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Genus two 3–manifolds are built from handle number one pieces

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Abstract Let M be a closed, irreducible, genus two 3–manifold, and $\mathcal{F}$ a maximal collection of pairwise disjoint, closed, orientable, incompressible surfaces embedded in M. Then each component manifold M_i of $M - \mathcal{F}$ has handle number at most one, i.e. admits a Heegaard splitting obtained by attaching a single 1–handle to one or two components of ∂M_i. This result also holds for a decomposition of M along a maximal collection of incompressible tori.

AMS Classification 57M99

Keywords 3–manifold, Heegaard splitting, incompressible surface

1 Introduction

Throughout this paper, all surfaces and 3–manifolds will be taken to be compact and orientable. Suppose a 3-manifold M contains an essential 2–sphere. The Haken lemma [4] tells us that each Heegaard surface for M intersects some essential 2–sphere in a single essential circle (see also [5]). As a consequence of this and the uniqueness of prime decompositions of 3-manifolds, Heegaard genus is additive under connected sum,

$$g(M_1 \# \cdots \# M_n) = g(M_1) + \cdots + g(M_n),$$

where $g(M)$ denotes the Heegaard genus of the manifold M.

How does Heegaard genus behave under decompositions of an irreducible manifold along incompressible surfaces? Clearly, we do not expect additivity of genus as before. Suppose that M contains an embedded, incompressible surface F that separates M into two components M_1 and M_2. The genus of the two component manifolds must be greater than the genus of their boundary component, $g(M_i) > g(F), i = 1, 2$. This is particularly relevant in light of the examples of Eudave-Muñoz [2], tunnel number one knots whose exteriors contain incompressible surfaces of arbitrarily high genus. (An appropriate Dehn surgery on such a knot results in a closed genus two manifold with an arbitrarily high genus incompressible surface).

However, we can build a Heegaard splitting for M from Heegaard splittings of the components M_1 and M_2. If done in an efficient manner, see for example [12], this yields an upper bound on the genus of M,

$$g(M) \leq g(M_1) + g(M_2) - g(F).$$

Upper bounds on the genus of the component manifolds (lower bounds on $g(M)$) are more difficult, and not even possible without additional assumptions. Consider the examples of Kobayashi [8], knots whose tunnel numbers degenerate arbitrarily under connected sum (decomposition along an annulus). Again, an appropriate Dehn surgery will yield a closed manifold containing an incompressible torus, and after cutting along the torus, the component manifolds have genus arbitrarily higher than that of the closed manifold. In contrast, Schultens [13] has demonstrated that for tunnel numbers, this phenomenon cannot occur in the absence of additional incompressible surfaces. We are led to adding the assumption that the closed manifold should be cut along a maximal embedded collection of incompressible surfaces (a slightly weaker assumption will suffice, see the definition of a complete collection of surfaces in the next section).

While it is true that the spine of a Heegaard splitting for M induces Heegaard splittings of the component manifolds, see Figure 3 and Section 4, the intersection between the Heegaard spine and incompressible surfaces could potentially be very complicated, and almost certainly depends on the genus of the incompressible surfaces. One approach to constructing upper bounds of the genus of the component manifolds is to bound the complexity of this intersection in terms of the genera of the incompressible surfaces and the Heegaard spine. This is the approach used by Johannson in [6].

In this paper, we adopt a different approach. Using ideas of Scharlemann and Thompson [11], we arrange the spine of the Heegaard splitting to intersect the collection of surfaces minimally. It is not hard to see that the induced Heegaard splitting of the component manifolds is weakly reducible. We then prove a generalization of a result of Casson and Gordon [1] to manifolds with boundary. (A similar theorem was proven by Lustig and Moriah [9].) A somewhat simplified version follows:

1.1 Theorem *Let M be an irreducible 3–manifold and $M = C_1 \cup_H C_2$ a weakly reducible Heegaard splitting of M. Then either M contains a closed, non-peripheral incompressible surface, or the splitting is not of minimal genus.*

This theorem allows us to make use of the assumption that we have taken the collection of surfaces to be complete. Additional refinements to this result show that for many of the component manifolds, the induced Heegaard splitting can be compressed to one that is induced by a single arc attached to the boundary.

1.2 Theorem *Let M be a closed, irreducible 3–manifold and $\mathcal{F}$ a complete collection of surfaces for M. If $\overline{M - N(\mathcal{F})}$ has n component manifolds, then at least $n + 2 - g(M)$ of these components have handle number at most 1.*

We give definitions in the next section, but, note here that a complete collection of surfaces applies both to maximal collections of incompressible surfaces and maximal collections of incompressible tori. Handle number one means that the component manifold has a Heegaard splitting that is induced by drilling out a single arc, this is a generalization of tunnel number one and the concepts are identical when the manifold has a single boundary component. While it is possible that a component manifold has handle number 0, this will occur only when M fibers over the circle, or unnecessary parallel copies of some surface occur in the collection. Handle number 0 implies that the component is a compression body, in fact a product, since its boundary is incompressible. This component is either bounded by disjoint parallel copies of a surface, or there is a single surface cutting M into a product, i.e., M fibers over the circle.

Unfortunately, we are unable to draw conclusions about every component manifold unless the genus of Γ, hence $g(M)$, is 2.

1.3 Corollary *If M is a closed, irreducible genus two 3–manifold and $\mathcal{F}$ is a complete collection of surfaces, then every component manifold of $\overline{M - N(\mathcal{F})}$ has handle number at most 1.*

Although the component manifolds have high genus Heegaard splittings, the fact that they are handle number one means that this is due precisely to the fact that these manifolds have boundary with high genus, and are otherwise very simple in terms of Heegaard structure. The genus of a handle number one component manifold is bounded above by

$$g(M_i) \leq g(\partial M_i) + 1,$$

where $g(\partial M_i)$ is the sum of the genera of the components of ∂M_i. In some cases, this allows one to precisely compute the genus of the component manifolds. For example, when $g(M) = 2$ and the complete collection $\mathcal{F}$ consists of a single separating surface F, we obtain the equality $g(M_i) = g(F) + 1, i = 1, 2$. By contrast, in this case Johannson obtains a bound of $g(M_1) + g(M_2) \leq 2g(F) + 10$.

Or, if $\mathcal{F}$ is a maximal embedded collection of tori, the single handle is either attached to one boundary component or is an arc joining two distinct torus boundary components. As a corollary, we obtain:

1.4 Corollary *If M is genus 2, and $\mathcal{F}$ is a maximal collection of tori then every component of $\overline{M - N(\mathcal{F})}$ has genus 2.*

Kobayashi [7] has proven a much stronger result regarding torus decompositions of genus 2 manifolds.

Most of the techniques presented here apply without assumption on the genus of M. The exception is Proposition 6.3 whose hypothesis on handle number will not be consistently met when the genus of M is greater than two. If the hypothesis on handle number can be removed, then a general upper bound on the handle numbers of the component manifolds is obtained. (It is likely that one must adopt the assumption that the collection $\mathcal{F}$ is in fact maximal). This would yield:

1.5 Conjecture *Let M be a closed, irreducible 3–manifold and $\mathcal{F}$ a maximal embedded collection of orientable, incompressible surfaces. If $M - N(\mathcal{F})$ has n components then*

$$\sum_{1}^{n} h(M_i) \leq g(M) + n - 2,$$

where $h(M_i)$ denotes the handle number of the component manifold M_i.

2 Preliminaries

We give brief definitions of concepts related to Heegaard splittings, the reader is referred to [10] for a more thorough treatment. Let S be a closed surface, $I = [-1, 1]$. A *compression body* C is a 3-manifold obtained by attaching 2–handles and 3–handles to $S \times I$, where no attachment is performed along $S \times \{1\}$. The boundary of a compression body is then viewed as having two parts, $\partial_+ C$ and $\partial_- C$, where $\partial_+ C = S \times \{1\}$ and $\partial_- C = \partial C - \partial_+ C$. Alternatively, we may construct a compression body C by attaching 1–handles to $S \times I$ where all attachments are performed along $S \times \{1\}$. In this case, $\partial_- C = S \times \{-1\}$, $\partial_+ C = \partial C - \partial_- C$, and the 1–handles are dual to the 2–handles of the former construction. In either construction, we adopt the convention that every 2–sphere boundary component of $\partial_- C$ is capped off with a ball. If $\partial_- C = \emptyset$

then C is called a *handlebody*. Note that a handlebody can also be defined as a connected manifold with boundary that possesses a *complete collection of compressing disks*, a properly embedded collection of disks (the cores of the 2–handles) which cut the handlebody into a disjoint union of balls.

