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A universal bound for surfaces in 3-manifolds with a given Heegaard genus

Mario Eudave-Muñoz
Jeremy Shor

Abstract It is shown that for given positive integers g and b, there is a number $\mathcal{C}(g,b)$, such that any orientable compact irreducible 3-manifold of Heegaard genus g has at most $\mathcal{C}(g,b)$ disjoint, nonparallel incompressible surfaces with first Betti number $b_1 < b$.

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1 Introduction

Let M be a compact, orientable, irreducible 3-manifold, possibly with boundary. As usual, by a surface we mean a connected, compact 2-manifold; if it is contained in a 3-manifold, then it is assumed to be properly embedded. There are several results which bound the number of disjoint incompressible surfaces in M.

The Haken–Kneser finiteness theorem says that given M, there exist an integer $c(M)$, such that any collection of pairwise disjoint, non-parallel, closed, incompressible surfaces in M has at most $c(M)$ components. This can be extended to surfaces with boundary, if the surfaces are taken to be incompressible and ∂-incompressible (for a proof see [3] and [5]).

However, this theorem is false if the surfaces are not ∂-incompressible. Howards [4] has shown that if a 3-manifold M has a boundary component of genus 2 or greater, then there are in M arbitrary large collections of disjoint, non-parallel, incompressible surfaces. This was proved previously by Sherman [11] for $F \times I$, where F is a closed surface of sufficiently high genus. But in these constructions, when there are large numbers of surfaces, the Betti numbers of the surfaces are also large. If we bound the Betti numbers, then the number of surfaces will be bounded by in the following theorem, proved by Freedman and Freedman [2].

Theorem 1 (Freedman and Freedman [2]) *Let M be a compact 3-manifold with boundary and b an integer greater than zero. There is a constant $c(M, b)$ so that if $F_1, \ldots, F_k$, $k > c$ is a collection of disjoint, incompressible surfaces such that all Betti numbers $b_1(F_i) < b$, $1 \leq i \leq k$, and no F_i, $1 \leq i \leq k$, is a boundary parallel annulus or a boundary parallel disk, then at least two members P_i and P_j are parallel.*

However in all of these theorems, the bound depends on a given manifold, i.e., the bound is not universal. We intend to look for some kind of universal bound. For example, if we restrict to the class of closed, irreducible 3-manifolds, then as said above, each manifold has a bounded number of incompressible surfaces, but clearly there is no universal bound, for it is easy to construct closed manifolds with a large number of incompressible surfaces, even with surfaces of a given genus. If we bound the Heegaard genus of the 3-manifolds, then there is no such bound, for in [1] a construction is given, so that for each integer N, we can find a closed, irreducible 3-manifold with Heegaard genus 2 and having more than N disjoint, non-parallel, incompressible surfaces. But in those examples, when the number of surfaces is high, the genus of the surfaces is also high. If we bound also the genus of the surfaces, say, if we restrict to tori, then it follows from [6] and [8], that there is an integer $N(g)$, depending only on g, such that any closed 3-manifold with Heegaard genus g, has at most $N(g)$ disjoint, non-parallel, incompressible tori. All this suggests that a result such as the following could exist. This is an extension of Theorem 1 to all manifolds of a given Heegaard genus.

Theorem 2 (Main theorem) *Consider the class $\mathcal{M}^3(g)$ of all compact, orientable and irreducible 3-manifolds with Heegaard genus g. Let b be an integer greater than zero. Then there is a constant $\mathcal{C}(g, b)$, depending only on g and b, such that for any M in $\mathcal{M}^3(g)$, if $P_1, \ldots, P_k$, $k > \mathcal{C}(g, b)$ is a collection of disjoint, incompressible surfaces in M, such that all Betti numbers $b_1(P_i) < b$, $1 \leq i \leq k$, and no P_i, $1 \leq i \leq k$, is a boundary parallel annulus or a boundary parallel disk, then at least two members P_i and P_j are parallel.*

