ON THE MULTIPLICITY OF A QUASI-HOMOGENEOUS ISOLATED SINGULARITY

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Abstract. We give a formula for the multiplicity of a quasi-homogeneous isolated singularity in terms of its weights.

Let \( f = f(x_1, \ldots, x_n) \in \mathbb{C}\{x_1, \ldots, x_n\} \) be a convergent power series. We call \( f \) an isolated singularity at the origin \( 0 \in \mathbb{C}^n \) if \( f(0) = 0 \) and \( 0 \in \mathbb{C}^n \) is an isolated solution of the system of equations \( \frac{\partial f}{\partial x_1} = \cdots = \frac{\partial f}{\partial x_n} = 0 \). By the multiplicity \( \text{ord} f \) of a series \( f \), we mean the lowest degree of a monomial which appears in \( f \) with nonzero coefficient. Moreover, let us recall that \( f \) is quasi-homogeneous of type \((w_1, \ldots, w_n)\) if it is a polynomial of the form

\[
f = \sum_{\frac{i_1}{w_1} + \cdots + \frac{i_n}{w_n} = 1} c_{i_1 \cdots i_n} x_1^{i_1} \cdots x_n^{i_n}
\]

for some positive rationals \( w_1, \ldots, w_n \).

The quasi-homogeneous isolated singularities have been studied by many authors. Milnor and Orlik ([2], Theorem 1) proved that the Milnor number of a quasi-homogeneous isolated singularity of type \((w_1, \ldots, w_n)\) equals \( \prod_{i=1}^{n} (w_i - 1) \). Thus this product is an integer, even though the \( w_i \)'s themselves may not be integers.

The main result of this note is

**Theorem.** If \( f \) is a quasi-homogeneous isolated singularity of type \((w_1, \ldots, w_n)\) then

\[
\text{ord} f = \min\{m \in \mathbb{N} : m \geq \min\{w_i : i = 1, \ldots, n\}\}.
\]

S. S.-T. Yau proved the above formula for \( n = 3 \) (see [4], Theorem 6). His proof is based on the classification of quasi-homogeneous isolated singularities given in [1] and in [3] and it does not generalize to the case of an arbitrary \( n \).
Proof. Since \( \text{ord} f \) is an integer, it suffices to show that
\[
\min \{ w_i : i = 1, \ldots, n \} \leq \text{ord} f < \min \{ w_i : i = 1, \ldots, n \} + 1.
\]
To check the first inequality, let us note that
\[
\text{ord} f = \min \{ i_1 + \cdots + i_n : c_{i_1 \ldots i_n} \neq 0 \}.
\]
For any \( i_1, \ldots, i_n \) such that \( c_{i_1 \ldots i_n} \neq 0 \), there holds
\[
1 = \frac{i_1}{w_1} + \cdots + \frac{i_n}{w_n} \leq \frac{i_1 + \cdots + i_n}{\min \{ w_i : i = 1, \ldots, n \}},
\]
and the first inequality follows.

In order to prove the inequality \( \text{ord} f < \min \{ w_i : i = 1, \ldots, n \} + 1 \), we need the following observation due to Arnold (see [1]).

Lemma. Fix an \( i \in \{ 1, \ldots, n \} \). For an isolated singularity \( f \), at least one of the monomials of the form \( x_1^a x_j, a \geq 1, j = 1, \ldots, n \) appears in the series \( f \) with a nonzero coefficient.

Proof. We may assume that \( i = 1 \). Let us write
\[
f(x_1, \ldots, x_n) = a_0(x_2, \ldots, x_n) + x_1 a_1(x_2, \ldots, x_n) + \cdots.
\]
There is \( \text{ord} a_0 \geq 2 \) and \( \text{ord} a_1 \geq 1 \) as \( \text{ord} f \geq 2 \). We will show that there exists a \( k \geq 1 \) such that \( \text{ord} a_k = 0 \) or \( \text{ord} a_k = 1 \).

To obtain a contradiction, suppose that \( \text{ord} a_k \geq 2 \) for all \( k \geq 1 \). This gives \( \text{ord} \frac{\partial a_k}{\partial x_j} \geq 1 \) for \( j = 2, \ldots, n \) and hence
\[
a_k(0, \ldots, 0) = 0 \quad \text{and} \quad \frac{\partial a_k}{\partial x_j}(0, \ldots, 0) = 0 \quad \text{for all} \quad k \geq 1 \quad \text{and} \quad j \geq 2,
\]
thus
\[
\frac{\partial f}{\partial x_1}(x_1, 0, \ldots, 0) = a_1(0) + 2 x_1 a_2(0) + \cdots = 0,
\]
\[
\frac{\partial f}{\partial x_j}(x_1, 0, \ldots, 0) = \frac{\partial a_0}{\partial x_1}(0) + x_1 \frac{\partial a_1}{\partial x_j}(0) + \cdots = 0 \quad \text{for} \quad j = 2, \ldots, n \quad \text{in} \quad \mathbb{C}\{x_1\}
\]
and this implies the inclusion \( \{ x_2 = \cdots = x_n = 0 \} \subset \left\{ \frac{\partial f}{\partial x_1} = \cdots = \frac{\partial f}{\partial x_n} = 0 \right\} \).

We get a contradiction because \( 0 \in \mathbb{C}^n \) is an isolated critical point of \( f \).

Now let us suppose that \( w_1 = \min \{ w_i : i = 1, \ldots, n \} \). According to Lemma, at least one of the monomials of the form \( x_1^a x_j, a \geq 1, j = 1, \ldots, n \)
appears in $f$ with nonzero coefficient. Thus $\text{ord} f \leq a + 1$ and for some $j \in \{1, \ldots, n\}$ there is $\frac{a}{w_1} + \frac{1}{w_j} = 1$. This gives
\[
\text{ord} f \leq w_1 \left( 1 - \frac{1}{w_j} \right) + 1 = w_1 + 1 - \frac{w_1}{w_j} < w_1 + 1
\]
and the proof is complete. □

References


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