A *Heegaard splitting* is a decomposition of a (closed or bounded) 3–manifold, $M' = C_1 \cup_S C_2$, where C_1 and C_2 are compression bodies with their positive boundaries identified, $S = \partial_+ C_1 = \partial_+ C_2$. In this case S will be a closed surface embedded in M' and will be called a *Heegaard surface* for M'. The genus of M' is

$$g(M') = \min\{g(S) | S \text{ is a Heegaard surface for } M'\}.$$

A Heegaard splitting will be called *weakly reducible* if there are non-empty properly embedded collections of compressing disks $\Delta_1 \subset C_1$ and $\Delta_2 \subset C_2$ so that $\partial D_1 \cap \partial \Delta_2 = \emptyset$ in the Heegaard surface S. If it exists, the collection $\Delta_0 = \Delta_1 \cup \Delta_2$ is called a *weak reducing system* for the Heegaard splitting.

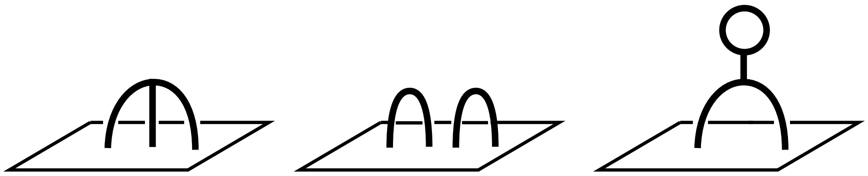

Figure 1: Graphs with handle number 2.

If Γ is a graph then we will refer to the vertices of valence 1, as the boundary of Γ, $\partial\Gamma$. A graph $\Gamma \subset M'$ will be said to be *properly embedded* if it is embedded in M and $\Gamma \cap \partial M' = \partial\Gamma$. For a properly embedded graph $\Gamma \subset M'$, we will define the genus of Γ to be

$$g(\Gamma) = rank\ H_1(\Gamma),$$

and define the *handle number* of Γ to be

$$h(\Gamma) = rank\ H_1(\Gamma, \partial\Gamma).$$

Equivalently, the handle number is the number of edges that need to be removed from Γ so that the resulting graph is empty or a collection of trees each attached to a boundary component of M' by a single vertex; or, $h(\Gamma) = -\chi(\Gamma) + |\partial\Gamma| = g(\Gamma) + |\partial\Gamma| - |\Gamma|$. Some handle number two graphs are pictured in Figure 1.

Typically we will keep track of a Heegaard splitting via a properly embedded graph in the manifold. A Heegaard splitting of a closed manifold M will necessarily consist of two handlebodies, and in this case, each of the handlebodies is isotopic to a regular neighborhood of a (non-unique) graph embedded in the handlebody, hence the manifold. Any such graph Γ, for either handlebody, will be called a *spine* of the Heegaard splitting. For bounded manifolds, Heegaard splittings come in two varieties, depending on whether or not one of the compression bodies is actually a handlebody. Correspondingly, there are two ways that a properly embedded graph can represent a Heegaard splitting of a bounded manifold. A *tunnel system* for M' is a properly embedded graph Γ so that $\overline{M' - N(\Gamma)}$ is a handlebody. The *tunnel number* of M' is

$$t(M') = \min\{h(\Gamma)|\Gamma \text{ is a tunnel system for } M'\}.$$

A *handle system* for M' is a properly embedded graph Γ so that $\overline{M' - N(\Gamma)}$ is a compression body C and $\partial_- C \subset \partial M'$. The *handle number* of M' is

$$h(M') = \min\{h(\Gamma)|\Gamma \text{ is a handle system for } M'\}.$$

In either case, if $\partial_1 M'$ denotes the boundary components of M' to which Γ is attached, then $\partial N(\Gamma \cup \partial_1 M')$ is a Heegaard surface for M'.

Whenever a Heegaard splitting is represented by an embedded graph, whether a spine, tunnel system, or handle system, then we may perform slides of edges of the graph along other edges of the graph without changing the isotopy class of the Heegaard surface, see Figure 2. Such moves are called *edge slides* or *handle slides.* When working with tunnel or handle system, we may also slide handles along the boundary of the manifold without changing the Heegaard splitting.

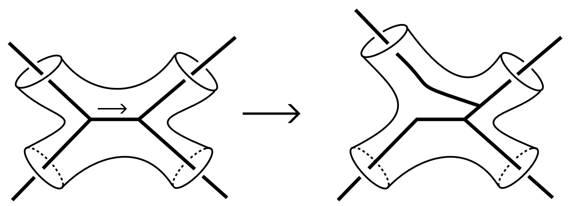

Figure 2: Edge slides do not change the Heegaard splitting.

In the case of a tunnel or handle system, Γ will be slide-equivalent to a collection of $h(\Gamma)$ properly embedded arcs in M'. So $t(M')$ and $h(M')$ should be thought of as the minimal number of arcs that need to be drilled out of M' so that the resulting manifold is a handlebody or compression body, respectively. The handle number is a strict generalization of the tunnel number and we have $h(M') \leq t(M')$. In general these quantities are different. For example the exterior of the Hopf link in S^3 is tunnel number one but handle number 0.

A bounded manifold M' will be said to be *indecomposable* if it contains no closed, orientable, non-peripheral incompressible surface whose genus is either less than or equal to the genus of a single boundary component of M' or strictly less than the sum of the genera of two distinct boundary components of M'. Let $\mathcal{F} = F_1 \cup F_2 \cup \ldots, F_k \subset M$ be an embedded collection of closed orientable incompressible surfaces. A *component manifold* is a component of the manifold $\overline{M - N(\mathcal{F})}$. If $\mathcal{F}$ is an embedded collection of closed, orientable, incompressible surfaces and each of the component manifolds is indecomposable, then we say that $\mathcal{F}$ is a *complete collection of surfaces.*

Clearly a maximal embedded collection of orientable, incompressible surfaces is complete. However, this is not required for the collection be complete. For example, a maximal embedded collection of tori in an irreducible manifold is complete as each of the component manifolds is indecomposable (any additional surface would have to be genus 1).

3 Proof of the Main Theorem

In this section we will give an outline of the proof of the main theorem, Theorem 1.2. The proofs of several important lemmas will be deferred to later sections. Throughout, M will denote a closed, orientable, irreducible 3–manifold, Γ will be the spine of an irreducible Heegaard splitting of M, and $\mathcal{F}$ will be a complete collection of incompressible surfaces.

Arranging Γ to intersect the decomposition minimally Embed in M two parallel copies of each of the incompressible surfaces in $\mathcal{F}$ and denote this collection by $2\ \mathcal{F}$. If there are k components of $\mathcal{F}$, $2\ \mathcal{F}$ decomposes M into $n+k$ pieces, k *product manifolds* $F_j \times I, j = 1..k$ and n *component manifolds* denoted $M_i, i = 1..n, n < k$, identical to those obtained by cutting along $\mathcal{F}$. See Figure 3.

Suppose that Γ is in general position with respect to $2\ \mathcal{F}$ and that we have chosen Δ to be a complete collection of compressing disks for the complementary handlebody $\overline{M - N(\Gamma)}$. The *complexity of* (Γ, Δ) is an ordered triple $(\cdot,\cdot,\cdot)$ of the following quantities:

(1) $\sum h(\Gamma \cap M_i)$ = the sum of handle numbers of the intersection of the spine Γ with each of the component manifolds M_i,

(2) $\sum h(\Gamma \cap (F_j \times I))$ = the sum of the handle numbers of the intersection of the spine Γ with each of the product manifolds $F_j \times I$,

(3) $|\Delta \cap 2\ \mathcal{F}|$ = the number of components in the intersection of the disk collection Δ and the surfaces $2\ \mathcal{F}$.

