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On 4–fold covering moves

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Abstract We prove the existence of a finite set of moves sufficient to relate any two representations of the same 3–manifold as a 4–fold simple branched covering of S^3. We also prove a stabilization result: after adding a fifth trivial sheet two local moves suffice. These results are analogous to results of Piergallini in degree 3 and can be viewed as a second step in a program to establish similar results for arbitrary degree coverings of S^3.

AMS Classification 57M12; 57M25

Keywords Branched covering, covering move, colored braid, colored link, 3–manifold

0 Introduction

It is known that every closed, orientable 3–manifold is a simple branched covering of S^3 and furthermore the degree can be chosen to be any number greater or equal to 3. Such a covering can be represented by a "colored link," that is, a link decorated with transpositions of some symmetric group. Of course the same 3–manifold admits many different representations as a colored link. In this paper we address the question of the existence of a finite set of moves sufficient to connect any two representations of the same 3–manifold as a colored link. We concentrate on coverings of degree 4, and give two answers.

In Theorem 16 we give a set of seven moves sufficient to relate any two presentations of the same manifold as a 4-sheeted simple branched covering of the 3-sphere.

In Theorem 19 we prove that two of those moves, the "local moves," together with a form of stabilization actually suffice.

These results and their proofs are modeled on similar results of Piergallini for branched coverings of degree 3 ([8], [9]). It is unknown at this time if similar results are true for coverings of degree greater than or equal to 5. Extensive computer calculations suggest that this is the case for degree 5 and we conjecture that indeed Theorem 19 is true for arbitrary degree. If this conjecture is true we will have a combinatorial description of 3–manifolds comparable to the one given by Kirby calculus. One application of such a calculus could be in the definition of new 3-manifold invariants or a geometric interpretation of existing ones.

The theorems mentioned above appear in Section 4. The plan of the proofs is the one introduced by Piergallini in [8] and [9]: Given L_1 and L_2, two colored link representations of the same 3–manifold M, first isotope the links so that they are in plat form. Each plat gives a Heegaard splitting of M whose gluing homeomorphism is determined by lifting the braid (viewed as a homeomorphism of the disk). A major part of our investigation therefore is to determine when two braids lift to the same gluing homeomorphism. This is done in Section 3 which culminates in Theorem 15, result that which gives normal generators of the kernel of the lifting homomorphism. The proof of this theorem is very similar to the proof of the corresponding theorem in [4].

In the preceding Section 2, the main technical section, we determine the domain of the lifting homomorphism using the fact that it is the stabilizer of the standard action of the braid group on the set of branched coverings of the 2-dimensional disk. The seemingly intractable calculations are made possible by the following two observations. First, the existence of the exceptional homomorphism $\mathcal{S}_4 \longrightarrow \mathcal{S}_3$, which we call "dimming the lights," means that one only needs to study the action of the 3–fold stabilizer already computed in [4] instead of the action of the full braid group. Second, the two "local moves," can already be considered at this level as moves between braids and therefore one only needs to compute in the "reduced" picture i.e. after taking quotient by the local moves.

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1 Preliminaries

We will always be working in the PL category. Thus in what follows manifold means PL manifold, homeomorphism means PL homeomorphism, submanifold means locally flat submanifold, etc.

We assume that the reader is familiar with the basic theory of branched coverings, especially in dimensions 2 and 3. An excellent background source is [2] and the references contained therein. We also assume the reader is familiar with the contents of [4] as well as with Piergallini's papers regarding covering moves [8], [9]. Finally we mention that the author's thesis [1] contains extensive background material.

A d-fold *branched covering* is a map $p\colon E \to B$ between manifolds, that is a d–fold covering outside a co-dimension 2 submanifold L of B. B is called the *base space*, E the *total space*, d is the degree and L the *branching locus* of p. In general, standard terminology of covering spaces (or more generally fiber bundles) will be used throughout the paper. Maps between branched coverings are defined as the obvious commutative diagrams and we insist that an *isomorphism* covers the identity map of the base space. If B has a basepoint $*$, assumed to lie outside the branching locus, then a *labeled* branched covering is a branched covering p together with a *labeling*, i.e. bijection of the fiber $p^{-1}(*)$ with $\{1, \dots, d\}$, where d is the degree of p. Maps between labeled coverings are required to preserve the labellings.

A covering of B branched over L is determined (up to isomorphism) by the corresponding (unbranched) covering of $B \setminus L$. Therefore isomorphism classes of labeled d-fold coverings of B branched over L are in one one correspondence with homomorphisms $\rho\colon \pi_1(B \setminus L, *) \to \mathcal{S}_d$ from the fundamental group of the complement of L to the symmetric group on d-letters. This correspondence will be referred to as *monodromy.* Quite often a covering will be confused with its monodromy; in particular the same symbol will be used for both.

If $p\colon E \to B$ be a d-fold branched covering and $x \in E$, the *branching index* of x is the "number of sheets coming together" at x, or more formally the degree of p restricted in a sufficiently small neighborhood of x. A branched covering will be called *simple* if the branching index of any point of E is 1 or 2. When B is simply connected the fundamental group $\pi_1(B \setminus L, *)$ is generated by *meridians*, simple closed paths that link L once. Then a branched covering is simple if the monodromy of each meridian is a transposition.

In the absence of an explicit statement to the contrary, "covering" means "simple, labeled, branched covering" throughout the article.

1.1 The Lifting Functor

If $p\colon E \to B$ is a covering branched over L and $f\colon X \to B$ is a homeomorphism then the pullback $f^*(p)$ is a covering of X branched over the preimage $f^{-1}(L)$.

Furthermore the equivalence class of $f^*(p)$ depends only on the isotopy class of f. In particular if L is a fixed co-dimension 2 submanifold of a manifold B, there is an action of the group of isotopy classes of homeomorphisms for (B,L) on the set of isomorphism classes of coverings of B branched over L. We are particularly interested in this action when the base space is the 2-dimensional sphere S^2; it is technically more convenient however to consider coverings over the 2 dimensional disk D^2.

In this case L will consist of a finite set of points, say $L = \{A_0, .., A_{n-1}\}$ and a d–fold (simple) covering ρ branched over L will be described by assigning a transposition $\sigma_i \in \mathcal{S}_d$ to each of the points A_i. Indeed the fundamental group of the punctured disk is freely generated by $\alpha_0, ..., \alpha_{n-1}$, where each α_i is the concatenation of a path from from the basepoint to the boundary of a small enough disc containing A_i but no other branching points in its interior, the boundary of that disc and the inverse of the path. Given such a covering the monodromy σ around the (positively oriented) boundary ∂D^2, will be equal to the product of all the monodromies $\sigma = \sigma_0 \cdots \sigma_{n-1}$. Coverings over the sphere S^2 correspond to coverings of D^2 with boundary monodromy $\sigma = \mathrm{id}$, indeed the total space of such a covering has d boundary components which can be "filled in" by d disks mapping with degree 1 to the base D^2 resulting in a covering of S^2.

It is well known that the mapping class group of $(D^2, L; \mathrm{rel}\, \partial D^2)$ is isomorphic to B_n, the braid group on n strands. The isomorphism is given by assigning to the Artin generator β_i the counterclockwise rotation about an interval x_i with endpoints A_i and A_{i+1}, for $i = 0, \ldots, n-2$. See [3] for details. So there is a (right) action of the braid group on the set of coverings with n branching values. The space of coverings branched over L decomposes into orbits of this action according to the number of connected components of the total space and the boundary monodromy. In particular this action is transitive on the set of coverings of S^2 with connected total space (see for example [2]).

A combinatorial description of the braid group action can be given as follows: Represent ρ as a coloring of L by transpositions as explained above.To see how a braid β acts on ρ draw a diagram of β with its top endpoints coinciding with L, and let the colors of ρ "flow down" through the diagram according to the rule that when a strand passes under another strand its color gets conjugated by the color of the over strand. In this way we get a new coloring of L at the bottom of the diagram. This bottom coloring represents $(\rho)\beta$. Figure 1 shows the action of an Artin generator in the cases that the top colors coincide, "interact," or are disjoint, respectively.

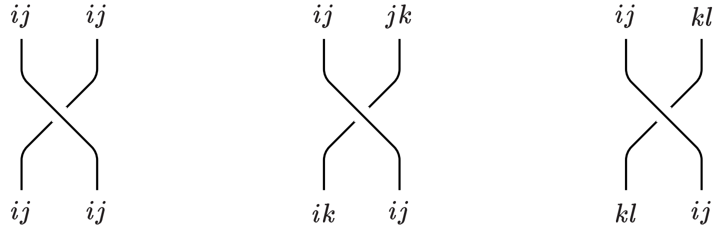


Figure 1: How a generator of B_n acts

The theory of group actions is equivalent to the theory of groupoids. We find it convenient to think of the action of the braid group on sets of branched coverings in this framework. So given a natural number d, one can consider the groupoid of d–colored braids whose objects are colorings of L by transpositions of the symmetric group $\mathcal{S}_d$ and its morphisms are colored braids. There is a *lifting functor* from the groupoid of colored braids to the mapping class groupoid of oriented surfaces. Restricting attention to a fixed object ρ one gets a lifting homomorphism

$$\lambda\colon\ \mathrm{Aut}(\rho) \to \mathfrak{M}(E(\rho)),$$

from the group of automorphisms of ρ (i.e. in the language of group actions the stabilizer of ρ) to the mapping class group of the total space of ρ. Note that the elements of $\mathrm{Aut}(\rho)$ are called *liftable braids* in the literature.

Throughout this paper the Artin generators of the braid group B_n will be denoted by β_i, for $i = 0, \dots, n-2$. Furthermore, for two braids β and γ we define $[\beta]\gamma := \gamma^{-1}\beta\gamma$.