Here is an outline of the proof. We consider a 3-manifold M with a Heegaard splitting of genus g. If the splitting is not strongly irreducible, we untelescope it [9], decomposing M into a union of submanifolds, glued along incompressible surfaces, and such that each submanifold has a strongly irreducible Heegaard splitting. Given a collection of surfaces in M, these can be isotoped to intersect the surfaces of the decomposition of M essentially. Then we can apply Theorem 1 to each of the pieces, which is a compression body of genus at most g, and by a counting argument, similar to the one in the Haken–Kneser Theorem which counts bad pieces in a tetrahedra, but now counting bad pieces in a compression body, we get the required bound.

For closed manifolds we deduce.

Corollary 3 *Consider the class $\mathcal{M}_c^3(g)$ of all closed, orientable and irreducible 3-manifolds with Heegaard genus g. Let h be an integer greater than zero. Then there is a constant $C(g,h)$ depending only on g and h, such that for any M in $\mathcal{M}_c^3(g)$, if $P_1, \dots, P_k$, $k > C(g,h)$ is a collection of disjoint, incompressible surfaces in M, all of genus less than h, then at least two members P_i and P_j are parallel.*

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2 Proof and estimates

A compression body is a 3-manifold W obtained from a connected closed orientable surface S by attaching 2-handles to $S \times \{0\} \subset S \times I$ and capping off any resulting 2-sphere boundary components. We denote $S \times \{1\}$ by $\partial_+ W$ and $\partial W - \partial_+ W$ by $\partial_- W$. Dually, a compression body is a connected, orientable 3-manifold obtained from a (not necessarily connected) closed orientable surface $\partial_- W \times I$ by attaching 1-handles. Define the index of W by $J(W) = \chi(\partial_- W) - \chi(\partial_+ W) \geq 0$. The genus of a compression body W is defined to be the genus of the surface $\partial_+ W$.

A Heegaard splitting of a 3-manifold M is a decomposition $M = V \cup_S W$, where V, W are compression bodies such that $V \cap W = \partial_+ V = \partial_+ W = S$. We call S the splitting surface or Heegaard surface. We say that the splitting is of genus g if the surface S is of genus g. We say that a 3-manifold has Heegaard genus g if M has a splitting of genus g, and any other splitting has genus $\geq g$.

A Heegaard splitting is reducible if there are essential disks D_1 and D_2, in V and W respectively, such that $\partial D_1 = \partial D_2$. A splitting is irreducible if it is not reducible. A Heegaard splitting is weakly reducible if there are essential disks D_1 and D_2, in V and W respectively, such that $\partial D_1 \cap \partial D_2 = \emptyset$. A Heegaard splitting is strongly irreducible if it is not weakly reducible.

A generalized Heegaard splitting of a compact orientable 3-manifold M is a structure $M = (V_1 \cup_{S_1} W_1) \cup_{F_1} (V_2 \cup_{S_2} W_2) \cup_{F_2} \ldots \cup_{F_{m-1}} (V_m \cup_{S_m} W_m)$. Each of the V_i and W_i is a union of compression bodies, $\partial_+ V_i = S_i = \partial_+ W_i$, (i.e., $V_i \cup_{S_i} W_i$ is a union of Heegaard splittings of a submanifold of M) and $\partial_- W_i = F_i = \partial_- V_{i+1}$. We say that a generalized Heegaard splitting is strongly irreducible if each Heegaard splitting $V_i \cup_{S_i} W_i$ is strongly irreducible and each F_i is incompressible in M. We will denote $\cup_i F_i$ by $\mathcal{F}$ and $\cup_i S_i$ by $\mathcal{S}$. The surfaces in $\mathcal{F}$ are called the thin levels and the surfaces in $\mathcal{S}$ the thick levels.

