 2 of GM-5}] & \Longleftrightarrow & [\text{Row 3 of GM-5}]
\end{array}
$$

Figure 115: $[x_0, x_1, x_2, x_3] \Longleftrightarrow (\,\mathbf{T}\,)$

and Row 2 of GM-5 is

$$(x_{123}, x_{023}, x_{013}, x_{012}).$$

The commutativity of Row 2 of GM-7 is, by definition of the composition in $\mathbb{B}(X_\bullet)$, just Equation $(\,\mathbf{T}\,)$; the commutativity of Row 2 of GM-5 is just the statement that $(x_{123}, x_{023}, x_{013}, x_{012}) \in X_3$, which we have abbreviated with the use of square brackets to $[x_{123}, x_{023}, x_{013}, x_{012}]$. Chaining these together as in Figure 115, with "[Row x of GM-y]" abbreviating "Row x of GM-y is commutative", we have the full equivalence of Theorem 7.45. *Q.E.D.*

As promised, we will now use Theorem 7.45 to verify the compatibility of ρ and λ with A.

7.46. Corollary. *In $\mathbb{B}(X_\bullet)$ the three compatibility axioms for ρ and λ with A are verified: With $\rho = \tilde{s}_0$, $\lambda = \tilde{s}_1$, and $I = s_0$, for all composable pairs of* 1*-cells,* $(x_{12}, x_{01}) \in \bigwedge_2^1(X_\bullet)$*, the three equalities,*

$$A(x_{12}, x_{01}, s_0(x_0)) \circ (x_{12} \otimes \tilde{s}_0(x_{01}) = \tilde{s}_0(x_{12} \otimes x_{01}) \Longleftrightarrow$$

$$A(x_{12}, x_{01}, I(x_0)) \circ (x_{12} \otimes \rho(x_{01}) = \rho(x_{12} \otimes x_{01}),$$

$$A(x_{12}, s_0(x_1), x_{01}) \circ (x_{12} \otimes \tilde{s}_1(x_{01})) = \tilde{s}_0(x_{12}) \otimes x_{01} \Longleftrightarrow$$

$$A(x_{12}, I(x_1), x_{01}) \circ (x_{12} \otimes \lambda(x_{01})) = \rho(x_{12}) \otimes x_{01},$$

and

$$A(s_0(x_2), x_{12}, x_{01}) \circ (\tilde{s}_1(x_{12} \otimes x_{01})) = \tilde{s}_1(x_{12}) \otimes x_{01} \Longleftrightarrow$$

$$A(I(x_2), x_{12}, x_{01}) \circ (\lambda(x_{12} \otimes x_{01})) = \lambda(x_{12}) \otimes x_{01}$$

are satisfied.

The three axioms are just the translations via Theorem 7.45 of our requirement that $X_\bullet$ be a subcomplex of its 2-Coskeleton, which requires that for any 2-simplex x, $s_0(x)$, $s_1(x)$, and $s_2(x)$ be commutative. If we take $x = \chi(x_{12}, x_{01})$, then

$$s_0(\chi(x_{12}, x_{01})) = (\chi(x_{12}, x_{01}), \chi(x_{12}, x_{01}), s_0(x_{12} \otimes x_{01}), s_0(x_{01})) \in K_3,$$

$$s_1(\chi(x_{12}, x_{01})) = (s_0(x_{12}, \chi(x_{12}, x_{01}), \chi(x_{12}, x_{01}), s_1(x_{01})) \in K_3,$$

and

$$s_2(\chi(x_{12}, x_{01})) = (s_1(x_{12}), s_1(x_{12} \otimes x_{01}), \chi(x_{12}, x_{01}), \chi(x_{12}, x_{01})) \in K_3$$

must all be in X_3, *i.e.*, must be commutative. But since we have already noted (Proposition 7.17) that as a 2-cell,

$$\mathrm{Int}(\chi(x_{12}, x_{01})) = \mathrm{Int}(\chi(x_{12}, x_{01})) = s_0(x_{12} \otimes x_{01}) = \mathrm{id}(x_{12} \otimes x_{01}) : x_{12} \otimes x_{01} \Longrightarrow x_{12} \otimes x_{01}.$$