We will denote by $\rho^n(3)$ the "standard" 3-fold covering with n branching points

$$(1,2),(1,2),(2,3),\dots,(2,3) \quad ,$$

while its automorphism group $\mathrm{Aut}(\rho^n)$ will be denoted by $L(n)$. Birman and Wajnryb studied $\rho^n(3)$ in [4] (see also [5]). They found that $L(n)$, is generated by

$$\beta_0, \beta_1^3, \beta_2, \beta_3, \dots, \beta_{n-2}, \quad \text{and if } n \geq 6, \quad \delta_4$$

where $\delta_4 = [\beta_4]\beta_3\beta_2\beta_1^2\beta_2\beta_3^2\beta_2\beta_1$ is the rotation around the interval d_4 shown in Figure 2.

We will refer to the above generators as "the Birman-Wajnryb generators" or simply as "the BW generators."

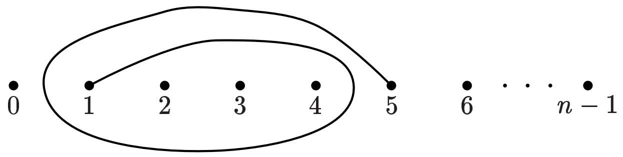


Figure 2: The interval d_4

1.2 Dimming the lights

There is an exact sequence of groups:

$$1 \longrightarrow V \longrightarrow \mathcal{S}_4 \xrightarrow{\kappa} \mathcal{S}_3 \longrightarrow 1$$

where $V = \{\mathrm{id}, (12)(34), (14)(23), (13)(24)\}$ is Klein's four-group. Symmetric groups are generated by transpositions. Therefore, to describe κ it suffices to describe its action on the set of transpositions of $\mathcal{S}_4$. Identify the edges of a numbered tetrahedron with the transpositions of $\mathcal{S}_4$ and the edges of its front face with the transpositions of $\mathcal{S}_3$ (see Figure 3). Then κ fixes the front edges and sends each back edge to its opposite edge.

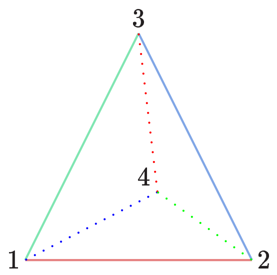


Figure 3: How to see the map κ

It is customary to identify, see [8] for example, the three transpositions of $\mathcal{S}_3$ with three colors Red, Green and Blue. The above exact sequence then suggests the identification of the transpositions of $\mathcal{S}_4$ with three colors that come in light and dark shades. Denoting by $\tilde{X}$ the dark shade of a color X this identification is given explicitly by: $R = (12)$, $\tilde{R} = (34)$, $B = (23)$, $\tilde{B} = (14)$, $G = (13)$ and $\tilde{G} = (24)$. The homomorphism κ can then be described as "dimming the lights" so that the two shades of the same color cannot be distinguished.

There is an induced functor from the groupoid of "bi-tricolored" braids to the groupoid of tricolored braids which will also be referred to as dimming the lights. Dimming the lights obviously commutes with the lifting functor. In particular if ρ is a 4–fold covering then

$$\mathrm{Aut}(\rho) \subset \mathrm{Aut}\,(\kappa(\rho))$$

and furthermore the orbit of ρ under $\mathrm{Aut}\,(\kappa(\rho))$ is contained in $\kappa^{-1}\,(\kappa(\rho))$, the set of 4–fold coverings that upon dimming the lights yield the same 3–fold covering as ρ.

1.3 The reduced groupoid

Consider the moves between colored braids shown in Figure 4, where i, j, k, l are distinct natural numbers. These moves are understood to happen locally: The figure shows only the parts of the braids that are altered by the moves.

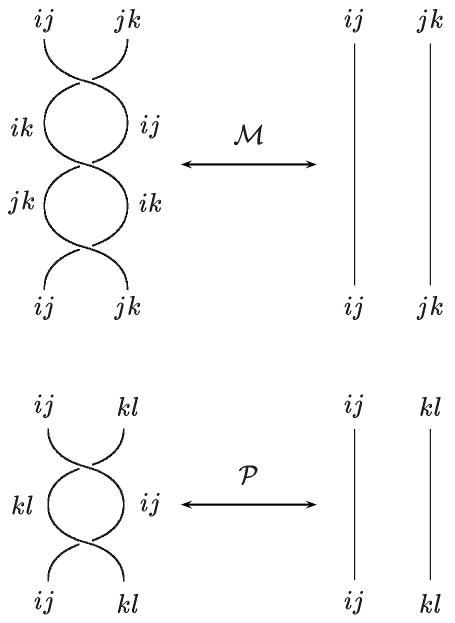


Figure 4: The two local moves

Let $\mathcal{N}$ denote the equivalence relation on (the set of morphisms of) the colored groupoid $\mathcal{B}$ generated by the two moves $\mathcal{M}$ and $\mathcal{P}$, that is two colored braids β_1 and β_2 are equivalent modulo $\mathcal{N}$ if and only if, they can be related by a finite sequence of these moves. By the local nature of the two moves it follows that $\mathcal{N}$ is compatible with the composition of colored braids and thus there exists a quotient groupoid which we call the *reduced groupoid* and denote by $\bar{\mathcal{B}}$. Furthermore it is known that

Lemma 1 *The lifting functor factors through the reduced groupoid.*

Proof It suffices to show that the braids on the left hand side of moves $\mathcal{M}$ and $\mathcal{P}$ lift to identity. Notice that the braid in the left hand side of move $\mathcal{M}$ (respectively $\mathcal{P}$) is the third (respectively, second) power of a rotation around an interval whose endpoints are labeled with interacting (respectively, disjoint) monodromies. Such an interval lifts to a disjoint union of intervals and the third (respectively second) power of the rotation around it lifts to a composition of the rotations around the lifted intervals. Such a composition is isotopic to the identity. See [2] for details. □

The induced functor on $\bar{\mathcal{B}}$ will be denoted by $\bar{\lambda}$ and called the *reduced lifting functor* or, when no confusion is likely, the lifting functor. If ρ is an object of the colored braid groupoid (i.e. a simple covering) we denote by $N(\rho)$ the subgroup of $\mathrm{Aut}(\rho)$ consisting of braids that are $\mathcal{N}$–equivalent to the identity morphism of ρ. Notice that the local nature of the moves $\mathcal{M}$ and $\mathcal{P}$ implies that $N(\rho)$ is a normal subgroup of $\mathrm{Aut}(\rho)$. The quotient group

$$\frac{\mathrm{Aut}(\rho)}{N(\rho)}$$

will be denoted by $\overline{\mathrm{Aut}}(\rho)$ and called the *reduced automorphism group* of ρ. Lemma 1 specializes to the fact that there is a reduced lifting homomorphism

$$\bar{\lambda}(\rho)\colon\ \overline{\mathrm{Aut}}(\rho) \to \mathfrak{M}\left(E(\rho)\right).$$

1.4 Coverings in dimension 3

In the case that the base space is the 3–dimensional sphere S^3 the branching locus will be a *link* L. A d–fold covering branched over L can be presented by assigning to each arc of a diagram of the link a transposition of $\mathcal{S}_d$, in a way compatible with the Wirtinger presentation. This means that at each crossing of the diagram exactly one of the three situations shown in Figure 1 occurs. A link diagram thusly decorated is called a d–colored link (diagram). One can easily trace the effect of each of the Reidemeister moves on a colored link diagram to get a finite set of "colored Reidemeister moves" with the property that two colored link diagrams represent isomorphic coverings if and only if, they can be related by a finite sequence of colored Reidemeister moves. The equivalence generated by colored Reidemeister moves will be called *colored isotopy*. We say that the total space of the covering is *represented* by the colored link. It is well known that every closed, oriented, 3–manifold can be represented as a d–colored link for all $d \geq 3$ (see [6] or [7]).

One way to understand the 3–manifold represented by a colored link is via Heegaard splittings. Represent the link by a plat diagram and split S^3 as the union of a ball B_{up} containing the caps of the plat, a tubular neighborhood of its boundary containing the braid of the plat, and a second ball B_{low} containing the cups of the plat. Each ball lifts to a handlebody and the braid lifts to a homeomorphism between the boundaries of the handlebodies, and so we get a Heegaard splitting of the covering manifold. (See Figure 5, for more details consult the references, for example [8]).