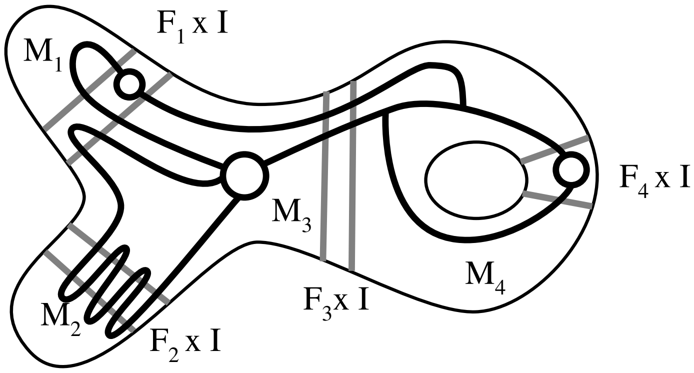

Figure 3: Letting Γ intersect the decomposition minimally.

Isotoping or manipulating Γ by edge–slides does not change the isotopy class of the Heegaard surface $\partial N(\Gamma)$, and we therefore consider such a spine to be *equivalent* to Γ. With no loss of generality, we will assume that a spine equivalent to Γ and a complete collection of compressing disks Δ have been chosen to minimize complexity with respect to lexicographic ordering. Specific properties of the intersection $(\Gamma \cup \Delta) \cap 2\,\mathcal{F}$ will be developed in Section 4; and are based on the arguments of Scharlemann and Thompson [11]. In particular we will prove:

3.1 Theorem *$\Gamma \cap M'$ is a tunnel system for each product or component manifold M'.*

Proof deferred to Section 4.

Ordering subdisks of $\Delta - 2\,\mathcal{F}$ By Lemma 4.3 we know that $\Delta - 2\,\mathcal{F}$ is a collection of disks. We will (non-uniquely) label these disks $d_1, \ldots, d_m$ according to the following rules:

(1) Label an outermost disk d_1,

(2) Assuming that the disks $d_1, \ldots, d_{l-1}$ have been labeled, give the label d_l to a subdisk of $\Delta - 2\,\mathcal{F}$ that is outermost *relative to* the subdisks $d_1, \ldots, d_{l-1}$. See Figure 4.

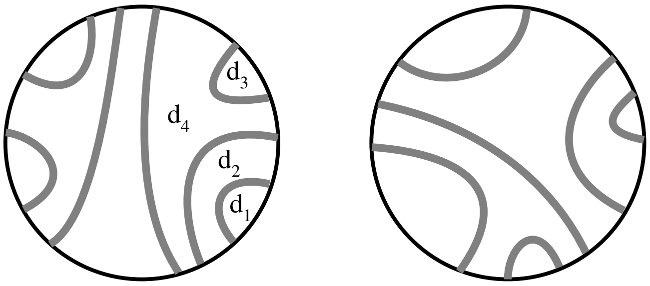

Figure 4: Labeling subdisks of $\Delta - 2\,\mathcal{F}$.

Note that each of the subdisks d_l is embedded in some component or product manifold M'. Moreover, it is a compressing disk for the handlebody that is the complement of the tunnel system induced by Γ, $\overline{M' - N(\Gamma)}$.

Let $\{d_{i_j}\}$ be a subcollection of the disks $\Delta - 2\,\mathcal{F}$. The *support of* $\{d_{i_j}\}$, denoted $supp(\{d_{i_j}\})$, is the sub–graph of Γ that is the spine of the handlebody obtained by maximally compressing $N(\Gamma)$ along compressing disks which are disjoint from $2\,\mathcal{F}$ and disjoint from the boundary of $\{d_{i_j}\} \subset \partial N(\Gamma)$ and throwing away any components which do not meet $\{d_{i_j}\}$. For each component manifold M_i let j be the least j so that $d_j \subset M_i$. The disk $D_i = d_j$ will be called *the relatively outermost disk* for M_i. The graph

$$\Omega_i = supp(D_i)$$

will be called *the relatively outermost graph* for M_j.

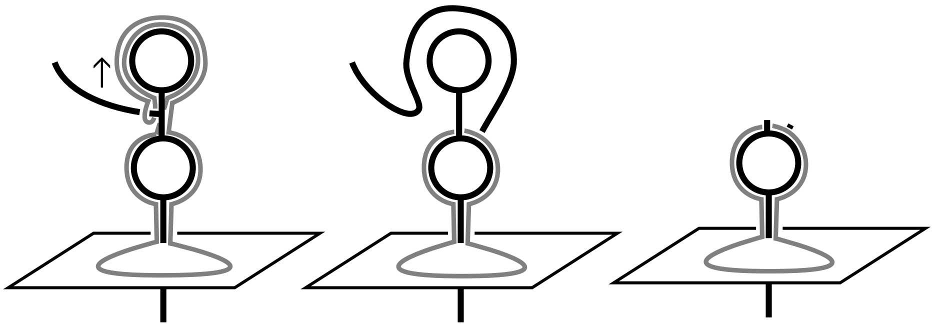

Figure 5: The support of a disk.

Remarks

(1) In the definition of support, it may be necessary to perform handle slides of Γ in the interior of some component manifolds in order to realize the maximal collection of compressing disks, see Figure 5.

(2) We have chosen a fixed numbering of the subdisks of $\Delta - 2\ \mathcal{F}$. Thus, the notions of the relatively outermost disk and the relatively outermost graph for a component manifold are well defined.

(3) We will consider the support of a relatively outermost disk $\Omega_i = supp(D_i)$ to be a graph that is properly embedded in the component manifold M_i. We will consider the support of a collection of subdisks $supp(\{d_{i_j}\})$ to be graph that is embedded in M.

We can reconstruct the spine Γ by building a sequence of graphs, each the support of a larger collection of ordered subdisks of $\Delta - 2\ \mathcal{F}$,

$$\Gamma_k = supp(\{d_i\}_{i=1..k}).$$

In particular,

$$\Gamma_m = \Gamma.$$

The relatively outermost graphs for each component manifold, $\Omega_1, \ldots, \Omega_n$, will be attached at some point in building Γ. Moreover, they are the support of the relatively outermost disks $D_1, \ldots, D_n$, and as we will see they are attached to the previous graph along all but at most one of their endpoints. This gives us a lower bound for the genus of Γ in terms of the handle number of the outermost graphs Ω_i.

3.2 Lemma *Let Γ be the spine of an irreducible Heegaard splitting. Then*

$$g(\Gamma) \geq \sum_1^n h(\Omega_i) - n + 2.$$

Proof deferred until Section 5.

However, it is our aim to develop a lower bound for the genus of Γ in terms of the handle numbers of the component manifolds, not just the handle numbers of the relatively outermost graphs Ω_i. In a special case ($h(\Omega_i) = 1$) we will show that Ω_i is in fact a handle system and obtain the desired bound.

3.3 Proposition *If $h(\Omega_i) = 1$ then Ω_i is a handle system for M_i. In particular, $h(M_i) \leq 1$.*

Proof deferred to Section 6.

Remark The restriction $h(\Omega_i) = 1$ in this proposition is what prevents us from making a more general statement connecting genus to the sum of handle numbers of the component manifolds. If Ω_i were always a handle system for the component manifold M_i then we would obtain the more general inequality $g(\Gamma) \geq \sum h(M_i) - n + 2$.

These lemma and proposition prove the main theorem. Let $j \leq n$ be the number of components M_i which have $h(M_i) > 1$. By Proposition 3.3, each of the corresponding outer handle systems Ω_i has $h(\Omega_i) > 1$. By Lemma 5.1 we have

$$g(\Gamma) - 2 \geq \sum h(\Omega_i) - n \geq n + j - n \geq j.$$

Therefore, the number of component manifolds with handle number one is at least

$$n - (g(\Gamma) - 2) = n + 2 - g(\Gamma). \qquad \square$$

4 Properties of the Minimal Intersection between $2\mathcal{F}$ and Γ

This section is devoted to developing the properties of the minimal intersection between the Heegaard complex $\Gamma \cup \Delta$ and the incompressible surfaces $2\mathcal{F}$. Many of these properties were either specified in the work of Scharlemann and Thompson [11], or follow from the same methods. They are included here, both for the sake of completeness, and due to the fact that the definition of minimality used here differs from that in [11]. We also apply these properties to characterize the support of outermost and relatively outermost disks.