Theorem 7.45 gives

$$s_0(\chi(x_{12}, x_{01})) = [\chi(x_{12}, x_{01}), \chi(x_{12}, x_{01}), s_0(x_{12} \otimes x_{01}), s_0(x_{01}] \Longleftrightarrow$$

$$A(x_{12}, x_{01}, s_0(x_0)) \circ (x_{12} \otimes \tilde{s}_0(x_{01}) = \tilde{s}_0(x_{12} \otimes x_{01}) \Longleftrightarrow$$

$$A(x_{12}, x_{01}, I(x_0)) \circ (x_{12} \otimes \rho(x_{01}) = \rho(x_{12} \otimes x_{01}),$$

$$s_1(\chi(x_{12}, x_{01})) = [s_0(x_{12}), \chi(x_{12}, x_{01}), \chi(x_{12}, x_{01}), s_1(x_{01})] \Longleftrightarrow$$

$$A(x_{12}, s_0(x_1), x_{01}) \circ (x_{12} \otimes \tilde{s}_1(x_{01})) = \tilde{s}_0(x_{12}) \otimes x_{01} \Longleftrightarrow$$

$$A(x_{12}, I(x_1), x_{01}) \circ (x_{12} \otimes \lambda(x_{01})) = \rho(x_{12}) \otimes x_{01},$$

and

$$s_2(\chi(x_{12}, x_{01})) = [s_1(x_{12}), s_1(x_{12} \otimes x_{01}), \chi(x_{12}, x_{01}), \chi(x_{12}, x_{01})] \Longleftrightarrow$$

$$A(s_0(x_2), x_{12}, x_{01}) \circ (\tilde{s}_1(x_{12} \otimes x_{01})) = \tilde{s}_1(x_{12}) \otimes x_{01} \Longleftrightarrow$$

$$A(I(x_2), x_{12}, x_{01}) \circ (\lambda(x_{12} \otimes x_{01})) = \lambda(x_{12}) \otimes x_{01},$$

as is evident in the pentagons of Figures 116, 117, and 118.

$$\begin{tikzcd}[column sep=small]
& & x_{12}\otimes x_{01} \arrow[dll, equal, "\mathrm{Int}(\chi(x_{12},x_{01}))=\mathrm{id}"'] \arrow[drr, Rightarrow, "\tilde{s}_0(x_{12}\otimes x_{01})=\rho(x_{12}\otimes x_{01})"] & & \\
x_{12}\otimes x_{01} \arrow[dr, Rightarrow, "x_{12}\otimes\tilde{s}_0(x_{01})=x_{12}\otimes\rho(x_{01})"'] & & & & (x_{12}\otimes x_{01})\otimes s_0(x_0) \arrow[dl, equal, "\mathrm{Int}(\chi(x_{12},x_{01}))\otimes s_0(x_0)=\mathrm{id}\otimes s_0(x_0)=\mathrm{id}"] \\
& x_{12}\otimes(x_{01}\otimes s_0(x_0)) \arrow[rr, Rightarrow, "\mathbf{A}(x_{12},x_{01},s_0(x_0))"'] & & (x_{12}\otimes x_{01})\otimes s_0(x_0) &
\end{tikzcd}$$

Figure 116: $s_0(\chi(x_{12}, x_{01})) = [\chi, \chi, s_0(x_{12} \otimes x_{01}), s_0(x_{01})]$

$$\begin{tikzcd}[column sep=small]
& & x_{12}\otimes x_{01} \arrow[dll, equal, "\mathrm{Int}(\chi(x_{12},x_{01}))=\mathrm{id}"'] \arrow[drr, equal, "\mathrm{Int}(\chi(x_{12},x_{01}))=\mathrm{id}"] & & \\
x_{12}\otimes x_{01} \arrow[dr, Rightarrow, "x_{12}\otimes\tilde{s}_1(x_{01})=x_{12}\otimes\lambda(x_{01})"'] & & & & x_{12}\otimes x_{01} \arrow[dl, Rightarrow, "\tilde{s}_0(x_{12})\otimes x_{01}=\rho(x_{12})\otimes x_{01}"] \\
& x_{12}\otimes(s_0(x_1)\otimes x_{01}) \arrow[rr, Rightarrow, "\mathbf{A}(x_{12},s_0(x_1),x_{01})"'] & & (x_{12}\otimes s_0(x_1))\otimes x_{01} &
\end{tikzcd}$$