Theorem A *Let M be a compact irreducible 3-manifold. Suppose the Heegaard genus of M is g. Then M has a strongly irreducible generalized Heegaard splitting $M = (V_1 \cup_{S_1} W_1) \cup_{F_1} (V_2 \cup_{S_2} W_2) \cup_{F_2} \ldots \cup_{F_{m-1}} (V_m \cup_{S_m} W_m)$, such that $\sum J(V_i) = \sum J(W_i) = 2g-2$.*

A proof of this Theorem can be found in [9] and [8]. The following Theorem is proved in [10]. A different proof is given in [7].

Theorem B *Let P be a properly embedded, not necessarily connected, incompressible surface in an irreducible 3-manifold M, and let $M = (V_1 \cup_{S_1} W_1) \cup_{F_1} (V_2 \cup_{S_2} W_2) \cup_{F_2} \ldots \cup_{F_{m-1}} (V_m \cup_{S_m} W_m)$ be a strongly irreducible generalized Heegaard splitting of M. Then $\mathcal{F} \cup \mathcal{S}$ can be isotoped to intersect P only in curves that are essential in both P and $\mathcal{F} \cup \mathcal{S}$.*

Proof of Theorem 2

Let M be an orientable, irreducible, compact 3-manifold of Heegaard genus g, that is, there are compression bodies V and W in M, with $S = \partial_+ V = \partial_+ W = V \cap W$, $M = V \cup_S W$, and genus$(S) = g$.

By Theorem A, M possesses a strongly irreducible Heegaard splitting $M = (V_1 \cup_{S_1} W_1) \cup_{F_1} (V_2 \cup_{S_2} W_2) \cup_{F_2} \ldots \cup_{F_{m-1}} (V_m \cup_{S_m} W_m)$, such that $\sum J(V_i) = \sum J(W_i) = 2g-2$. For a fixed i, $V_i \cup_{S_i} W_i$ is a union of strongly irreducible Heegaard splittings, only one component of it is not a product, this is called the active component. If V_{ij} is a component of V_i, then $J(V_{ij}) = 0$, only if V_{ij} is a product, and $J(V_{ij}) \geq 2$ otherwise. It follows that $m \leq g-1$, and that there are at most $g-1$ active components.

Let $P_1, P_2, \ldots, P_n$ be a collection of disjoint, properly embedded, incompressible surfaces in M. Suppose that $b_1(P_i) < b$, for all i. By Theorem B, $\mathcal{F} \cup S$ can be isotoped to intersect $P_1, P_2, \ldots, P_n$ only in curves which are essential in both $P_1, P_2, \ldots, P_n$ and $\mathcal{F} \cup S$. If some component of $V_i \cup_{S_i} W_i$ is a product, say, it is of the form $S \times [-1, 1]$, then by taking the product sufficiently small, we can suppose that $(S \times [-1,1]) \cap P_j = ((S \times \{0\}) \cap P_j) \times [-1,1]$ for any surface P_j. So we can disregard the product $S \times [-1,1]$, and consider only the surface $S \times \{1\}$. Doing this with all product components, we can assume, without loss of generality, that the decomposition of M consist only of the active components. So M is decomposed into at most $2g-2$ compression bodies.

Consider $P_j \cap V_i$; this is a collection of surfaces. Each of the components has $b_1 < b$, for P_j intersect $\mathcal{S}$ and $\mathcal{F}$ in simple closed curves which are essential in P_j. We can assume that no component of $P_j \cap V_i$ is a boundary parallel annulus in V_i, for if there is one, then by an isotopy of P_i, it can be pushed into another compression body, reducing the number of intersections between P_j and $\mathcal{F} \cup \mathcal{S}$.

For each V_i (and W_i), by Theorem 1, there is a constant $c(V_i, b)$ depending only on V_i and b, such that any collection of incompressible surfaces in V_i, with $b_1 < b$, no two of them parallel, has at most $c(V_i, b)$ components.

