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An indecomposable PD_3-complex : II

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Abstract We show that there are two homotopy types of PD_3-complexes with fundamental group $S_3 *_{Z/2Z} S_3$, and give explicit constructions for each, which differ only in the attachment of the top cell.

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In [3] we showed that $\pi = S_3 *_{Z/2Z} S_3$ satisfies the criterion of [5] and thus is the fundamental group of a PD_3-complex. As π has infinitely many ends but is indecomposable, this illustrates a divergence from the known properties of 3-manifolds, and provides a counter-example to an old question of Wall [6]. In particular, the Sphere Theorem does not extend to all PD_3-complexes.

Here we shall give an explicit description of a finite PD_3-complex Y realizing this group. The construction is modelled on a similar construction for a PD_3-complex X with fundamental group S_3. In each case the cellular chain complex of the universal cover has the striking property that it is self-dual. In §2 we show a PD_3-complex with fundamental group π must be orientable, and we use Turaev's work to show there are two homotopy types of such PD_3-complexes. The 2-fold cover of Y is homotopy equivalent to $L(3,1)\sharp L(3,1)$, while a simple modification of our construction (suggested by the referee) gives a PD_3-complex with 2-fold cover homotopy equivalent to $L(3,1)\sharp{-L(3,1)}$. (This group was first suggested as a test case in [2].)

1 A finite complex with group $S_3 *_{Z/2Z} S_3$

Let G be a group and let $\Gamma = \mathbb{Z}[G]$, $\varepsilon : C_1 = \Gamma \to \mathbb{Z}$ and $I(G) = \mathrm{Ker}(\varepsilon)$ be the integral group ring, the augmentation homomorphism and the augmentation ideal, respectively. If M is a left Γ-module $\overline{M}$ shall denote the conjugate right module, with G-action given by $m.g = g^{-1}m$ for all $g \in G$ and $m \in M$, and similarly $\overline{N}$ shall denote the conjugate left module structure on a right Γ-module N. If C_* is a chain complex over Γ with an augmentation $\varepsilon : C_0 \to \mathbb{Z}$

a *diagonal approximation* is a chain homomorphism $\Delta : C_* \to C_* \otimes_{\mathbb{Z}} C_*$ (with diagonal G-action) such that $(\varepsilon \otimes 1)\Delta = id_{C_*} = (1 \otimes \varepsilon)\Delta$.

The cellular chain complex $C_*(\widetilde{K})$ for the universal covering space of a finite 2-complex K determined by a presentation for a group is isomorphic to the Fox-Lyndon complex of the presentation, via an isomorphism carrying generators corresponding to based lifts of cells of K to the standard generators.

The symmetric group S_3 has a presentation $\langle a, b \mid a^2, abab^{-2}\rangle$. Let $\pi = S_3 *_{Z/2Z} S_3$, with presentation $\langle a, b, c \mid r, s, t\rangle$, where $r = a^2$, $s = abab^{-2}$ and $t = acac^{-2}$. The two obvious embeddings of S_3 into π admit retractions, as $\pi/\langle\langle b\rangle\rangle \cong \pi/\langle\langle c\rangle\rangle \cong S_3$. Let A, B and C be the cyclic subgroups generated by the images of a, b and c, respectively. The inclusions of A into S_3 and π induce isomorphisms on abelianization, while the commutator subgroups are $S_3' = B$ and $\pi' = B * C$. Thus these groups are semidirect products: $S_3 \cong B \rtimes (Z/2Z)$ and $\pi \cong (B * C) \rtimes Z/2Z$. In particular, π is virtually free, and so has infinitely many ends. However it follows easily from the Grushko-Neumann Theorem that π is indecomposable. (See [3]).

The above presentations determine finite 2-complexes K and L, with fundamental groups S_3 and π, respectively. There are two obvious embeddings of K as a retract in L, with retractions $r_b, r_c : L \to K$ given by collapsing the pair of cells $\{c, t\}$ and $\{b, s\}$, respectively.