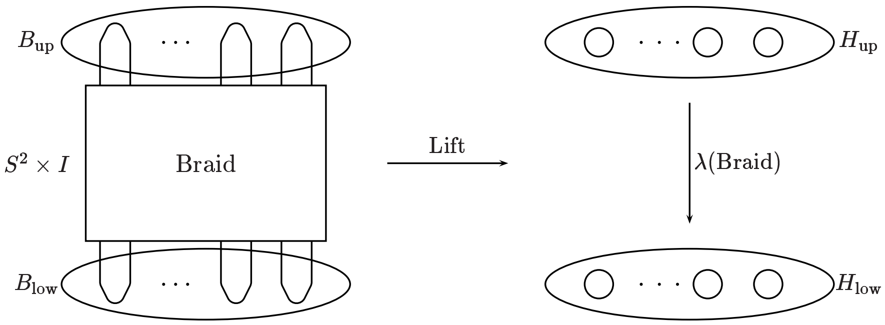


Figure 5: How a colored plat gives a Heegaard splitting

Moves $\mathcal{M}$ and $\mathcal{P}$ of Figure 4 (recall that i, j, k, l are distinct natural numbers) can be regarded as moves on colored links. From now on they will collectively referred to as *the local moves*. Since these two moves "are in the kernel" of the lifting functor it is clear that two colored links that differ by a finite sequence of moves represent the same 3–manifold. There is also a *stabilization* move which does not change the represented manifold but increases the degree of the covering by 1. If L is a d–colored link, stabilization adds an unknotted unlinked component colored with a transposition of $\mathcal{S}_{d+1}$ involving $d+1$. That this move does not change the represented manifold is seen as follows. Let M_d (respectively M_{d+1}) be the total space of the d–fold (respectively $(d+1)$–fold) covering. In M_{d+1} the preimage of S^3 with a ball about the added component removed consists of M_d with d balls removed union S^3 with one ball removed. The preimage of the ball containing the added component consists of $d-1$ balls (filling in $d-1$ of the balls removed from M_d) union $S^2 \times I$, joining the last puncture in M_d to the punctured S^3, thus filling in the last puncture in M_d. This move and its inverse will also be referred to as an *addition* or *deletion of a trivial sheet.*

2 The complex

In this section we study the "presentation complex" associated with the action of $L(n)$, the automorphism group of the standard 3–fold covering $\rho^n(3)$, on $\kappa^{-1}(\rho^n(3))$ the set of 4–fold coverings that upon dimming the lights yield $\rho^n(3)$ (recall Section 1.2). First we establish some notation:

Definition 2 For $\rho \in \kappa^{-1}(\rho^n(3))$ we denote by $\mathcal{C}(\rho)$ the $L(n)$–orbit of ρ. The set of 4-fold, *connected* coverings in $\kappa^{-1}(\rho^n(3))$ that have their first two monodromies equal to $(1,2)$ will be denoted by $\mathcal{C}^n$, that is

$$\mathcal{C}^n := \{\rho \in \kappa^{-1}(\rho^n(3)) : \rho(\alpha_0) = \rho(\alpha_1) = (1,2) \quad \text{and } \rho \text{ is surjective}\}.$$

Also for $\sigma \in \mathcal{S}_n$ we denote by $\mathcal{C}^n_\sigma$ the subset of $\mathcal{C}^n$ consisting of coverings with boundary monodromy equal to σ, that is

$$\mathcal{C}^n_\sigma := \{\rho \in \mathcal{C}^n : \mu(\rho) = \sigma\},$$

where $\mu(\rho) = \rho(\alpha_0\alpha_1\cdots\alpha_{n-1})$ denotes the boundary monodromy of ρ. Finally, elements of $\mathcal{C}^n$ are denoted according to which monodromies are equal to $(2,3)$, or alternatively which monodromies are equal to $(1,4)$:
for $I \subseteq \{2,3,\ldots,n-1\}$ define the 4-fold coverings ρ^n_I, $\tilde{\rho}^n_I$ of D^2 via:

$$\rho^n_I(\alpha_i) := \begin{cases} (12) & \text{if } i = 0,1 \\ (14) & \text{if } i \in I \\ (23) & \text{if } i \notin I \end{cases} \qquad \tilde{\rho}^n_I(\alpha_i) := \begin{cases} (12) & \text{if } i = 0,1 \\ (23) & \text{if } i \in I \\ (14) & \text{if } i \notin I \end{cases}$$

Next we look at the action of the Birman-Wanjryb generators of $L(n)$. The only generator whose action on $\mathcal{C}^n$ is not obvious is δ_4.

Lemma 3 *δ_4 acts on $\mathcal{C}^6$ as follows: it fixes $\rho_\emptyset$, $\tilde{\rho}_\emptyset$, ρ_{ij} and $\tilde{\rho}_{ij}$, while $(\rho_i)\delta_4 = \tilde{\rho}_i$.*

Proof Since $\delta_4 \in L(6)$, it fixes $\rho_\emptyset$ and $\tilde{\rho}_\emptyset$. The remaining cases are easily checked by direct calculation. Two characteristic calculations are shown in Figure 6. ◻

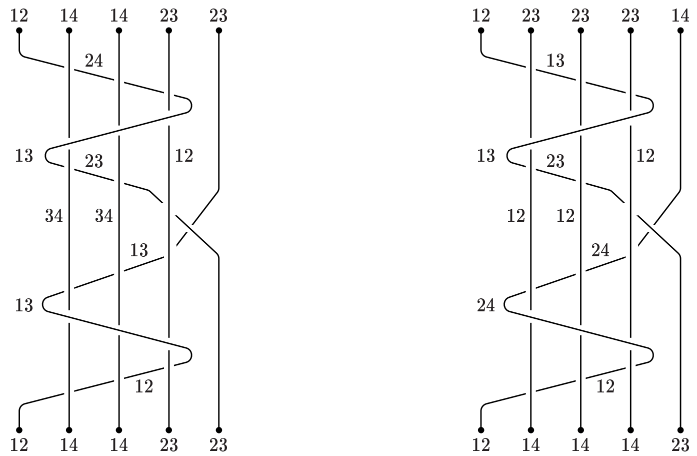


Figure 6: How δ_4 acts

Corollary 4 *The $L(n)$–action preserves $\mathcal{C}^n$.*

Now we can describe the orbits of the $L(n)$–action on $\mathcal{C}^n$:

Proposition 5 *For $\rho \in \mathcal{C}^n$,*

$$\mathcal{C}(\rho) = \mathcal{C}^n_{\mu(\rho)}.$$

Proof Since the boundary monodromy is preserved by the braid action, $\mathcal{C}(\rho) \subset \mathcal{C}^n_{\mu(\rho)}$.
To prove the reverse inclusion notice that for $\rho \in \mathcal{C}^n$, $\mu(\rho)$ is equal to

- id, if n is even, and there is an even number of monodromies equal to $(1,4)$,
- $(1,4)(2,3)$, if n is even and there is an odd number of monodromies equal to $(1,4)$,
- $(2,3)$, if n is odd and there is an even number of monodromies equal to $(1,4)$,
- $(1,4)$, if n is odd and there is an odd number of monodromies equal to $(1,4)$.

To see, for example, that every covering in $\mathcal{C}^n_{\mathrm{id}}$ is in the $L(n)$-orbit of ρ^n_{23} (where, of course, n is even), observe that the subgroup $\langle \beta_2, \cdots, \beta_{n-2} \rangle$ of $L(n)$, acts transitively on the subset of $\mathcal{C}^n_{\mathrm{id}}$ consisting of coverings with any fixed even number of monodromies equal to (14), while δ_4, acting on suitably chosen

coverings, increases or decreases the number of monodromies equal to (14) by two.
The same argument works in the other cases as well. □

Next we enhance the orbits into 2–complexes in a standard way:

Definition 6 For $\rho \in \mathcal{C}^n$ define a 2-complex as follows

- The 0-cells are the elements of $\mathcal{C}(\rho)$,
- the 1-cells are labeled by the Birman-Wajnryb generators of $L(n)$. There is one 1-cell labeled by β for each *unordered* pair $\{\rho', \rho''\} \subseteq \mathcal{C}(\rho)$ with the property $\rho'\beta = \rho''$,
- a path on the 1–skeleton gives a word on the Birman-Wajnryb generators of $L(n)$. There is a 2-cell attached along each closed path whose word lies in $N(\rho')$ for some ρ' in the path.

The resulting 2–complex will be denoted by $\mathbf{C}(\rho)$. In general we will use bold font to signify the transition from $L(n)$–sets to the corresponding complexes.

Note that since all BW generators act as involutions we don't need to consider oriented edges. Clearly the fundamental group $\pi_1(\mathbf{C}(\rho), \rho)$ is isomorphic to $\overline{\mathrm{Aut}}(\rho)$.

We are going to use the following terminology: a *vertex* is a 0–cell, an *edge* is a 1–cell attached to two *distinct* vertices and a *loop* is a 1–cell attached to a single vertex. A *simple* path is a path without closed subpaths. A *lasso* is a closed path of the form wlw^{-1} where l is a loop and w is a simple path, w is the *tail* of the lasso and l is its *head.*

Conjugation by (12)(34) (i.e interchanging the dark and light shades of Blue) defines an involution $\tilde{}$ on $\mathcal{C}^n$. Clearly $\tilde{}$ interchanges $\mathcal{C}^n_{(23)}$ and $\mathcal{C}^n_{(14)}$ and preserves $\mathcal{C}^n_{\mathrm{id}}$ and $\mathcal{C}^n_{(14)(23)}$. In particular $\mathbf{C}^n_{(23)}$ and $\mathbf{C}^n_{(14)}$ are homeomorphic. In what follows we will refer to $\tilde{}$ simply as "symmetry."

Given a covering with n branch values one can "add" one more point with prescribed monodromy at the end to get a covering with $n+1$ branch values. More formally, for each n there are two embeddings:

$$i^n_{(23)} : \mathcal{C}^n \hookrightarrow \mathcal{C}^{n+1},$$

and

$$i^n_{(14)} : \mathcal{C}^n \hookrightarrow \mathcal{C}^{n+1}$$

which are obviously $L(n)$-equivariant if B_n is (as usual) considered to be the subgroup of B_{n+1} that fixes the last point. Therefore these embeddings extend to embeddings of the corresponding complexes which we will still denote by $i^n_{(23)}$ and $i^n_{(14)}$.

Proposition 7 *For even n:*

(a) *At the level of 0–skeletons:*

$$\mathcal{C}^n_{id} = i^{n-1}_{(23)}(\mathcal{C}^{n-1}_{(23)}) \bigsqcup i^{n-1}_{(14)}(\mathcal{C}^{n-1}_{(14)}),$$

and each vertex of $i_{(23)}i_{(14)}(\mathbf{C}^{n-2}_{(14)(23)})$ is joined by an edge labeled by β_{n-2} to the corresponding vertex of $i_{(14)}i_{(23)}(\mathbf{C}^{n-2}_{(14)(23)})$. These are the only new edges of $\mathbf{C}^n_{id}$.