Throughout this section, we assume that the spine Γ of the irreducible Heegaard splitting and compressing disks Δ for its complement have been chosen to intersect the surfaces $2\mathcal{F}$ minimally, as defined in the previous section. However, it is not necessary to place any restrictions on the surface collection $\mathcal{F}$.

First we will demonstrate that Γ induces Heegaard splittings of each of the component and product manifolds.

4.1 Lemma *Let F be a component of $2\mathcal{F}$. Then the punctured surface $F' = \overline{F - N(\Gamma)}$ is incompressible in the handlebody $\overline{M - N(\Gamma)}$.*

Proof If some component of the punctured surface were compressible, then there would be a compressing disk D for F', a perhaps distinct component of the punctured surface, embedded in some component or product manifold M'. The boundary of D bounds a disk D' in F. As M is irreducible, D and D' cobound a ball B, and B must be contained in M', for otherwise the incompressible surface F would lie in the ball B. We can therefore isotope F through B, thereby pushing a portion of $\Gamma \cap M'$ into an adjacent product or component manifold. See Figure 6. Since Γ is the spine of an irreducible splitting, by a theorem of Frohman [3], $B \cap \Gamma$ does not contain any loops of Γ, it is merely a collection of trees. As there is no loop of Γ in the ball, this move does not raise the induced handle number of the adjacent manifold, while it does reduce the handle number of $\Gamma \cap M'$. This contradicts the minimality of the intersection. □

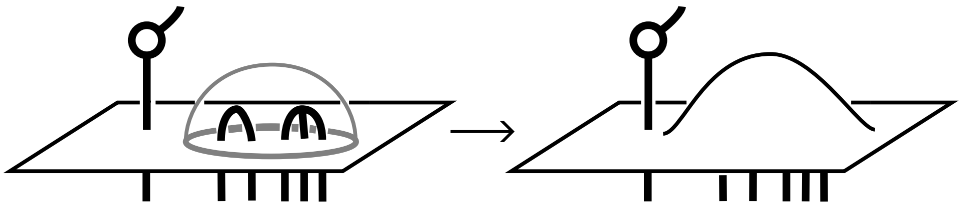

Figure 6: If F' is compressible the intersection is not minimal.

4.2 Theorem *$\Gamma \cap M'$ is a tunnel system for each product or component manifold M'.*

Proof The manifold $\overline{M' - N(\Gamma)}$ is a component of the handlebody $\overline{M - N(\Gamma)}$ after it is cut along the properly embedded collection of punctured incompressible surfaces $2\,\mathcal{F}' = \overline{2\,\mathcal{F} - N(\Gamma)}$. It is well known that when a handlebody is cut along a collection of incompressible surfaces, the resulting pieces are handlebodies. So $\overline{M' - N(\Gamma)}$ is a handlebody and $\Gamma \cap M'$ is the corresponding tunnel system. □

The intersection of the 2-complex $\Gamma \cup \Delta$ with the incompressible surfaces $2\,\mathcal{F}$ is a graph $G \subset 2\,\mathcal{F}$. See Figure 7. A component of intersection with the spine, $\Gamma \cap 2\,\mathcal{F}$ is called a *vertex*. Since handlebodies do not contain closed incompressible surfaces, there is at least one vertex in each component of $2\,\mathcal{F}$. A component of the intersection with the compressing disks, $\Delta \cap 2\,\mathcal{F}$ is called a *circle* if it is an intersection with the interior of Δ and an *edge* otherwise. An edge joining distinct vertices will be called an *arc* and an edge joining a vertex v to itself is called a *loop based at* v.

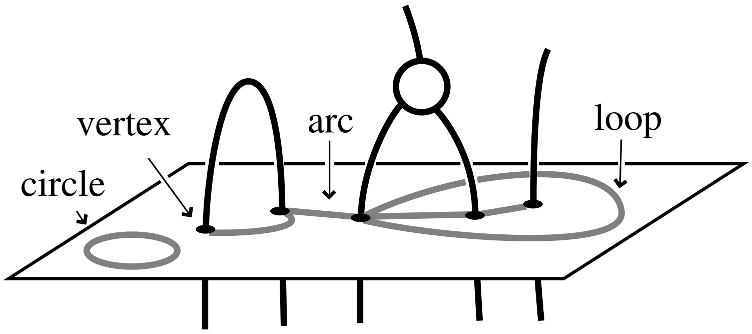

Figure 7: The intersection of $\Gamma \cup \Delta$ with $2\,\mathcal{F}$ is a graph in $2\,\mathcal{F}$.

4.3 Lemma *There are no circles in* G.

Proof This follows from the minimality of $\Delta \cap 2\,\mathcal{F}$, using an innermost disk argument and Lemma 4.1. □

4.4 Lemma [11] *There are no isolated vertices (every vertex belongs to some edge).*

Proof If some vertex is isolated then it defines a compressing disk D for the handlebody $N(\Gamma)$ (or the vertex cuts off a tree, contradicting minimality). Moreover, the boundary of D is disjoint from the complete collection of disks Δ. After compressing the handlebody $\overline{M - N(\Gamma)}$ along Δ we obtain a collection of balls, and ∂D is a loop on the boundary of one of these balls. It therefore also bounds a disk in the handlebody $\overline{M - N(\Gamma)}$. This implies that Γ is the spine of a reducible Heegaard splitting. □

We rely heavily on the notion of outermost edges [11]. Every edge e of G separates some disk $D \subset \Delta$ into two subdisks, D_1 and D_2. If one of the subdisks does not contain any other edges of G then e is called an *outermost edge* of G. Suppose that an edge e is joined to the vertex v and that one of the two subdisks D_1 or D_2 does not contain an edge of G which is joined to v. Then, e is an *outermost edge with respect to* v.

Note that by passing to subdisks, every vertex v has some edge e which is outermost with respect to it. Also, an outermost edge is outermost with respect to its vertices (or vertex), but not (in general) vice-versa.

4.5 Lemma [11] *Let e be an outermost edge with respect to one of its vertices v. Then e is a loop based at v that is essential in $2\,\mathcal{F}$.*

Proof Suppose that e is an arc and joins v to a distinct vertex w. See Figure 8. (The edge e may or may not be outermost for w.) The edge e cuts off a disk $D' \subset \Delta$ which does not contain any edge joined to v. Let M' be the adjacent manifold into which D' starts, and let M'' be adjacent manifold. Let $\gamma \subset \Gamma$ denote the handle containing v.

We will now perform a *broken edge slide* [11] which shows that the intersection is not minimal. See Figure 8. Add a new vertex to γ that lies slightly into M', this breaks γ into two handles, γ_1 and γ_2. Use the disk D' to guide an edge-slide of γ_1, which pulls it back into M''. This edge-slide is permissible precisely because e is outermost for v, we did not ask the handle γ_1 to slide along itself. It does not increase the handle number of $\Gamma \cap M''$, while it strictly decreases the handle number of $\Gamma \cap M'$ (and possibly others, if γ_2 runs through other manifolds). This contradicts the minimality of the intersection between Γ and $2\,\mathcal{F}$.

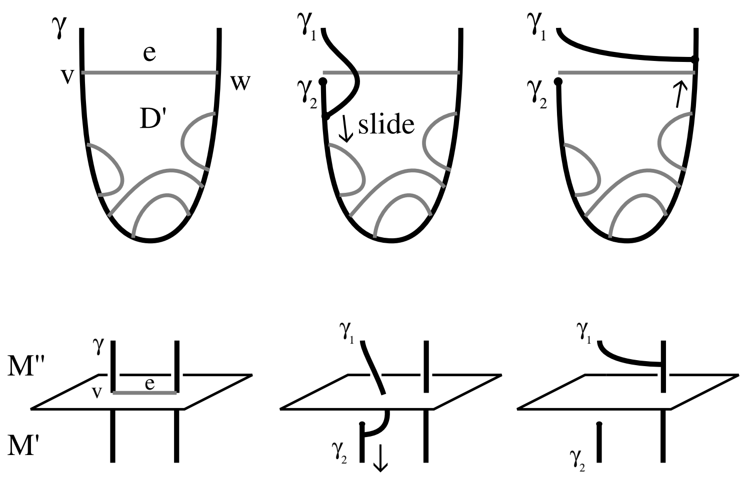

Figure 8: A broken edge slide.