The chain complex $C_*(\widetilde{K})$ has the form

$$\mathbb{Z}[S_3]^2 \xrightarrow{\partial_2} \mathbb{Z}[S_3]^2 \xrightarrow{\partial_1} \mathbb{Z}[S_3],$$

where $\partial_1(1,0) = a - 1$, $\partial_1(0,1) = b - 1$, $\partial_2(1,0) = (a+1, 0)$ and $\partial_2(0,1) = (b^2a + 1, a - b - 1)$. The 2-chain $\psi = (a-1, -ba + a + b^2 - b)$ is a 2-cycle, and so determines an element of $\pi_2(K) = H_2(\widetilde{K};\mathbb{Z})$, by the Hurewicz Theorem. Let $X = K \cup_\psi e^3$, and let C_* be the cellular chain complex for the universal cover $\widetilde{X}$. (Thus $C_i = C_i(\widetilde{K})$ for $i \leq 2$ and $C_3 \cong \mathbb{Z}[S_3]$). The dual cochain complex $C^* = Hom_\Gamma(C_*, \mathbb{Z}[S_3])$ is a complex of right $\mathbb{Z}[S_3]$-modules.

We shall define new bases which display the structure of C_* to better advantage, as follows. Let $e_1 = (1,0)$ and $e_2 = (-ba - b^2, 1)$ in C_1 and $f_1 = (1,0)$ and $f_2 = (0, -a)$ in C_2, and let g be the generator of C_3 corresponding to the top cell. Then $\partial_1 e_1 = a - 1$, $\partial_1 e_2 = -b^2a + ba + b^2 - 1$, $\partial_2 f_1 = (a+1)e_1$, $\partial_2 f_2 = (b^2a + a - 1)e_2$, and $\partial_3 g = \psi = (a-1)f_1 + (-b^2a + ba + b - 1)f_2$. The matrix for ∂_2 with respect to the bases $\{\tilde{e}_i\}$ and $\{\tilde{f}_j\}$ is diagonal, and is hermitian with respect to the canonical involution of $\mathbb{Z}[S_3]$, while the matrix for ∂_3 is the conjugate transpose for that of ∂_1. Hence the chain complex $\overline{C^{3-*}}$

obtained by conjugating and reindexing the cochain complex C^* is isomorphic to C_*.

Let $\beta = b^2 + b + 1$ and $\nu = \Sigma_{s \in S_3} s = \beta(a+1)$.

Lemma 1 *The complex X is a PD_3-complex with $\widetilde{X} \simeq S^3$.*

Proof Since C_* is the cellular chain complex of a 1-connected cell complex $H_0(C_*) \cong \mathbb{Z}$ and $H_1(C_*) = 0$. If $\partial_2(rf_1 + sf_2) = 0$ then $r(a+1) = 0$ and $s(b^2a + a - 1) = 0$. Now the left annihilator ideals of $a+1$ and $b^2a + a - 1$ in $\mathbb{Z}[S_3]$ are principal left ideals, generated by $a-1$ and $(b-1)(ba-1)$, respectively. Hence $r = p(a-1)$ and $s = q(b-1)(ba-1)$ for some $p, q \in \mathbb{Z}[B]$. A simple calculation gives $\partial_3((p(ba+b+1) + q(ba+b))g) = rf_1 + sf_2$ and so $H_2(C_*) = 0$.

If $\partial_3 hg = 0$ then $h(a-1) = 0$, so $h = h_1(a+1)$ for some $h_1 \in \mathbb{Z}[B]$, and $h(b^2a - ba - b + 1) = 0$. Now $h(b^2a - ba - b + 1) = h_1(1-b)(a+b+1)$, so $h_1(1-b) = 0$. Therefore $h_1 = m\beta$ for some $m \in \mathbb{Z}$, so $h = m\nu$ and $H_3(C_*) = \mathbb{Z}[S_3]\nu g \cong \mathbb{Z}$. Hence $\widetilde{X} \simeq S^3$. Now $H_3(X;\mathbb{Z}) = H_3(\mathbb{Z} \otimes_{\mathbb{Z}[S_3]} C_*) = \mathbb{Z}[1 \otimes g]$ and $tr([1 \otimes g]) = \nu g$, where $tr : H_3(X;\mathbb{Z}) \to H_3(\widetilde{X};\mathbb{Z})$ is the transfer homomorphism. The homomorphisms from $H^q(\overline{C^*})$ to $H_{3-q}(C_*)$ determined by cap product with $[X] = [1 \otimes g]$ may be identified with the Poincaré duality isomorphisms for $\widetilde{X}$, and so X is a PD_3-complex. $\square$

The verification that $\widetilde{X} \simeq S^3$ is essentially due to [4] and the fact that X is a PD_3-complex is due to [6]. The only novelty here is the diagonalization of ∂_2, which was a guiding feature in the study of $\pi = S_3 *_{Z/2Z} S_3$.