Figure 117: $s_1(\chi(x_{12}, x_{01})) = [s_0(x_{12}), \chi, \chi, s_1(x_{01})]$

$$
\begin{tikzcd}[column sep=small]
 & & x_{12}{\otimes}x_{01} \arrow[dll, Rightarrow, "\tilde{s}_1(x_{12}\otimes x_{01})=\lambda(x_{12}\otimes x_{01})"'] \arrow[drr, equal, "\mathrm{Int}(\chi(x_{12},x_{01}))=\mathrm{id}"] & & \\
s_0(x_2){\otimes}(x_{12}{\otimes}x_{01}) \arrow[dr, equal, "s_0(x_2)\otimes\mathrm{Int}(\chi(x_{12},x_{01}))=s_0(x_2)\otimes\mathrm{id}=\mathrm{id}"'] & & & & x_{12}{\otimes}x_{01} \arrow[dl, Rightarrow, "\tilde{s}_1(x_{12})\otimes x_{01}=\lambda(x_{12})\otimes x_{01}"] \\
 & s_0(x_2){\otimes}(x_{12}{\otimes}x_{01}) \arrow[rr, Rightarrow, "\mathbf{A}(s_0(x_2),x_{12},x_{01})"'] & & (s_0(x_2){\otimes}x_{12}){\otimes}x_{01} &
\end{tikzcd}
$$

Figure 118: $s_2(\chi(x_{12}, x_{01})) = [s_1(x_{12}), s_1(x_{12} \otimes x_{01}), \chi, \chi]$

Finally, note that Proposition 7.18 showed that the interior of any invertible 2-simplex in $X_\bullet$ was an isomorphism in $\mathbb{B}(X_\bullet)$ and Theorem 7.45 now shows that if $\tilde{x}_{012}$ is an isomorphism in $\mathbb{B}(X_\bullet)$, then x_{012} was an invertible 2-simplex, since isomorphisms in $\mathbb{B}(X_\bullet)$ all satisfy the horn filling criteria which define invertibility of a 2-simplex in $X_\bullet$. This completes the proof of the converse of Proposition 7.18: A 2-simplex x_{012} in $X_\bullet$ is invertible if, and only if, $\tilde{x}_{012}$ is an isomorphism in $\mathbb{B}(X_\bullet)$. We leave it to the reader to show that a 1-simplex is weakly invertible if, and only if, it is fully faithful as a functorial action and invertible if, and only if, this functorial action is an equivalence.

This completes the proof that the category $\mathbb{B}(X_\bullet)$ with tensor product and structural components as defined above is a bicategory $\mathbf{Bic}(X_\bullet)$ in which 2-simplices are invertible in $X_\bullet$ if, and only if, their interiors are isomorphisms in $\mathbb{B}(X_\bullet)$ and in which 1-simplices are invertible in $X_\bullet$ if, and only if, they are equivalences in $\mathbb{B}(X_\bullet)$.

8. Summary Theorem : The Equivalence of Bicategories and their Postnikov Complex Nerves

Referring to the construction of the nerve of a bicategory as constructed in Section 6, Proposition 7.16 gives a simplicial bijection

$$\mathbf{Ner}(\mathbf{Bic}(X_\bullet))]_0^2 \xrightarrow{\sim} X_\bullet]_0^2$$

(and an equality on the 1-truncation) and Theorem 7.45 gives a bijection

$$\mathbf{Ner}(\mathbf{Bic}(X_\bullet))]_0^3 \xrightarrow{\sim} X_\bullet]_0^3$$

so that since both complexes are 3-coskeletal, we have the

8.1. Theorem. *The above definitions and construction defines a canonical* simplicial *isomorphism*

$$\mathbf{Ner}(\mathbf{Bic}(X_\bullet)) \xrightarrow{\sim} X_\bullet.$$

If one goes in the other direction and takes a bicategory $\mathbb{B}$ and constructs its nerve $\mathbf{Ner}(\mathbb{B})$ as in Section 6, then by definition,

$$\mathbf{Cosk}^3(\mathbf{Ner}(\mathbb{B})) = \mathbf{Ner}(\mathbb{B}) \subseteq \mathbf{Cosk}^2(\mathbf{Ner}(\mathbb{B})),$$

so that the simplicial set $\mathbf{Ner}(\mathbb{B})$ is a 2-dimensional Postnikov complex. Moreover, the set of 2-simplices whose interiors are isomorphisms in $\mathbb{B}$ are all invertible in the sense of Definition 7.7. Similarly, 1-simplices with fully faithful functorial actions as 1-cells in $\mathbb{B}$ are weakly invertible in the sense of Definition 7.8 and 1-simplices which are 1-cell equivalences are invertible in the sense of Definition 7.9. In particular, by definition,