We have established that an outermost edge for a vertex must be a loop. If it were inessential then we can find an innermost inessential loop bounding a disk D. If D contains a vertex v then an outermost edge for v is an arc, contradicting the previous conclusion of this lemma. If D does not contain a vertex, then we can reduce the number of intersections of $\Delta \cap 2\,\mathcal{F}$ by boundary compressing Δ along D. See Figure 9. □

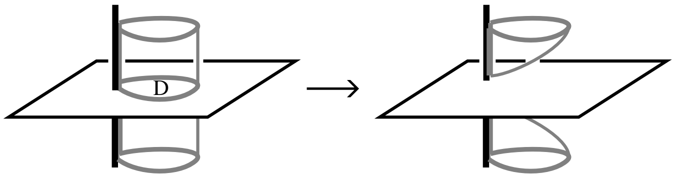

Figure 9: Boundary compressing Δ reduces the number of intersections.

4.6 Lemma *The support of an outermost disk, $supp(d_j)$ is connected, has a single boundary vertex, and $h(supp(d_j)) = g(supp(d_j)) > 0$.*

Proof There is a single edge $e \subset G$ cutting off the outermost disk d_j from Δ. By Lemma 4.5, this edge is an essential loop in some component F of $2\,\mathcal{F}$. This implies that $supp(d_j)$ has a single boundary vertex and is connected. Now, if $h(supp(d_j) = 0$, then the subarc $\alpha = \partial d_j - e \subset \partial N(\Gamma)$ of the boundary of d_j does not cross any compressing disk of $\Gamma \cap M_i$ other than the disk corresponding to the vertex. This means that we can perform edge slides of Γ that allow us to pull the arc α back to F, creating an essential circle of intersection in the process. This is a contradiction, a subdisk of d_j becomes a compressing disk for F, see Figure 10.

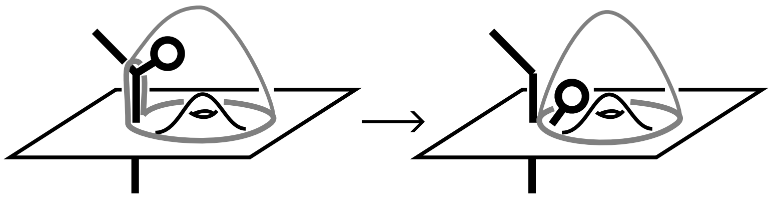

Figure 10: An outermost disk with handle number 0.

Since, $supp(d_j)$ has a single boundary vertex, all of its handles must be realized by genus, i.e., $g(supp(d_j)) = h(supp(d_j))$. □

4.7 Lemma *The support Ω_i of a relatively outermost disk D_i for a component manifold M_i is connected and has $h(\Omega_i) \geq 1$.*

Proof We first show that Ω_i is connected. The boundary of the relatively outermost disk D_i consists of arcs on Ω_i and edges lying in $2\,\mathcal{F}$. Each arc in Ω_i lies in a single component of Ω_i. All but at most one of the edges cuts off a disk which does not return to M_i. Each of these edges is therefore outermost for its vertices, and by Lemma 4.5 an essential loop in some component F of $2\,\mathcal{F}$. Loops do not join distinct components of Ω_i. This means that Ω_i is connected, for any edge leaving a component there must be an additional edge that returns to that component, and we have at most one edge that is not a loop.

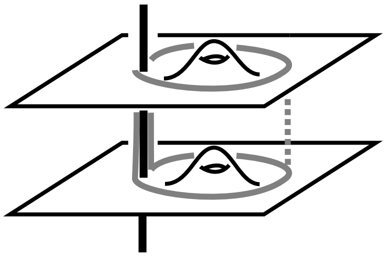

Figure 11: A relatively outermost disk that joins distinct boundary components.

Now, suppose that $h(\Omega_i) = 0$. We know that all but at most one of the edges is a loop. While in general it is possible that the remaining edge e is an edge, this does not occur when $h(\Omega_i) = 0$. A single edge implies that the boundary of D_i joins two distinct vertices in the graph and therefore traverses a handle in the component manifold.

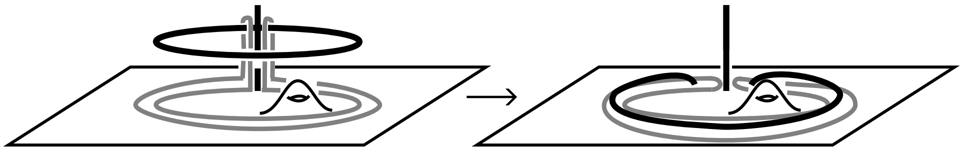

Figure 12: A relatively outermost disk whose support has handle number 0.

Since $h(\Omega_i) = 0$ we may perform edge slides so that a sub disk of D_i intersects some component F of $2\,\mathcal{F}$ in a circle that bounds a disk in M_i. See Figure 12. This may raise the handle number of an adjacent product manifold. Since F is incompressible, the boundary of this disk bounds a disk in F, the two disks bound a ball, and as in Lemma 4.1 we can perform an isotopy of the graph that reduces the induced handle number of the component manifold M_i. This contradicts minimality of the intersection of Γ and $2\,\mathcal{F}$. □

There is one situation contradicting minimality that cannot be detected from the intersection of Γ and $2\,\mathcal{F}$ and the knowledge that an edge is outermost. It is possible that there is a loop based at a vertex v that cuts off a disk lying in a component manifold which runs along a handle exactly once, see for example Figure 13. In this case, the handle can be slid into the product manifold reducing complexity. This is also the motivation for working with the collection $2\mathcal{F}$ instead of $\mathcal{F}$ and choosing our definition of complexity. If we were working with a single copy, $\mathcal{F}$, this move would not decrease complexity, it raises the induced handle number of the adjacent component manifold. This situation will be detected by using the machinery of Casson and Gordon [1] and is analyzed in Section 6.

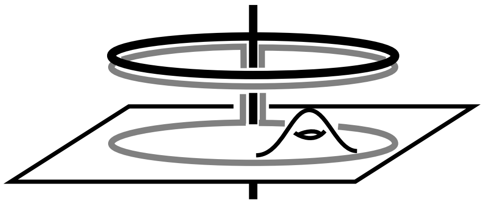

Figure 13: A handle that is parallel to a component of $\mathcal{F}$.

5 Estimating the Genus of Γ

The setup for this section is that of the proof of the main theorem: Γ is the spine of an irreducible Heegaard splitting, Δ is a complete collection of compressing disk for $\overline{M - N(\Gamma)}$, both chosen to intersect $2\mathcal{F}$ minimally; and $\Omega_1, \ldots, \Omega_n$ are the support of relatively outermost disks, $D_1, \ldots, D_n$, for the component manifolds, $M_1, \ldots, M_n$. We demonstrate that the sum of the handle numbers of the supports gives us a lower bound on the genus of Γ.

5.1 Lemma

$$g(\Gamma) \geq \sum_{1}^{n} h(\Omega_i) - n + 2$$

Proof Recall that we have defined

$$\Gamma_k = supp(\{d_i\}_{i=1..k}),$$

where $d_1, \ldots, d_m$ is an outermost ordering of the subdisks of $\Delta - 2\mathcal{F}$. The proof is an inductive one, demonstrating that when $\Gamma_k \subsetneq \Gamma$,

$$g(\Gamma_k) - |\Gamma_k| \geq \sum_{\Omega_i \subset \Gamma_k} (h(\Omega_i) - 1), \tag{1}$$

where $|\Gamma_k|$ denotes the number of components of Γ_k. We then analyze the final attachment, when $\Gamma_k = \Gamma$. Note that at each stage we are attaching some portion of the spine Γ; the right hand side of the inequality can only increase when this portion is actually the support of the relatively outermost disk Ω_i for some component manifold M_i.