Let $\Pi = \mathbb{Z}[\pi]$. The cellular chain complex for the universal covering space $\widetilde{L}$ has the form

$$\Pi^3 \xrightarrow{\partial_2} \Pi^3 \xrightarrow{\partial_1} \Pi.$$

The differentials are given by $\partial_1(1,0,0) = a - 1$, $\partial_1(0,1,0) = b - 1$ and $\partial_1(0,0,1) = c - 1$, $\partial_2(1,0,0) = (a+1,0,0)$, $\partial_2(0,1,0) = (b^2a + 1, a - b - 1, 0)$ and $\partial_2(0,0,1) = (c^2a + 1, 0, a - c - 1)$. In particular, $H_2(\widetilde{L};\mathbb{Z}) = \mathrm{Ker}(\partial_2)$.

Let $\theta = (a - 1, -ba + a + b^2 - b, -ca + a + c^2 - c)$. Then $\partial_2(\theta) = 0$, and so θ determines an element of $\pi_2(L) = H_2(\widetilde{L};\mathbb{Z})$, by the Hurewicz Theorem. Let $Y = L \cup_\theta e^3$ and let D_* be the cellular chain complex for the universal covering space $\widetilde{Y}$.

Let	Then
$\tilde{e}_1 = (1,0,0)$	$\partial_1\tilde{e}_1 = a-1$
$\tilde{e}_2 = (-ba-b^2,1,0)$	$\partial_1\tilde{e}_2 = ba-b^2a+b^2-1$
$\tilde{e}_3 = (-ca-c^2,0,1)$	$\partial_1\tilde{e}_3 = ca-c^2a+c^2-1$
$\tilde{f}_1 = (1,0,0)$	$\partial_2\tilde{f}_1 = (a+1)\tilde{e}_1$
$\tilde{f}_2 = (0,-a,0)$	$\partial_2\tilde{f}_2 = (b^2a+a-1)\tilde{e}_2$
$\tilde{f}_3 = (0,0,-a)$.	$\partial_2\tilde{f}_3 = (c^2a+a-1)\tilde{e}_3$.

Moreover $\theta = (a-1)\tilde{f}_1 + (-b^2a+ba+b-1)\tilde{f}_2 + (-c^2a+ca+c-1)\tilde{f}_3$. Let $D^* = Hom_\Gamma(D_*,\Pi)$ be the cochain complex dual to D_*. Then it is easily seen that $\overline{D^*} \cong D_{3-*}$.

Theorem 2 *The complex Y is a PD_3-complex.*

Proof Clearly $H_0(D_*) \cong \mathbb{Z}$ and $H_1(D_*) = 0$. The argument of the first part of Lemma 1 extends immediately to show that the kernel of ∂_2 is generated by $(a-1)\tilde{f}_1$, $(b-1)(ba-1)\tilde{f}_2$ and $(c-1)(ca-1)\tilde{f}_3$. Hence these elements represent generators for $H_2(D_*)$. Let $\tilde{g}$ be the generator for D_3 corresponding to the top cell, so that $\partial_3\tilde{g} = \theta$. Note that the image of g in $\mathbb{Z}\otimes_\varepsilon D_3$ is a cycle, and represents a generator for $H_3(Y;\mathbb{Z}) = H_3(\mathbb{Z}\otimes_\varepsilon D_*)$. If $h\theta = 0$ then (as in Lemma 1) $h = h_1(a+1)$ for some $h_1 \in \mathbb{Z}[B*C]$ such that $h_1(b-1) = h_1(c-1) = 0$. It follows that $h_1 = 0$. Hence ∂_3 is injective and so $H_3(D_*) = 0$.