$$\mathrm{Int}(\chi(x_{12}, x_{01})) = \mathrm{id}(x_{12} \otimes x_{01}),$$

$$\mathrm{Int}(s_0(x_{01})) = \rho(x_{01}) : x_{01} \overset{\sim}{\Longrightarrow} x_{01} \otimes I(x_0),$$

and

$$\mathrm{Int}(s_1(x_{01})) = \lambda(x_{01}) : x_{01} \overset{\sim}{\Longrightarrow} I(x_1) \otimes x_{01}$$

are isomorphisms and hence $\chi(x_{12}, x_{01})$ and the degeneracies $s_0(x_{01})$ and $s_1(x_{01})$ are all invertible. Consequently, $\mathbf{Ner}(\mathbb{B})$ meets the "two basic assumptions" of Section 7.10. The pseudo-identity $I(x_0)$ is an equivalence in $\mathbb{B}$ and hence $s_0(x_0) = I(x_0)$ is invertible as a 1-simplex for any $x_0 \in \mathbf{Ner}(\mathbb{B})_0$, the set of 0-cells of $\mathbb{B}$.

8.2. Theorem. $\mathbb{P}(\mathbf{Ner}(\mathbb{B}))$ *is the nerve of a category which, we claim, is always* isomorphic *to the category*

$$\mathbb{B}^2 = \coprod_{(x_0,x_1)\in X_0\times X_0} \mathbb{B}(x_0, x_1)$$

of 2*-cells of* $\mathbb{B}$ *and, in fact,* equal *to* $\mathbb{B}^2$ *if the bicategory* $\mathbb{B}$ *is unitary.*

In effect, Theorem 7.11 shows that $\mathbb{P}(\mathbf{Ner}(\mathbb{B}))$ is a category, since the degenerate 2-simplices are isomorphisms and hence are invertible in $\mathbf{Ner}(\mathbb{B})$.

Now for $\alpha : x_{01} \Longrightarrow y_{01}$ a 2-cell in $\mathbb{B}$, define a 2-simplex $S_0^{\mathbb{B}}(\alpha)$ by

$$S_0^{\mathbb{B}}(\alpha) = (\partial(S_0^{\mathbb{B}}(\alpha)), \mathrm{Int}(S_0^{\mathbb{B}}(\alpha))) =$$

$$((y_{01}, x_{01}, s_0(x_0)), \rho(y_{01}) \circ \alpha : x_{01} \Longrightarrow y_{01} \otimes s_0(x_0)),$$

where $\rho(y_{01}) \circ \alpha$ is composition in $\mathbb{B}$. $S_0^{\mathbb{B}}(\alpha) \in \mathbb{P}(\mathbf{Ner}(\mathbb{B}))_1$ and $S_0^{\mathbb{B}}(\alpha) : x_{01} \Longrightarrow y_{01}$ in the category (whose nerve is) $\mathbb{P}(\mathbf{Ner}(\mathbb{B}))$. Then we have the

$$\begin{tikzcd}
x_{01} \arrow[r, Rightarrow, "\beta"] \arrow[d, Rightarrow, "\gamma"'] & y_{01} \arrow[r, Rightarrow, "\tilde{s}_0(y_{01})"] \arrow[dl, Rightarrow, "\alpha"'] & y_{01} \otimes s_0(x_0) \arrow[d, Rightarrow, "\alpha \otimes s_0(x_0)"] \\
z_{01} \arrow[rr, Rightarrow, "\tilde{s}_0(z_{01})"] \arrow[d, Rightarrow, "\tilde{s}_0(z_{01})"'] & & z_{01} \otimes s_0(x_0) \arrow[d, Rightarrow, "\tilde{s}_0(z_{01}) \otimes s_0(x_0)"] \\
z_{01} \otimes s_0(x_0) \arrow[rr, Rightarrow, "\tilde{s}_0(z_{01} \otimes s_0(x_0))"] \arrow[dr, Rightarrow, "z_{01} \otimes \tilde{s}_0(s_0(x_0))"'] & & (z_{01} \otimes s_0(x_0)) \otimes s_0(x_0) \\
& z_{01} \otimes (s_0(x_0) \otimes s_0(x_0)) \arrow[ur, Rightarrow, "A"'] &
\end{tikzcd}$$