(b) *At the level of 0–skeletons:*

$$\mathcal{C}^n_{(14)(23)} = i^{n-1}_{(23)}(\mathcal{C}^{n-1}_{(14)}) \bigsqcup i^{n-1}_{(14)}(\mathcal{C}^{n-1}_{(23)})$$

and each vertex of $i_{(23)}i_{(14)}(\mathbf{C}^{n-2}_{id} \sqcup \{\rho^{n-2}_{\emptyset}\} \sqcup \{\tilde{\rho}^{n-2}_{\emptyset}\})$ is joined by an edge labeled by β_{n-2} to the corresponding vertex of $i_{(14)}i_{(23)}(\mathbf{C}^{n-2}_{id} \sqcup \{\rho^{n-2}_{\emptyset}\} \sqcup \{\tilde{\rho}^{n-2}_{\emptyset}\})$. These are the only new edges of $\mathbf{C}^n_{(14)(23)}$.

For odd n:

At the level of 0–skeletons

$$\mathcal{C}^n_{(23)} = i^{n-1}_{(23)}(\mathcal{C}^{n-1}_{id} \sqcup \{\tilde{\rho}^{n-1}_{\emptyset}\}) \bigsqcup i^{n-1}_{(14)}(\mathcal{C}^{n-1}_{(14)(23)}),$$

and each vertex of $i_{(23)}i_{(14)}(\mathbf{C}^{n-2}_{(14)} \sqcup \{\tilde{\rho}^{n-2}_{\emptyset}\})$ is joined by an edge labeled by β_{n-2} to the corresponding vertex of $i_{(14)}i_{(23)}(\mathbf{C}^{n-2}_{id} \sqcup \{\tilde{\rho}^{n-2}_{\emptyset}\})$. These are the only new edges of $\mathbf{C}^n_{(23)}$.

Proof The proof is straightforward: remove the last point and examine the result. □

The complexes $\mathbf{C}(\rho)$ have the property that every closed simple path is made up from squares.

Proposition 8 *Every edge of $\mathbf{C}(\rho)$ that is not part of an isolated chain, that is a path of the form $\rho_1 \text{———} \rho_2 \text{———} \cdots \text{———} \rho_m$ where all non displayed 1–cells are loops, is part of a square*

$$\begin{tikzcd}
\rho_1 \arrow[r, dash, "y"] \arrow[d, dash, "x"'] & \rho_2 \arrow[d, dash, "x"] \\
\rho_3 \arrow[r, dash, "y"'] & \rho_4
\end{tikzcd}$$

Every such square bounds a 2–cell.

Proof The first statement follows from the fact that all BW generators act as involutions. For the second statement, observe that for such a square either x and y are non-adjacent β's or one of them is δ_4 and the other is different from β_5. In all these cases, except when $x = \delta_4$ and $y = \beta_4$ (or vice versa), x and y commute in the braid group. In the remaining case, according to Lemma 9, x and y commute modulo the subgroup N. □

Lemma 9 *δ_4 commutes with β_4 modulo the move $\mathcal{M}$.*

Proof This is proven (with a different formulation) in [4], in the middle of page 37. For a pictorial proof consult [1]. □

2.1 Generators

In this section we derive generators for the reduced automorphism group of the standard 4–fold covering ρ^n_{23}. Namely, we prove the following main technical result:

Theorem 10 *For all even $n \geq 6$, $\overline{\mathrm{Aut}}(\rho^n_{23})$ is generated by β_0, β_2, β_4, β_5, $\ldots, \beta_{n-2}$, δ_4 and, if $n \geq 8$, δ_6, where,*

$$\delta_6 = [\delta_4]\beta_5^{-1}\beta_4^{-1}\beta_3^{-1}\beta_2^{-1}\beta_6^{-1}\beta_5^{-1}\beta_4^{-1}\beta_3^{-1}.$$

Note that δ_6 is rotation around the interval d_6 shown in figure 7. Actually the

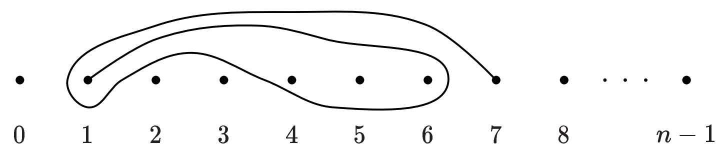


Figure 7: The interval d_6

following stronger statement will be proved:

Theorem 11 (a) *For each $n \geq 6$, $\overline{\mathrm{Aut}}(\rho^n_{23})$ is generated by β_0, β_2, β_4, β_5, $\ldots, \beta_{n-2}$, δ_4 and, if $n \geq 8$, δ_6.*

(b) *For each even $n \geq 6$, $\overline{\mathrm{Aut}}(\rho^n_{2,3,n-1})$ is generated by β_0, β_2, $\beta_4, \ldots, \beta_{n-3}$, $[\beta_{n-2}]\beta_{n-3}^{-1}\cdots\beta_3^{-1}$ and if $n \geq 8$, δ_4 and, if $n \geq 10$, δ_6.*

We start by establishing the following

Lemma 12 *Let σ be any of id, $(2,3)$, $(1,4)(2,3)$. The fundamental group of $\mathbf{C}^n_\sigma$ is generated by lassos. Furthermore the tail of a lasso is not important: two lassos based at the same vertex and having the same head (at the same vertex labeled by the same element) are equal.*

Proof Attach 2–cells to kill all loops in $\mathbf{C}^n_\sigma$. It suffices to prove that the resulting complex is simply connected. Proceed by induction on the number of branch points n.

For $n = 6$ a simple inspection of $\mathcal{C}^6_{\mathrm{id}}$ and $\mathcal{C}^6_{(1,4)(2,3)}$ shown in Figures 8 and 9 confirms the truth of the claim. (Recall Lemma 9.)
For the induction step, notice that by Proposition 7, $\mathbf{C}^n_\sigma$ is the union of simply connected complexes with connected intersection. Thus the claim follows from Van Kampen's theorem. □

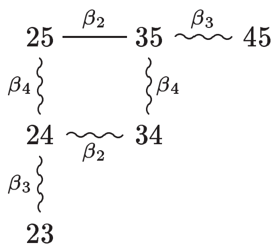


Figure 8: $\mathcal{C}(\rho^6_{23})$ with a maximal tree

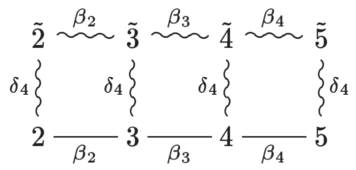


Figure 9: $\mathcal{C}(\tilde{\rho}^6_4)$ with a maximal tree

Proof of Theorem 11 Let G be the subgroup of $\overline{\mathrm{Aut}}(\rho)$ generated by the listed elements. By Lemma 12 it suffices to prove that all lassos belong to G. Proceed by induction on the number of branch points n. For $n = 6$, two simple applications of the Reidemeister-Schreier method using the transversals shown

in Figures 8 and 9 prove the claim.
Assume then that the result has already been proved for all smaller values of n.

If n is odd,

$$\begin{aligned}\mathcal{C}(\rho^n_{23}) &= \mathcal{C}^n_{(23)}\\ &= i_{(23)}(\mathcal{C}^{n-1}_{\mathrm{id}} \sqcup \{\tilde{\rho}^{n-1}_{\emptyset}\}) \bigsqcup i_{(14)}(\mathcal{C}^{n-1}_{(14)(23)})\end{aligned}$$

and the lassos fall into three cases:

Case a Lassos with heads at $i_{(23)}(\mathbf{C}^{n-1}_{\mathrm{id}})$

In this case, by the induction hypothesis, one needs only to check lassos with head labeled by β_{n-2}. After writing

$$\mathcal{C}^{n-1}_{\mathrm{id}} = i_{(23)}(\mathcal{C}^{n-2}_{(23)}) \bigsqcup i_{(14)}(\mathcal{C}^{n-2}_{(14)})$$

one observes that these lassos actually reside in $i_{(23)}i_{(23)}(\mathbf{C}^{n-2}_{(23)})$ and thus their tail can be chosen to commute with β_{n-2}. Therefore, all of these lassos are equal to β_{n-2}.

Case b Lassos with heads at $i_{(14)}(\mathbf{C}^{n-1}_{(14)(23)})$

By the induction hypothesis one needs only to check the lassos with heads at $\rho_{2,3,n-2,n-1}$, the lasso with head at $\rho_{2,n-3,n-2,n-1}$ labeled by β_{n-3}, and the lasso with head at $\rho_{5,6,n-2,n-1}$ labeled by δ_4.

For the lassos with head at $\rho_{2,3,n-2,n-1}$ labeled by β_i with $i = 2, 4, 5, \ldots, n-5$ or by δ_4 one can use the path $\beta_{n-2}\beta_{n-3}$ to pull them at $\rho_{2,3,n-3,n-2} \in i_{(23)}(\mathbf{C}^{n-1}_{\mathrm{id}})$ with the same label and refer to case a). For the lasso with head at $\rho_{2,3,n-2,n-1}$ labeled by β_{n-2} use the path $\beta_{n-2}\beta_{n-3}$ to pull it at $\rho_{2,3,n-3,n-2} \in i_{(23)}(\mathbf{C}^{n-1}_{\mathrm{id}})$ with label β_{n-3} and refer to case a).

For the lasso with head at $\rho_{2,3,n-2,n-1}$ labeled by β_{n-4} one can use the path $\beta_{n-3}\beta_{n-2}\beta_{n-4}\beta_{n-3}$ to pull it at $\rho_{2,3,n-4,n-3} \in i_{(23)}(\mathbf{C}^{n-1}_{\mathrm{id}})$ with label β_{n-2} and refer to case a).