Let $\hat{1}$, $\hat{e}^*$, $\hat{f}^*$ and $\hat{g}$ denote the bases of D^* dual to the above bases for D_*. Let Δ be a diagonal approximation for D_* and suppose that $\Delta(\tilde{g}) = \Sigma_{0\le q\le 3}\Sigma_{i\in I(q)}x_i\otimes y_i$, where $x_i \in D_q$ and $y_i \in D_{3-q}$, for all $i \in I(q)$ and $0 \le q \le 3$. Then $\Sigma_{i\in I(3)}x_i = \tilde{g}$. Let $r_i = \hat{g}(x_i)$ for $i \in I(3)$ and let $\tilde{\xi}$ denote the image of $\tilde{g}$ in $H_3(Y;\mathbb{Z}) = \mathbb{Z}\otimes_\varepsilon D_3$. Then $\varepsilon(\hat{g}\cap\tilde{\xi}) = \varepsilon(\Sigma_{i\in I(3)}\overline{r_i}y_i) = \varepsilon(\Sigma_{i\in I(3)}\overline{r_i}) = \varepsilon(\overline{\hat{g}(\tilde{g})}) = 1$, and so $\hat{g}\cap\tilde{\xi}$ generates $H_0(D_*)$. Since $H_1(D_*) = H_3(D_*) = H^0(\overline{D^*}) = H^2(\overline{D^*}) = 0$, $-\cap\tilde{\xi}$ induces isomorphisms $H^q(\overline{D^*}) \cong H_{3-q}(D_*)$ for all $q \ne 1$. The remaining case follows as in [5] from the facts that $\overline{D^*} \cong D_{3-*}$ and Δ is chain homotopic to $\tau\Delta$, where $\tau : D_*\otimes D_* \to D_*\otimes D_*$ is the transposition defined by $\tau(\alpha\otimes\omega) = (-1)^{pq}\omega\otimes\alpha$ for all $\alpha \in D_p$ and $\omega \in D_q$. Thus Y is a PD_3-complex. □

Can the last step of this argument be made more explicit? The work of Handel [1] on diagonal approximations for dihedral groups may be adapted to give the following formulae for a diagonal approximation for the truncation to degrees ≤ 2 of D_* which is compatible with the above two embeddings of K as a retract in L:

$$
\begin{aligned}
&\Delta(1) = 1\otimes 1\\
&\Delta(\tilde{e}_1) = \tilde{e}_1\otimes a + 1\otimes \tilde{e}_1,\\
&\Delta(\tilde{e_2}) = \tilde{e_2}\otimes 1 - ba\tilde{e_1}\otimes(b-1) - b^2\tilde{e_1}\otimes(b^2a-1) - (ba-b)\otimes ba\tilde{e_1}\\
&\qquad - (b^2-b)\otimes b^2\tilde{e_1} + b\otimes\tilde{e_2},\\
&\Delta(\tilde{e_3}) = \tilde{e_3}\otimes 1 - ca\tilde{e_1}\otimes(c-1) - c^2\tilde{e_1}\otimes(c^2a-1) - (ca-c)\otimes ca\tilde{e_1}\\
&\qquad - (c^2-c)\otimes c^2\tilde{e_1} + c\otimes\tilde{e_3},\\
&\Delta(\tilde{f_1}) = \tilde{f_1}\otimes 1 + \tilde{e}_1\otimes a\tilde{e}_1 + 1\otimes\tilde{f_1},\\
&\Delta(\tilde{f_2}) = \tilde{f_2}\otimes a + (b^2+b)\tilde{f_1}\otimes(a-ba) + (b^2a+b^2)\tilde{f_2}\otimes(a-ba)\\
&\qquad + ((ba+b^2-1)\tilde{e}_1+\tilde{e}_2)\otimes((b^2a)\tilde{e}_1+ba\tilde{e}_2)\\
&\qquad - ((b^2a+1)\tilde{e}_1+ba\tilde{e}_2)\otimes((ba+a+b^2+b)\tilde{e}_1+(b^2a+a)\tilde{e}_2)\\
&\qquad - ((a+b)\tilde{e}_1+b^2a\tilde{e}_2)\otimes((ba+b^2)\tilde{e}_1+a\tilde{e}_2) - (a+1)\tilde{e}_1\otimes\tilde{e}_1\\
&\qquad + (a-b)\otimes(b^2+b)\tilde{f_1} + (a-b)\otimes(b^2a+b^2)\tilde{f_2} + a\otimes\tilde{f_2} \qquad \text{and}\\
&\Delta(\tilde{f_3}) = \tilde{f_3}\otimes a + (c^2+c)\tilde{f_1}\otimes(a-ca) + (c^2a+c^2)\tilde{f_3}\otimes(a-ca)\\
&\qquad + ((ca+c^2-1)\tilde{e}_1+\tilde{e}_3)\otimes((c^2a)\tilde{e}_1+ca\tilde{e}_3)\\
&\qquad - ((c^2a+1)\tilde{e}_1+ca\tilde{e}_3)\otimes((ca+a+c^2+c)\tilde{e}_1+(c^2a+a)\tilde{e}_3)\\
&\qquad - ((a+c)\tilde{e}_1+c^2a\tilde{e}_3)\otimes((ca+c^2)\tilde{e}_1+a\tilde{e}_3) - (a+1)\tilde{e}_1\otimes\tilde{e}_1\\
&\qquad + (a-c)\otimes(c^2+c)\tilde{f_1} + (a-c)\otimes(c^2a+c^2)\tilde{f_3} + a\otimes\tilde{f_3}
\end{aligned}
$$