Figure 119: Interior of $[S_0^{\mathbb{B}}(\alpha), S_0^{\mathbb{B}}(\gamma), S_0^{\mathbb{B}}(\beta), s_0^2(x_0)]$

8.3. Lemma. **($S_0^{\mathbb{B}}$ is a Functor.)** *(a) For any 1-cell x_{01} in $\mathbb{B}$, $S_0^{\mathbb{B}}(\mathrm{id}(x_{01})) = s_0(x_{01})$ in* $\mathbf{Ner}(\mathbb{B})$. *(b) For any composable pair (β, α) of 2-cells in $\mathbb{B}$, $\alpha \circ \beta = \gamma$ in $\mathbb{B}$ is equivalent to*

$$[S_0^{\mathbb{B}}(\alpha), S_0^{\mathbb{B}}(\gamma), S_0^{\mathbb{B}}(\beta), s_0^2(x_0)]$$

is commutative in $\mathbf{Ner}(\mathbb{B})$.

(a) is immediate, since $\tilde{S}_0^{\mathbb{B}}(\mathrm{id}(x_{01}) : x_{01} \Longrightarrow x_{01}) = \tilde{s}_0(x_{01}) \circ \mathrm{id}(x_{01})$ and $s_0(x_{01})$ is the identity on the object x_{01} in $\mathbb{P}(\mathbf{Ner}(\mathbb{B}))_1$. (b) is evident from an analysis of the commutativity of the diagram in Figure 119, which is the definition of composition in $\mathbb{P}(\mathbf{Ner}(\mathbb{B}))_2$.

Finally, since $\rho(x_{01}) = \tilde{s}_0(x_{01})$ is an isomorphism in $\mathbb{B}$, $S_0^{\mathbb{B}}$ is a functorial isomorphism and is the identity if ρ is the identity, *i.e.,* if $\mathbb{B}$ is unitary. Thus we have Theorem8.2

The tensor product of 1-cells in the *bicategory* $\mathrm{Bic}(\mathbf{Ner}(\mathbb{B}))$, the nerve of whose underlying category of 2-cells is $\mathbb{P}(\mathbf{Ner}(\mathbb{B}))$ is identical to that in $\mathbb{B}$. That the remainder of the bicategorical structure on $\mathbb{P}(\mathbf{Ner}(\mathbb{B}))$ is also preserved by $S_0^{\mathbb{B}}$ is an immediate consequence of the following

8.4. Lemma. *(a) For any composable pair*

$$(x_2 \xleftarrow{x_{12}} x_1 \xleftarrow{x_{01}} x_0)$$

of 1-cells in $\mathbb{B}$, let $\chi(x_{12}, x_{01})$ be the 2-simplex in $\mathbf{Ner}(\mathbb{B})$ which has boundary

$$\partial(\chi(x_{12}, x_{01})) = (x_{12}, x_{12} \otimes x_{01}, x_{01})$$

and interior

$$\tilde{\chi}(x_{12}, x_{01}) = \mathrm{id}(x_{12} \otimes x_{01}) : x_{12} \otimes x_{01} \Longrightarrow x_{12} \otimes x_{01}$$

and let x_{012} be a 2-simplex in $\mathbf{Ner}(\mathbb{B})$ with boundary

$$\partial(x_{012}) = (x_{12}, x_{02}, x_{01})$$

and interior

$$\tilde{x}_{012} : x_{02} \Longrightarrow x_{12} \otimes x_{01}.$$

Then the tetrahedron

$$[\mathrm{a}] \qquad [\chi(x_{12}, x_{01}), x_{012}, S_0^{\mathbb{B}}(\tilde{x}_{012}), s_0(x_{01})]$$

is commutative in $\mathbb{B}$.