For the lasso with head at $\rho_{2,n-3,n-2,n-1}$ labeled by β_{n-3}, one can use the path $\beta_{n-2}\beta_{n-3}\beta_{n-4}$ to pull it at $\rho_{2,n-4,n-3,n-2} \in i_{(23)}(\mathbf{C}^{n-1}_{\mathrm{id}})$ with label β_{n-4} and refer to case a).

Finally for the lasso with head at $\rho_{5,6,n-2,n-1}$ labeled by δ_4, use the path $\beta_{n-2}\beta_{n-3}$ to pull it at $\rho_{5,6,n-3,n-2} \in i_{(23)}(\mathbf{C}^{n-1}_{\mathrm{id}})$ with label δ_4 and refer to case a).

Case c Lassos with heads at $\tilde{\rho}^n_{n-1}$

For those with heads labeled by δ_4 or β_i with $i = 1, 2, \ldots, n-4$ use β_{n-2} to pull them at $\tilde{\rho}^n_{n-2} \in i_{(14)}(\mathbf{C}^{n-1}_{(14)(23)})$ with the same label, and refer to case b). For the one with head labeled by β_{n-3} use $\beta_{n-3}\beta_{n-2}$ to pull it at $\tilde{\rho}^n_{n-3} \in i_{(14)}(\mathbf{C}^{n-1}_{(14)(23)})$ with label β_{n-2}, and refer to case b).
This concludes the proof in the odd n case.

If n is even we have to check two different boundary monodromies. If the boundary monodromy is equal to identity then

$$\begin{aligned}\mathcal{C}(\rho^n_{23}) &= \mathcal{C}^n_{\mathrm{id}} \\ &= i_{(23)}(\mathcal{C}^{n-1}_{(23)}) \bigsqcup i_{(14)}(\mathcal{C}^{n-1}_{(14)})\end{aligned}$$

and the lassos fall into two cases:

Case a Lassos with heads at $i_{(23)}(\mathbf{C}^{n-1}_{(23)})$.

In this case, by the induction hypothesis, one needs only to check lassos with head labeled by β_{n-2}. After writing

$$\mathcal{C}^{n-1}_{(23)} = i_{(23)}(\mathcal{C}^{n-2}_{\mathrm{id}} \sqcup \{\tilde{\rho}^{n-2}_{\emptyset}\}) \bigsqcup i_{(14)}(\mathcal{C}^{n-2}_{(14)(23)})$$

one observes that those lassos actually reside in $i_{(23)}i_{(23)}(\mathbf{C}^{n-2}_{\mathrm{id}} \sqcup \{\tilde{\rho}^{n-2}_{\emptyset}\})$ and those lassos with heads at a vertex of $i_{(23)}i_{(23)}(\mathbf{C}^{n-2}_{\mathrm{id}})$ labeled by β_{n-2}, are equal to β_{n-2}. So one needs only to check the lasso with head at $\tilde{\rho}^n_{n-2,n-1}$.
If $n = 8$ use the path $w = \beta_5\delta_4\beta_6\beta_5$ to connect ρ^8_{56} to $\tilde{\rho}^8_{56}$ and observe that $w\beta_6 w^{-1} = \delta_4$. Thus the remaining lasso is equal to to the lasso with head at ρ^8_{56} labeled by δ_4. This is by definition equal to δ_6 (since the path $\beta_3\beta_4\beta_5\beta_6\beta_2\beta_3\beta_4\beta_5$ connects ρ^8_{23} to ρ^8_{56}).

For the $n \geq 10$ case we need to consider the following braid which is a generalization of δ_4 and δ_6: for even $n \geq 4$ let

$$\delta_n := [\beta_n]\beta_{n-1}\beta_{n-2}\beta_{n-3}\cdots\beta_1^2\beta_2^{-1}\cdots\beta_{n-4}^{-1}\beta_{n-3}\beta_{n-2}^2\beta_{n-3}\cdots\beta_1.$$

Lemma 13 *δ_{n-2} belongs to G.*

Proof Notice that for $n-2 \geq 6$, δ_{n-2} fixes pointwise a small disc containing A_2 and A_3 but no other branch value. Connect this disc to ∂D^2 by an arc that misses the points A_i, the loops α_i and the intervals x_i. Remove the disc and the arc to get a *3–fold* covering of a disc which is equivalent to $\rho(n-2)$ via an equivalence that sends d_6 to d_4. Therefore δ_{n-2} is a word in the BW generators. Glue the disc and the arc back in to get an expression of δ_{n-2} as a word in the generators of G. □

Returning to the $n \geq 10$ case, the path $w = \delta_{n-2}\beta_{n-4}\beta_{n-5}\beta_{n-3}\beta_{n-4}\beta_{n-2}\beta_{n-3}$ connects $\tilde{\rho}_{n-2,n-1}$ to $\tilde{\rho}^n_{n-5,n-3,n-2,n-1} \in \tilde{\rho}^n_{n-5,n-3,n-2,n-1}$. Now observe that $w\beta_{n-2}w^{-1} = \delta_{n-2}\beta_{n-5}\delta_{n-2}^{-1}$. Therefore, by the induction hypothesis, the lasso with head at $\tilde{\rho}_{n-2,n-1}$ labeled by β_{n-2} is indeed in G.

Case b Lassos with heads at $i_{(14)}(\mathbf{C}^{n-1}_{(14)})$.

By symmetry and Case a) one needs to check only lassos with head at $\tilde{\rho}^n_{23}$ and when $n \geq 10$ the lasso with head at $\tilde{\rho}^n_{56}$ labeled by β_{n-2} (corresponding to δ_6). For the lassos based at $\tilde{\rho}^n_{23}$ and head labeled by β_2, β_4, $\beta_5, \ldots,$ β_{n-2} notice that the path $w = \beta_{n-3}\beta_{n-4}\cdots\beta_2\beta_{n-2}\beta_{n-3}\cdots\beta_3$ connects $\tilde{\rho}^n_{n-2,n-1}$ to $\tilde{\rho}^n_{23}$. Straightforward but tedious calculations then give:

$$w\beta_2 w^{-1} = \beta_{n-2} \quad \text{and,}$$
$$w\beta_i w^{-1} = \beta_{i-2} \quad \text{for} \quad i = 4, \ldots, n-2.$$

All these lassos therefore belong to G.
For the lasso with head at $\tilde{\rho}^n_{56}$ labeled by β_{n-2} notice that the path $w = \beta_{n-2}\beta_{n-3}\cdots\beta_7$ connects $\tilde{\rho}^n_{5,n-1}$ to $\tilde{\rho}^n_{56}$ and that $w\beta_{n-2}w^{-1} = \beta_{n-3}$ and thus, since $\tilde{\rho}^n_{5,n-1} \in i_{(23)}(\mathbf{C}^{n-1}_{(23)})$ this lasso is in G by Case a).
This concludes the proof for $\overline{\mathrm{Aut}}(\rho^n_{23})$ when n is even.

Continuing with n even:

$$\begin{aligned} \mathcal{C}(\rho^n_{2,3,n-1}) &= \mathcal{C}^n_{(23)(14)} \\ &= i_{(23)}(\mathcal{C}^{n-1}_{(14)} \sqcup \{\rho^{n-1}_{\emptyset}\}) \bigsqcup i_{(14)}(\mathcal{C}^{n-1}_{(23)} \sqcup \{\tilde{\rho}^{n-1}_{\emptyset}\}) \end{aligned}$$

and the lassos fall into four cases:

Case a Lassos with heads at $i_{(14)}(\mathbf{C}^{n-1}_{(23)})$.

In this case, by the induction hypothesis, one only needs to check lassos with head labeled by β_{n-2}. But these heads actually lie inside $i_{(14)}i_{(14)}(\mathbf{C}^{n-2}_{(14)(23)})$, which is connected and all of its edges commute with β_{n-2}. It therefore follows that all of these lassos are equal to the lasso with head at $\rho_{2,n-2,n-1}$ labeled by β_{n-2}. Noticing that the path $w = \beta_3\beta_4\cdots\beta_{n-3}$ connects $\rho^n_{2,3,n-1}$ to $\rho^n_{2,n-2,n-1}$, one sees that the later lasso equals $[\beta_{n-2}]\beta_{n-3}^{-1}\cdots\beta_3^{-1}$.

Case b Lassos with heads at ρ^n_{n-1}.

Using as tail the path $w = \beta_{n-2}\beta_{n-3}\cdots\beta_5\delta_4\beta_4\beta_5\cdots\beta_{n-2}$, one (straightforwardly, but tediously) calculates:

$$w\beta_i w^{-1} = \beta_i \quad \text{for} \quad i \neq 3, 4,$$

$$w\beta_4 w^{-1} = \delta_4\,,$$
$$w\delta_4 w^{-1} = \beta_4\,.$$

Finally,

$$\begin{aligned} w\beta_3 w^{-1} &= [\beta_3]\beta_{n-2}^{-1}\beta_{n-3}^{-1}\cdots\beta_5^{-1}\beta_4^{-1}\delta_4^{-1}\beta_5^{-1}\cdots\beta_{n-3}^{-1}\beta_{n-2}^{-1} \\ &= [\beta_3]\beta_4^{-1}\delta_4^{-1}\beta_5^{-1}\cdots\beta_{n-3}^{-1}\beta_{n-2}^{-1} \\ &= [\beta_3]\delta_4^{-1}\beta_4^{-1}\beta_5^{-1}\cdots\beta_{n-3}^{-1}\beta_{n-2}^{-1} \\ &= [\beta_3]\beta_4^{-1}\beta_5^{-1}\cdots\beta_{n-3}^{-1}\beta_{n-2}^{-1} \end{aligned}$$

which, as seen in Figure 10, equals $[\beta_{n-2}]\beta_{n-3}^{-1}\beta_{n-4}^{-1}\cdots\beta_4^{-1}$.