These formulae were derived from the work of Handel by using the canonical involution of $\mathbb{Z}[S_3]$ to switch right and left module structures and showing that C_* is a direct summand of a truncation of the Wall-Hamada resolution for S_3. (In Handel's notation $a=y$, $b=x$, $e_1=c_1^2$, $e_2=-c_1^1-c_1^2(x+xy)$, $f_1=c_2^3$, $f_2=-c_2^1y+c_2^2x^2-c_2^3y$ and $g=-(c_3^1+c_3^3)(x+y)-c_3^4y)$. Handel's work also leads to a formula for $\Delta(g)$, but it is not clear what $\Delta(\tilde{g})$ should be.

2 Other PD_3-complexes with this group

Having constructed one PD_3-complex with group π one may ask how many there are. Any such PD_3-complex must be orientable. For let $w_1:\pi\to\{\pm1\}$ be a homomorphism and define an involution on Γ by $\bar{g}=w_1(g)g^{-1}$, for all $g\in\pi$. Let $w=w_1(a)$ and $R=\mathbb{Z}[\pi/\pi']=\mathbb{Z}[a]/(a^2-1)$. Let $J=\mathrm{Coker}(\overline{\partial_2}^{tr})$, where $\partial_2 : \Pi^3 \to \Pi^3$ is the presentation matrix for $I(\pi)$ given in §1. Then $R \otimes_\Gamma I(\pi) \cong R/(a+1) \oplus (R/(a+1,3))^2$, while $R \otimes_\Gamma J \cong R/(a+w) \oplus (R/(a+w,3))^2$. If the pair (π, w_1) is realized by a PD_3-complex then $I(\pi)$ and J are projective homotopy equivalent [5]. But then $R \otimes_\Gamma I(\pi)$ and $R \otimes_\Gamma J$ are projective homotopy equivalent R-modules, and so we must have $w = 1$.

If W is an oriented PD_3-complex with fundamental group G and $c_W : W \to K(G,1)$ is a classifying map let $\mu(W) = c_{W*}[W] \in H_3(W;\mathbb{Z})$. Two such PD_3-complexes W_1 and W_2 are homotopy equivalent if and only $\mu(W_1)$ and $\mu(W_2)$ agree up to sign and the action of $Out(G)$. Turaev constructed an isomorphism ν_C from $H_3(G;\mathbb{Z})$ to a group $[F^2(C), I(G)]$ of projective homotopy classes of module homomorphisms and showed that $\mu \in H_3(G;\mathbb{Z})$ is the image of the orientation class of a PD_3-complex if and only if $\nu_C(\mu)$ is the class of a self-homotopy equivalence [5].