(b) For any composable triplet

$$(x_3 \overset{x_{23}}{\longleftarrow} x_2 \overset{x_{12}}{\longleftarrow} x_2 \overset{x_{01}}{\longleftarrow} x_0)$$

of 1*-cells in* $\mathbb{B}$*, let* $\langle \mathrm{B}(x_{23}, x_{12}, x_{01}) \rangle$ *be the* 2*-simplex in* $\mathbf{Ner}(\mathbb{B})$ *which has boundary*

$$\partial(\langle \mathrm{A}(x_{23}, x_{12}, x_{01}) \rangle) = (x_{23} \otimes x_{12}, x_{23} \otimes (x_{12} \otimes x_{01}), x_{01})$$

and interior

$$\mathrm{Int}(\langle \mathrm{A}(x_{23}, x_{12}, x_{01}) \rangle) = \mathrm{A}(x_{23}, x_{12}, x_{01}),$$

where

$$\mathrm{A}(x_{23}, x_{12}, x_{01}) : x_{23} \otimes (x_{12} \otimes x_{01}) \Longrightarrow (x_{23} \otimes x_{12}) \otimes x_{01}$$

is the 2*-cell associativity isomorphism in* $\mathbb{B}$*. Then the tetrahedron*

$$[\mathrm{b}] \qquad [\chi(x_{23}, x_{12}), \chi(x_{23}, x_{12} \otimes x_{01}), \langle \mathrm{A}(x_{23}, x_{12}, x_{01}) \rangle, \chi(x_{12}, x_{01})]$$

is commutative in $\mathbb{B}$.

(c) Let $x_{01} : x_0 \longrightarrow x_1$*,* $y_{01} : x_0 \longrightarrow x_1$*, and* $x_{12} : x_1 \longrightarrow x_2$ *be* 1*-cells and* $\beta : x_{01} \Longrightarrow y_{01}$ *be a* 2*-cell in* $\mathbb{B}$ *and let* $x_{12} \otimes \beta : x_{12} \otimes x_{01} \Longrightarrow x_{12} \otimes y_{01}$ *be the result of the left tensor action of* 1*-cells on* 2*-cells in* $\mathbb{B}$*. Then the tetrahedron*

$$[\mathrm{c}] \qquad [\chi(x_{12}, y_{01}), \chi(x_{12}, x_{01}), S_0^{\mathbb{B}}(x_{12} \otimes \beta), S_0^{\mathbb{B}}(\beta)]$$

is commutative in $\mathbb{B}$.

(d) Let $x_{01} : x_0 \longrightarrow x_1$*,* $x_{12} : x_1 \longrightarrow x_2$*, and* $y_{12} : x_1 \longrightarrow x_2$ *be* 1*-cells in* $\mathbb{B}$ *and* $\alpha : x_{12} \Longrightarrow y_{12}$ *a* 2*-cell and let* $\alpha \otimes x_{01} : x_{12} \otimes x_{01} \Longrightarrow x_{12} \otimes y_{12}$ *be the result of the right tensor action of* 1*-cells on* 2*-cells in* $\mathbb{B}$*. Let* $\{\alpha \otimes x_{12}\}$ *be the* 2*-simplex in* $\mathbf{Ner}(\mathbb{B})$ *which has boundary*

$$\partial(\{\alpha \otimes x_{12}\}) = (y_{12}, x_{12} \otimes x_{01}, x_{01})$$

and interior

$$\mathrm{Int}(\{\alpha \otimes x_{12}\}) = \alpha \otimes x_{01} : x_{12} \otimes x_{01} \Longrightarrow y_{12} \otimes x_{01}.$$

Then the tetrahedron

$$[\mathrm{d}] \qquad [S_0^{\mathbb{B}}(\alpha), \{\alpha \otimes x_{12}\}, \chi(x_{12}, x_{01}), s_1(x_{01})]$$

is commutative in $\mathbb{B}$.

$$\begin{tikzcd}
x_{02} \arrow[rr, Rightarrow, "\tilde{S}^{\mathbb{B}}_0(\tilde{x}_{012})"] \arrow[d, Rightarrow, "\tilde{x}_{012}"'] & & (x_{12}\otimes x_{01})\otimes s_0(x_0) \arrow[d, Rightarrow, "\mathrm{id}=\mathrm{id}\otimes s_0(x_0)"] \\
x_{12}\otimes x_{01} \arrow[r, Rightarrow, "x_{12}\otimes \tilde{s}_0(x_{01})"'] \arrow[urr, Rightarrow, "\tilde{s}_0(x_{12}\otimes x_{01})"] & x_{12}\otimes(x_{01}\otimes s_0(x_0)) \arrow[r, Rightarrow, "\mathrm{A}"'] & (x_{12}\otimes x_{01})\otimes s_0(x_0)
\end{tikzcd}$$