Let W be a compression body. Let $F_1, \ldots, F_k$ a collection of incompressible surfaces in W. A component of $W - (F_1 \cup \ldots \cup F_k)$ is good, if it is of the form $F_i \times I$, $I = [0,1]$, where $F \times \{0\}$ and $F \times \{1\}$ are surfaces in the collection, and $F \times \partial I \subset \partial W$. A component of $W - (F_1 \cup \ldots \cup F_k)$ is bad, if it is not good. If all the $F_i' s$ satisfy that $b_1(F_i) < b$, then it follows from Theorem 1, that for any such collection, $W - (F_1 \cup \ldots \cup F_k)$ has at most $c(W, b) + 1$ bad components.

Let $C = max\{c(V, b); V$ a compression body of genus $\leq g\}$. And let

$$\mathcal{C}(g, b) = 4g + (2g-2)(C+1)$$

Note that $\mathcal{C}(g, b)$ depends only on g and b. For the collection of surfaces $P_1, \ldots, P_n$, suppose further that $n > \mathcal{C}(g, b)$. As M has Heegaard genus g, then $b_1(M) \leq 2g$. This implies that $M - (P_1 \cup P_2 \cup \ldots \cup P_n)$ has at least $\mathcal{C}(g, b) - 2g + 1$ components. In each V_i and W_i there are at most $C+1$ bad pieces, so there are in total at most $(2g-2)(C+1)$ bad pieces. Then at least $2g+1$ components of $M - (P_1 \cup P_2 \cup \ldots \cup P_n)$ are made of good pieces. A component made of good pieces is a I-bundle over a surface, but again $b_1(M, Z_2) \leq 2g$, so at most $2g$ of them can be I-bundles over a nonorientable surface. Then at least a component of $M - (P_1 \cup P_2 \cup \ldots \cup P_n)$ is a product, which implies that some P_i and P_j are parallel. This completes the proof of Theorem 2. □

Estimates

A rough estimate for $\mathcal{C}(g, b)$ can be obtained from [2]. From that paper it follows that if V is a compression body of genus g, which is not a product, then a bound $c(V, b)$ is given by

$$c(V, b) = \frac{14}{3} c_0 3^{2b-2} + 5|\chi(V)| + T + b_1(V, Z_2)$$

where c_0 is the constant given by the Kneser-Haken Theorem, and T is the number of tori boundary components. An incompressible and ∂-incompressible surface in a compression body V is either an essential disk, or an annulus with one boundary component on $\partial_+ V$ and the other in $\partial_- V$. It is not difficult to see that if V has genus g then $c_0(v) = 3g - 3$. Also $T \leq g$, $|\chi(V)| \leq 2g - 3$, and $b_1(V, Z_2) \leq 2g$.

It follows that

$$c(V, b) \leq 14(g-1)3^{2b-2} + 5(2g-3) + g + 2g = 14(g-1)3^{2b-2} + 13g - 15.$$

Define $C = 14(g-1)3^{2b-2} + 13g - 15$. So $c(W, b) \leq C$ for any compression body W of genus $\leq g$. Then C can be taken to be the constant defined in the proof of Theorem 2. From this follows that the constant $\mathcal{C}(g, b)$ can be expressed as

$$\mathcal{C}(g, b) = p_1(g) + p_2(g)3^{2b-2}$$

where $p_1(g)$ and $p_2(g)$ are quadratic polynomials in g.

Remarks Theorem 2 can be extended to arbitrary orientable compact 3-manifolds. This follows from the existence of a decomposition on prime 3-manifolds, the additivity of the Heegaard genus, and from the fact that given any collection of incompressible surfaces in a 3-manifold M, then there is a complete collection of decomposing spheres in M which are disjoint from the given surfaces.

An extension of Theorem 2 for nonorientable manifolds would follow from a version of Theorems A and B for nonorientable 3-manifolds.

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Instituto de Matemáticas, UNAM, Circuito Exterior, Ciudad Universitaria
04510 México D.F., MEXICO

and

250 E. Houston St. no. 3J, New York City, NY 10002, U.S.A.

Email: `mario@matem.unam.mx`

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