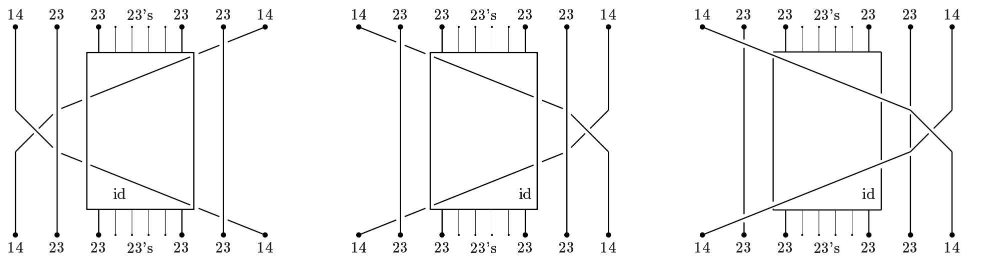


Figure 10: First shift to the right by an isotopy and then use move $\mathcal{P}$ to pass over the strands in the box.

Case c Lassoes with heads at $i_{(23)}(\mathbf{C}_{(14)}^{n-1})$.

By symmetry and Case a) one needs only to check lassoes with head at $\tilde{\rho}_{2,3,n-1}^{n}$, the lasso with head at $\tilde{\rho}_{2,n-2,n-1}$ labelled by β_{n-2} (which corresponds to $[\beta_{n-2}]\beta_{n-3}^{-1}\cdots\beta_3^{-1}$), and for $n \geq 10$ the lasso with head at $\tilde{\rho}_{2,3,n-1}^{n}$ labelled by δ_4 (which corresponds to δ_6).

For the lassos with head at $\tilde{\rho}_{2,3,n-1}^{n}$ labelled by δ_4 or β_i for $i = 2, 4, 5, \ldots, n-4$ notice that β_{n-2} connects $\tilde{\rho}_{2,3,n-1}^{n}$ to $\tilde{\rho}_{2,3,n-2}^{n} \in i_{(14)}(\mathbf{C}_{(12)}^{n-1})$ and so they have been checked in Case a). For the one with head labelled by β_{n-3} "pull further into $i_{(14)}(\mathbf{C}_{(12)}^{n-1})$" using the path $\beta_{n-2}\beta_{n-3}$.

For the lasso with head at $\tilde{\rho}_{2,n-2,n-1}$ labelled by β_{n-2} use again the path $\beta_{n-2}\beta_{n-3}$ to "pull it" to the lasso with head at $\tilde{\rho}_{2,n-3,n-2} \in i_{(14)}(\mathbf{C}_{(12)}^{n-1})$, labelled by β_{n-3}.

Finally for the lasso with head at $\tilde{\rho}_{2,3,n-1}^{n}$ labelled by δ_4 use β_{n-2} to "pull" it to $\tilde{\rho}_{2,3,n-1}^{n} \in i_{(14)}(\mathbf{C}_{(23)}^{n-1})$ with label δ_4.

Case d Lassoes with heads at $\tilde{\rho}_{n-1}^{n}$.

Analogously to Case b), the check for these lassos reduces to Case c).
This conclues the proof for $\overline{\mathrm{Aut}}(\rho^n_{2,3,n-1})$ and the even n case. □

3 The kernel

In this section we compute the kernel of the (reduced) lifting homomorphism for the standard 4–fold covering ρ^n_{23}, which, for simplicity, will be denoted by ρ. The genus of the total space of ρ according to the Riemann-Hurwitz formula (see [2]) is given by $g = \frac{n-6}{2}$.

Theorem 14 *The reduced lifting homomorphism*

$$\bar{\lambda} \colon \overline{\mathrm{Aut}}(\rho) \to \mathfrak{M}_g$$

is given on the generators of Theorem 10 by

$$\bar{\lambda}(x) = \begin{cases} id & \text{if} \quad x = \beta_0, \beta_2 \\ a_{i-1} & \text{if} \quad x = \beta_{2i} \text{ for } i \geq 2 \\ b_{i-1} & \text{if} \quad x = \beta_{2i+1} \text{ with } i \geq 2 \\ a_1 & \text{if} \quad x = \delta_4 \\ d & \text{if} \quad x = \delta_6 \end{cases} \tag{3.1}$$

where a_i, b_i, and d are Wajnryb's generators of $\mathcal{M}_g$, (see [10]).

Proof Since all the generators given in Theorem 10 are rotations around intervals, their image under $\bar{\lambda}$ can be determined by lifting those intervals (see [2]). Figure 11 depicts a model of ρ constructed by "cutting and pasting," the 2–sphere S^2 is cut open along intervals x_i for even i, and then four copies of the cut sphere are glued together according to the specified monodromies. As an example, in Figure 11 the interval δ_4 is shown and how it lifts to the disjoint union of a curve isotopic to a_1 and two arcs. □

Now one can prove the following

Theorem 15 *The kernel of $\bar{\lambda}$ is the smallest normal subgroup of $\overline{\mathrm{Aut}}(\rho_{2g+6})$ containing the elements β_0, β_2, B, D, and all words obtained from B or D by replacing some appearances of β_4 with δ_4, where:*

$$B = (\beta_4\beta_5\beta_6)^4([\delta_6^{-1}]\beta_6^{-1}\beta_5^{-1}\beta_4^{-2}\beta_5^{-1}\beta_6^{-1}\beta_7^{-1})\delta_6^{-1},$$
$$D = \beta_{2g+4}\chi\beta_{2g+4}^{-1}\chi^{-1},$$

where:

$$\chi = \beta_{2g+3}\beta_{2g+2}\cdots\beta_5\beta_4^2\beta_5\cdots\beta_{2g+2}\beta_{2g+3},$$

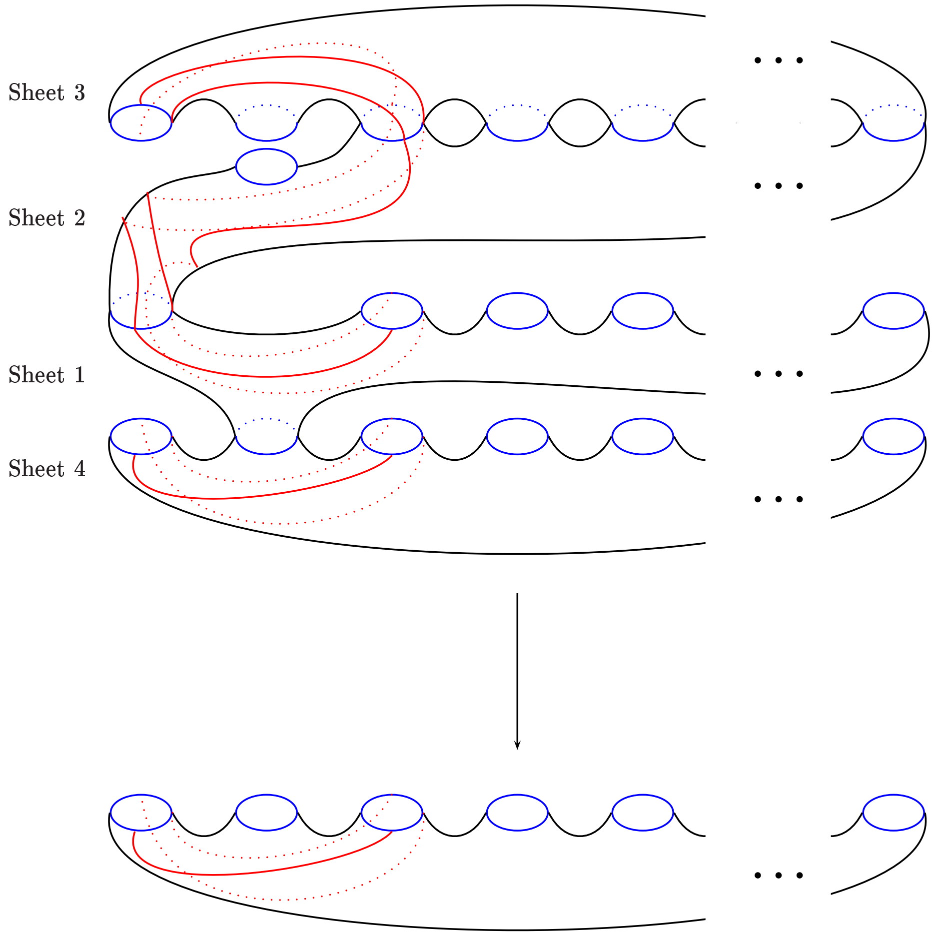


Figure 11: The covering ρ and how d_4 lifts

Proof Given the Wajnryb's presentation of $\mathcal{M}_g$ and the fact that $\bar{\lambda}$ is given by (3.1) the proof reduces to checking that the words obtained by the relators in [10], after replacing the generators of $\mathcal{M}_g$ with their preimages to generators of $\overline{\mathrm{Aut}}(\rho_{2g+6})$ in all possible ways, are in the normal closure of the given elements.

The calculations are identical to those in Theorems 5.1 and 6.1 of [4] after replacing x_i in [4] with β_{i+2} for $i \geq 2$, and δ_4 in [4] with δ_6, and then replacing β_4 with δ_4 in all possible ways. $\square$

4 Moves

In this section we prove the two theorems mentioned in the introduction. Recall that "bi-tricolored" means "colored by transpositions of $\mathcal{S}_4$."

Theorem 16 *Two bi-tricolored links represent the same 3–manifold if and only if they can be related using a finite number of moves from the following list:*

- *bi-tricolored Reidemeister moves*
- *moves $\mathcal{M}$, $\mathcal{P}$, (see Figure 4)*
- *moves I,. . . ,V shown in Figures 12, 13.*

Remark 17 Notice the similarity between moves II, III, and IV and the "homonymous" moves in [8]. This similarity will be exploited in the proof of this and the next theorems.