Figure 120: Interior of $[\chi(x_{12},x_{01}),x_{012},S^{\mathbb{B}}_0(\tilde{x}_{012}),s_0(x_{01})]$

$$\begin{tikzcd}
x_{23}\otimes(x_{12}\otimes x_{01}) \arrow[rr, Rightarrow, "\mathrm{A}"] \arrow[d, Rightarrow, "\mathrm{id}"'] & & (x_{23}\otimes x_{12})\otimes x_{01} \arrow[d, Rightarrow, "\mathrm{id}=\mathrm{id}\otimes x_{01}"] \\
x_{23}\otimes(x_{12}\otimes x_{01}) \arrow[r, Rightarrow, "x_{23}\otimes\mathrm{id}=\mathrm{id}"'] & x_{23}\otimes(x_{12}\otimes x_{01}) \arrow[r, Rightarrow, "\mathrm{A}"] & (x_{23}\otimes x_{12})\otimes x_{01}
\end{tikzcd}$$

Figure 121: Interior of $[\chi(x_{23},x_{12}),\chi(x_{23},x_{12}\otimes x_{01}),\langle \mathrm{A}(x_{23},x_{12},x_{01})\rangle,\chi(x_{12},x_{01})]$

$$\begin{tikzcd}
x_{12}\otimes x_{01} \arrow[rr, Rightarrow, "\tilde{S}^{\mathbb{B}}_0(x_{12}\otimes\beta)"] \arrow[dr, Rightarrow, "x_{12}\otimes\beta"'] \arrow[dd, Rightarrow, "\mathrm{id}"'] & & (x_{12}\otimes y_{01})\otimes s_0(x_0) \arrow[dd, Rightarrow, "\mathrm{id}\otimes s_0(x_0)=\mathrm{id}"] \\
& x_{12}\otimes y_{01}) \arrow[ur, Rightarrow, "\tilde{s}_0(x_{12}\otimes y_{01})"'] \arrow[d, Rightarrow, "x_{12}\otimes\tilde{s}_0(y_{01})"'] \arrow[dr, Rightarrow, "\tilde{s}_0(x_{12}\otimes y_{01}"] & \\
x_{12}\otimes x_{01} \arrow[ur, Rightarrow, "x_{12}\otimes\beta"] \arrow[r, Rightarrow, "x_{12}\otimes\tilde{S}^{\mathbb{B}}_0(\beta)"'] & x_{12}\otimes(y_{01}\otimes s_0(x_0)) \arrow[r, Rightarrow, "\mathrm{A}"'] & (x_{12}\otimes y_{01})\otimes s_0(x_0)
\end{tikzcd}$$

Figure 122: Interior of $[\chi(x_{12},y_{01}),\chi(x_{12},x_{01}),S^{\mathbb{B}}_0(x_{12}\otimes\beta),S^{\mathbb{B}}_0(\beta)]$

$$\begin{tikzcd}
x_{12}\otimes x_{01} \arrow[rr, Rightarrow, "\mathrm{id}"] \arrow[dr, Rightarrow, "\alpha\otimes x_{01}"'] \arrow[dd, Rightarrow, "\alpha\otimes x_{01}"'] & & x_{12}\otimes x_{01} \arrow[dl, Rightarrow, "\alpha\otimes x_{01}"] \arrow[dd, Rightarrow, "\tilde{S}^{\mathbb{B}}_0(\alpha)\otimes x_{01}"] \\
& y_{12}\otimes x_{01} \arrow[d, Rightarrow, "y_{12}\otimes\tilde{s}_1(x_{01})"'] \arrow[dr, Rightarrow, "\tilde{s}_0(y_{12})\otimes x_{01}"] & \\
y_{12}\otimes x_{01} \arrow[ur, Rightarrow, "\mathrm{id}"] \arrow[r, Rightarrow, "y_{12}\otimes\tilde{s}_1(x_{01})"'] & y_{12}\otimes(s_0(x_1)\otimes x_{01}) \arrow[r, Rightarrow, "\mathrm{A}"'] & (y_{12}\otimes s_0(x_0))\otimes x_{01}
\end{tikzcd}$$

Figure 123: Interior of $[S^{\mathbb{B}}_0(\alpha),\{\alpha\otimes x_{12}\},\chi(x_{12},x_{01}),s_1(x_{01})]$

The interior "pentagon" diagrams for the tetrahedra in Lemma 8.4 are the exterior rectangles in Figures 120, 121 , 122 , and 123. They are tautologically commutative given the definitions and the compatibility of $s_0 = \rho$ and $s_1 = \lambda$ with A.