Let $k = 1$. The graph Γ_1 is the support of the outermost disk d_1 which is embedded in either a component or product manifold M'. By Lemma 4.6, Γ_1 is connected, has a single boundary vertex, and has positive genus. This means that $g(\Gamma_1) = h(\Gamma_1)$. If M' is a component manifold we have $g(\Gamma_1) - 1 \geq h(\Omega_1) - 1$ and if M' is a product manifold we have $g(\Gamma_1) - 1 \geq 0$. This establishes Inequality 1 for $k = 1$.

Now, suppose that $k > 1$, $\Gamma_k \subsetneq \Gamma$, and that Γ_{k-1} satisfies the inductive hypothesis. If d_k is not an relatively outermost disk for a component manifold, then we merely need to observe that the left-hand side of Inequality 1 does not decrease when we attach $supp(d_k)$. It will decrease only if the number of components increases, which means that some component of $supp(d_k)$ is not attached to Γ_{k-1}. But, this happens only if d_k is an outermost disk, in which case $supp(d_k)$ has a single component and there is an increase of genus to compensate for the additional component.

We are left in the case that d_k is a relatively outermost disk D_i for some component manifold M_i, then $supp(d_k) = \Omega_i$. By Lemma 4.7, Ω_i is connected. Again, if Ω_i is not actually attached to Γ_{k-1}, an additional component is added, but then d_k is actually an outermost disk, $supp(d_k)$ is connected, has positive genus, and $h(\Omega_i) - 1 = g(\Omega_i) - 1$ is added to both sides.

If d_k is a relatively outermost disk D_i for M_i, but not absolutely outermost (for example that in Figure 11), then all but at most one boundary vertex of Ω_i is attached to Γ_{k-1}. As noted in the proof of Lemma 4.7, all but at most one of the vertices of Ω_i, has an outermost loop in G attached to it that cuts off a subdisk of Δ containing only disks with labels d_i, where $i < k$. Each such edge of the disk is attached to a disk with strictly smaller labels, so all but one boundary vertex is attached to Γ_{k-1}.

So, for all but the first vertex attached, each attached vertex adds to the genus by one or reduces the number of components by 1, see Figure 14. Moreover, any genus of Ω_i is added to the genus of Γ_{k-1}. We have added $g(\Omega_i) + |\partial\Omega_i| - 2$ to the left-hand side of 1, this is the same as $h(\Omega_i) - 1$, which is added to the right side.

A similar analysis pertains for the final attachment, when $\Gamma_k = \Gamma$. However, in this case *every* vertex of $supp(d_k)$ is attached to Γ_{k-1}, for there can be no

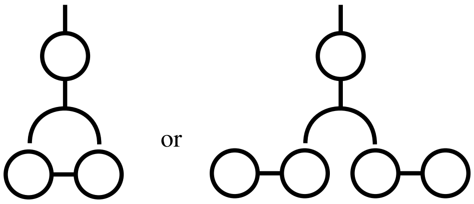

Figure 14: Attaching Ω_i, a relatively outermost graph with handle number 3, adds 2 to $g(\Gamma_{k-1}) - |\Gamma|$.

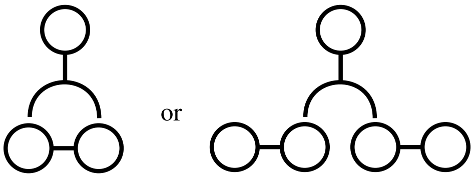

Figure 15: The final attachment.

unattached vertices, see Figure 15. When $supp(d_k)$ is not some Ω_i this adds at least 1 to left side of Inequality 1 and nothing to the right hand side. When $supp(d_k) = \Omega_i$ for some i, this adds $h(\Omega_i)$ to the left hand side and $h(\Omega_i) - 1$ to the right side. In either case, the inequality will still hold even if we add an additional 1 to the right side. This yields

$$g(\Gamma) - |\Gamma| \geq \sum_1^n (h(\Omega_i) - 1) + 1.$$

Since Γ is connected, we have

$$g(\Gamma) \geq \sum_1^n h(\Omega_i) - n + 2. \qquad \square$$

6 Weakly Reducible Heegaard splittings of Manifolds with Boundary

In [1] Casson and Gordon introduced the notion of a weakly reducible Heegaard splitting of a closed 3–manifold, and showed that such a splitting is either reducible or the manifold contains an incompressible surface. We first state and prove an extension of their theorem to manifolds with boundary; a similar theorem was proven by Lustig and Moriah [9]. We will then apply these techniques to the Heegaard splittings of the component manifolds M_i that are induced by the Heegaard spine Γ. These splittings are typically weakly reducible.

First, we introduce some notation. Suppose that $H \subset M$ is a closed embedded surface and $\Delta \subset M$ is an embedded collection of disks so that $\Delta \cap H = \partial \Delta$. Let $\sigma(H, D)$ denote the surface obtained by performing an ambient 1-surgery of H along D (i.e., compression). We use the notion of complexity introduced in [1], the *complexity* of a surface is defined to be,

$$c(surface) = \sum (1 - \chi(S)),$$

where the sum is taken over all non-sphere components of the surface S. Note that if D is a single disk with essential boundary then

$$c(\sigma(H, D)) = \begin{cases} c(H) - 1 & \text{if } D \text{ is separating or compresses a torus} \\ & \text{component of } H\text{, and,} \\ c(H) - 2 & \text{otherwise.} \end{cases}$$

6.1 Theorem *Let M be a compact, orientable, irreducible 3-manifold and $M = C_1 \cup_H C_2$ a Heegaard splitting of M. If $\Delta_0 = \Delta_1 \cup \Delta_2$ is a weak reducing system for the Heegaard splitting then either*

1. *M contains an orientable, non-peripheral incompressible surface S so that $c(S)$ is less than or equal to the complexity of $\sigma(H, \Delta_0)$, or*
2. *there is an embedded collection of disks $\widehat{\Delta_1} \subset C_1$ so that $\Delta_1 \subset \widehat{\Delta_1}$ and some component of $\sigma(H, \widehat{\Delta_1})$ is a Heegaard surface for M, or*
3. *there is an embedded collection of disks $\widehat{\Delta_2} \subset C_2$ so that $\Delta_2 \subset \widehat{\Delta_2}$ and some component of $\sigma(H, \widehat{\Delta_2})$ is a Heegaard surface for M.*

In particular, conclusions (2) and (3) imply that H is not of minimal genus.

Proof Define the complexity of a weak reducing system $\Delta_0 = \Delta_1 \cup \Delta_2$ for the Heegaard splitting $C_1 \cup_H C_2$ to be

$$c(\Delta_0) = c(\sigma(H, \Delta_0)).$$

Let the surfaces $H_i = \sigma(H, \Delta_i), i = 0, 1, 2$, be obtained by compressing H along the corresponding disk collections. See the schematic in Figure 16, it is essential to understanding the arguments of this section. Note that the surface H_0 separates M into two components, denote these by X_1 and X_2. If we compress the compression body C_1 along the disk system Δ_1 we obtain a compression body Y_1 which we will think of as sitting slightly inside X_1. Its complement $X_1 - Y_1$ can be thought of as $(H_1 \times I) \cup N(\Delta_2)$ - a product with 2-handles attached, and is therefore a compression body. Symmetrically, we also have that $X_2 - Y_2$ is a compression body. Thus, the surfaces $H_1 = \partial Y_1$ and $H_2 = \partial Y_2$ are Heegaard surfaces for the (possibly disconnected) manifolds X_1 and X_2, respectively.