Proof Since the braids on the left side of moves I-V belong to the kernel of the lifting homeomorphism, these indeed do not change the represented manifold. The proof of the reverse direction is completely analogous to Piergallini's proof of the main theorem of [8]. We just comment on how each step of that proof goes through in the present situation:

- Each bi-tircolored link has a normalized diagram that is a plat diagram whose top and bottom are colored by the standard 4-fold covering. This is easily seen by observing that the proof of the transitivity of the braid action in [2] can be carried out by colored isotopy, much as Piergallini does in the 3-fold case.
- The first step (Heegaard stabilization) can be realized in exactly the same way.
- For the second step (Heegaard equivalence), in order to get braids that lift to the Suzuki generators one just has to add two trivial strands colored by $(1, 4)$ to the braids Piergallini uses.
- For the third step, we have to show that one can add each element of $\ker\bar{\lambda}$ to the top (and bottom) of normalized diagrams using the moves. This will be accomplished if we show that one can insert in a normalized diagram, in any position with the right colors, each of the normal generators of $\ker\bar{\lambda}$ given in Theorem 15.[1]

[1]This proof was corrected 20 January 2004. I am grateful to Riccardo Piergallini for pointing out the mistake.

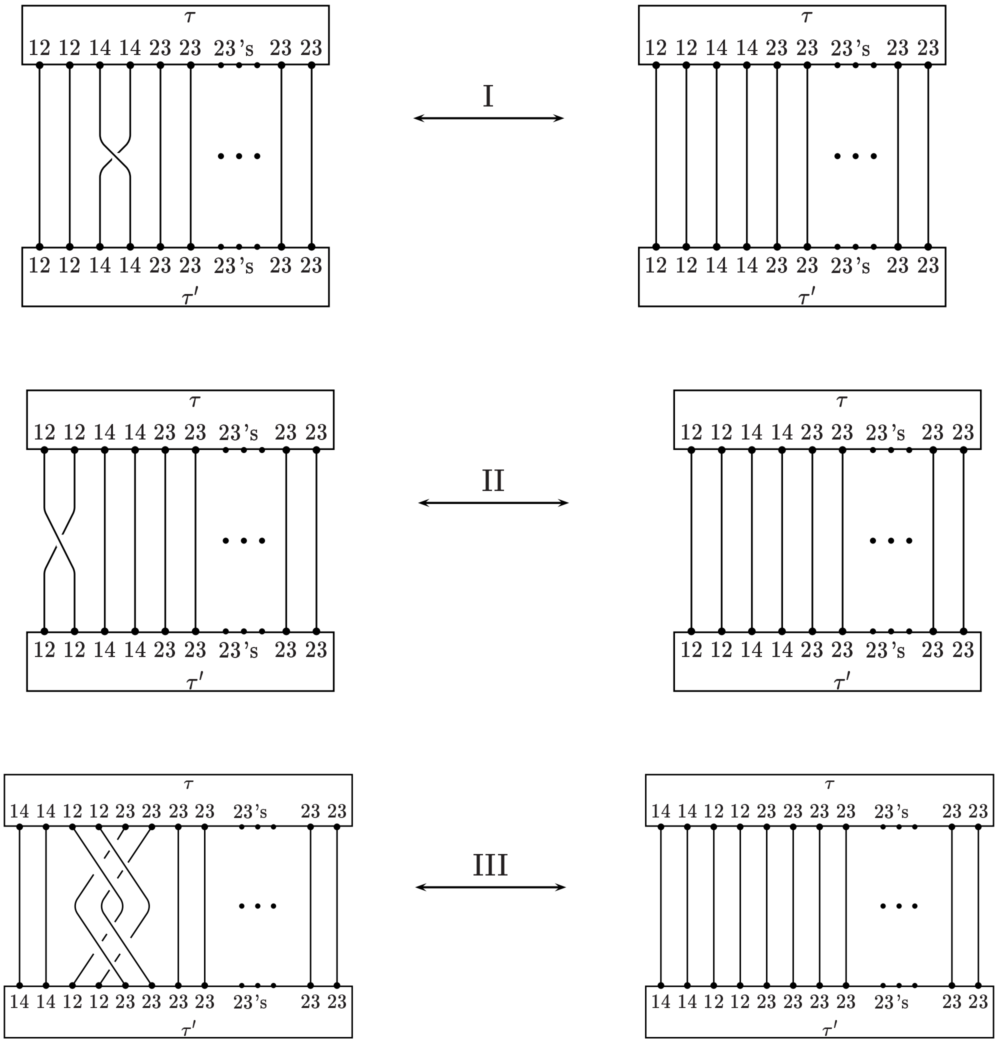


Figure 12: The first three non-local moves

Braids β_0 and β_2 can be obviously added using moves I and II, respectively. Also by slight modification of Piegallini's proof (by just adding two trivial strands colored by $(1,4)$), B_1 and D_1 can be added using moves $\mathcal{M}$, I-IV.
Now move V transforms δ_4 to β_4 and therefore using move V we can transform the remaining normal generators of $\ker \bar{\lambda}$ to B or D (it is eas-

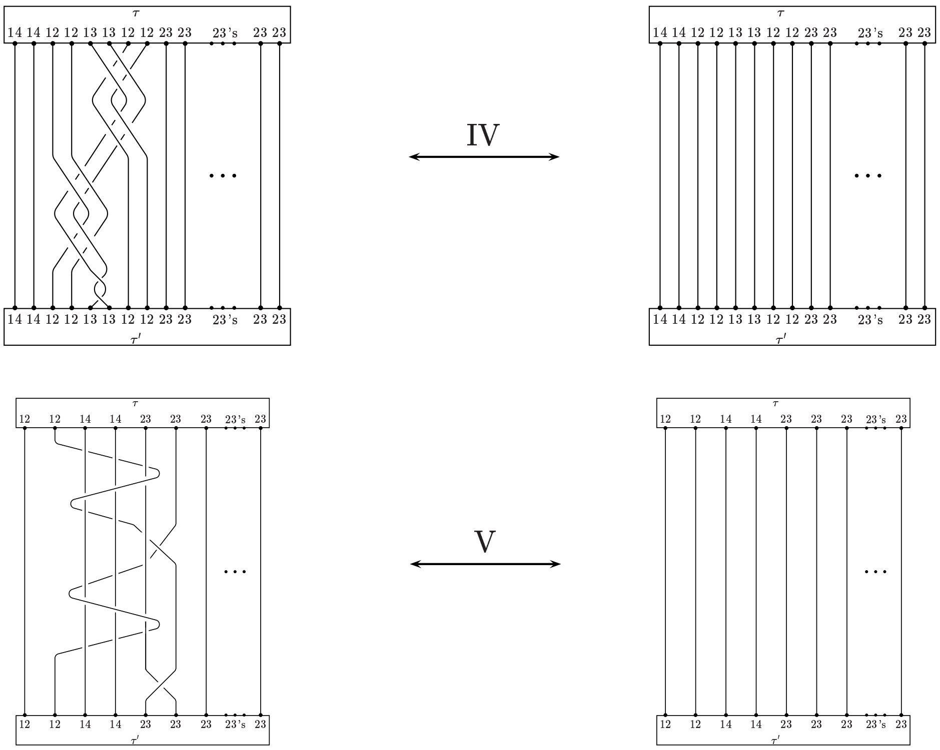


Figure 13: The last two non-local moves

ily checked that the colors are right). Thus any element of $\ker\bar{\lambda}$ can be added on the top and the bottom of a normalized diagram using the given moves. □

The following combination of move $\mathcal{M}$ and isotopy will be used throughout what follows, so we give it a name.

Lemma 18 *The "circumcision" move shown in Figure 14 is a consequence of move* $\mathcal{M}$*.*

Proof Perform $\mathcal{M}$ moves inside the dotted circles and then isotope as in figure 15. □

Theorem 19 *Moves* $\mathcal{M}$*,* $\mathcal{P}$ *and stabilization suffice. Actually one needs only five sheets.*

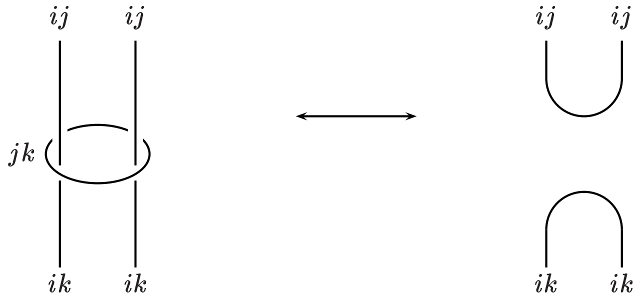


Figure 14: The circumcision move

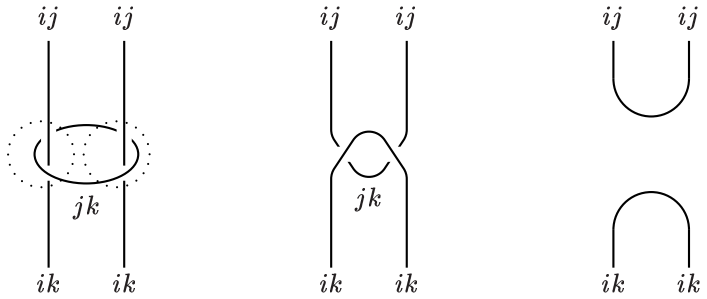
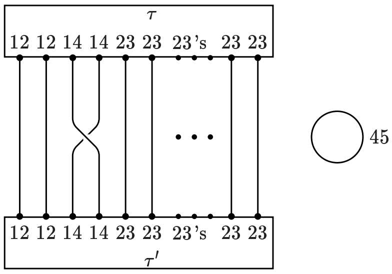
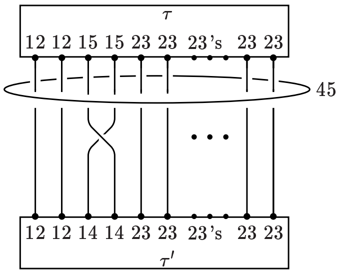