In Lemma 8.4 the commutative tetrahedra involved are the same ones used in the 2-dimensional Postnikov Complex $X_\bullet$ to *define* the 2-cell (in $\mathbb{P}(X_\bullet)$) which is [a] the interior of a 2-simplex, [b] the associativity isomorphism, and [c] the left, and [d] the right tensor actions of 1-cells on 2-cells. Taken by itself, the lemma shows that the mapping $S_0^{\mathbb{B}} : \mathbb{B} \longrightarrow \mathbf{Bic}(\mathbf{Ner}(\mathbb{B}))$, for which the isomorphism of the underlying categories of 2-cells, $S_0^{\mathbb{B}} : \mathbb{B}_1^2 \longrightarrow \mathbf{Bic}(\mathbf{Ner}(\mathbb{B}))_1^2$, is the functorial part ($\mathbf{Ner}(\mathbf{Bic}(\mathbf{Ner}(\mathbb{B}))_1^2) = \mathbb{P}(\mathbf{Ner}(\mathbb{B}))$), does indeed preserve the additional bicategorical tensor product structure as well as the 2-cell categorical structure and we have the

8.5. Theorem. *With $S_0^{\mathbb{B}}$ defined as in Lemma8.3*

$$S_0^{\mathbb{B}} : \mathbb{B} \xrightarrow{\sim} \mathrm{Bic}(\mathbf{Ner}(\mathbb{B}))$$

is a strictly unitary strict (iso) homomorphism of bicategories.

Theorem8.1 together with Theorem8.5 complete our "geometric description" of bicategories and their nerves.

8.6. Theorem. **(Simplicial Characterization of the Nerves of Bicategories and Bigroupoids)** *For a simplicial complex $X_\bullet$,*

- *$X_\bullet$ is the nerve of a bicategory if, and only if, $X_\bullet$ is a 2-dimensional Postnikov complex, i.e.,*

  $$\mathbf{Cosk}^3(X_\bullet) = X_\bullet \subseteq \mathbf{Cosk}^2(X_\bullet),$$

  and the conditions,

  (a) for all 1-simplices $x_{01} \in X_1$, the degenerate 2-simplices $s_0(x_{01})$ and $s_1(x_{01})$ are invertible (Definition 7.7), and

  (b) for all pairs of 1-simplices $(x_{12}, \_, x_{01}) \in \bigwedge_2^1(X_\bullet)$, there exists an invertible 2-simplex $\chi_2^1(x_{12}, x_{01})$ such that $\mathrm{pr}_{\hat{1}}(\chi_2^1(x_{12}, x_{01})) = (x_{12}, \_, x_{01})$, are satisfied.

- *$X_\bullet$ is the nerve of a strict bicategory (2-category) if, and only if, $X_\bullet$ is a 2-dimensional Postnikov complex with an invertible section χ_2^1 which satisfies the two additional conditions*

  (a) for all 1-simplices $x_{01} : x_0 \longrightarrow x_1$,

  $$s_0(x_{01}) = \chi(x_{01}, s_0(x_0)) \text{ and } s_1(x_{01}) = \chi(s_0(x_1), x_{01}), \text{ and}$$

  (b) for all composable triplets of 1-simplices (x_{12}, x_{02}, x_{01}),

  $$[\chi(x_{23}, x_{12}), \chi(x_{23}, x_{12} \otimes x_{01}), \chi(x_{23} \otimes x_{12}, x_{01}), \chi(x_{12}, x_{01})].$$

  (i.e., $(\chi(x_{23}, x_{12}), \chi(x_{23}, x_{12} \otimes x_{01}), \chi(x_{23} \otimes x_{12}, x_{01}), \chi(x_{12}, x_{01}))$ is a 3-simplex of $X_\bullet$.)

- *$X_\bullet$ is the nerve of a bicategory in which every 2-cell is an isomorphism if, and only if, $X_\bullet$ is a weak Kan complex in which the weak Kan conditions are satisfied exactly in all dimensions > 2.*

- *$X_\bullet$ is the nerve of a bigroupoid (i.e., all 2-cells are isomorphisms and all 1-cells are equivalences) if, and only if, $X_\bullet$ is a Kan complex in which the Kan conditions are satisfied exactly in all dimensions > 2 (i.e., $X_\bullet$ is a 2-dimensional hypergroupoid).*

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