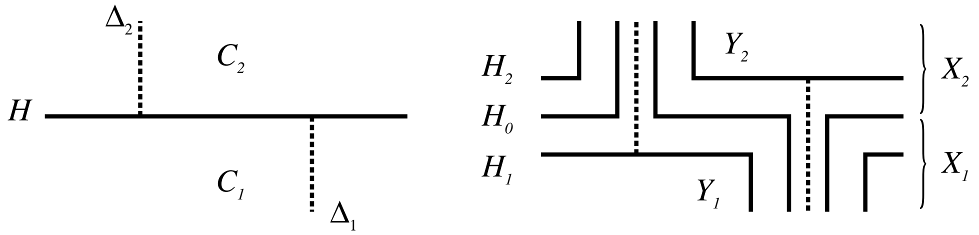

Figure 16: Compressing a Heegaard surface along a weak reducing system.

Suppose that some positive genus component of H_0 is compressible, say into X_1. The compressing disk D for H_0 is a boundary reducing disk for the manifold X_1. As H_1 is a Heegaard surface for X_1, the Haken lemma (see [1]) implies that we may isotope D to intersect H_1 in a single circle. It also says that we may choose a new collection of compressing disks Δ_2' for $X_1 - Y_1$, hence for C_2, which is disjoint from D. The collection $\Delta_0' = (\Delta_1 \cup D) \cup \Delta_2'$ is a weak reducing system with lower complexity than Δ_0 because we have compressed H along an additional disk. A symmetric phenomenon occurs if H_0 is compressible into X_2.

In fact, we may continue to compress H_0, finding new disk collections of strictly decreasing complexity, until each component of H_0 is a 2-sphere or incompressible surface. Denote the final weak reducing system by Δ_0', and the corresponding surfaces and sub-manifolds indicated in Figure 16 by $H_1', X_1', Y_1', \dots$, etc.. Now Δ_0' may or may not contain the original disk collections Δ_1 and Δ_2. However, the compression bodies Y_1' and Y_2' are obtained by compressing the compression bodies Y_1 and Y_2. These in turn were obtained by compressing C_1 and C_2 along the original collections Δ_1 and Δ_2. So we may also think of Y_1' and Y_2' as being obtained by compressing C_1 and C_2 along a collection of disks $\widehat{\Delta_1} \subset C_1$ and $\widehat{\Delta_2} \subset C_2$ where $\Delta_1 \subset \widehat{\Delta_1}$ and $\Delta_2 \subset \widehat{\Delta_2}$. In general $\widehat{\Delta_1}$ and $\widehat{\Delta_2}$ do not have disjoint boundary on H and cannot be taken to be part of a weak reducing system.

If some component S of H_0' is an incompressible and non-peripheral surface, then we have conclusion (1) of the theorem. Moreover, we have that $c(S) \leq$

$c(\sigma(H,\Delta_0))$, for S is a component of H_0' which was obtained by compressing $H_0 = c(\sigma(H,\Delta_0))$.

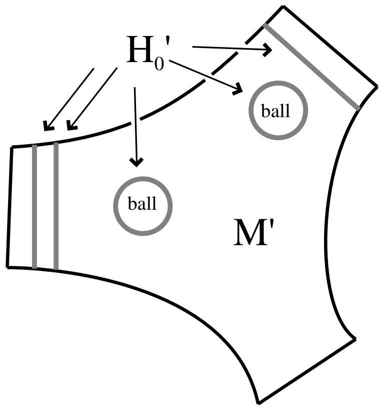

Figure 17: A component M' is essentially the same as M.

We therefore assume that each component of H_0' is a 2-sphere bounding a ball (M is irreducible) or a peripheral incompressible surface. See Figure 17. Then some component of $M - N(H_0')$, call it M', is a copy of M perhaps with some balls and product neighborhoods of boundary components of M removed. Since H_0' separates M into X_1' and X_2', M' must actually be a component of either X_1' or X_2', say X_1'. Recall that the surface H_1' is a Heegaard surface for X_1'. This means that some component $H'' \subset H_1'$ is a Heegaard surface for M'. In fact, H'' is also a Heegaard surface for M; filling in balls and product neighborhoods of the appropriate boundary components does not change the property that H'' bounds compression bodies to both sides. Moreover, the Heegaard surface H'' is a component of the boundary of Y_1' and it follows from our earlier remarks, that it is a component of the surface $\sigma(H,\widehat{\Delta_1})$, where $\Delta_1 \subset \widehat{\Delta_1}$. Symmetrically, if $M' \subset X_2'$ then the Heegaard surface H'' is a component of the surface $\sigma(H,\widehat{\Delta_2})$, where $\Delta_2 \subset \widehat{\Delta_2}$. □

There is one major difference between the case of closed manifolds addressed by Casson and Gordon and the case of bounded manifolds addressed in Theorem 6.1. Conclusions (2) and (3) in the above theorem do *not* imply that the splitting is reducible. A Heegaard splitting defines a partition of the boundary components of the manifold. Reducing (destabilizing) a Heegaard splitting does not change this partition of the boundary components, whereas compression along $\widehat{\Delta_1}$ or $\widehat{\Delta_2}$ may change the partition. This is seen in the following example.

6.2 Example Consider M, the exterior of the three component chain pictured in Figure 18. (The manifold M may also be though of as $P \times S^1$, where P is a pair of pants).

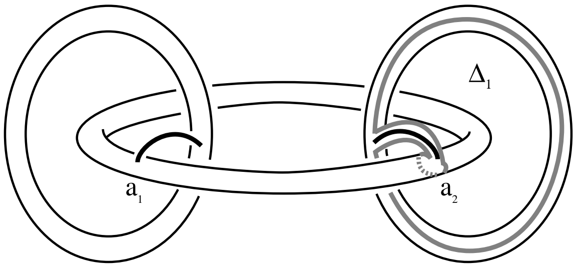

Figure 18: A weakly reducible tunnel system for the exterior of the three component chain.

It is not difficult to see that when a neighborhood of the arcs a_1 and a_2 are tunneled out of M, a handlebody is the result. Thus, $\{a_1 \cup a_2\}$ is a tunnel system for M. Moreover, this system is weakly reducible: let $\Delta_1 = D_1$ be the cocore of a_1 and Δ_2 be the compressing disk for $M - N(a_1 \cup a_2)$ whose boundary is indicated in the figure, running over a_2 twice. Since, M does not contain any closed non-peripheral incompressible surfaces, Theorem 6.1 implies that this splitting can be compressed to a splitting of lower genus. However, the tunnel system $\{a_1 \cup a_2\}$ cannot possibly be reducible, three is the minimal genus of a Heegaard splitting for which all three boundary components of M are contained in the same compression body. In fact, either of the arcs a_1 or a_2 taken alone are a handle system for M. This induces a genus 2 Heegaard splitting of M where one compression body contains two boundary components of M and the other compression body contains one boundary component of M.

We now refine these methods to address the problem outlined in Section 3. The setup is the same as in that section: M is a closed manifold, $\mathcal{F}$ is a complete collection of surfaces, Γ is the spine of a Heegaard splitting that has been arranged to intersect 2 $\mathcal{F}$ minimally, and Ω_i is the support of a relatively outermost disk D_i for some component manifold M_i. The proof uses the notation and closely follows the proof of Theorem 6.1.

6.3 Proposition *If $h(\Omega_i) = 1$ then Ω_i is a handle system for M_i. In particular, $h(M_i) \leq 1$.*

Proof By Theorem 4.2 we know that Γ_i is a tunnel system for M_i; $H = \partial N(\Gamma_i \cup \partial M_i)$ is a Heegaard surface for M_i. We may write $M_i = C_1 \cup_H C_2$

where C_1 is a compression body containing *all* components of ∂M_i and C_2 is a handlebody.

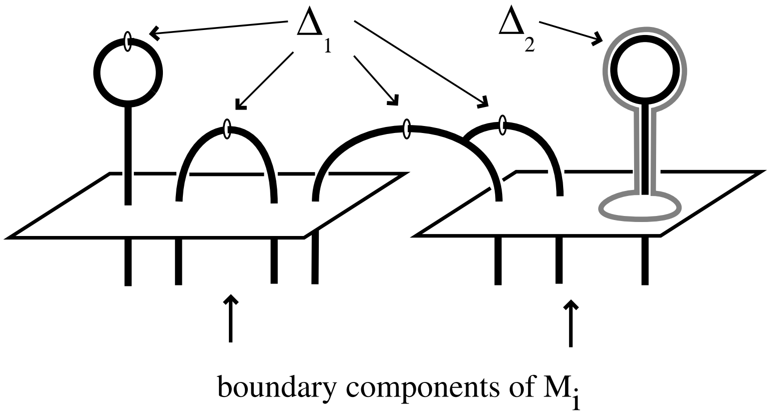

Figure 19: The tunnel system $\Gamma \cap M_i$ is weakly reducible.