Figure 15: Proof of lemma 18

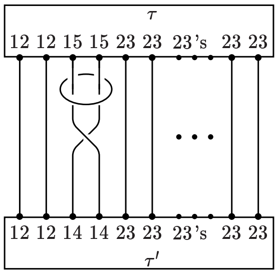
Proof It suffices to show how the non-local moves I,...,V can be realized. For moves I and V this shown in Figures 16 and 17 respectively.
For moves II, III and IV we use the similarity with the homonymous moves of [8]: the proofs for these are very similar to Piergallini's proofs in [9]. The new element is that one can get a circumcising fourth sheet "for free" and thus five sheets suffice. This is done in detail for move II in Figure 18 leaving the remaining cases to the reader. □



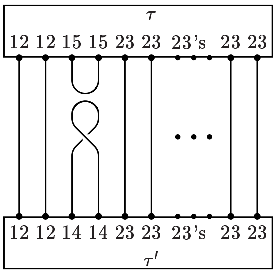
(a) Start by adding a trivial sheet with monodromy $(4,5)$



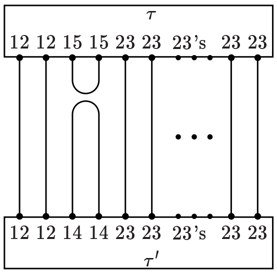
(b) flip it over the whole diagram…



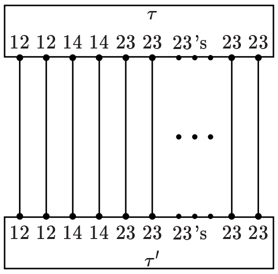
(c) and apply a sequence of $\mathcal{P}$ moves…



(d) circumcise …



(e) and isotope to get rid of the crossing.



(f) Now reverse the process starting at (d) to complete move I.

Figure 16: Proof of Theorem 19 for move I

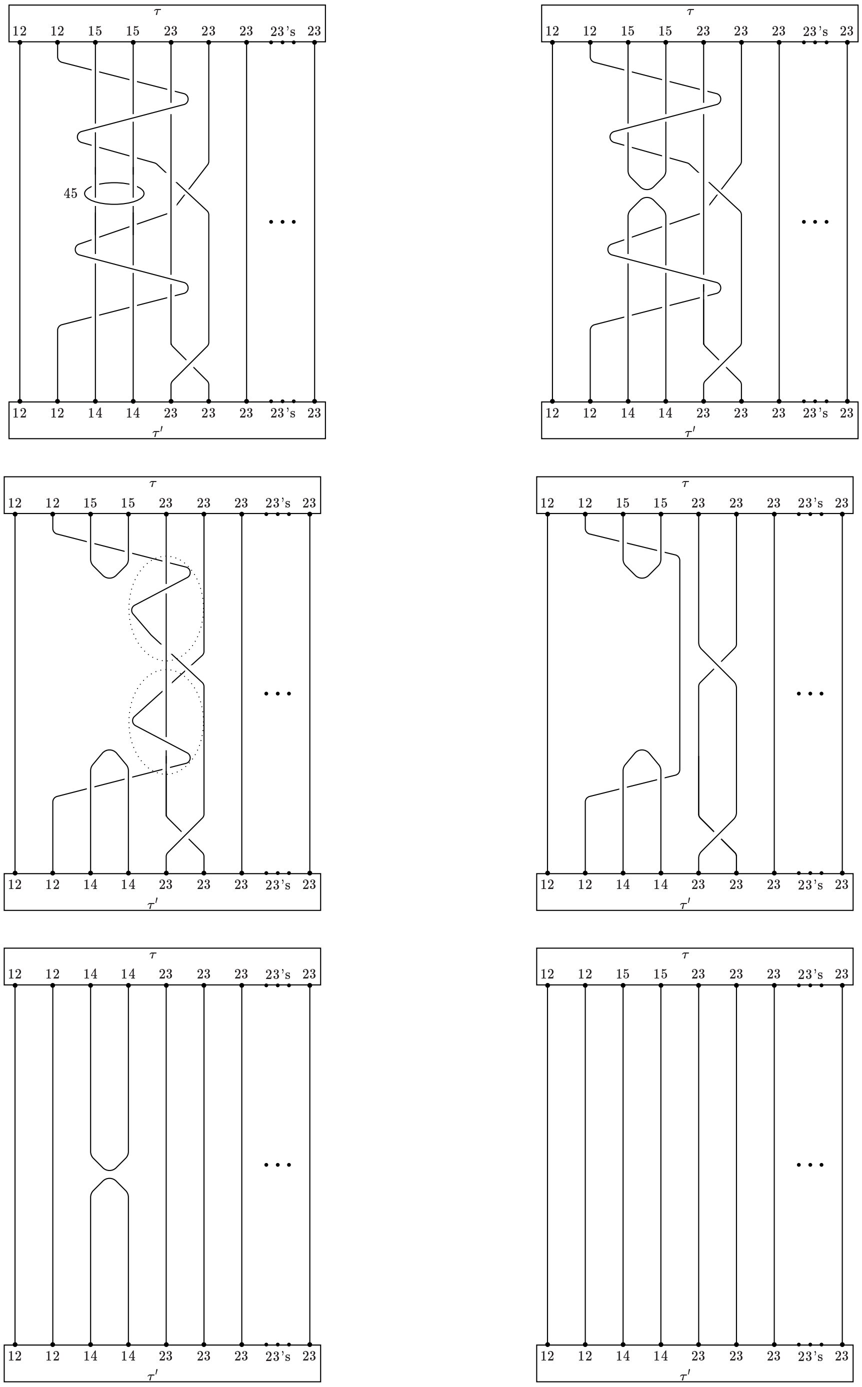
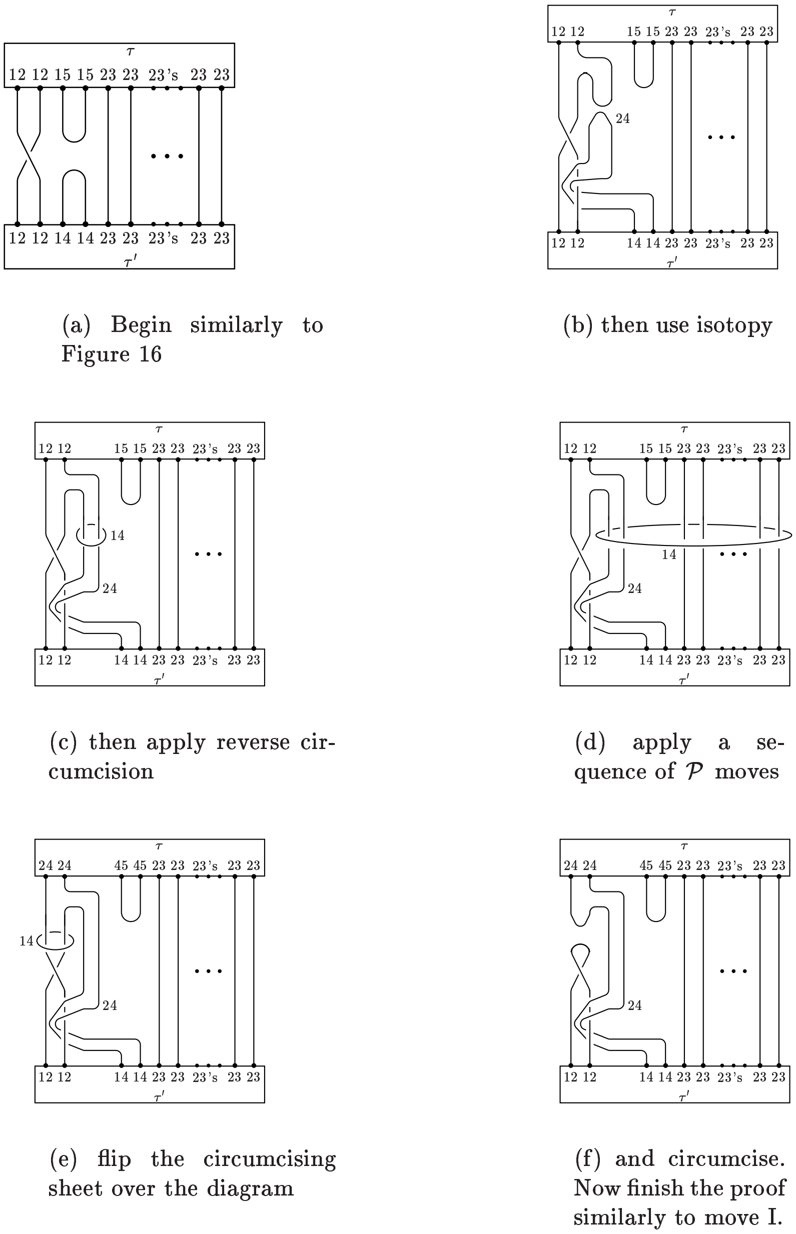


Figure 17: Proof of Theorem 19 for move V



(a) Begin similarly to Figure 16

(b) then use isotopy

(c) then apply reverse circumcision

(d) apply a sequence of $\mathcal{P}$ moves

(e) flip the circumcising sheet over the diagram

(f) and circumcise. Now finish the proof similarly to move I.

Figure 18: Proof of Theorem 19 for move II

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