When $G = \pi = S_3 *_{Z/2Z} S_3$ we have $F^2(C) \cong I(\pi)$, and $H_3(\pi;\mathbb{Z}) \cong H_3(\pi';\mathbb{Z}) \oplus H_3(Z/2Z;\mathbb{Z}) \cong (Z/3Z)^2 \oplus (Z/2Z)$. Let W' be the double cover of W, with fundamental group $\pi' \cong (Z/3Z) * (Z/3Z)$. Then W' is a connected sum, by Theorem 1 of [5], and so it is homotopy equivalent to one of the 3-manifolds $L(3,1)\sharp L(3,1)$ and $L(3,1)\sharp - L(3,1)$. (These may be distinguished by the torsion linking forms on their first homology groups). In particular, $\mu(W')$ has nonzero entries in each summand. Since $\mu(W')$ is the image of $\mu(W)$ under the transfer to $H_3(\pi';\mathbb{Z}) \cong (Z/3Z)^2$ the image of $\mu(W)$ in each $Z/3Z$-summand must be nonzero. Let $u \in H^1(W;\mathbb{F}_2)$ correspond to the abelianization homomorphism. Since $\beta_2(W;\mathbb{F}_2) = \beta_1(W;\mathbb{F}_2) = 1 = \beta_2(\pi;\mathbb{F}_2)$ we have $u^2 \neq 0$, and so $u^3 \neq 0$, by Poincaré duality. It follows easily that the image of $\mu(W)$ in the $Z/2Z$-summand must be nonzero also. (Note that W' is $\mathbb{Z}_{(2)}$-homology equivalent to S^3 and so W is $\mathbb{Z}_{(2)}[Z/2Z]$-homology equivalent to RP^3). Since reversing the orientation of W reverses that of W', we may conclude that there are at most two distinct homotopy types of PD_3-complexes with fundamental group π, and that they may be detected by their double covers.

The retractions r_b and r_c of L onto K extend to maps $r_b, r_c : Y \to X$. These maps induce the same isomorphism $H_3(Y;\mathbb{Z}) \cong H_3(X;\mathbb{Z})$, and so their lifts to the double covers induce the same isomorphism $H_3(Y';\mathbb{Z}) \to H_3(X';\mathbb{Z})$. Hence $Y' \simeq L(3,1)\sharp L(3,1)$, rather than $L(3,1)\sharp{-L(3,1)}$. The referee has pointed out that if we use $\xi = (a-1)\tilde{f}_1 + (-b^2a + ba + b - 1)\tilde{f}_2 - (-c^2a + ca + c - 1)\tilde{f}_3$ instead of θ (changing only the sign of the final term) then $Z = L \cup_\xi e^3$ is another PD_3-complex with $\pi_1(Z) \cong \pi$, and a similar argument shows that the double cover is now $Z' \simeq L(3,1)\sharp{-L(3,1)}$.

The question of whether every aspherical PD_3-complex is homotopy equivalent to a 3-manifold remains open. The recent article [7] gives a comprehensive survey of Poincaré duality in dimension 3, emphasizing the role of the JSJ decomposition in relation to this question.

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References

[1] Handel, D. On products in the cohomology of the dihedral groups, Tôhoku Math. J. 45 (1993), 13-42.

[2] Hillman, J.A. On 3-dimensional Poincaré duality complexes and 2-knot groups, Math. Proc. Cambridge Phil. Soc. 114 (1993), 215-218.

[3] Hillman, J.A. An indecomposable PD_3-complex whose group has infinitely many ends, Math. Proc. Cambridge Phil. Soc., to appear (2005).

[4] Swan, R.G. Periodic resolutions for finite groups. Ann. of Math. 72 (1960), 267-291.

[5] Turaev, V.G. Three-dimensional Poincaré complexes: homotopy classification and splitting. (Russian) Mat. Sb. 180 (1989) 809–830. (Math. USSR-Sb. 67 (1990), 261-282.)

[6] Wall, C.T.C. Poincaré complexes: I. Ann. of Math. 86 (1967), 213-245.

[7] Wall, C.T.C. Poincaré duality in dimension 3, in *Proceedings of the Casson Fest (Arkansas and Texas 2003)* (edited by C.McA.Gordon and Y.Rieck), Geom. Topol. Monogr. 7 (2004), 1-26.

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