If $h(\Gamma_i) = 1$ then $\Omega_i = \Gamma_i$ and the result holds trivially. So we assume that $h(\Gamma_i) > 1$. This implies that $M_i = C_1 \cup_H C_2$ is weakly reducible: let $\Delta_1 \subset C_1$ be a complete collection of compressing disks for $N(\Gamma_i - \Omega_i)$, and $\Delta_2 = D_i$, the relatively outermost disk. See Figure 19. We may choose Δ_1 so that Y_1 does not contain any balls, every component is attached to ∂M.

Following the proof of Theorem 6.1, by further compressing H_0 we obtain a sequence of weak reducing systems, with strictly decreasing complexity. Let $\Delta'_0 = \Delta'_1 \cup \Delta'_2$ be the final disk system; for this system the surface H'_0 consists of 2-spheres and incompressible surfaces.

Claim Every incompressible component of H'_0 is peripheral.

Otherwise, by Theorem 6.1 M_i contains an orientable, non-peripheral incompressible surface S. Recall that H_1 is the surface obtained by compressing H along Δ_1, in this case it consists of copies of boundary components (possibly none), and either one or two boundary components with a handle, $\partial N(\Omega_i)$, attached. To obtain S, we further compress along $\Delta_2 = D_i$ and perhaps along additional disks. Therefore S has genus less than or equal to the genus of a boundary component when Ω_i is attached to a single boundary component, or strictly less than the sum of the genera of two boundary components when Ω_i is attached to two boundary components. This violates our assumption that the decomposition along $\mathcal{F}$ was complete. This completes the proof of the claim.

We conclude, as in Theorem 6.1, that some component $M' \subset X_1'$ or $M' \subset X_2'$ is a copy of M_i with some balls removed.

Claim If $M' \subset X_1'$ then Ω_i is a handle system for M_i.

In this case some component of $H_1' = \partial Y_1'$ is a Heegaard surface for M_i. Recall that Y_1' was obtained by first compressing $C_1 = N(\partial M_i \cup \Gamma_i)$ along Δ_1 yielding $Y_1 = (\partial M_i \times I) \cup N(\Omega_i)$, and then perhaps compressing further. But, the only compressions remaining are along the cocore of $N(\Omega_i)$ or the separating disk which is the double of the cocore (only if Ω_i is attached to a single component of ∂M_i). But, we could not have compressed along either of these: compressing along the cocore leaves $Y_1' = \partial M_i \times I$ whose boundary cannot include a Heegaard surface (M_i is not a compression body), and compressing along the double would imply that the Heegaard surface is the boundary of a solid torus, in particular M_i has genus 1. This is not possible since ∂M_i has positive genus. Thus Δ_0 is the final weak reducing system and $H_1' = H_1$. The Heegaard surface is the boundary of the component of Y_1 that includes $N(\Omega_i)$. In other words, Ω_i is a handle system for M_i. (Recall Figure 16). This completes the claim.

The theorem will follow from the proof of the following claim. In it we argue that in fact, the initial disk system Δ_0 is "almost" the final system Δ_0'. There may be one additional compression, but, it can be controlled.

Claim If $M' \subset X_2'$ then the intersection of Γ with 2 $\mathcal{F}$ is not minimal (a contradiction).

Since M' is isotopic to M (modulo balls), for each boundary component, M' either contains that boundary component or a parallel copy of that boundary component. In fact, each must be a parallel copy: the original Heegaard surface H separated the handlebody C_2 from the boundary components, and then so must H_0' separate X_2' from the boundary components. (Figure 16).

Since $M' \subset X_2'$ we have that $H_0' = \partial X_2'$ contains at least one parallel copy of each boundary component of M_i. It follows that

$$c(H_0') \geq c(\partial M_i).$$

We now show that the surface H_0 contains a parallel copy of each component of ∂M_i. Denote the component(s) to which the handle Ω_i is attached by $\partial_1 M_i$, and the others by $\partial_2 M_i$. The surface H_1 was obtained by compressing along

Δ_1 and therefore contains a copy of each component of $\partial_1 M_i$ and a copy of $\partial N(\Omega_i) \cup \partial_2 M_i$. It follows that

$$c(H_1) = \begin{cases} c(\partial M_i) + 2 & \text{if } |\partial_1 M_i| = 1, \\ c(\partial M_i) + 1 & \text{if } |\partial_1 M_i| = 2. \end{cases}$$

The complexity of H_0 is less by one if $\Delta_2 = D_i$ separates and less by two if $\Delta_2 = D_i$ does not separate. Unless Ω_i is attached to a single component and D_i separates, we have $c(H_0) \leq c(\partial M_i)$. But we know that $c(\partial M) \leq c(H_0') \leq c(H_0)$. This implies that $H_0 = H_0'$ and by previous comments contains a copy of each boundary component.

The remaining case is that $\partial_1 M_i$ is a single component and Δ_2 is separating. In this case H_0 contains $\partial_2 M_i$ and two other components, S' and S'', each with positive genus. In particular, $c(H_0) = c(\partial M_i) + 1$. There can only be one additional compression, say to S'. But this implies that S'' is incompressible and hence is a copy of $\partial_1 M_i$. In this case S'' is a compressible torus.

We now know that the surface H_0 is a single parallel copy of each boundary component and possibly a single compressible torus (see the proof of the above claim). The disks Δ_1 were chosen so that C_1, and hence X_1, contains at most one component for each component of ∂M_i. Therefore the torus, if it exists, is contained in a product neighborhood of the boundary and some component of X_2 is a manifold M' isotopic to M. See Figure 20.

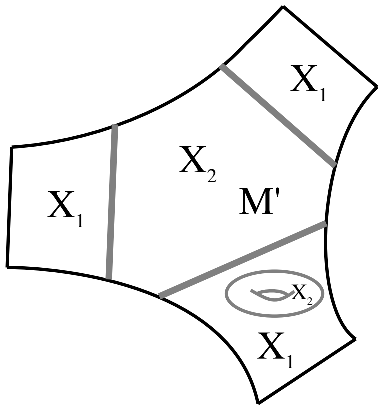

Figure 20: A component of X_2 is isotopic to M.

How does the spine Γ intersect M' ? As M' is a component of X_2 it is formed by attaching the 2-handles $N(\Delta_1)$ to a handlebody C_2 which is disjoint from Γ, again Figure 16. But Γ intersects X_2 in a single arc for each disk in Δ_1 and the number of these is strictly less than $h(\Gamma \cap M_i)$. Then Γ intersects $M' \subset X_2'$ in a subset of these arcs (perhaps all). By shrinking M_i to M' we reduce $h(\Gamma \cap M_i)$ to $|\Delta_1|$ which is less by at least one (the handle Ω_i). This contradicts the fact that $\sum_1^n h(\Gamma \cap M_i)$ was chosen to be minimal. Note that this may increase the handle number of the intersection of Γ with the product manifold adjacent to M_i. Also note that the situation in this claim is precisely the situation that occurs when the outermost disk demonstrates that its support is parallel to the boundary surface, recall Figure 13. This completes the proof of the claim and the theorem. $\square$

Remark It is in the last claim of the proof that we are using the fact that $h(\Omega_i) = 1$. It works because we have one handle, Ω_i, and one compressing disk for the complement, $\Delta_2 = D_i$. This implies that the original disk system Δ_0 is "almost" the final compressing system Δ_0'. If the handle number were greater than one then we would have a discrepancy between the handle number of Ω_i and the number of compressions in Δ_2, we would need to compress C_2 further, and be forced to change from the original disk collection Δ_1 to a new disk collection Δ_1'. We cannot directly conclude that Γ intersects each disk of Δ_1' exactly once and the intersection of Γ with M' might not be lower than that with M_i.

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